Linear Programming: Geometry, Duality, and Sensitivity Analysis

Varun Kumar
Master of Operational Research
University of Delhi

1 Introduction

Linear programming (LP) is a fundamental methodology in optimization and operational research with applications in resource allocation, production planning, and logistics. This report presents a concise study of a two-variable LP problem, emphasizing geometric interpretation, primal–dual theory, and sensitivity analysis. The focus is on conceptual understanding supported by computational verification.

2 Problem Formulation

Consider the following linear programming problem:

$$\begin{aligned} \text{Maximize} \quad & Z = 3x_1 + 5x_2 && (1)\\ \text{subject to} \quad & x_1 + 2x_2 \le 8, && (2)\\ & 3x_1 + 2x_2 \le 12, && (3)\\ & x_1, x_2 \ge 0. && (4) \end{aligned}$$

This problem admits a graphical solution due to the presence of two decision variables.

3 Detailed Solution

3.1 Graphical Solution

To obtain the feasible region, each inequality constraint is first expressed as an equality.

The graphical representation of the feasible region and the optimal solution is shown in Figure 1.

Intercepts

For the constraint $x_1 + 2x_2 = 8$:

$$(x_1, x_2) = (0, 4),\ (8, 0).$$

For the constraint $3x_1 + 2x_2 = 12$:

$$(x_1, x_2) = (0, 6),\ (4, 0).$$

Since all constraints are of the form "$\le$" and the variables are restricted to be non-negative, the feasible region lies in the first quadrant below both lines. (See the corresponding graphical plot generated in the accompanying computational notebook.)

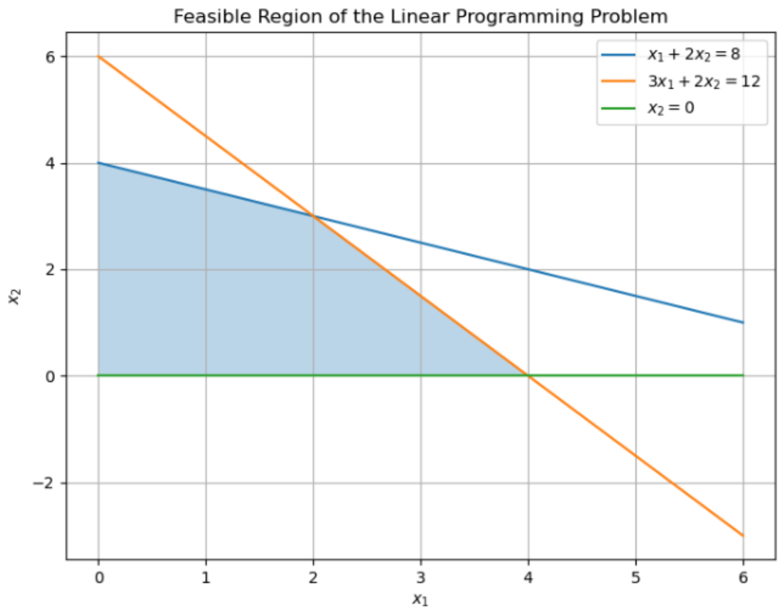

Figure 1: Feasible region defined by the constraints and the optimal point $(2, 3)$.

Corner Points

The extreme points of the feasible region are:

$$(0, 0), \quad (0, 4), \quad (4, 0),$$

and the intersection of the two constraint lines.

Solving

$$x_1 + 2x_2 = 8, \tag{5}$$
$$3x_1 + 2x_2 = 12, \tag{6}$$

subtracting the first equation from the second gives

$$2x_1 = 4 \Rightarrow x_1 = 2.$$

Substituting into the first equation yields

$$2 + 2x_2 = 8 \Rightarrow x_2 = 3.$$

Hence, the intersection point is $(2, 3)$.

Evaluation of the Objective Function

Point	$Z = 3x_1 + 5x_2$
$(0, 0)$	0
$(0, 4)$	20
$(4, 0)$	12
$(2, 3)$	21

The maximum value is attained at $(x_1, x_2) = (2, 3)$, with

$$Z^* = 21.$$

3.2 Primal–Dual Formulation

The primal problem in matrix form:

$$\max\ c^T x \quad \text{subject to } Ax \le b,\ x \ge 0,$$

where

$$c = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 8 \\ 12 \end{bmatrix}.$$

The corresponding dual problem is

$$\begin{aligned} \text{Minimize} \quad W &= 8y_1 + 12y_2 && (7) \\ \text{subject to} \quad y_1 + 3y_2 &\ge 3, && (8) \\ 2y_1 + 2y_2 &\ge 5, && (9) \\ y_1, y_2 &\ge 0. && (10) \end{aligned}$$

3.3 Solution of the Dual Problem

Dividing the second constraint by 2 yields

$$y_1 + y_2 \ge 2.5.$$

Solving

$$y_1 + 3y_2 = 3, \tag{11}$$
$$y_1 + y_2 = 2.5, \tag{12}$$

we obtain

$$y_2 = 0.25, \quad y_1 = 2.25.$$

The dual objective value is therefore

$$W^* = 8(2.25) + 12(0.25) = 21.$$

3.4 Strong Duality and Complementary Slackness

Since

$$Z^* = W^* = 21,$$

the strong duality theorem is verified.

Both primal constraints are binding at $(2, 3)$, and both dual variables are positive:

$$y_1 = 2.25 > 0, \quad y_2 = 0.25 > 0.$$

Thus, the complementary slackness conditions are satisfied.

3.5 Sensitivity Interpretation

The dual variables represent shadow prices of the constraints. An increase of one unit in the right-hand side of the first constraint increases the optimal objective value by approximately 2.25, while a similar increase in the second constraint increases it by approximately 0.25, within allowable ranges.

These results demonstrate how resource availability influences the optimal objective value and highlight the economic interpretation of dual variables as the marginal worth of resources.

3.6 Final Solution

$$(x_1^*, x_2^*) = (2, 3), \quad Z^* = 21,$$
$$(y_1^*, y_2^*) = (2.25, 0.25), \quad W^* = 21.$$

Thus, both graphical and analytical approaches confirm the same optimal solution, validating the theoretical results through computational verification.