CHAPTER 9

Sampling Theory and Pulse Modulation

Inside this Chapter

9.1. Introduction 485
9.2. The Sampling Theorem 486
9.3. Proof of Sampling Theorem 486
9.4. Nyquist Rate and Nyquist Interval 489
9.5. Reconstruction Filter (Low-Pass Filter) 492
9.6. Signal Reconstruction: The Interpolation Formula 493
9.7. Effect of Under Sampling: Aliasing 495
9.8. Sampling of Bandpass Signals 495
9.9. Sampling Techniques 500
9.10. Aperture Effect 507
9.11. Comparison of Various Sampling Techniques 509
9.12. Analog Pulse Modulation Methods 511
9.13. Pulse Amplitude Modulation (PAM) 512
9.14. Pulse Time Modulation 519
9.15. Comparison of Various Pulse Analog Modulation Methods 526

9.1. Introduction

As we know that broadly, there are two types of signals, continuous time signal and discrete-time signals. Due to some recent advance development in digital technology over the past few decades, the inexpensive, light weight, programmable and easily reproducible discrete-time systems are available. Therefore, the processing of discrete-time signals is more flexible and is also preferable to processing of continuous-time signals.

This means that in practice, although we have a large number of continuous-time signals, but we prefer processing of discrete-time signals. For this purpose we should be able to convert a continuous-time signal into discrete-time signal.

This problem is solved by a fundamental mathematical tool known as sampling theorem. The sampling theorem is extremely important and useful in signal processing. With the help of sampling theorem, a continuous-time signal may be completely represented and recovered from the knowledge of samples taken uniformly. This means that sampling theorem provides a mechanism for representing

Scanned by CamScanner

a continuous-time signal by a discrete-time signal. Therefore, sampling theorem may be viewed as a bridge between continuous-time signals and discrete-time signals.

The concept of sampling provides a widely used method for using discrete-time system technology to implement continuous-time systems and process the continuous-time signals. We utilize sampling to convert a continuous-time signal to a discrete-time signal, process the discrete-time signal using a discrete-time system and then convert back to continuous-time signals.

9.2. The Sampling Theorem

(U.P. Tech-Semester Exam. 2002-2003)

As discussed earlier, sampling of the signals is the fundamental operation in signal-processing. A continuous time signal is first converted to discrete-time signal by sampling process. The sufficient number of samples of the signal must be taken so that the original signal is represented in its samples completely. Also, it should be possible to recover or reconstruct the original signal completely from its samples. The number of samples to be taken depends on maximum signal frequency present in the signal. Sampling theorem gives the complete idea about the sampling of signals. Different types of samples are also taken like ideal samples, natural samples and flat-top samples.

Let us discuss the sampling theorem first and then we shall discuss different types of sampling processes. The statement of sampling theorem can be given in two parts as:

(*i*) A bandlimited signal of finite energy, which has no frequency-component higher than f_m *Hz*, is completely described by its sample values at uniform intervals less than or equal to $\frac{1}{2f_m}$ second apart.

(*ii*) A bandlimited signal of finite energy, which has no frequency components higher than f_m Hz, may be completely recoverd from the knowledge of its samples taken at the rate of 2 f_m samples per second.

The first part represents the representation of the signal in its samples and minimum sampling rate required to represent a continuous-time signal into its samples.

The second part of the theorem represents reconstruction of the original signal from its samples. It gives sampling rate required for satisfactory reconstruction of signal from its samples.

Combining the two parts, the sampling theorem may be stated as under:

"A continuous-time signal may be completely represented in its samples and recovered back if the sampling frequency is $f_s \geq 2 f_m$. Here, f_s is the sampling frequency and f_m is the maximum frequency present in the signal".

9.3. Proof of Sampling Theorem

(U.P. Tech-Semester Examination 2002-2003)

To prove the sampling theorem, we shall show that a signal whose spectrum is bandlimited to f_m *Hz*, can be reconstructed exactly without any error from its samples taken uniformly at a rate $f_s > 2 f_m$ *Hz*.

Let us consider a continuous time signal $x(t)$ whose spectrum is bandlimited to f_mHz. This means that the signal $x(t)$ has no frequency components beyond f_mHz. Therefore, $X(w)$ is zero for $|w| > w_m$, *i.e.*,

$$X(w) = 0 \text{ for } |w| > w_m$$

where $$w_m = 2\pi f_m$$

Figure 9.1 (*a*) shows this continuous-time signal $x(t)$. Let $X(w)$ represents its Fourier transform or frequency spectrum as shown in figure 9.1(*b*).

Sampling of $x(t)$ at a rate of f_s*Hz* (f_s samples per second) may be achieved by multiplying $x(t)$ by an impulse train $\delta_{Ts}(t)$. The impulse train $\delta_{Ts}(t)$ consists of unit impulses repeating periodically every T_s seconds, where $T_s = \frac{1}{f_s}$.

Scanned by CamScanner

Figure 9.1(*c*) shows this impulse train. This multiplication results in the sampled signal $g(t)$ shown in figure 9.1(*e*).

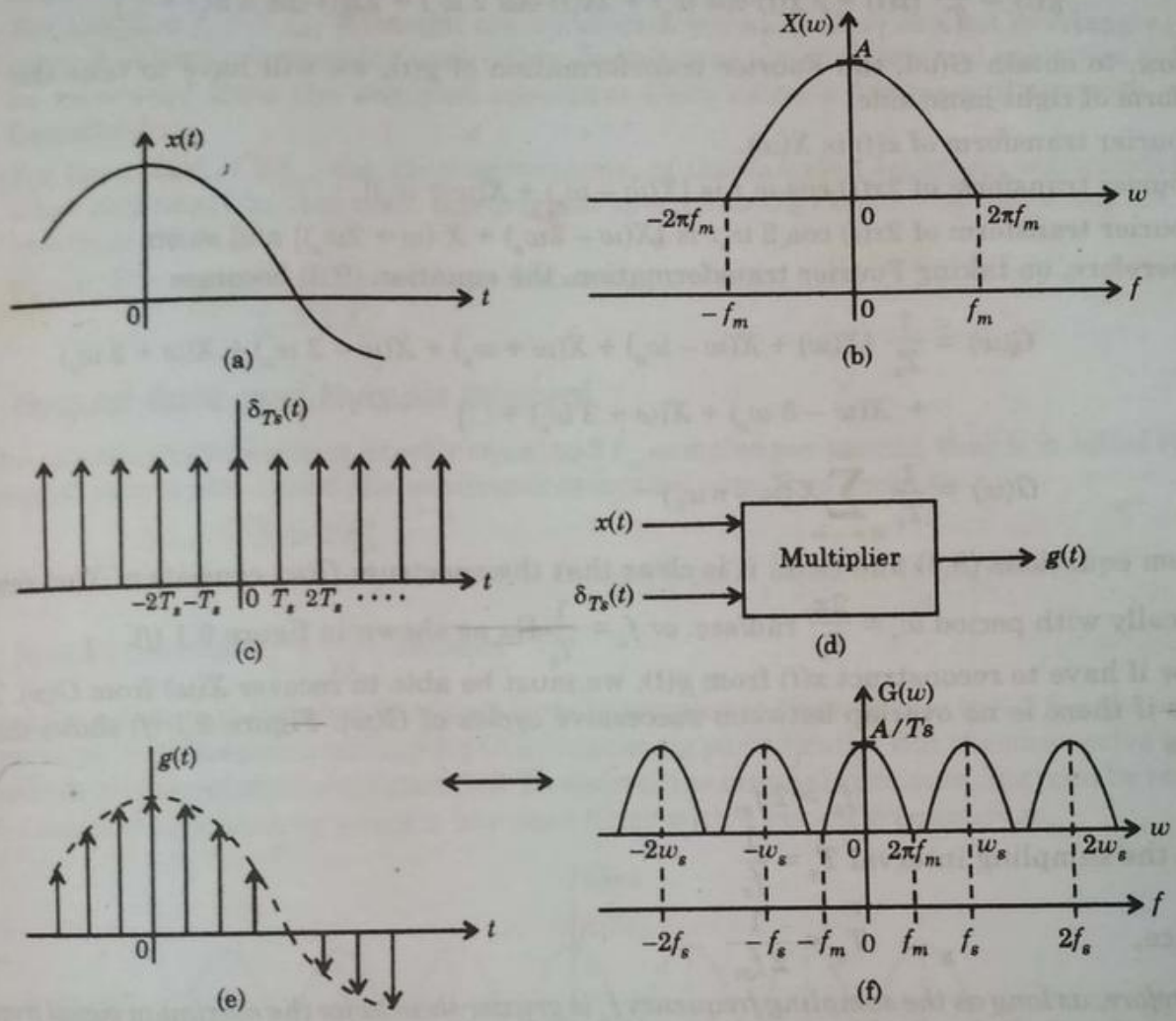


Fig. 9.1. *(a) A continuous-time signal.*
(b) Spectrum of continuous-time signal.
(c) Impulse train as sampling function.
(d) Multiplier.
(e) Sampled signal.
(f) Spectrum of sampled signal.

This sampled signal consists of impulses spaced every T_s seconds (the sampling interval). The resulting or sampled signal may be written as

$$g(t) = x(t)\,\delta_{T_s}(t) \qquad ...(9.1)$$

Again, since the impulse train $\delta_{T_s}(t)$ is a periodic signal of period T_s, it may be expressed as a Fourier series.

The trigonometric Fourier series expansion of impulse-train $\delta_{T_s}(t)$ is expressed as

$$\delta_{T_s}(t) = \frac{1}{T_s}\,[1 + 2\cos w_s t + 2\cos 2\,w_s t + 2\cos 3\,w_s t + ...] \qquad ...(9.2)$$

Here $\quad w_s = \dfrac{2\pi}{T_s} = 2\,\pi f_s$

Scanned by CamScanner

Putting the values of $\delta_{Ts}(t)$ from equation (9.2) in equation (9.1), the sampled signal is

$$g(t) = \frac{1}{T_s}\,[x(t) + 2\,x(t)\cos w_s t + 2x(t)\cos 2\,w_s t + 2x(t)\cos 3\,w_s t + \ldots] \qquad \ldots(9.3)$$

Now, to obtain $G(w)$, the Fourier transformation of $g(t)$, we will have to take the Fourier transform of right hand side.

Fourier transform of $x(t)$ is $X(w)$.

Fourier transform of $2x(t)\cos w_s t$ is $[X(w - w_s) + X(w + w_s)]$.

Fourier transform of $2x(t)\cos 2\,w_s t$ is $[X(w - 2w_s) + X\,(w + 2w_s)]$ and so on.

Therefore, on taking Fourier transformation, the equation (9.3) becomes

$$G(w) = \frac{1}{T_s}\,[X(w) + X(w - w_s) + X(w + w_s) + X(w - 2\,w_s) + X(w + 2\,w_s)$$
$$+\ X(w - 3\,w_s) + X(w + 3\,w_s) + \ldots] \qquad \ldots(9.4)$$

or
$$G(w) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X\,(w - n\,w_s) \qquad \ldots(9.5)$$

From equations (9.4) and (9.5), it is clear that the spectrum $G(w)$ consists of $X(w)$ repeating periodically with period $w_s = \frac{2\pi}{T_s}$ rad/sec. *or* $f_s = \frac{1}{T_s}$ Hz as shown in figure 9.1 (*f*).

Now if have to reconstruct $x(t)$ from $g(t)$, we must be able to recover $X(w)$ from $G(w)$. This is possible if there is no overlap between successive cycles of $G(w)$. Figure 9.1 (*f*) shows that this requires

$$f_s > 2\,f_m \qquad \ldots(9.6)$$

But the sampling interval $T_s = \frac{1}{f_s}$

Hence,
$$T_s < \frac{1}{2\,f_m} \qquad \ldots(9.7)$$

Therefore, as long as the sampling frequency f_s is greater than twice the maximum signal frequency f_m (signal, bandwidth, f_m), $G(w)$ will consist of non-overlapping repetitions of $X(w)$. If this is true, figure 9.1 (f) shows that $x(t)$ can be recovered from its samples $g(t)$ by passing the sampled signal $g(t)$ through an ideal law-pass filter of bandwidth f_m Hz. This proves the sampling theorem.*

9.3.1. Few Points about Sampling Theorem

(*i*) Figure 9.1 (*f*) shows the spectrum of sampled signal. According to the figure, as long as, the signal is sampled at rate $f_s > 2\,f_m$, the spectrum $G(w)$ will repeat periodically without overlapping.

(*ii*) The spectrum of sampled signal extends upto infinity and the ideal bandwidth of sampled signal is infinite. But here our purpose is to extract our original spectrum $X(w)$ out of the spectrum $G(w)$.

(*iii*) The original *or* desired spectrum $X(w)$ is centred at $w = 0$ and is having bandwidth *or* maximum frequency equal to w_m. The desired spectrum may be recovered by passing the sampled signal with spectrum $G(w)$ through a low pass filter with cut-off frequency w_m. This means that since a low-pass filter allows to pass only low frequencies up to cut-off frequency (w_m) and rejects all other higher frequencies, the original spectrum $X(w)$ extended upto w_m will be selected and all other successive higher frequency cycles in the sampled-spectrum will be rejected. Therefore, in this way, original spectrum $X(w)$ will be extracted out of spectrum $G(w)$. This original spectrum $X(w)$ can now be converted into time-domain signal $x(t)$.

* Maximum signal frequency f_m means that the bandwidth of this signal is simply f_m.

Scanned by CamScanner

(*iv*) It may also be observed from figure that for the case $f_s > 2f_m$, the successive cycles of $G(w)$ are not overlapping each other. Hence in this case, there is no problem in recovering the original spectrum $X(w)$.

(*v*) For the case $f_s = 2\,f_m$, although the successive cycles of $G(w)$ are not overlapping each other, but they are touching each other. In this case also, the original spectrum $X(w)$ can be recovered from the sampled spectrum $G(w)$ using a low-pass filter with cut-off frequency w_m.

(*vi*) For the case $f_s < 2\,f_m$, the successive cycles, of the sampled spectrum will overlap each other and hence in this case, the original spectrum $X(w)$ cannot be extracted out of the spectrum $G(w)$.

Hence, For reconstruction without distortion, we must have

$$f_s \geq 2\,f_m$$

9.4. Nyquist Rate and Nyquist Interval

When the sampling rate becomes exactly equal to $2\,f_m$ samples per second, then it is called Nyquist rate. Nyquist rate is also called the *minimum* sampling rate. It is given by

$$f_s = 2\,f_m \qquad ...(9.8)$$

Similarly, maximum sampling interval is called *Nyquist interval*. It is given by

$$\text{Nyquist interval } T_s = \frac{1}{2f_m} \text{ seconds} \qquad ...(9.9)$$

When the continuous-time bandlimited signal is sampled at Nyquist rate ($f_s = 2\,f_m$), the sampled-spectrum $G(w)$ contains non-overlapping $G(w)$ repeating periodically. But the successive cycles of $G(w)$ touch each other as shown in figure 9.2. Therefore, the original spectrum $X(w)$ can be recovered from the sampled spectrum by using a low pass filter with a cut-off frequency w_m.

