

GATE 2022 -AE 63

EE23BTECH11057 - Shakunaveti Sai Sri Ram Varun

Question: Which one of the following is the closed form for the generating function of the sequence $\{a_n\}_{n \geq 0}$ defined below?

$$a_n = \begin{cases} n+1 & , n \text{ is odd} \\ 1 & \text{otherwise} \end{cases} \quad (1)$$

(A) $\frac{x(1+x)^2}{(1-x^2)^2} + \frac{1}{1-x}$

(B) $\frac{x(3-x^2)}{(1-x^2)^2} + \frac{1}{1-x}$

(C) $\frac{2x}{(1-x^2)^2} + \frac{1}{1-x}$

(D) $\frac{x}{(1-x^2)^2} + \frac{1}{1-x}$

(GATE CS 2022 QUESTION 36)

Solution:

For the given sequence:

Parameter	Description	Value
$X(z)$	Generating function for a sequence $\{a_n\}$?
a_n	n^{th} term of the sequence	$(n+1)u(n)$ (when odd) $u(n)$ (when even)

TABLE I
INPUT VALUES

$$X(z) = \sum_{k=-\infty}^{\infty} u(2k) z^{-2k} + \sum_{k=-\infty}^{\infty} ((2k+2)u(2k+1)) z^{-(2k+1)} \quad (2)$$

$$\Rightarrow X(z) = (1 + z^{-2} + z^{-4} + \dots) + (2z^{-1} + 4z^{-3} + 6z^{-5} + \dots) \quad (3)$$

$$\Rightarrow X(z) = \frac{1}{1-z^{-2}} + (2z^{-1} + 4z^{-3} + 6z^{-5} \dots) \quad (4)$$

$$\Rightarrow X(z) = \frac{1}{1-z^{-2}} + 2z^{-1} \left(\frac{1}{1-z^{-2}} + \frac{z^{-2}}{(1-z^{-2})^2} \right) \quad (5)$$

$$\therefore X(z) = \frac{1}{1-z^{-1}} + \frac{z^{-1}(1+z^{-2})}{(1-z^{-2})^2} \quad (6)$$

(6) is the closed form of generating function required in the question.

Hence, option (A) is correct.

$$X(z) = X_1(z) + X_2(z) \quad (7)$$

$$X_1(z) = \frac{1}{1-z^{-1}} \quad (8)$$

$$\Rightarrow x_1(n) = \frac{1}{1-z^{-1}} \quad (9)$$

$$\Rightarrow a_n = x_1(n) + x_2(n) \quad (10)$$

To find inverse z-transform of $X_2(z)$ we use contour integration technique:

$$x_2(n) = \frac{1}{2\pi j} \oint_C x_2(A; z^{-1}) z^{n-1} dz \quad (11)$$

$$= \frac{1}{2\pi j} \oint_C \frac{z^n (z^2 + 1)}{(z^2 - 1)^2} dz \quad (12)$$

We can observe that we have two poles at $z = 1, -1$. And poles are repeated twice, thus by applying residue theorem two times for poles 1 and -1:

$$x_2(n) = \frac{1}{(1)!} \lim_{z \rightarrow 1} \frac{d}{dz} \left((z-1)^2 X_2(z) \right) + \frac{1}{(1)!} \lim_{z \rightarrow -1} \frac{d}{dz} \left((z+1)^2 X_2(z) \right) \quad (13)$$

$$\Rightarrow x_2(n) = \lim_{z \rightarrow 1} \frac{d}{dz} \left((z-1)^2 \frac{z^n (z^2 + 1)}{(z^2 - 1)^2} \right) + \lim_{z \rightarrow -1} \frac{d}{dz} \left((z+1)^2 \frac{z^n (z^2 + 1)}{(z^2 - 1)^2} \right) \quad (14)$$

$$\begin{aligned} \Rightarrow x_2(n) &= \lim_{z \rightarrow 1} \frac{(z+1)^2 (nz^{n-1} + (n+2)z^{n+1}) - 2z^n (1+z^2)(z+1)}{(z+1)^4} \\ &\quad + \lim_{z \rightarrow -1} \frac{(z-1)^2 (nz^{n-1} + (n+2)z^{n+1}) - 2z^n (1+z^2)(z-1)}{(z-1)^4} \end{aligned} \quad (15)$$

on simplification, we get

$$x_2(n) = \frac{n + n(-1)^{n-1}}{2} \quad (16)$$

$$\therefore a_n = u(n) + \frac{n + n(-1)^{n-1}}{2} u(n) \quad (17)$$

Which is the sequence given in the Question.