GATE 2022 -AE 63

EE23BTECH11057 - Shakunaveti Sai Sri Ram Varun

Question: Which one of the following is the closed form for the generating function of the sequence $\{a\}_{n\geq 0}$ defined below?

$$a_n = \begin{cases} n+1 & \text{,n is odd} \\ 1 & \text{otherwise} \end{cases} \tag{1}$$

(A) $\frac{x(1+x)^2}{\left(1-x^2\right)^2} + \frac{1}{1-x}$
(B) $\frac{x\left(3-x^2\right)}{\left(1-x^2\right)^2} + \frac{1}{1-x}$
(C) $\frac{2x}{\left(1-x^2\right)^2} + \frac{1}{1-x}$
(D) $\frac{x}{\left(1-x^2\right)^2} + \frac{1}{1-x}$

(GATE CS 2022 QUESTION 36)

Solution:
For any given sequence $\{s_n\}$ generating function is

Parameter	Description	Value
$G\left(S;z^{-1}\right)$	Generating function for a sequence $\{s_n\}$	$G\left(A;z^{-1}\right)$
$G\left(A;z^{-1}\right)$	Generating function for a sequence $\{a_n\}$	?
a_n	n^{th} term of the sequence	$(n+1)\,u(n)$ (when odd)
		$u(n)$ (when even)

TABLE I
INPUT VALUES

defined as:

$$G\left(S;z^{-1}\right) = \sum_{n=0}^{\infty} s_n z^{-n} \tag{2}$$

For the given sequence:

$$G\left(A;z^{-1}\right) = \sum_{k=-\infty}^{\infty} u(2k)\, z^{-2k} + \sum_{k=-\infty}^{\infty} \left((2k+2)\,u(2k+1)\right) z^{-(2k+1)} \tag{3}$$

$$\implies G\left(A;z^{-1}\right) = \left(1+z^{-2}+z^{-4}+\dots\right) + \left(2z^{-1}+4z^{-3}+6z^{-5}+\dots\right) \tag{4}$$

$$\implies G\left(A;z^{-1}\right) = \frac{1}{1-z^{-2}} + \left(2z^{-1}+4z^{-3}+6z^{-5}\dots\right) \tag{5}$$

$$\implies G\left(A;z^{-1}\right) = \frac{1}{1-z^{-2}} + 2z^{-1}\left(\frac{1}{1-z^{-2}} + \frac{z^{-2}}{(1-z^{-2})^2}\right) \tag{6}$$

$$\therefore G\left(A;z^{-1}\right) = \frac{1}{1-z^{-1}} + \frac{z^{-1}\left(1+z^{-2}\right)}{(1-z^{-2})^2} \tag{7}$$

(7) is the closed form of generating function required in the question.
Hence, option (A) is correct. Taking it into partial fractions