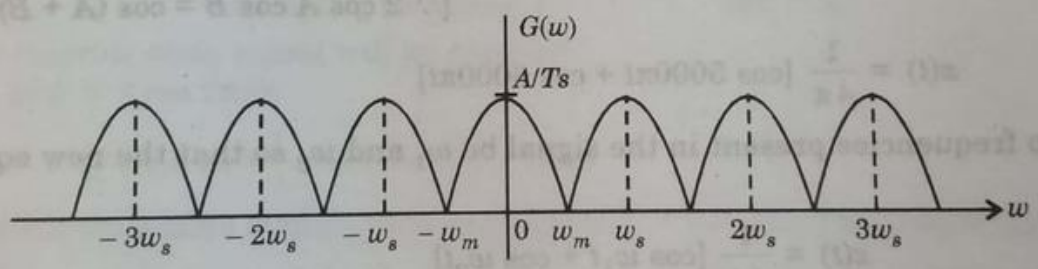


Fig. 9.2. *Sampled spectrum at Nyquist rate.*

Example 9.1. An analog signal is expressed by the equation $x(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t$. Calculate the Nyquist rate for this signal.

Solution: The given signal is expressed as

$$x(t) = 3\cos 50\pi t + 10\sin 300\pi t - \cos 100\pi t \qquad ...(i)$$

Let three frequencies present be w_1, w_2 and w_3

So that the new equation for signal,

$$x(t) = 3\cos w_1 t + 10\sin w_2 t - \cos w_3 t \qquad ...(ii)$$

Comparing equations (*i*) and (*ii*) we have

$$w_1 t = 50\,\pi t;\ w_1 = 50\,\pi$$

or $$2\,\pi f_1 = 50\,\pi \quad \text{or} \quad 2\,f_1 = 50$$

$\therefore$ $$f_1 = 25 \text{ Hz}$$

Similarly, for second factor

$$w_2 t = 300\pi t \quad \text{or} \quad w_2 = 300\pi$$

2πf = 50π
f = 50/2
fs = 300.

Scanned by CamScanner

or $$2\pi f_2 = 300\pi \quad \text{or} \quad 2\pi f_2 = 300\pi$$

$\therefore$ $$f_2 = 150 \text{ Hz.}$$

Again, for third factor

$$w_3 t = 100\pi t \quad \text{or} \quad 2\pi f_3 t = 100\pi t$$

or $$2\pi f_3 = 100\pi$$

$\therefore$ $$f_3 = 50 \text{ Hz}$$

Therefore, the maximum frequency present in $x(t)$ is,

$$f_2 = 150 \text{ Hz}$$

Nyquist rate is given as

$$f_s = 2 f_m$$

where f_m = Maximum frequency present in the signal.

Here $$f_m = f_2 = 150 \text{ Hz}$$

Therefore, Nyquist rate

$$f_s = 2 f_2 = 2 \times 150 = 300 \text{ Hz} \qquad \textbf{Ans.}$$

Example 9.2. Find the Nyquist rate and the Nyquist interval for the signal

$$x(t) = \frac{1}{2\pi} \cos(4000\pi t) \cos(1000\pi t)$$

Solution: Given signal is

$$x(t) = \frac{1}{2\pi} \cos(4000\pi t) \cos(1000\pi t) = \frac{1}{4\pi} [2 \cos(4000\pi t) \cos 1000\pi t)]$$

or $$x(t) = \frac{1}{4\pi} [\cos(4000\pi t + 1000\pi t) + \cos(4000\pi t - 1000\pi t)]$$

$$[\because 2 \cos A \cos B = \cos(A + B) + \cos(A - B)]$$

or $$x(t) = \frac{1}{4\pi} [\cos 5000\pi t + \cos 3000\pi t] \qquad ...(i)$$

Let the two frequencies present in the signal be w_1 and w_2 so that the new equation for the signal will be

$$x(t) = \frac{1}{4\pi} [\cos w_1 t + \cos w_2 t] \qquad ...(ii)$$

Comparing equations (*i*) and (*ii*), we have

$$w_1 t = 5000\pi t$$

or $$2\pi f_1 t = 5000\pi t \quad \text{or} \quad 2 f_1 = 5000$$

$\therefore$ $$f_1 = 2500 \text{ Hz}$$

Similarly, for second factor

$$w_2 t = 3000\pi t$$

or $$2\pi f_2 t = 3000\pi t \quad \text{or} \quad 2\pi f_2 = 3000$$

$\therefore$ $$f_2 = 1500 \text{ Hz}$$

Therefore, the maximum frequency present in $x(t)$ is

$$f_1 = 2500 \text{ Hz}$$

Nyquist rate is given as

$$f_s = 2 f_m$$

where f_m = Maximum frequency present in the signal.

Here $$f_m = f_1 = 2500 \text{ Hz}$$

Scanned by CamScanner

Therefore Nyquist rate

$$f_s = 2 f_m = 2 \times 2500 = 5000 \text{ Hz} = 5 \text{ kHz} \quad \textbf{Ans.}$$

Nyquist interval is given as

$$T_s = \frac{1}{2f_m} = \frac{1}{2 \times 2500} = \frac{1}{5000}$$

or

$$T_s = 0.2 \times 10^{-3} \text{ seconds} = 0.2 \text{ m sec.} \quad \textbf{Ans.}$$

Example 9.3. A continuous-time signal is given below:

$$x(t) = 8 \cos 200\,\pi t$$

Determine:

(*i*) Minimum sampling rate *i.e.*, Nyquist rate required to avoid aliasing.

(*ii*) If sampling frequency $f_s = 400$ Hz. What is the discrete-time signal $x[n]$ or $x[nT_s]$ obtained after sampling?

(*iii*) If sampling frequency $f_s = 150$ Hz. What is the discrete-time signal $x[n]$ or $x[nT_s]$ obtained after sampling?

(*iv*) What is the frequency $0 < f < f_s/2$ of sinusoidal that yields samples identical to those obtained in part (*iii*)?

Solution:

(*i*) The highest frequency component of continuous-time signal is $f = 100$ Hz. Hence minimum sampling rate required to avoid aliasing is called Nyquist rate and is given as

$$\text{Nyquist rate} = 2f = 2 \times 100 = 200 \text{ Hz} \quad \textbf{Ans.}$$

(*ii*) The continuous-time signal $x(t)$ is sampled at $f_s = 400$ Hz. The frequency of the discrete-time signal will be

$$F = \frac{\text{Frequency of continuous-time signal } f}{\text{Sampling frequency, } f_s} = \frac{100}{400} = \frac{1}{4}$$

Then the discrete-time signal will be given as

$$x[n] = 8 \cos 2\pi F n$$

$$= 8 \cos 2\pi \times \frac{1}{4} n = 8 \cos \frac{\pi n}{2} \quad \textbf{Ans.}$$

(*iii*) The continuous-time signal $x(t)$ is sampled at $f_s = 150$ Hz. The frequency of discrete-time will be

$$F = \frac{f}{f_s} = \frac{100}{150} = \frac{2}{3}$$

Then, the discrete-time signal will be given as

$$x[n] = 8 \cos 2\pi f n$$

$$= 5 \cos 2\pi \left(\frac{2}{3}\right) n = 8 \cos \frac{4\pi}{3} n$$

$$= 8 \cos \left(2\pi - \frac{2\pi}{3}\right) n$$

$$= 8 \cos \frac{2\pi n}{3} = 8 \cos \frac{2\pi n}{3} \quad \textbf{Ans.}$$

(*iv*) For sampling rate of $f_s = 150$ Hz

$$F = \frac{f}{f_s} \text{ or } f = f_s \times F = \frac{1}{3} \times 150 = 50 \text{ Hz}$$

Scanned by CamScanner

Then, the sinusoidal signal will be

$$y(t) = 8 \cos 2\pi ft = 8 \cos 2\pi \times 50 \times t = 8 \cos 100\,\pi t$$

Sampling at $f_s = 150$ Hz, yields identical samples hence $f = 100$ Hz is an alias of $f = 50$ Hz for sampling rate $f_s = 150$ Hz. **Ans.**

Example 9.4. Determine the Nyquist rate for a continuous-time signal

$$\mathbf{x(t) = 6 \cos 50\,\pi t + 20 \sin 300\,\pi t - 10 \cos 100\,\pi t}$$

Solution: In a general form, any continuous-time signal may be expressed as

$$x(t) = A_1 \cos \omega_1 t + A_2 \cos \omega_2 t + A_3 \cos \omega_3 t \quad \text{...(i)}$$

And the given signal is

$$x(t) = 6 \cos 50\pi t + 2 \sin 300\pi t - 10 \cos 100\,\pi t \quad \text{...(ii)}$$

On comparing given signal (*ii*) with standard form of a signal (*i*), we obtain the frequencies for the given signal as

$$f_1 = \frac{\omega_1}{2\pi} = \frac{50\pi}{2\pi} = 25 \text{ Hz}$$

$$f_2 = \frac{\omega_2}{2\pi} = \frac{300\pi}{2\pi} = 150 \text{ Hz}$$

$$f_3 = \frac{\omega_3}{2\pi} = \frac{100\pi}{2\pi} = 50 \text{ Hz}$$

Thus, the highest frequency component of the given message signal will be

$$f_{max} = 150 \text{ Hz}$$

Therefore, Nyquist rate $= 2\,f_{max} = 2 \times 150 = 300$ Hz **Ans.**

9.5. Reconstruction Filter (Low-Pass Filter)

The low-pass filter is used to recover original signal from its samples. This is also known as **interpolation filter**.

A low-pass filter is that type of filter which passes only low-frequencies upto a specified cut-off frequency and rejects all other frequencies above cut-off frequency. Figure 9.3 shows the frequency response of low-pass filter.

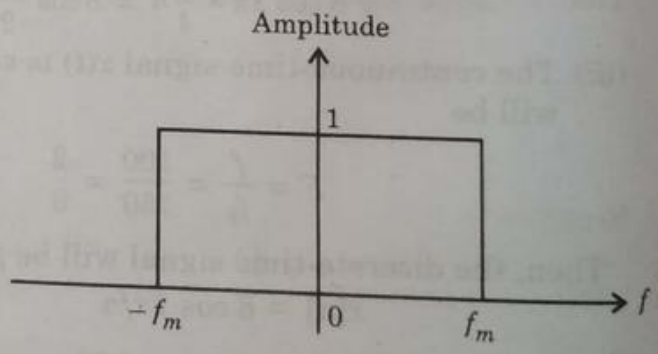


Fig. 9.3. *Ideal low-pass filter (LPF).*

From figure 9.3, it may be observed that in case of low-pass filter, there is sharp-change in response at cut-off frequency, that is amplitude response becomes suddenly zero at cut-off frequency which is not possible practically. This means that an ideal low-pass filter is not physically realizable. In place of ideal-low pass filter, we use practical filter.

Figure 9.4 shows the frequency response of practical low-pass filter. From figure 9.4, it may be observed that in case of practical filter, the amplitude response decreases slowly to become zero. This means that there is a transition band in case of practical filter. Figure 9.5 shows the use of practical low-pass filter in reconstruction of original signal from its sample.

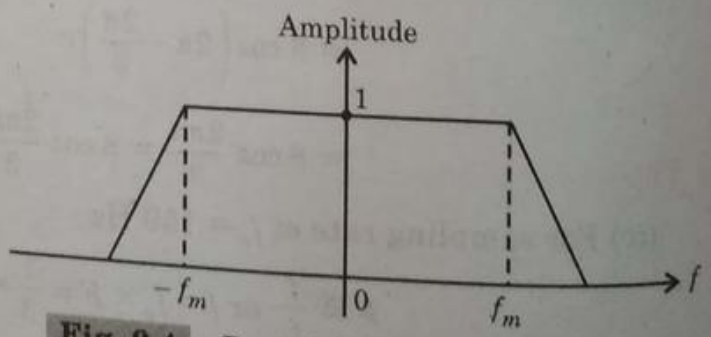


Fig. 9.4. *Practical low-pass filter (LPF).*

Scanned by CamScanner

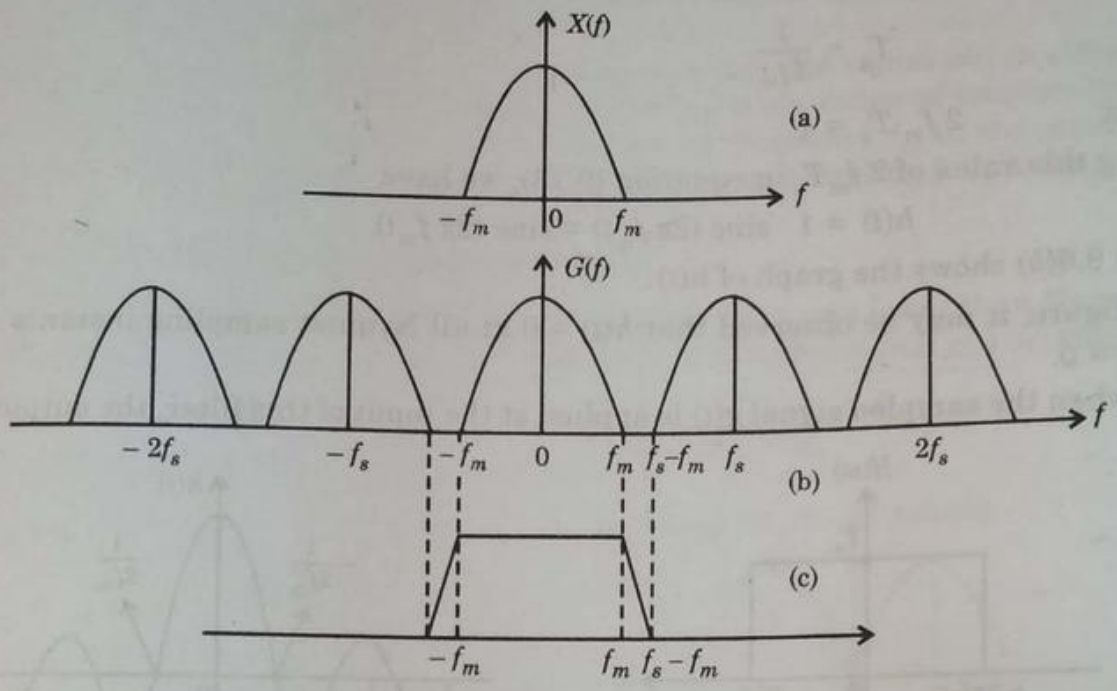


Fig. 9.5. *(a) Spectrum of original signal*
(b) Spectrum of sampled signal
(c) Amplitude response of practical low-pass filter.

9.6. Signal Reconstruction: The Interpolation Formula

The process of reconstructing a continuous-time signal $x(t)$ from its samples is called as **interpolation.**

As discussed earlier, a signal $x(t)$ bandlimited to f_m *Hz* can be reconstructed (interpolated) completely from its samples. This is achieved by passing the sampled signal through an ideal low-pass filter (LPF) of cut-off frequency f_m Hz.

The expression for sampled signal is written as

$$g(t) = x(t).\delta_{T_s}(t) \qquad ...(9.10)$$

$$g(t) = \frac{1}{T_s}\,[x(t) + 2\,x(t)\cos w_s t + 2\,x(t)\cos 2w_s t + ...] \qquad ...(9.11)$$

From above equation, it may be observed that the sampled signal contains a component $\frac{1}{T_s} \times x(t)$.

To recover $x(t)$ or $X(w)$, the sampled signal must be passed through an ideal low-pass filter of bandwidth of f_mHz and gain T_s.

