GATE 2022 -AE 63

EE23BTECH11057 - Shakunaveti Sai Sri Ram Varun

Question: A two degree of freedom spring-mass system undergoing free vibration with generalized coordinates x_1 and x_2 has natural frequencies $\omega_1 = 233.9$ rad/s and $\omega_2 = 324.5$ rad/s , respectively. The corresponding mode shapes $\phi_1 = \begin{bmatrix} 1 \\ -3.16 \end{bmatrix}$ and $\phi_2 = \begin{bmatrix} 1 \\ 3.16 \end{bmatrix}$. If the system is disturbed with certain deflections and zero initial velocities, then which of the following statement(s) is/are true?

(A) An initial deflection of $x_1(0) = 6.32$cm and $x_2(0) = -3.16$cm would make the system oscillate with only the second natural frequency.
(B) An initial deflection of $x_1(0) = 2$cm and $x_2(0) = -6.32$cm would make the system oscillate with only the first natural frequency.
(C) An initial deflection of $x_1(0) = 2$cm and $x_2(0) = -2$cm would make the system oscillate with linear combination of first and second natural frequency.
(D) An initial deflection of $x_1(0) = 1$cm and $x_2(0) = -6.32$cm would make the system oscillate with only the first natural frequency.

(GATE AE 2021 QUESTION 32)

Solution:

Parameter	Description	Value
m_1, m_2	mass of block attached to springs	m_1
k_1, k_c, k_2	spring constants of springs	k_1, k_c, k_2
$x_1(0)$	Initial vibration of first spring	?
$x_2(0)$	Initial vibration of second spring	?
A_{11}, A_{12}	Amplitudes of block 1 under natural conditions	?
A_{21}, A_{22}	Amplitudes of block 2 under natural conditions	?
ω_1	First natural frequency of the system	233.9 rad/s
ω_2	Second natural frequency of the system	324.5 rad/s
$\boldsymbol{\phi_1}$	mode shape for first natural frequency	$\begin{bmatrix} 1 \\ -3.16 \end{bmatrix}$
$\boldsymbol{\phi_2}$	mode shape for second natural frequency	$\begin{bmatrix} 1 \\ 3.16 \end{bmatrix}$

TABLE I
INPUT VALUES

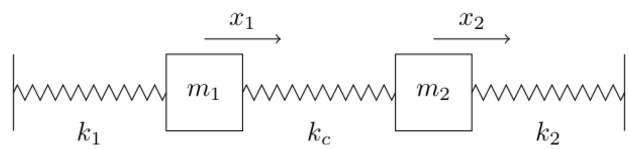

Fig. 1. System with D.O.F =2

The F.B.D for above system is written as:

$$m_1 \frac{d^2 x_1}{dt^2} - k_c (x_2 - x_1) + k_1 x_1 = 0 \tag{1}$$

$$m_2 \frac{d^2 x_2}{dt^2} + k_c (x_2 - x_1) + k_2 x_2 = 0 \tag{2}$$

Which can be written in the form of matrices as:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} \frac{d^2 x_1}{dt^2} \\ \frac{d^2 x_2}{dt^2} \end{pmatrix} = - \begin{bmatrix} k_1 + k_c & -k_c \\ -k_c & k_2 + k_c \end{bmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \tag{3}$$

Taking laplace transform:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} s^2 X_1(s) - s x_1(0) \\ s^2 X_2(s) - s x_2(0) \end{pmatrix} = - \begin{bmatrix} k_1 + k_c & -k_c \\ -k_c & k_2 + k_c \end{bmatrix} \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} \tag{4}$$

$$\implies \begin{pmatrix} s^2X_1(s) - sx_1(0) \\ s^2X_2(s) - sx_2(0) \end{pmatrix} = -\begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} k_1 + k_c & -k_c \\ -k_c & k_2 + k_c \end{bmatrix} \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} \tag{5}$$

$$\implies \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \frac{\begin{bmatrix} s^2 + \frac{k_1+k_c}{m_1} & \frac{-k_c}{m_1} \\ \frac{-k_c}{m_2} & s^2 + \frac{k_2+k_c}{m_2} \end{bmatrix} \begin{pmatrix} sx_1(0) \\ sx_2(0) \end{pmatrix}}{\left(s^2 + \frac{k_1+k_c}{m_1}\right)\left(s^2 + \frac{k_c+k_2}{m_2}\right) - \frac{k_c^2}{m_1 m_2}} \tag{6}$$

Assuming the solutions to the equations are:

$$x_1(t) = A_1 \cos(\omega t) \tag{7}$$
$$x_2(t) = A_2 \cos(\omega t) \tag{8}$$

Substituting (7) and (8) in (3), we get:

$$\begin{bmatrix} k_1 + k_c - m_1\omega^2 & -k_c \\ -k_c & k_2 + k_c - m_2\omega^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \sin(\omega t + \phi) = 0 \tag{9}$$

$$\implies \det\left(\begin{bmatrix} k_1 + k_c - m_1\omega^2 & -k_c \\ -k_c & k_2 + k_c - m_2\omega^2 \end{bmatrix}\right) = 0 \tag{10}$$

Let the roots of this equation be ω_1 and ω_2. Which are the two modes of the system.
Substituting ω_1 in (3):
we obtain

$$\boldsymbol{\phi_1} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}_1 \tag{11}$$

and

Substituting ω_2 in (3):
we obtain

$$\boldsymbol{\phi_2} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}_2 \tag{12}$$

These are called mode shapes.
So any oscillation can be represented as:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \{\phi\}_1 \cos(\omega_1 t) + \{\phi\}_2 \cos(\omega_2 t) \tag{13}$$

From Table I:

$$\implies x_1(t) = A_{11}\cos(\omega_1 t) + A_{21}\cos(\omega_1 t) \tag{14}$$
$$\implies x_2(t) = A_{12}\cos(\omega_2 t) + A_{22}\cos(\omega_2 t) \tag{15}$$
$$\therefore x_1(0) = A_{11} + A_{12} \tag{16}$$
$$\therefore x_2(0) = A_{21} + A_{22} \tag{17}$$

1) For first natural frequency:

$$\frac{x_1(0)}{x_2(0)} = \frac{A_{11}}{A_{21}} \tag{18}$$
$$\implies \frac{x_1(0)}{x_2(0)} = \frac{1}{-3.16} \tag{19}$$

2) For second natural frequency:

$$\frac{x_1(0)}{x_2(0)} = \frac{A_{12}}{A_{22}} \tag{20}$$
$$\implies \frac{x_1(0)}{x_2(0)} = \frac{1}{3.16} \tag{21}$$

So, option (B) is correct.

3) For linear combination of first and second natural frequencies:

$$x_1(0) = A_{11} + A_{12}x_2(0) \quad = A_{21} + A_{22} \tag{22}$$

a) If $\boldsymbol{\phi_1} \neq \boldsymbol{\phi_2}$ solution always exists
b) If $\boldsymbol{\phi_1} = \boldsymbol{\phi_2}$ solution exists only if $x_1(0) = x_2(0)$

So, option (C) is also correct.