GATE 2022 -AE 63

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Question: Which one of the following is the closed form for the generating function of the sequence $\{a\}_{n\geq 0}$ defined below?

$$a_n = \begin{cases} n+1 & \text{, n is odd} \\ 1 & \text{otherwise} \end{cases} \tag{1}$$

(A) $\frac{x(1+x)^2}{\left(1-x^2\right)^2} + \frac{1}{1-x}$
(B) $\frac{x\left(3-x^2\right)}{\left(1-x^2\right)^2} + \frac{1}{1-x}$
(C) $\frac{2x}{\left(1-x^2\right)^2} + \frac{1}{1-x}$
(D) $\frac{x}{\left(1-x^2\right)^2} + \frac{1}{1-x}$

(GATE CS 2022 QUESTION 36)

Solution:

Parameter	Description	Value
$G(S;x)$	Generating function for a sequence S	$G(A;x)$
$G(a;x)$	Generating function for a sequence a	?
a_n	n^{th} term of the sequence	$n+1$ (when odd)
		1 (when even)

TABLE I
INPUT VALUES

For a sequence($\{a\}$) generating function is defined as,

$$G(a;x) = \sum_{n=0}^{\infty} a_n x^n \tag{2}$$

$$\implies G(a;x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \tag{3}$$

For the given sequence,

$$G(a;x) = 1 + (1+1)x + x^2 + (1+3)x^3 + \dots \tag{4}$$

$$\implies G(a;x) = \left(1 + x^2 + x^4 + \dots\right) + \left(2x + 4x^3 + 6x^5 + \dots\right) \tag{5}$$

$$\implies G(a;x) = \frac{1}{1-x^2} + \frac{d}{dx}\left(x^2 + x^4 + x^6 \dots\right) \tag{6}$$

$$\implies G(a;x) = \frac{1}{1-x^2} + \frac{d}{dx}\left(\frac{x^2}{1-x^2}\right) \tag{7}$$

$$\implies G(a;x) = \frac{1}{1-x^2} + \frac{2x}{1-x^2} \tag{8}$$

$$\implies G(a;x) = \frac{1}{1-x^2} + 2x\left(\frac{1}{1-x^2} + \frac{x^2}{1-x^2}\right) \tag{9}$$

$$\implies G(a;x) = \frac{1+x}{1-x^2} + \frac{x}{1-x^2} + \frac{2x^3}{\left(1-x^2\right)^2} \tag{10}$$

$$\therefore G(a;x) = \frac{1}{1-x} + x\frac{1+x^2}{\left(1-x^2\right)^2} \tag{11}$$

(11) is the closed form of generating function required in the question.
Hence, option (A) is correct.