Therefore, the reconstruction or interpolating filter transfer function may be expressed as

$$H(w) = T_s \times \text{rect}\left(\frac{w}{4\pi f_m}\right) \qquad ...(9.12)$$

The impulse response $h(t)$ of this filter is the inverse Fourier transform of $H(w)$, *i.e.*

$$h(t) = F^{-1}[H(w)] = F^{-1}\left[T_s\,\text{rect}\left(\frac{w}{4\pi f_m}\right)\right]$$

$$h(t) = 2\,f_m\,T_s\,\text{sinc}\,(2\pi f_m t) \qquad ...(9.13)$$

Scanned by CamScanner

Assuming that sampling is done at Nyquist rate, then

$$T_s = \frac{1}{2f_m}$$

So that $\quad 2 f_m T_s = 1 \quad$ ...(9.13a)

Putting this value of $2 f_m T_s$ in equation (9.13), we have

$$h(t) = 1 \cdot \text{sinc}\,(2\pi f_m t) = \text{sinc}\,(2\pi f_m t) \quad \text{...(9.14)}$$

Figure 9.6(*b*) shows the graph of $h(t)$.

From figure, it may be observed that $h(t) = 0$ at all Nyquist sampling instants $t = \pm\, n/2 f_m$ except at $t = 0$.

Now, when the sampled signal $g(t)$ is applied at the input of this filter, the output will be $x(t)$.

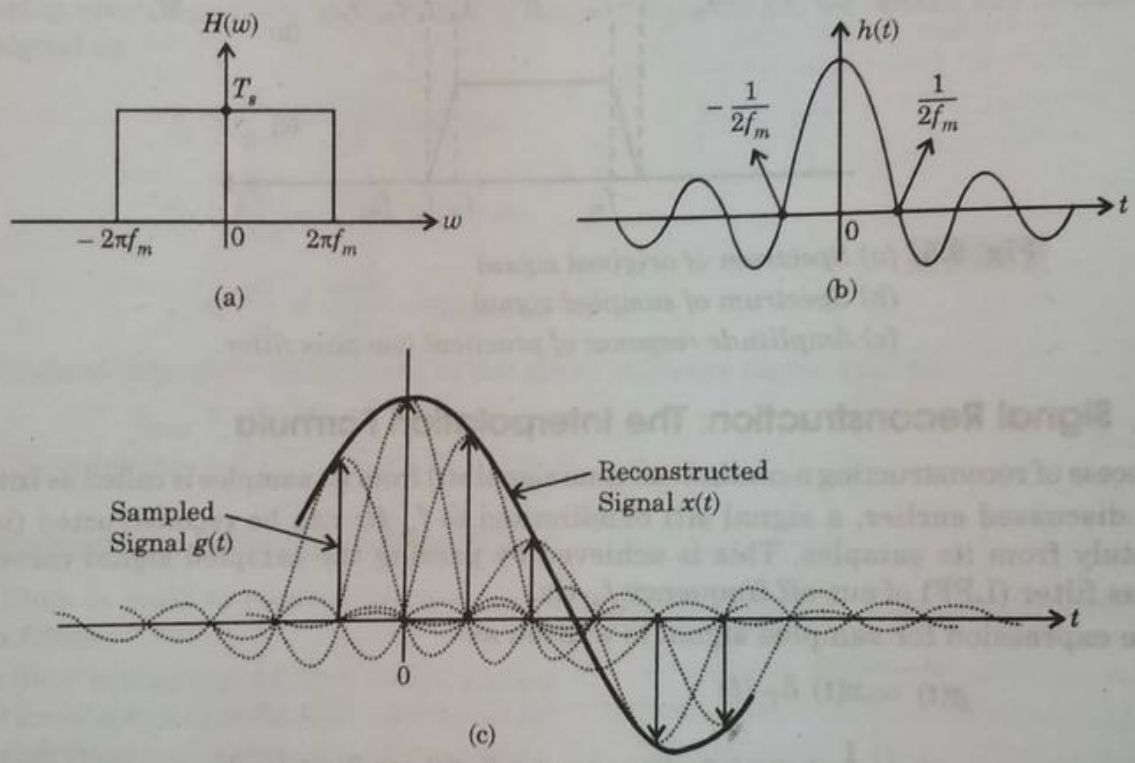


Fig. 9.6.

Each sample in $g(t)$, being an impulse, produces a sinc pulse of height equal to the strength of the sample.

Addition of the sinc pulses produced by all the samples results in $x(t)$.

For instant, the kth sample of the input $g(t)$ is the impulse $x(k T_s)\, \delta(t - kT_s)$.

The filter output of this impulse will be $x(kT_s)\, h(t - kT_s)$.

Therefore, the filter output to $g(t)$, which is $x(t)$, may be expressed as a sum

$$x(t) = \sum_k x(kT_s)\, h(t - kT_s) \quad \text{...(9.15)}$$

$$= \sum_k x(kT_s)\, \text{sinc}\,[2\pi f_m (t - kT_s)] \quad \text{...(9.16)}$$

$$x(t) = \sum_k x(kT_s)\, \text{sinc}\,(2\pi f_m - k\pi) \quad \left[\because T_s = \frac{1}{2f_m}\right] \quad \text{...(9.17)}$$

Scanned by CamScanner

Equation (9.17) is known as the **interpolation formula**, which provides values of $x(t)$ between samples as a weighted sum of all the sample values.

In the proof of sampling theorem, it is assumed that the signal $x(t)$ is strictly bandlimited. But, in general, an information signal may contain a wide range of frequencies and cannot be strictly bandlimited. This means that the maximum frequency f_m in the signal $x(t)$ cannot be predictable. Therefore, it is not possible to select suitable sampling frequency f_s.

9.7. Effect of Under Sampling: Aliasing

When a continuous-time bandlimited signal is sampled at a rate lower than Nyquist rate $f_s < 2f_m$, then the successive cycles of the spectrum $G(w)$ of the sampled signal $g(t)$ overlap with each other as shown in figure 9.7.

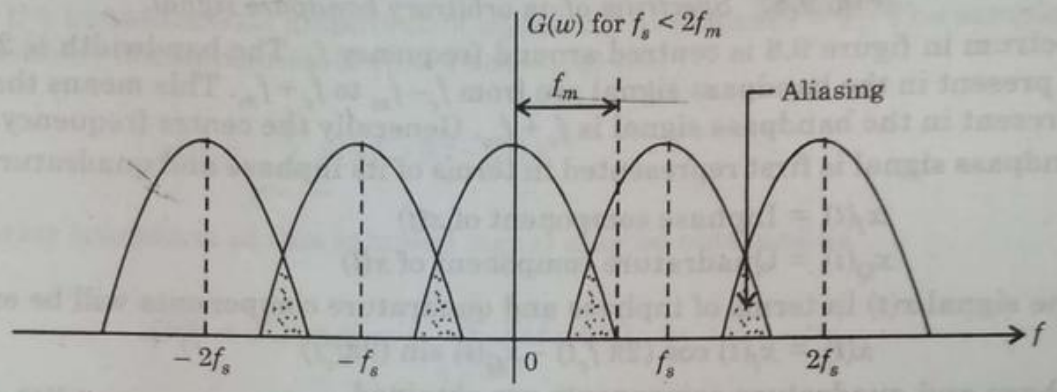


Fig. 9.7. *Spectruym of the sampled signal for the case $f_s < 2f_m$.*

Hence, the signal is under-slampled in this case ($f_s < 2f_m$) and some amount of aliasing is produced in this under-sampling process. In fact, aliasing is the phenomenon in which a high frequency component in the frequency-spectrum of the signal takes identity of a lower-frequency component in the spectrum of the sampled signal.

From figure 9.7, it is obvious that because of the overlap due to aliasing phenomenon, it is not possible to recover original signal $x(t)$ from sampled signal $g(t)$ by low-pass filtering since the spectral components in the overlap regions add and hence the signal is distorted.

Since any information signal contains a large number of frequencies, so, to decide a sampling frequency is always a problem. Therefore, a signal is first passed through a low-pass filter. This low-pass filter blocks all the frequencies which are above f_m Hz. This process is known as band-limiting of the original signal $x(t)$. This low-pass filter is called **prelias filter** because it is used to prevent aliasing effect. After bandlimiting, it becomes easy to decide sampling frequency since the maximum frequency is fixed at f_m Hz.

In short, to avoid aliasing:

(*i*) Prealias filter must be used to limit band of frequencies of the signal to f_mHz.

(*ii*) Sampling frequency 'f_s' must be selected such that

$$f_s > 2f_m$$

9.8. Sampling of Bandpass Signals

In previous sections, we discussed sampling theorem for low-pass signals. However, when the given signal is a bandpass signal, then a different criteria must be used to sample the signal. Therefore, the sampling theorem for bandpass signals may be expressed as under:

The bandpass signal $x(t)$ whose maximum bandwidth is $2f_m$ can be completely represented into and recovered from its samples if it is sampled at the minimum rate of twice the bandwidth. Here, f_m is the maximum frequency component present in the signal.

Scanned by CamScanner

Hence if the bandwidth is $2f_m$, then the minimum sampling rate for bandpass signal must be $4\,f_m$ samples per second. Figure 9.8 shows the spectrum of an arbitrary bandpass signal.

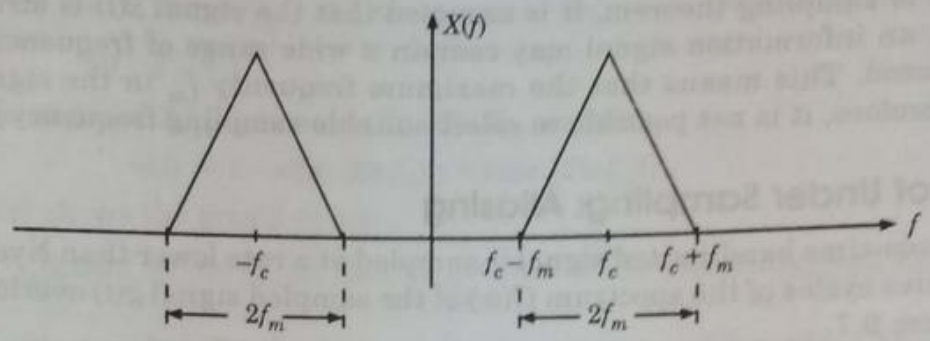


Fig. 9.8. *Spectrum of an orbitrary bandpass signal.*

The spectrum in figure 9.8 is centred around frequency f_c. The bandwidth is $2f_m$. Thus, the frequencies present in the bandpass signal are from $f_c - f_m$ to $f_c + f_m$. This means that the highest frequency present in the bandpass signal is $f_c + f_m$. Generally the centre frequency $f_c > f_m$.

This bandpass signal is first represented in terms of its inphase and quadrature components

Let $\quad x_I(t)$ = Inphase component of $x(t)$

and $\quad x_Q(t)$ = Quadrature component of $x(t)$

Thus, the signal $x(t)$ in terms of inphase and quadrature components will be expressed as

$$x(t) = x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t) \quad ...(9.18)$$

The inphase and quadrature components are obtained by multiplying $x(t)$ by $\cos(2\pi f_c t)$ and $\sin(2\pi f_c t)$ and then suppressing the sum frequencies by means of low-pass filters. Thus, inphase $x_I(t)$ and quadrature $x_Q(t)$ components contain only low frequency components. The spectrum of these components is limited between $-f_m$ to $+f_m$. This is shown in figure 9.9.

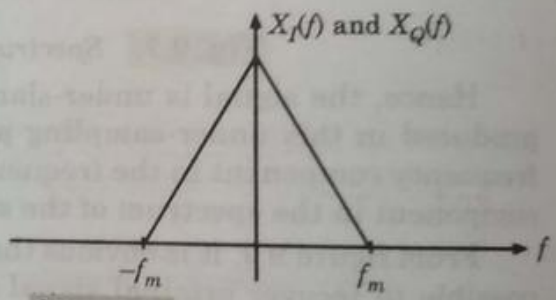


Fig. 9.9. *Spectrum of inphase and quadrature components of bandpass signal x(t).*

After few mathematical manipulation in equation (9.18), we obtain the reconstruction formula as

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{4f_m}\right) \sin c\left(2f_m t - \frac{n}{2}\right) \cos\left[2\pi f_c\left(t - \frac{n}{4f_n}\right)\right] \quad ...(9.19)$$

Comparing this reconstruction formula with that of low-pass signals given in equation (9.17), we observe that $x(t)$ is replaced by $x\left(\frac{n}{4f_m}\right)$.

Here, $\quad x\left(\frac{n}{4f_m}\right) = x(nT_s)$ = Sampled version of bandpass signal

and $\quad T_s = \frac{1}{4f_m}$

Thus, if $4f_m$ samples per second are taken, then the bandpass signal of bandwidth $2f_m$ can be completely recovered from its samples.

Hence, for bandpass signals of bandwidth $2f_m$,

Minimum Sampling rate = Twice of bandwidth

$= 4\,f_m$ samples per second.

Scanned by CamScanner

Example 9.5. Show that a bandlimited signal of finite energy which has no frequency components higher than f_m Hz is completely described by specifying values of the signals at instants of time separated by $1/2f_m$ seconds and also show that if the instantaneous values of the signal are separated by intervals larger than $1/2\ f_m$ seconds, they fail to describe the signal. A bandpass signal has spectral range that extends from 20 to 82 kHz.

Find the acceptable range of sampling frequency f_s.

Solution: Let $x(t)$ be the bandlimited signal which has no frequency components higher than f_m Hz. Let this signal be sampled by a sampling function given as

$$\delta_{Ts}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

The sampling function is the train of impulses with T_s as distance between successive impulses. Let $x(nT_s)$ be the instantaneous amplitude of signal $x(t)$ at instant $t = T_s$. The sampled version of $x(t)$ may be given as multiplication of $x(nT_s)$ and $\delta_{Ts}(t)$ *i.e.*,

$$g(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(t - nT_s) \qquad ...(i)$$

Now, Fourier transform of this sampled signal may be obtained as

$$G(f) = FT\{g(t)\} = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

Here, f_s is the sampling rate which is given as

$$f_s = \frac{1}{T_s}$$

and $\qquad X(f - nf_s) = X(f)$ at $nf_s = 0, \pm f_s, \pm 2f_s \pm 3f_s + ...$

Hence, the same spectrum $X(f)$ appears at $f = 0$,

$$f = \pm f_s, f = \pm 2f_s \quad \text{etc.,}$$

This means that a periodic spectrum with period equal to f_s is generated in frequency domain because of sampling $x(t)$ in time-domain.

Thus, equation (*i*) may be written as

$$G(f) = f_s \times (f) + f_s \times (f \pm f_s) + f_s \times (f \pm 2f_s) + f_s \times (f \pm 3f_s) + f_s \times (f \pm 4f_s) + ... \qquad ...(ii)$$

or $$G(f) = f_s \times (f) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} f_s . \times (f - nf_s) \qquad ...(iii)$$

By definition of Fourier transform, we have

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$$

For sampled version of $x(t)$, we have $t = nT_s$.

Then the equation (*iii*) becomes

$$G(f) = \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi fnT_s} \qquad ...(iv)$$

Scanned by CamScanner

Now, given that the signal is bandlimited to f_m Hz and

$$T_s = \frac{1}{2f_m} \text{ seconds}$$

Therefore, $$f_s = \frac{1}{T_s} = 2f_m \quad \text{...(v)}$$

From equation (*ii*), it may be observed that $G(f)$ is periodic with a period f_s.
Thus the spectrum $X(f)$ and $G(f)$ are shown in figure 9.10.

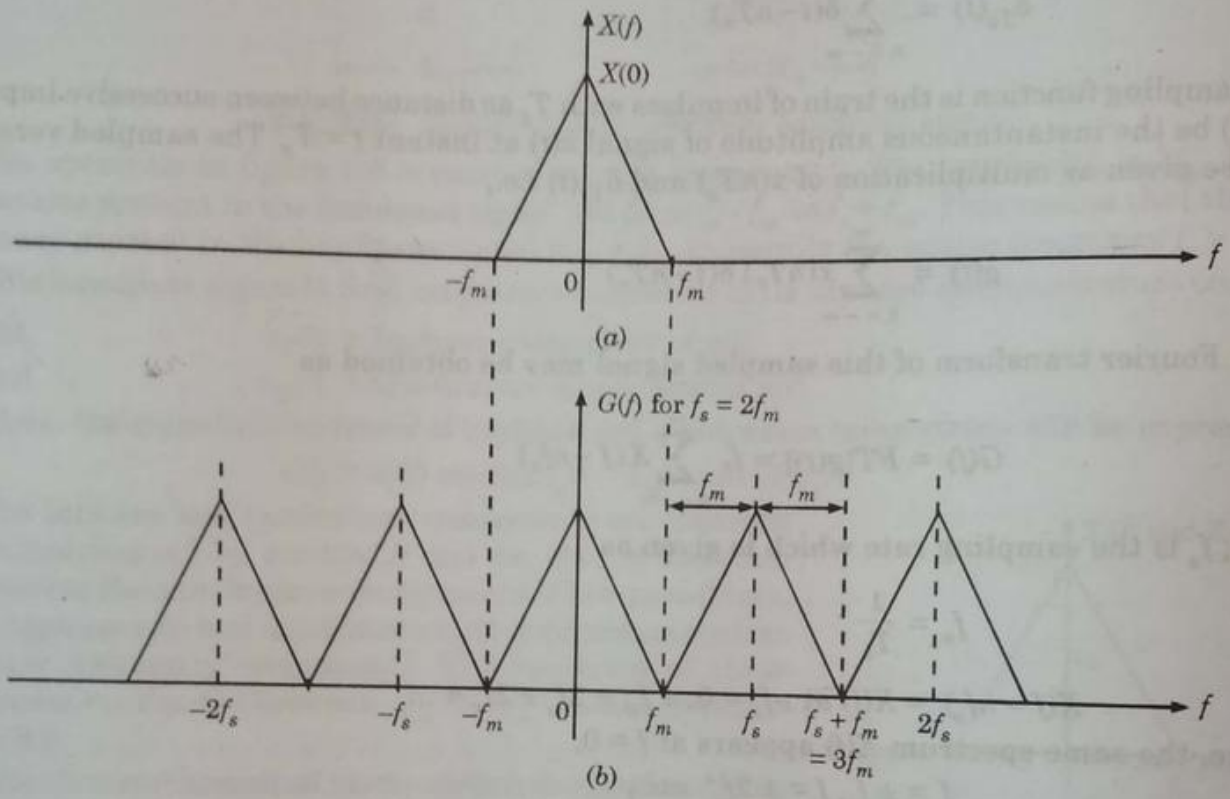


Fig. 9.10. *(a) Spectrum of x(t) (b) Spectrum of g(t) with $f_s = 2f_m$.*

Now, since $f_s = 2f_m$

therefore, $f_s - f_m = f_m$ and $f_s + f_m = 3\,f_m$

Hence, the periodic spectrums $X(f)$ just touch $\pm f_m, \pm 3f_m, \pm 5f_m$ ... etc.

Thus, there is no aliasing.

Using equation (*iii*), we write

$$X(f) = \frac{1}{f_s} G(f) - \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} X(f - nf_s) \quad \text{...(vi)}$$

Substituting $f_s = 2f_m$ in equation (*vi*), we get

$$X(f) = \frac{1}{2f_m} G(f) - \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} X(f - nf_s)$$

or $$X(f) = \frac{1}{2f_m} G(f) \quad \text{for } -f_m \le f \le f_m \quad \text{...(vii)}$$

Scanned by CamScanner

Now putting the value of $G(f)$ from equation (*iv*) to (*vii*), we get

$$X(f) = \frac{1}{2f_m} \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi fn.T_s}$$

Since $$T_s = \frac{1}{2f_m}$$

i.e., $$X(f) = \frac{1}{2f_m} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) e^{-j\pi fn/f_m} \quad ...(viii)$$

$x(t)$ may, be recovered from $X(f)$ by taking Inverse Fourier transform of last equation, *i.e.*,

$$x(t) = IFT\left\{\frac{1}{2f_m} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) e^{-j\pi f_n/f_m}\right\} \quad ...(ix)$$

This equation indicates that $x(t)$ is represented completely by its samples $x\left(\frac{n}{2f_m}\right)$ for $-\infty < n < \infty$. This means that the sequence $x\left(\frac{n}{2f_m}\right)$ has all the information contained in $x(t)$.

Reconstruction of Signal form Samples

Let us consider equation (*ix*) as

$$x(t) = IFT\left\{\frac{1}{2f_m} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) e^{-j\pi fn/f_m}\right\} = \int_{-f_m}^{f_m} \frac{1}{2f_m} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) e^{-j\pi fn/f_m} e^{j2\pi ft}\, dt$$

Interchanging the order of summation and integration, we get

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) \frac{1}{2f_m} \int_{-f_m}^{f_m} e^{j2\pi f\left(t-\frac{n}{2f_m}\right)} dt$$

or $$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) \frac{\sin(2\pi f_m t - n\pi)}{(2\pi f_m t - n\pi)} = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) \frac{\sin \pi(2f_m t - n\pi)}{\pi(2f_m t - n)}$$

Since, $$\text{sinc}\, \theta = \frac{\sin \pi\theta}{\pi\theta}$$

Therefore, $$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2f_m}\right) \text{sinc}\,(2f_m t - n) \quad -\infty < n < \infty$$

Hence, this is the interpolation formula to reconstruct $x(t)$ from its samples $x(nT_s)$. Therefore, from all above, it is clear that the signal may be completely represented into and recovered from its samples if the spacing between the successive samples is $\frac{1}{2f_m}$ seconds *i.e.*, $f_s = 2f_m$ samples per second.

Scanned by CamScanner

Sampling Frequency for Bandpass Signal

Since the spectral range of the bandpass signal is 20 kHz to 82 kHz

Therefore

$$\text{Bandwidth} = 2f_m = 82 \text{ kHz} - 20 \text{ kHz} = 62 \text{ kHz}$$

Hence, Minimum Sampling rate = 2 × bandwidth = 2 × 62 = 124 kHz

Generally, the range of minimum sampling frequencies is specified for bandpass signals. It lies between $4f_m$ to $8 f_m$ samples per second.

Therefore,

Range of minimum sampling frequencies

$$= (2 \times \text{bandwidth}) \text{ to } (4 \times \text{bandwidth})$$

$$= 2 \times 62 \text{ kHz to } 4 \times 62 \text{ kHz} = 124 \text{ kHz to } 248 \text{ kHz}$$ **Ans.**

9.9. Sampling Techniques

In the last article, we discussed how sampling of a continuous-time signal is done. This sampling of a signal is done in several ways. Therefore, in this section, we shall discuss different types of sampling *i.e.*, sampling techniques.

Basically, there are three types of sampling techniques as under:

(*i*) Instantaneous sampling

(*ii*) Natural sampling

(*iii*) Flat-top sampling.

Out of these three, instantaneous sampling is called ideal sampling whereas natural sampling and flat-top sampling are called practical sampling methods. Now, let us discuss three different types of sampling techniques in detail.

9.9.1. Ideal Sampling or Instantaneous Sampling or Impulse Sampling

In the proof of sampling theorem, we used ideal or impulse sampling. In this type of sampling, the sampling function is a train of impulses. Figure 9.11(*b*) shows this sampling function.

$x(t)$ is the input signal (*i.e.*, signal to be sampled) as shown in figure 9.11(*a*).

Figure 9.11(*c*) shows a circuit to produce instantaneous or ideal sampling. This circuit is known as the **switching sampler**.

The working principle of this circuit is quite easy. The circuit simply consists of a switch. Now if we assume that the closing time '*t*' of the switch approaches zero, then the output $g(t)$ of this circuit will contain only instantaneous value of the input signal $x(t)$. Since the width of the pulse approaches zero, the instantaneous sampling gives a train of impulses of height equal to the instantaneous value of the input signal $x(t)$ at the sampling instant.

We know that the train of impulses may be represented as

$$\delta T_s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \qquad \text{...(9.20)}$$

This is known as sampling function and its waveform is shown in figure 9.11(*b*).

The sampled signal $g(t)$ is expressed as the multiplication of $x(t)$ ans $\delta_{Ts}(t)$.

Thus,

$$g(t) = x(t) \cdot \delta_{Ts}(t) \qquad \text{...(9.21)}$$

$$= x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \qquad \text{...(9.22)}$$

Scanned by CamScanner

or
$$g(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\cdot\delta(t-nT_s) \quad ...(9.23)$$

The Fourier transform of the ideally sampled signal given by above equation may be expressed as

$$G(f) = f_s \sum_{n=-\infty}^{\infty} X(f-nf_s) \quad ...(9.24)$$

Note: *This equation gives the spectrum of ideally sampled signal. It shows that the spectrum X(f) is periodic in f_s and weighted by f_s. However, it may be noted that ideal or instantaneous sampling is possible only in theory since it is impossible to have a pulse whose width approaches zero. Ideal sampling was used in last article to prove sampling theorem. Practically flat-top sampling and nature sampling are used.*

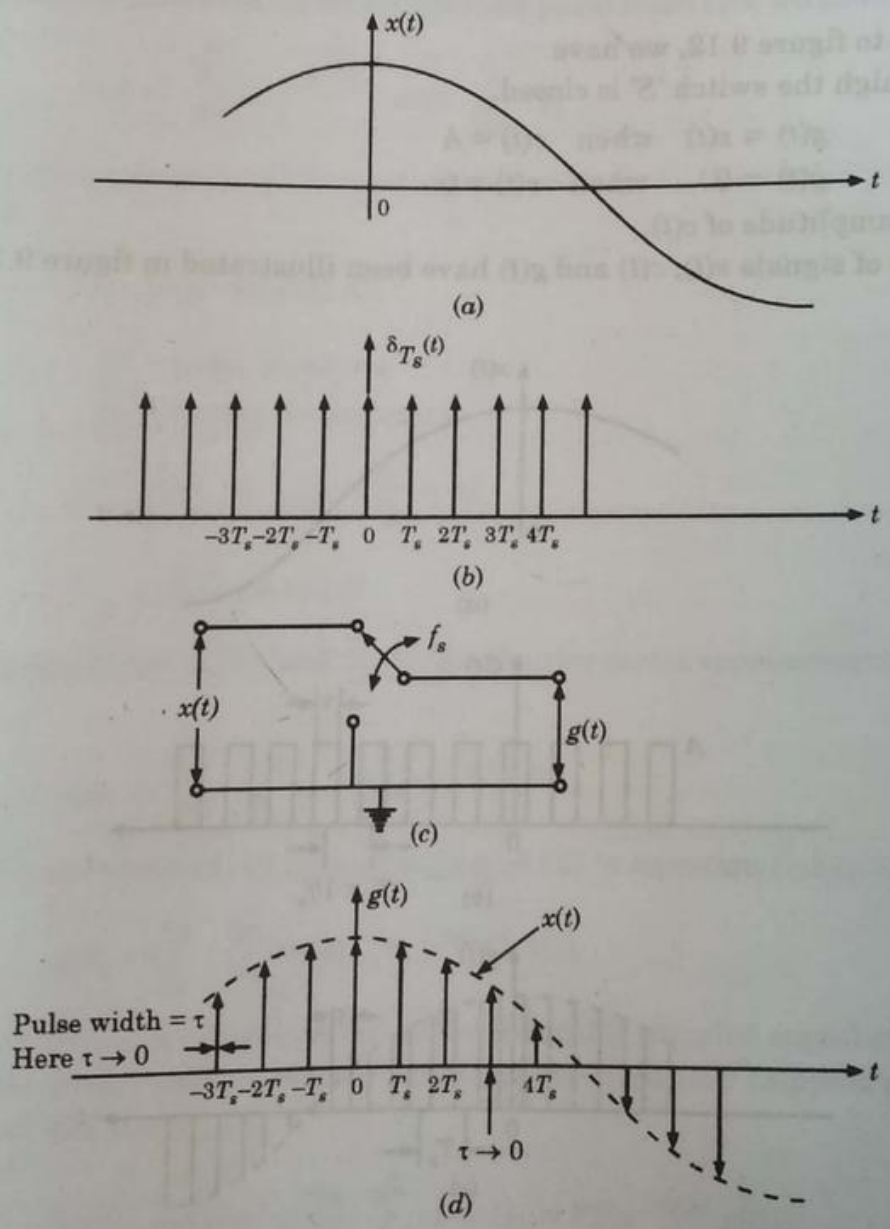


Fig. 9.11. *(a) Baseband signal (b) Impulse train (c) Functional diagram of a switching sampler (d) Sampled signal*

Scanned by CamScanner

Now, the sampled signal $g(t)$ may also be described mathematically as

$$g(t) = c(t) \cdot x(t) \quad ...(9.27)$$

Here, $c(t)$ is the periodic train of pulse of width τ and frequency f_s.

We know that the Exponential Fourier series for any periodic waveform is expressed as

$$c(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi nt/T_0} \quad ...(9.28)$$

Also, for the periodic pulse train of $c(t)$, we have

$$T_0 = T_s = \frac{1}{f_s} = \text{period of } c(t)$$

or

$$f_0 = f_s = \frac{1}{T_0} = \frac{1}{T_s} = \text{frequency of } c(t)$$

Therefore according to equation (9.28) for periodic pulse train $c(t)$, we have

$$c(t) = \sum_{n=-\infty}^{\infty} C_n . e^{j2\pi f_s nt} \text{ with } \frac{1}{T_0} = f_s \quad ...(9.29)$$

Now, it may be noted that since $c(t)$ is a rectangular pulse train, therefore C_n for this waveform will be expressed as

$$C_n = \frac{TA}{T_0} \sin c\,(f_n . T) \quad ...(9.30)$$

here T = pulse width = τ

and f_n = harmonic frequency

But here, $f_n = nf_s$ or $f_s = \frac{n}{T_0} = nf_0$

Hence,

$$C_n = \frac{\tau . A}{T_s} \cdot \sin c(f_n . \tau) \quad ...(9.31)$$

Therefore, using equations (9.29) and (9.30) the Fourier series representation for $c(t)$ will be given as

$$c(t) = \sum_{n=-\infty}^{\infty} \frac{\tau . A}{T_s} \cdot \sin c(f_n . \tau)\, e^{j2\pi f_s \cdot nt} \quad ...(9.32)$$

Now, substituting the value of $c(t)$ from equation (9.32) to equation (9.27), we get

$$g(t) = \frac{\tau A}{T_s} \cdot \sum_{n=-\infty}^{\infty} \sin c(f_n . \tau) . e^{j2\pi f_s nt} . x(t) \quad ...(9.33)$$

This is required time-domain representation for naturally sampled signal $g(t)$.

Now, to get the frequency-domain representation of the naturally sampled signal $g(t)$, let us take its Fourier transform as

$$G(f) = FT[g(t)] = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \text{sinc}\,(f_n . \tau)\, FT[e^{j2\pi f_s nt} . x(t)] \quad ...(9.34)$$

Recall the frequency-shifting property of Fourier transform which states that

$$e^{j2\pi f_s n(t)} . x(t) \longleftrightarrow X(f - f_s . n) \quad ...(9.35)$$

Scanned by CamScanner

Therefore, $$G(f) = \frac{\tau A}{T_s} \cdot \sum_{n=-\infty}^{\infty} \text{sinc}\,(f_n.\tau)\, X(f - f_s n) \quad ...(9.36)$$

Now, since $f_n = f_s n$ = harmonic frequency
Therefore, equation (9.36) becomes

$$G(f) = \frac{\tau A}{T_s} \cdot \sum_{n=-\infty}^{\infty} \text{sinc}\,(nf_s.\tau)\, X\,(f - nf_s) \quad ...(9.37)$$

Hence, we write
Spectrum of Naturally Sampled signal:

$$G(f) = \frac{\tau A}{T_s} \cdot \sum_{n=-\infty}^{\infty} \text{sinc}\,(nf_s.\ \tau)\, X(f - nf_s) \quad ...(9.38)$$

This equation shows that the spectra of $x(t)$ *i.e.*, $X(f)$ are periodic in f_s and are weighed by the sin c function.

Figure 9.14, illustrates some arbitray spectra for $x(t)$ and corresponding spectrum $G(f)$.

Note: *Thus form figure 9.14, it may be noted that unlike the spectrum of instantaneously sampled signal shown in figure 9.1(f), the spectrum of a naturally sampled signal is weighted by a sin c function whereas the spectrum of an instantaneously sampled signal (figure 9.1(f)) remains constant throughout the frequency range.*

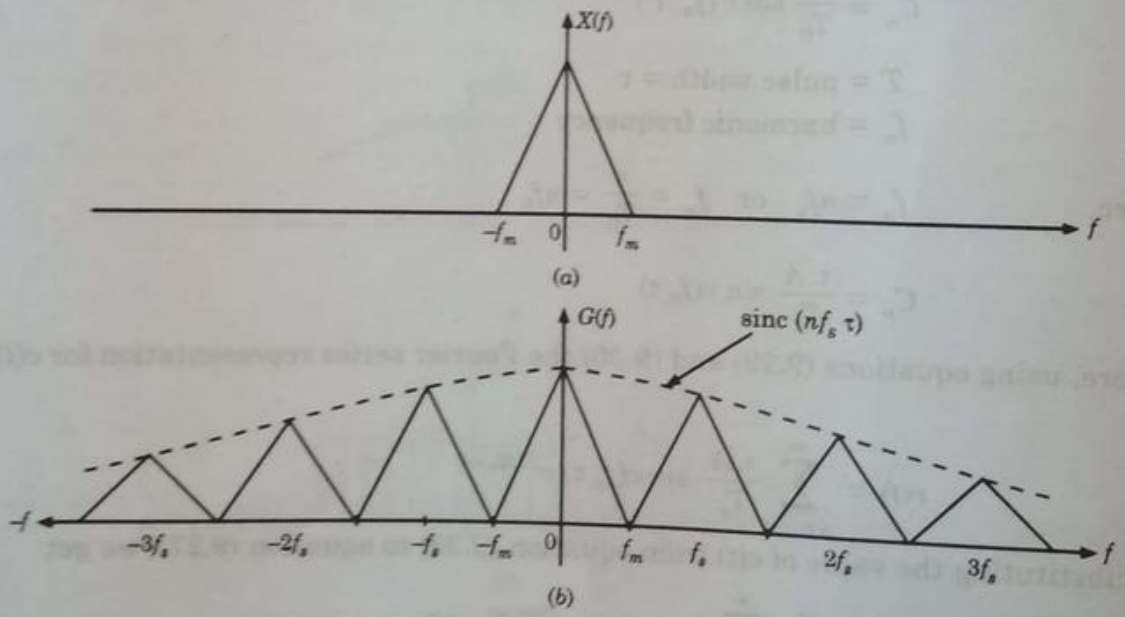


Fig. 9.14. *(a) Spectrum of continuous-time signal x(t),*
(b) Spectrum of naturally sampled signal

9.9.3. Flat Top Sampling or Rectangular Pulse Sampling

(U.P. Tech., Semester Examination, 2003-2004)

Flat top sampling like natural sampling is also a practically possible sampling method. But natural sampling is little complex whereas it is quite easy to get flat top samples.

In flat-top sampling or rectangular pulse sampling, the top of the samples remains constant and is equal to the instantaneous value of the baseband signal $x(t)$ at the start of sampling. The duration or width of each sample is τ and sampling rate is equal to $f_s = \frac{1}{T_s}$. Figure 9.15(*a*) shows

Scanned by CamScanner

the functional diagram of a sample and hold circuit which is used to generate the flat top samples. Figure 9.15(*b*) shows the general waveform of flat top samples.

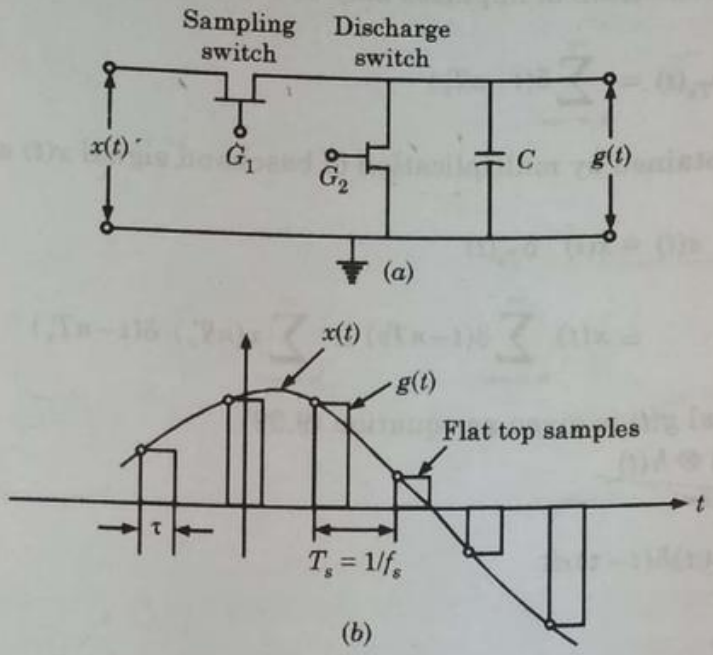


Fig. 9.15. *(a) A sample and hold circuit to generate flat top samples.*
(b) A general waveform of flat top sampling.

From figure 9.15(*b*), it may be noted that only starting edge of the pulse represents instantaneous value of the baseband signal $x(t)$. Also the flat top pulse of $g(t)$ is mathematically equivalent to the convolution of instantaneous sample and a pulse $h(t)$ as depicted in figure 9.16.

This means that the width of the pulse in $g(t)$ is determined by the width of $h(t)$ and the sampling instant is determined by delta function.

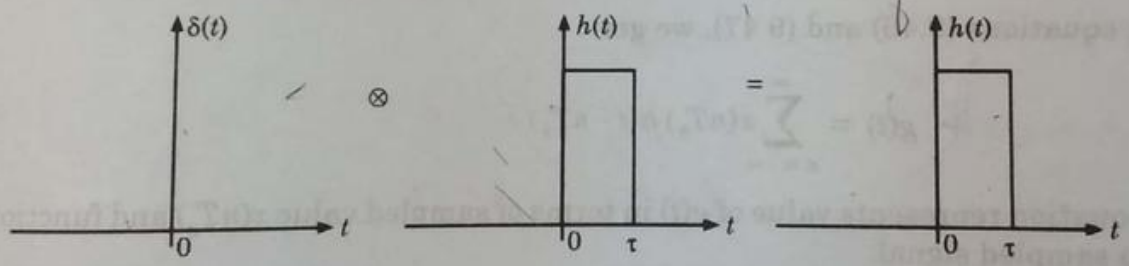


Fig. 9.16. *Convolution of any function with delta function is equal to that function.*

In figure 9.15(*b*), the starting edge of the pulse represents the point where baseband signal is sampled and width is determined by function $h(t)$. Therefore $g(t)$ will be expressed as

$$g(t) = s(t) \otimes h(t) \qquad ...(9.39)$$

This equation has been explained in figure 9.17.

Now, from the property of delta function, we know that for any function $f(t)$

$$f(t) \otimes \delta(t) = f(t) \qquad ...(9.40)$$

This property is used to obtain flat top samples. It may be noted that to obtain flat top sampling, we are not applying the equation (9.40) directly here *i.e.*, we are applying a modified form of equation (9.40). This modified equation is equation (9.39).

Thus, in this modified equation, we are taking $s(t)$ in place of delta function $\delta(t)$. Observe that $\delta(t)$ is a constant amplitude delta function whereas $s(t)$ is a varying amplitude train of impulses. This means that we are taking $s(t)$ which is an istantaneously sampled signal and this is convolved with function $h(t)$ as in equation (9.39).

Scanned by CamScanner

Therefore, on convolution of $s(t)$ and $h(t)$, we get a pulse whose duration is equal to $h(t)$ only but amplitude is defined by $s(t)$.

Now, we know that the train of impulses may be represented mathematically as

$$\delta_{Ts}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \quad \text{...(9.41)}$$

The signal $s(t)$ is obtained by multiplication of baseband signal $x(t)$ and $\delta_{Ts}(t)$.

Thus,

$$s(t) = x(t) \cdot \delta_{Ts}(t) \quad \text{...(9.42)}$$

$$= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nTs) = \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \delta(t - nT_s) \quad \text{...(9.43)}$$

Now, sampled signal $g(t)$ is given as equation (9.39)

$$g(t) = s(t) \otimes h(t) \quad \text{...(9.44)}$$

or

$$g(t) = \int_{-\infty}^{\infty} s(\tau) h(t - \tau)\, d\tau \quad \text{...(9.45)}$$

or

$$g(t) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(\tau - nT_s) h(t - \tau)\, d\tau = \sum_{n=-\infty}^{\infty} x(nT_s) \int_{-\infty}^{\infty} \delta(\tau - nT_s) h(t - \tau)\, d\tau \quad \text{...(9.46)}$$

According to shifting property of delta function, we know that

$$\int_{-\infty}^{\infty} f(t)\, \delta(t - t_0) = f(t_0) \quad \text{...(9.47)}$$

Using equations (9.46) and (9.47), we get

$$g(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, h(t - nT_s)$$

This equation represents value of $g(t)$ in terms of sampled value $x(nT_s)$ and function $h(t - nT_s)$ for flat top sampled signal.

Now, again from equation (9.39), we have

$$g(t) = s(t) \otimes h(t)$$

Taking Fourier transform of both sides of above equation, we get

$$G(f) = S(f)\, H(f) \quad \text{...(9.48)}$$

We know that $S(f)$ is given as

$$S(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \quad \text{...(9.49)}$$

Therefore, equation (9.48) becomes

$$G(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \cdot H(f) \quad \text{...(9.50)}$$

Scanned by CamScanner

Thus, spectrum of flat top sampled signal:

$$G(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)\, H(f) \qquad ...(9.51)$$

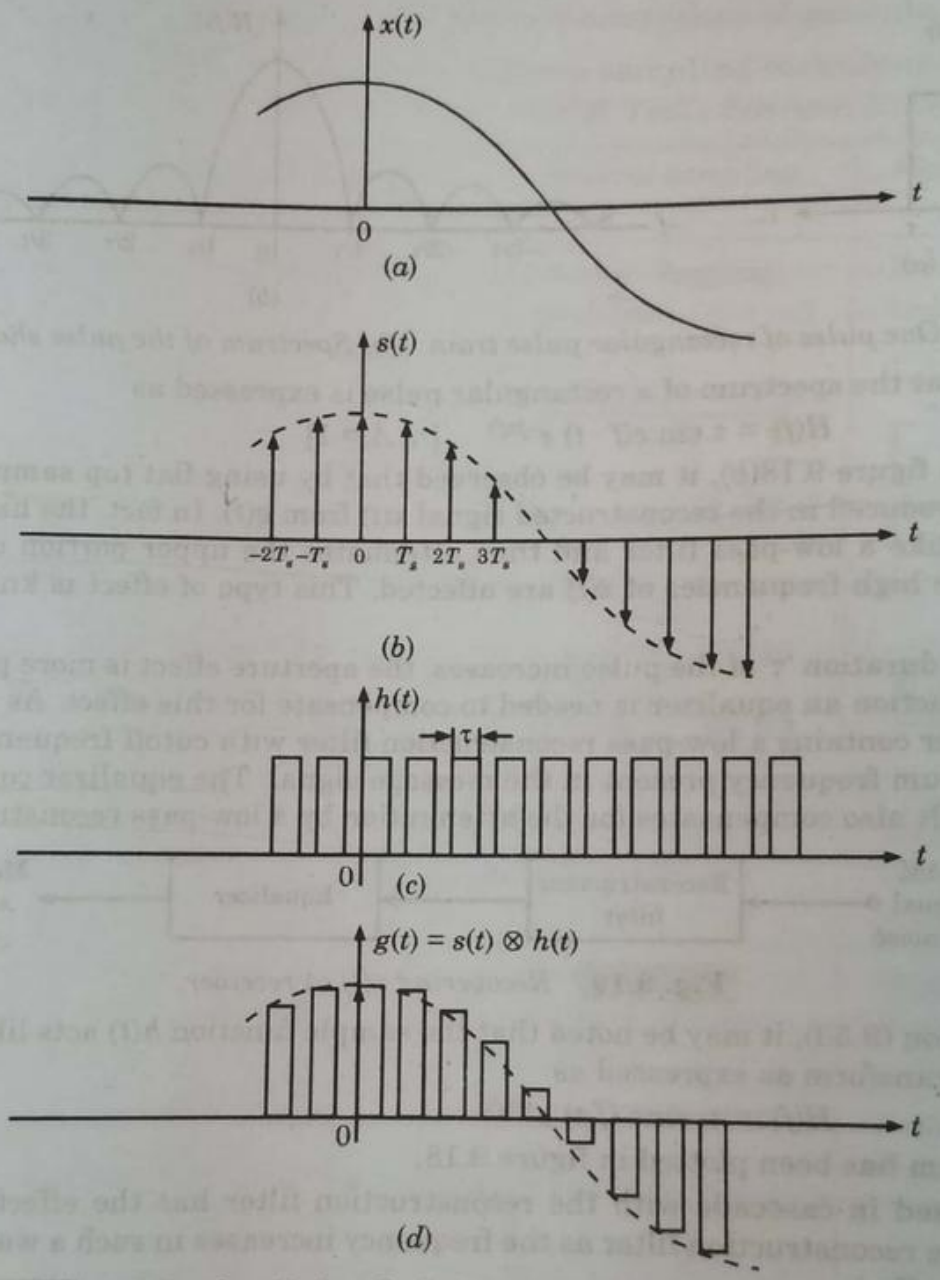


Fig. 9.17. *(a) Baseband signal x(t) (b) Instantaneously sample single s(t) (c) Constant pulse width function h(t) (d) Flat top sampled signal g(t) obtained through convolution of h(t) and s(t)*

9.10. Aperture Effect

The spectrum of flat top sampled signal is expressed as

$$G(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \cdot H(f) \qquad ...(9.52)$$

This equation shows that the signal $g(t)$ is obtained by passing the signal $s(t)$ through a filter having transfer function $H(f)$. The corresponding impulse response $h(t)$ in time-domain has been

Scanned by CamScanner

shown in figure 9.18(*a*). This is one pulse of rectangular pulse train shown in figure 9.17(*c*). Each sample of $x(t)$ [*i.e.*, $s(t)$] is convolved with this pulse. Equation (9.52) represents that the spectrum of this rectangular pulse is multiplied with that of $s(t)$.

Figure 9.18(*b*) shows the spectrum of one rectangular pulse of $h(t)$.

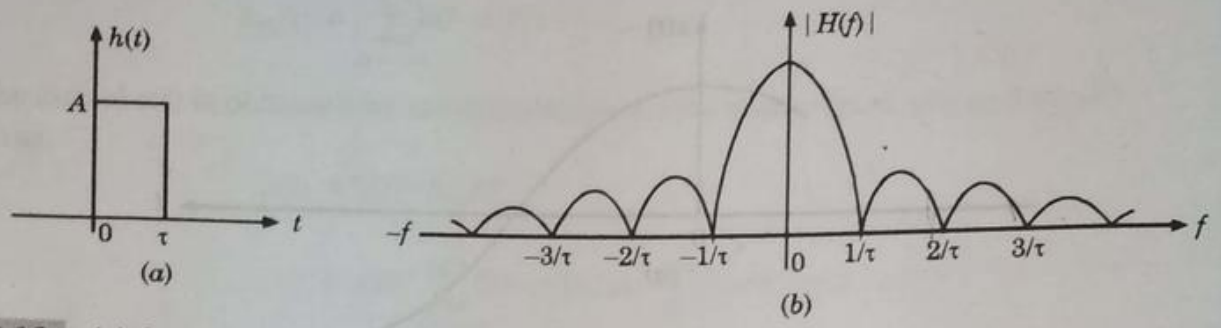


Fig. 9.18. *(a) One pulse of rectangular pulse train (b) Spectrum of the pulse shown in figure (a).*

We know that the spectrum of a rectangular pulse is expressed as

$$H(f) = \tau.\sin c(f \cdot t)\, e^{-j\pi f\tau} \quad [\because A = 1] \qquad ...(9.53)$$

Hence, from figure 9.18(*b*), it may be observed that by using flat top samples an amplitude distortion is introduced in the reconstructed signal $x(t)$ from $g(t)$. In fact, the high frequency roll-off of $H(f)$ acts like a low-pass filter and thus attenuates the upper portion of message signal spectrum. These high frequencies of $x(t)$ are affected. This type of effect is known as **aperture effect**.

Now, as the duration 'τ' of the pulse increases, the aperture effect is more prominent. Hence, during reconstruction an equalizer is needed to compensate for this effect. As depicted in figure 9.19, the receiver contains a low-pass reconstruction filter with cutoff frequency slightly higher than the maximum frequency present in the message signal. The equalizer compensates for the aperture effect. It also compensates for the attenuation by a low-pass reconstruction filter .

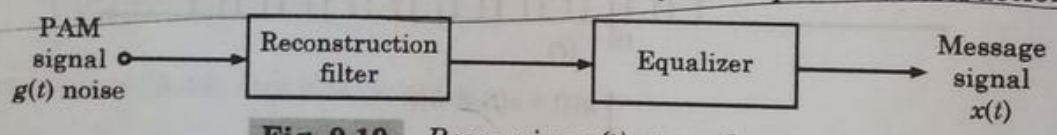


Fig. 9.19. *Recovering x(t) at receiver.*

From equation (9.53), it may be noted that the sample function $h(t)$ acts like a low-pass filter where Fourier transform as expressed as

$$H(f) = \tau.\ \text{sinc}\,(f.\tau)\, e^{-j\pi f\tau} \qquad ...(9.54)$$

This spectrum has been plotted in figure 9.18.

Equalizer used in casecade with the reconstruction filter has the effect of decreasing the inband loss of the reconstruction filter as the frequency increases in such a way as to compensate for the aperture effect.

Also, the transfer function of the equalizer is expressed as

$$H_{eq}(f) = \frac{K \cdot e^{-j2\pi f t_d}}{H(f)} \qquad ...(9.55)$$

Here 't_d' is known as the delay introduced by low-pass filter which is equal to τ/2.

Therefore,
$$H_{eq}(f) = \frac{K \cdot e^{-j\pi f\tau}}{\tau\,\text{sinc}\,(f\tau)\, e^{-j\pi f\tau}} \qquad ...(9.56)$$

or
$$H_{eq}(f) = \frac{K}{\tau\,\text{sinc}\,(f\tau)} \qquad ...(9.57)$$

which is the transfer function of an equalizer.

Scanned by CamScanner

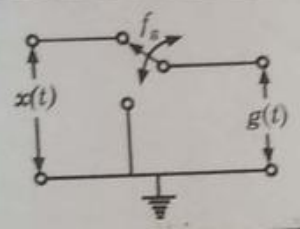
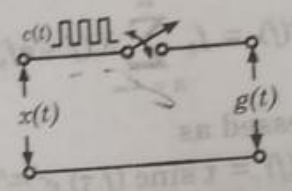
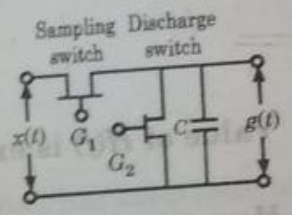
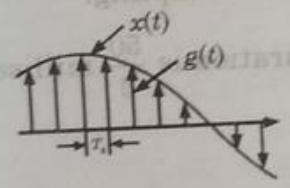
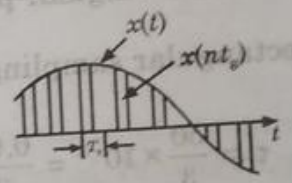
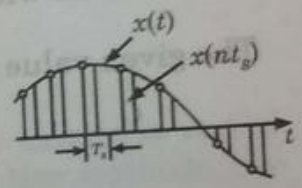
9.11. Comparison of Various Sampling Techniques

We can compare various sampling techniques on the basis of their method, noise interference and spectral properties etc.

The Table 9.1 lists some of the important points of comparison of generation.

Table 9.1. Comparison of three sampling techniques

(U.P. Tech., Semester, Examination, 2003-04)

S. No.	*Parameter of comparison*	*Ideal or instantaneous sampling*	*Natural sampling*	*Flat top sampling*
1.	Sampling principle	It uses multiplication	It uses chopping principle	It uses sample and hold circuit
2.	Generation circuit			
3.	Waveforms involved			
4.	Feasibility	This is not a practically possible method	This method is used practically	This method is also used practically
5.	Sampling rate	Sampling rate tends to infinity	Sampling rate satisfies Nyquist criteria	Sampling rate satisfies Nyquist criteria
6.	Noise interference	Noise interference is maximum	Noise interference is minimum	Noise interference is maximum
7.	Time domain representation	$g(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$	$g(t) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} x(t) \sin c(nf_s\tau) e^{j2\pi nf_s t}$	$g(t) = \sum_{n=-\infty}^{\infty} x(nT_s)h(t - nT_s)$
8.	Frequency domain representation	$G(f) = f_x \sum_{n=-\infty}^{\infty} X(f - nf_s)$	$G(f) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \sin c(nf_s\tau) X(f - nf_s)$	$G(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) H(f)$

Example 9.6. Figure 9.20 shows the spectrum of an arbitary signal $x(t)$. This signal is sampled at the Nyquist rate with a periodic train of rectangular pulses of duration $\frac{50}{3}$ milliseconds.

Scanned by CamScanner

Determine the spectrum of the sampled signal for frequencies upto 50 Hz giving relevant expression.

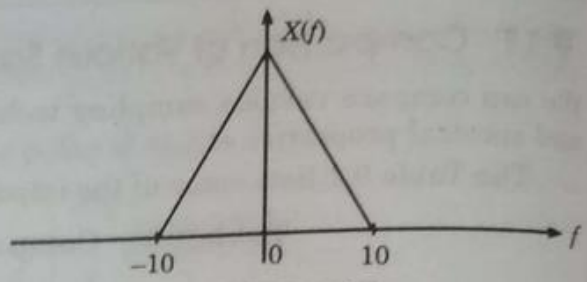


Fig. 9.20.

Solution: From figure 9.20, it may be observed that the signal is bandlimited to 10 Hz.

thus $f_m = 10$ Hz

So the Nyquist rate is $= 2 f_m$

$= 2 \times 10 = 20$ Hz

Since the signal is sampled at the Nyquist rate, the sampling frequency would be

$$f_s = 20 \text{ Hz}$$

Given that the rectangular pulses are used for sampling *i.e.*, flat top sampling is used. The spectrum of the flat top sampled signal is given as

$$G(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) H(f) \qquad ...(i)$$

Value of $H(f)$ is expressed as

$$H(f) = \tau \text{ sinc } (f\tau)\, e^{-j\pi f\tau} \qquad ...(ii)$$

Here τ is the width of the rectangular pulse used for sampling.

The given value of rectangular sampling pulse duration is $\frac{50}{3}$ milliseconds *i.e.*,

$$\tau = \frac{50}{3} \times 10^{-3} = \frac{0.05}{3} \text{ sec}$$

Substituting the value of τ in equation (*ii*), we get

$$H(f) = \frac{0.05}{3} \text{ sinc}\left(\frac{0.05f}{3}\right) e^{-j0.05\pi f/3}$$

Again, putting this value of $H(f)$ and f_s in equation (*i*), we get

$$G(f) = 20 \sum_{n=-\infty}^{\infty} X(f - 20n) \frac{0.05}{3} \text{ sinc}\left(\frac{0.05f}{3}\right) e^{-j0.05\pi f/3} \qquad (\because f_s = 20)$$

$$G(f) = \frac{1}{3} \sum_{n=-3}^{3} X(f - 20n) \text{ sinc}\left(\frac{0.05f}{3}\right) e^{-j0.05\pi f/s}$$

This expression gives the spectrum up to 60 Hz (since $n = \pm 3$) for the sampled signal. **Ans.**

Example 9.7. A flat top sampling system samples a signal of maximum frequency 1 Hz with 9.5 Hz sampling frequency. The duration of the pulse is 0.2 seconds. Compute the amplitude distortion due to aperture effect at the highest signal frequency. Also determine the equalization characteristic.

Solution: Given that sampling frequency

$$f_s = 2.5 \text{ Hz}$$

Maximum signal frequency,

$$f_{max} = 1 \text{ Hz}$$

and pulse width $\tau = 0.2$ seconds.

We know that the aperture effect is expressed by a transfer function $H(f)$ as

$$H(f) = \tau \text{ sin c } (f\tau)\, e^{-j\pi f\tau}$$

Scanned by CamScanner

The magnitude of this equation will be

$$|H(f)| = \tau \text{ sinc } (f\tau) \quad ...(i)$$

$$|H(f)| = 0.2 \text{ sinc } (f \times 0.2)$$

Now, aperture effect at the highest frequency will be obtained by putting $f = f_{max} = 1$ Hz in equation (*i*) *i.e.*,

$$|H(1)| = 0.2 \text{ sinc } (0.2) = 0.18709$$

or $$|H(1)| = 18.70\% \quad \textbf{Ans.}$$

Also, the equalizer characteristic is expressed as

$$H_{eq}(f) = \frac{K}{\tau \text{ sinc } (f\tau)}$$

Substituting, $\tau = 0.2$ second and assuming $K = 1$, the last equation becomes

$$H_{eq}(f) = \frac{1}{0.2 \text{ sinc } (0.2f)}$$

This equation is the plot of $H_{eq}(f)$ versus f and it represents the equalization characteristic to overcome aperture effect.

9.12. Analog Pulse Modulation Methods

We know that in analog modulation systems, some parameter of a sinusoidal carrier is varied according to the instantaneous value of the modulating signal. In pulse modulation methods, the carrier is no longer a continuous signal but consists of a pulse train. Some parameter of which is varied according to the instantaneous value of the modulating signal. There are two types of pulse modulation systems as under:

(*i*) Pulse Amplitude Modulation (PAM)

(*ii*) Pulse Time Modulation (PTM)

In pulse amplitude modulation (PAM), the amplitude of the pulses of the carrier pulse train is varied in accordance with the modulating signal whereas in pulse time modulation (PTM), the timing of the pulses of the carrier pulse train is varied.

They are two types of PTM as under:

(*i*) Pulse width modulation (PWM)*

(*ii*) Pulse position modulation (PPM)

In Pulse width modulation, the width of the pulses of the carrier pulse train is varied in accordance with the modulating signal whereas in Pulse position modulation (PPM), the position of pulses of the carrier pulse train is varied. Figure 9.21 shows three types of pulse analog modulation methods.

According to the sampling theorem, if a modulating signal is bandlimited to f_m Hz,** the sampling frequency must be at least $2f_m$ Hz and, hence the frequency of the carrier pulse train must also be least $2f_m$ Hz.

At this point, it may be noted that all the above pulse modulation methods (*i.e.*, PAM, PWM and PPM) are called analog Pulse modulation methods because the modulating signal is analog in nature in PAM, PWM and PPM.

---

* Pulse width modulation is also known as Pulse duration modulation (PDM).

** If a signal is said to be bandlimited to f_m Hz, then it means that the maximum frequency component in this signal is f_m Hz.

Scanned by CamScanner

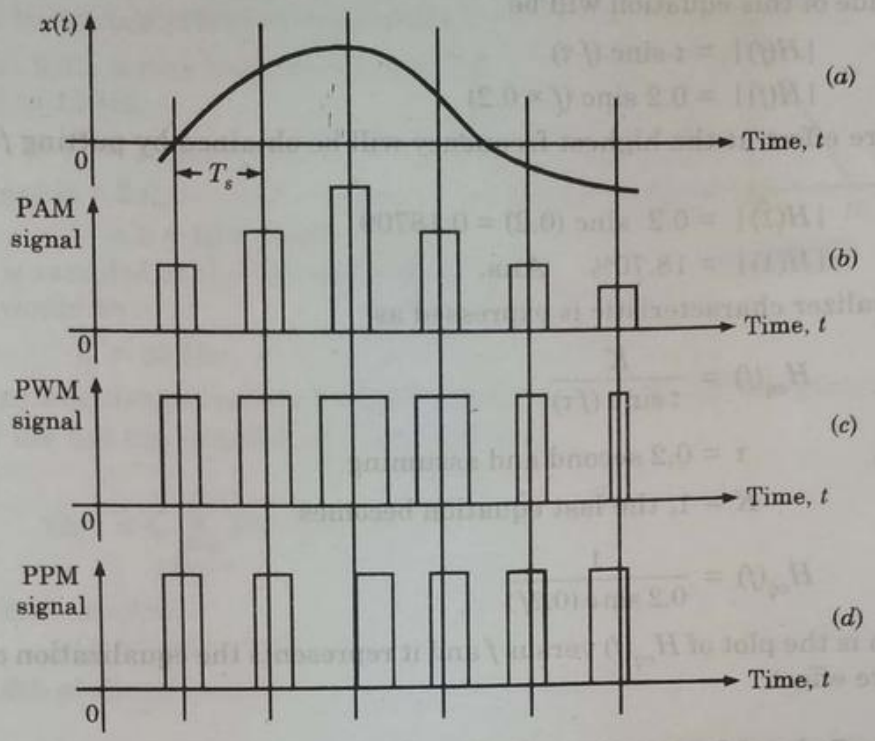


Fig. 9.21. *Different types of pulse analog modulation methods.*

9.13. Pulse Amplitude Modulation (PAM)

Pulse amplitude modulation may be defined as that type of modulation in which the amplitudes of regularly spaced rectangular pulses vary according to instantaneous value of the modulating or message signal. In fact, the pulses in a PAM signal may be of flat top type or natural type or ideal type. Actually all the sampling methods which have been discussed in last sections are basically pulse amplitude modulation methods.

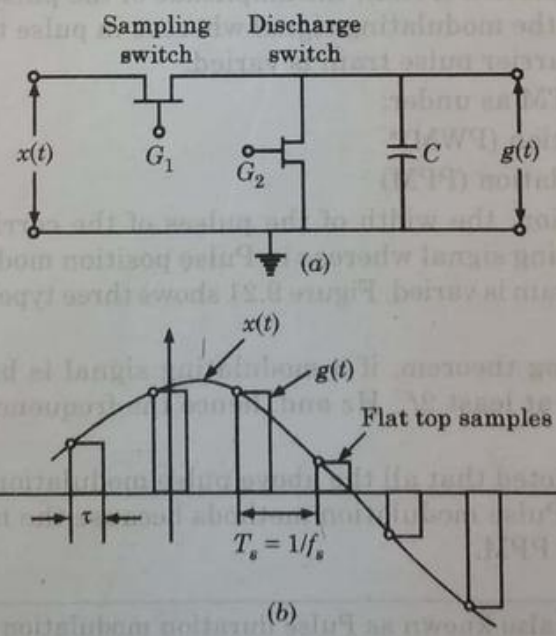


Fig. 9.22. *(a) Sample and hold circuit generating flat top sampled PAM*
(b) Waveforms of flat top sampled PAM

Scanned by CamScanner

Out of these three pulse amplitude modulation methods, the Flat top PAM is most popular and is widely used. The reason for using flat top PAM is that during the transmission, the noise interferes with the top of the transmitted pulses and this noise can be easily removed if the PAM pulse has flat top.

However, in case of natural samples PAM signal, the pulse has varying top in accordance with the signal variation. Now, when such type of pulse is received at the receiver, it is always contaminated by noise. Then it becomes quite difficult to determine the shape of the top of the pulse and thus amplitude detection of the pulse is not exact. Due to this, errors are introduced in the received signal.

Therefore, flat top sampled PAM is widely used.

Figure 9.22 shows the sample and hold circuit to produce flat top sampled PAM and the waveform for flat top sampled PAM.

Working Principle

A sample and hold circuit shown in figure 9.22 is used to produce flat top sampled PAM. The working principle of this circuit is quite easy. The sample and hold (S/H) circuit consists of two field

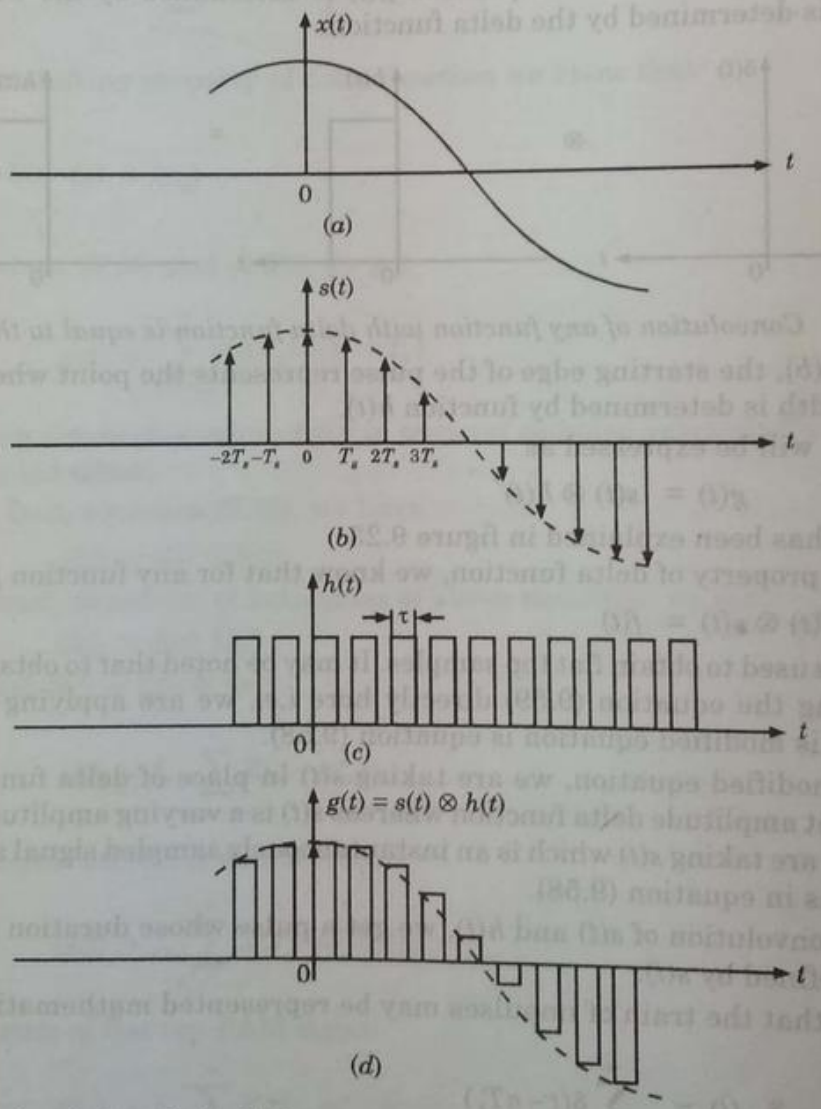


Fig. 9.23. *(a) Baseband signal x(t) (b) Instantaneously sample single s(t) (c) Constant pulse width function h(t) (d) Flat top sampled PAM signal g(t) obtained through convolution of h(t) and s(t).*

effect transistors (FET) switches and a capacitor. The sampling switch is closed for a short duration by a short pulse applied to the gate G_1 of the transistor. During this period, the capacitor 'C' is

Scanned by CamScanner

quickly charged upto a voltage equal to the instantaneous sample value of the incoming signal $x(t)$. Now, the sampling switch is opened and the capacitor 'C' holds the charge. The discharge switch is then closed by a pulse applied to gate G_2 of the other transistor. Due to this, the capacitor 'C' is discharged to zero volts. The discharges switch is then opened and thus capacitor has no voltage.

Hence, the output of the sample and hold circuit consists of a sequence of Flat top samples as shown in figure 9.23.

Mathematical Analysis

In a Flat top PAM, the top of the samples remains constant and is equal to the instantaneous value of the baseband signal $x(t)$ at the start of sampling. The duration or width of each sample is τ and sampling rate is equal to $f_s = \frac{1}{T_s}$. From figure 9.22 (*b*), it may be noted that only starting edge of the pulse represents instantaneous value of the baseband signal $x(t)$. Also, the flat top pulse of $g(t)$ is mathematically equivalent to the convolution of instantaneous sample and a pulse $h(t)$ as depicted in figure 9.24.

This means that the width of the pulse in $g(t)$ is determined by the width of $h(t)$ and the sampling instant is determined by the delta function.

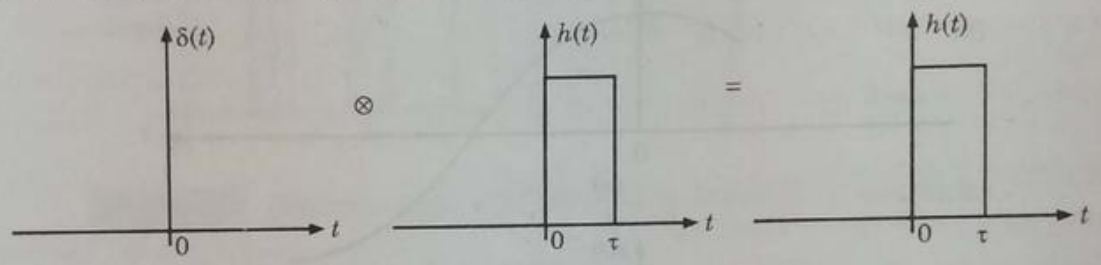


Fig. 9.24. *Convolution of any function with delta function is equal to that function.*

In figure 9.22 (*b*), the starting edge of the pulse represents the point where baseband signal is sampled and width is determined by function $h(t)$.

Therefore, $g(t)$ will be expressed as

$$g(t) = s(t) \otimes h(t) \qquad ...(9.58)$$

This equation has been explained in figure 9.23.

Now, from the property of delta function, we know that for any function $f(t)$

$$f(t) \otimes \delta(t) = f(t) \qquad ...(9.59)$$

This property is used to obtain flat top samples. It may be noted that to obtain flat top sampling, we are not applying the equation (9.59) directly here *i.e.*, we are applying a modified from of equation (9.59). This modified equation is equation (9.58).

Thus, in this modified equation, we are taking $s(t)$ in place of delta functions $\delta(t)$. Observe that $\delta(t)$ is a constant amplitude delta function whereas $s(t)$ is a varying amplitude train of impulses. This means that we are taking $s(t)$ which is an instantaneously sampled signal and this is convolved with function $h(t)$ as in equation (9.58).

Therefore, on convolution of $s(t)$ and $h(t)$, we get a pulse whose duration is equal to $h(t)$ only but amplitude is defined by $s(t)$.

Now, we know that the train of impulses may be represented mathematically as

$$\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \qquad ...(9.60)$$

The signal $s(t)$ is obtained by multiplication of baseband signal $x(t)$ and $\delta_{T_s}(t)$.

Thus,

$$s(t) = x(t) \cdot \delta_{T_s}(t) \qquad ...(9.61)$$

Scanned by CamScanner

or $$s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \delta(t - nT_s) \quad ...(9.62)$$

Now, sampled signal $g(t)$ is given as equation (9.58)

$$g(t) = s(t) \otimes h(t) \quad ...(9.63)$$

or $$g(t) = \int_{-\infty}^{\infty} s(\tau)\, h(t-\tau)\, d\tau \quad ...(9.64)$$

or $$g(t) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(\tau - nT_s)\, h(t-\tau)\, d\tau$$

or $$g(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \int_{-\infty}^{\infty} \delta(\tau - nT_s)\, h(t-\tau)\, d\tau \quad ...(9.65)$$

According to shifting property of delta function we know that

$$\int_{-\infty}^{\infty} f(t)\, \delta(t - t_0) = f(t_0) \quad ...(9.66)$$

Using equations (9.65) and (9.66), we get

$$g(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, h(t - nT_s)$$

This equation represents value of $g(t)$ in terms of sampled value $x(nT_s)$ and function $h(t - nT_s)$ for flat top sampled signal.

Now, again from equation (9.58), we have

$$g(t) = s(t) \otimes h(t)$$

Taking Fourier transform of both sides of above equation, we get

$$G(f) = S(f)\, H(f) \quad ...(9.67)$$

We know that $S(f)$ is given as

$$S(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \quad ...(9.68)$$

Therefore, equation (9.67) will become

$$G(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)\, H(f) \quad ...(9.69)$$

Thus, spectrum of flat top PAM signal:

$$G(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)\, H(f) \quad ...(9.70)$$

Here, $H(f)$ is the Fourier transform of the rectangular pulse. The spectrum of this rectangular pulse is shown in figure 9.18(*b*). Let the spectrum of $s(t)$ be the rectangular pulse train as shown in figure 9.25(*a*) and the spectrum of $h(t)$ *i.e.*, $H(f)$ is shown in figure 9.25(*b*).

Scanned by CamScanner

By equation (9.67), we know that

$$G(f) = S(f) \cdot H(f)$$

Thus, according to above equation, we can plot the spectrum $G(f)$ as shown in figure 9.25(*b*).

Note: *It may be observed in figure 9.25(b) that higher frequencies in S(f) are attenuated due to roll-off characteristics of the 'sinc' pulse.* ***This effect is popularly known as aperture effect.***

An equalizer is needed to overcame this effect.

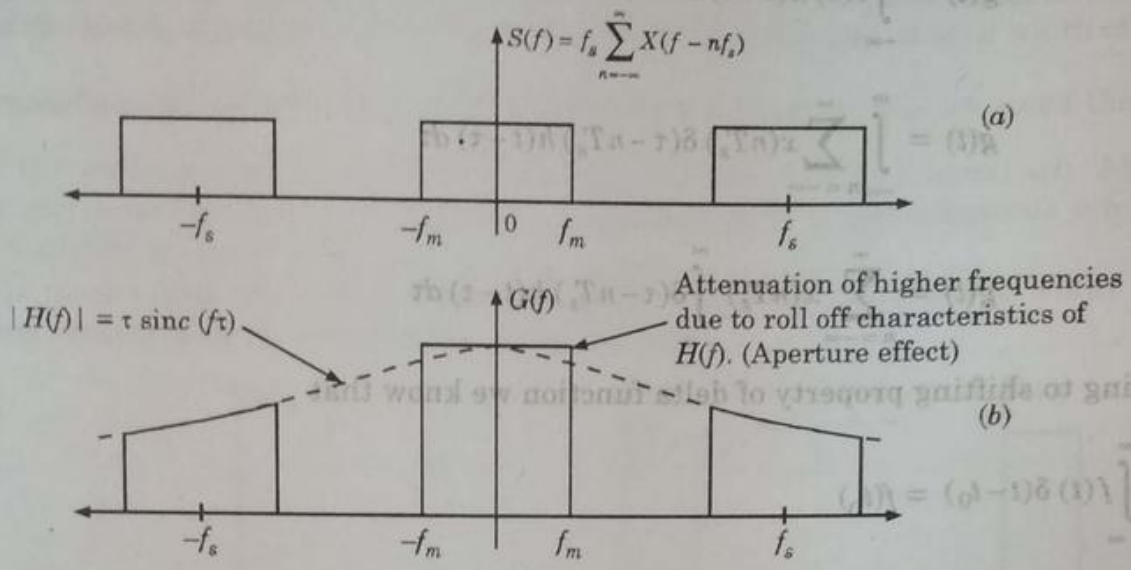


Fig. 9.25. *(a) Spectrum of some arbitrary signal. The signal is sampled at f_s and maximum frequency in the signal is f_m,*
(b) Spectrum of flat top signal. The dotted curve is $H(f) = \tau \sin c\ (f\tau)$

9.13.1. Naturally Sampled Pulse Amplitude Modulated (PAM) Signal

We have discussed natural sampling in article 9.9. This natural sampling is basically pulse amplitude modulation (PAM). Therefore, it is called naturally sampled PAM signal.

Thus, time-domain representation of a naturally-sampled PAM signal will be given as

$$g(f) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} x(t), \sin c(f_n \tau)\, e^{j2\pi n f_s t} \quad ...(9.71)$$

and the frequency-domain representation, *i.e.* frequency-spectrum of a naturally-sampled PAM signal will be given as

$$G(f) = \frac{\tau A}{T_s} \sum_{n=-\infty}^{\infty} \sin c\,(nf_s\tau) \cdot X(f - nf_s) \quad ...(9.72)$$

9.13.2. Instantaneous or Idealy Sampled Pulse Amplitude Modulated (PAM) Signal

We have discussed ideal or instantaneous sampling in article 9.9. This instantaneous sampling is basically pulse amplitude modulation (PAM). Therefore, it is called ideally or instantaneously sampled PAM signal.

Thus, time-domain representation of a ideally or instantaneously sampled PAM signal will be given as

$$g(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \delta(t - nT_s) \quad ...(9.73)$$

Scanned by CamScanner

and the frequency-domain representation *i.e.*, frequency-spectrum of a ideally or instantaneously sampled PAM sinal will be given as

$$G(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \qquad ...(9.74)$$

9.13.3. Transmission Bandwidth in Pulse Amplitude Modulation (PAM)

In a pulse amplitude modulated (PAM) signal the pulse duration 'τ' is considered to be vary very small in comparison to time period (*i.e.*, sampling period) T_s between any two samples *i.e.*,

$$\tau << T_s \qquad ...(9.75)$$

Now, if the maximum frequency in the modulating signal $x(t)$ is f_m, then according to sampling theorem, the sampling frequency f_s must be equal to or higher than the Nyquist rate, *i.e.*

$$f_s \geq 2f_m \qquad ...(9.76)$$

or $$\frac{1}{T_s} \geq 2f_m \qquad (\because f_s = \frac{1}{T_s})$$

or $$T_s \leq \frac{1}{2f_m}$$

But according to equation (9.75), we have

$$\tau << T_s$$

therefore $$\tau << T_s \leq \frac{1}{2f_m} \qquad ...(9.77)$$

Now, if the 'ON' and 'OFF' time of the pulse amplitude modulated (PAM) pulse is same as shown in figure 9.26(*a*) then maximum frequency of the PAM pulse will be

$$f_{max} = \frac{1}{\tau + \tau} = \frac{1}{2\tau} \qquad ...(9.78)$$

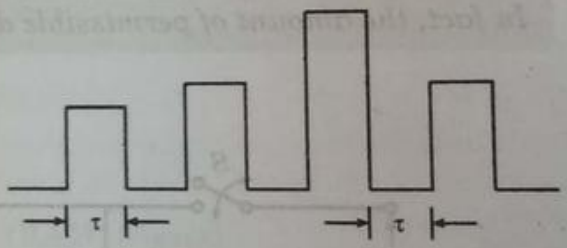


Fig. 9.26. *(a) Illustration of maximum frequency in PAM signal.*

Therefore, the bandwidth required for the transmission of a PAM signal would be equal to the maximum frequency f_{max} given by the equation (9.78)

Thus, we have

Transmission bandwidth

$$BW \geq f_{max} \qquad ...(9.79)$$

But $$f_{max} = \frac{1}{2\tau}$$

Hence $$BW \geq \frac{1}{2\tau}$$

Again, since $$\tau << \frac{1}{2f_m}$$

Therefore $$BW \geq \frac{1}{2\tau} >> f_m \qquad ...(9.80)$$

or $$BW >> f_m \qquad ...(9.81)$$

Scanned by CamScanner

9.13.4. Demodulation of PAM Signals

As discussed earlier, demodulation is the reverse process of modulation in which the modulating signal is recovered back from a modulated signal. For pulse-amplitude modulated (PAM) signals, the demodulation is done using a holding circuit. Figure 9.26(*b*) shows the block diagram of a PAM demodulator.

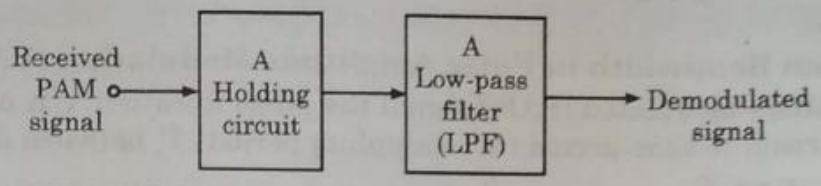


Fig. 9.26. *(b) A block diagram of PAM demodulator*

In this method, the received PAM signal is allowed to pass through a holding circuit and a low pass filter (LPF) as shown in above figure. Now, figure 9.27(*a*) illustrates a very simple holding circuit. Here the switch '*S*' is closed after the arrival of the pulse and it is opened at the end of the pulse. In this way, the capacitor *C* is charged to the pulse amplitude value and it holds this value during the interval between the two pulses. Hence, the sampled values are held as shown in figure 9.27(*c*). After this the holding circuit output is smothened in Low Pass filter as shown in figure 9.27(*c*). It may be observed that some kind of distortion is introduced due to the holding circuit. In fact the circuit of figure 9.27(*b*) is known as **zero-order holding circuit.** This zero-order holding circuit considers only the previous sample to decide the value between the two pulses.

Note: *It may be noted the first order hold circuit considers the previous two samples whereas a second order holding circuit considers the previous three samples and so on. However, as the order of the holding circuit increases, the distortion decreases at the cost of the circuit complexity. In fact, the amount of permissible distortion decides the order of the holding circuit.*

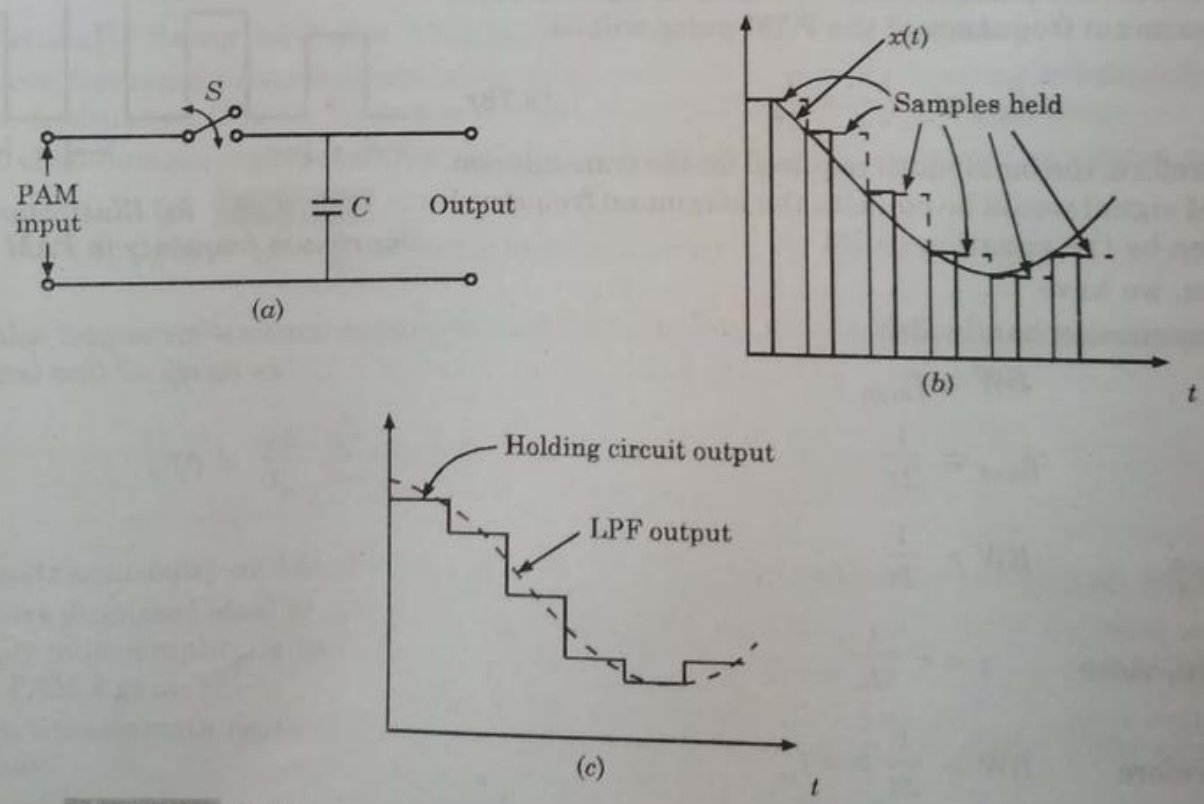


Fig. 9.27. *(a) A zero-order holding circuit (b) The output of holding circuit (c) The output of a low pass filter (LPF)*

Scanned by CamScanner

9.13.5. Transmission of PAM Signals

If the PAM signals are to be transmitted directly *i.e.*, over a pair of wires then no further signal processing is necessary. However, if they are to be transmitted through the space using an antenna, they must first be amplitude or frequency or phase modulated by a high frequency carrier and only then they can be transmitted. Thus, the overall system will be then known as PAM-AM or PAM-FM or PAM-PM respectively. At the receiving end, AM or FM or PM detection is first employed to get the PAM signal and then the message signal is recovered from it.

Example 9.8. For a pulse-amplitude modulated (PAM) transmission of voice signal having maximum frequency equal to f_m = 3 kHz, calculate the transmission bandwidth. It is given that the sampling frequency f_s = 8 kHz and the pulse duration $\tau = 0.1T_s$.

Solution: We know that the sampling period T_s is expressed as

$$T_s = \frac{1}{f_s} = \frac{1}{8 \times 10^3} \text{ seconds}$$

$$T_s = 0.125 \times 10^{-3} \text{ seconds} = 125\ \mu \text{ seconds} \qquad ...(i)$$

Also, τ is given that

$$\tau = 0.1\ T_s$$

Using (*i*), we get

$$\tau = 0.1 \times 125 = 12.5\ \mu \text{ seconds} \qquad ...(ii)$$

Now, we know that the transmission bandwidth for PAM signal is expressed as

$$BW \geq \frac{1}{2\tau}$$

Using equation (*ii*), we get

$$BW \geq \frac{1}{2 \times 12.5 \times 10^{-6}} \geq \frac{1 \times 10^6}{25}$$

$$BW \geq 40 \text{ kHz} \qquad \textbf{Ans.}$$

9.13.6. Drawbacks of Pulse-Amplitude Modulated (PAM) Signal

Following are the drawbacks of a PAM signal:

(*i*) The bandwidth required for the transmission of a PAM signal is very very large in comparison to the maximum frequency present in the modulating signal.

(*ii*) Since the amplitude of the PAM pulses varies in accordance with the modulating signal therefore the interference of noise is maximum in a PAM signal. This noise cannot be removed easily.

(*iii*) Since the amplitude of the PAM signal varies, therefore, this also varies the peak power required by the transmitter with modulating signal.

9.14. Pulse Time Modulation

In pulse time modulation, the signal to be transmitted is sampled as in pulse amplitude modulation (PAM). In pulse time modulation, amplitude of pulse is held constant, whereas position of pulse or width of pulse is made proportional to the amplitude of signal at the sampling instant. There are two types of pulse time modulation, viz. Pulse Width Modulation (PWM) and Pulse Position Modulation (PPM). Because in both PWM and PPM, amplitude is held constant and does not carry any information, therefore amplitude limiters can be used. The amplitude limiters, similar to those used in FM, will clip off the portion of the signal corrupted by noise and hence provide a good degree of noise immunity.

Scanned by CamScanner

9.14.1. Pulse Width Modulation

Let us first discuss Pulse Width Modulation (PWM). This is also known as Pulse Duration Modulation (PDM). Three variations of pulse width modulation are possible. In one variation, the leading edge of the pulse is held constant and change in pulse width with signal is measured with respect to the leading edge. In other variation, the tail edge is held constant and with respect to it, pulse width is measured. In the third variation, centre of the pulse is held constant and pulse width changes on either side of the centre of the pulse. This has been illustrated in figure 9.28.

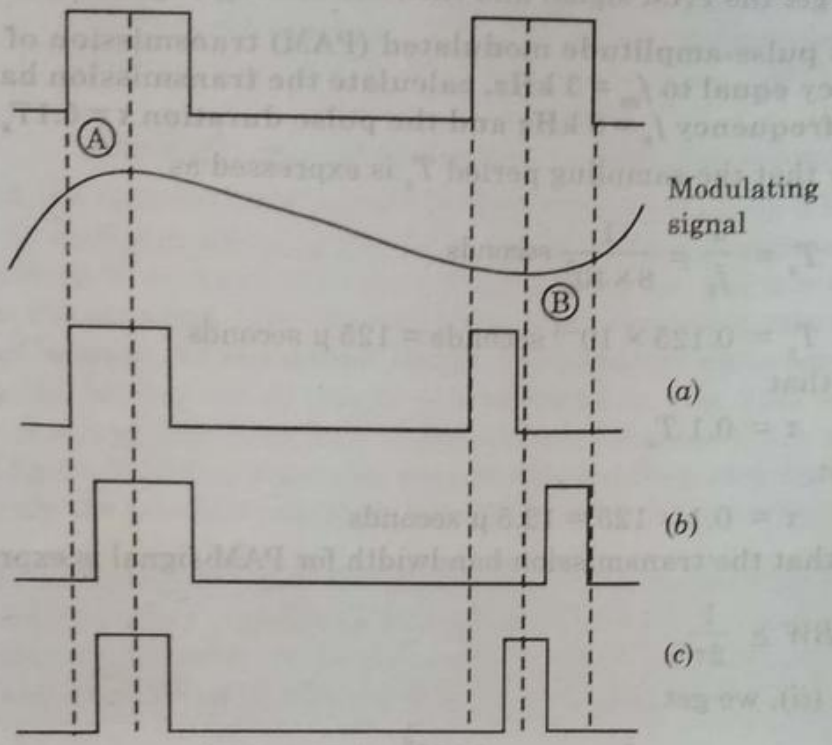


Fig. 9.28. *PWM waveforms*

The modulating signal is at its positive peak at point (A) and at its negative peak at (B). In figure 9.28 (*a*), the leading edge of pulse is kept constant and pulse width is measured from the lead edge. As shown, pulse width is maximum corresponding to point (A), while it is minimum at point (B).

In figure 9.28 (*b*), the tail edge of the pulse is kept constant and pulse width is measured from the tail end of the pulse. As before, pulse width is maximum corresponding to positive peak of the modulating signal and minimum at the negative peak.

As shown in figure 9.28 (*c*), the center of the pulse is kept constant and pulse extends on either side of the center of the pulse, depending upon the modulating signal.

9.14.2. Frequency Spectrum for PWM Wave

With a sinusoidal modulating signal at frequency f_m, the spectrum of PWM signal consists the modulating signal frequency f_m along with several harmonics. This is shown in figure 9.29.

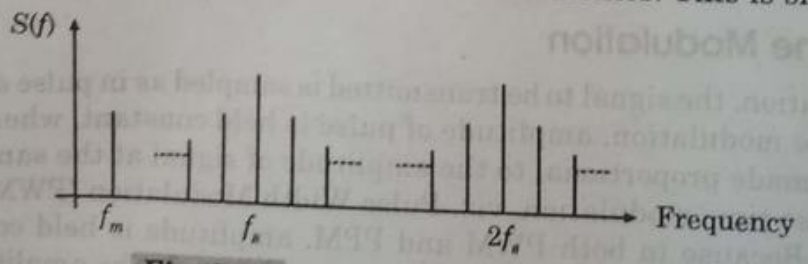


Fig. 9.29. *Spectrum of PWM signal*

To have a better separation with respect to frequency, between highest frequency of baseband signal [in Fig. 9.29, f_m] and lower sidebands of f_s (sampling frequency), a higher sampling frequency which is more than Nyquist rate is used; and pulse width deviation is kept small.

Scanned by CamScanner

9.14.3. Modulation of PWM Signal or PWM Generation

Figure 9.30 shows pulse width modulator. It is basically a monostable multivibrator with a modulating input signal applied at the control voltage input. Internally, the control voltage is adjusted to the 2/3 V_{CC}. Externally applied modulating signal changes the control voltage, and hence the threshold voltage level. As a result, the time period required to charge the capacitor up to threshold voltage level changes, giving pulse modulated signal at the output, as shown in the figure 9.30 (*b*).

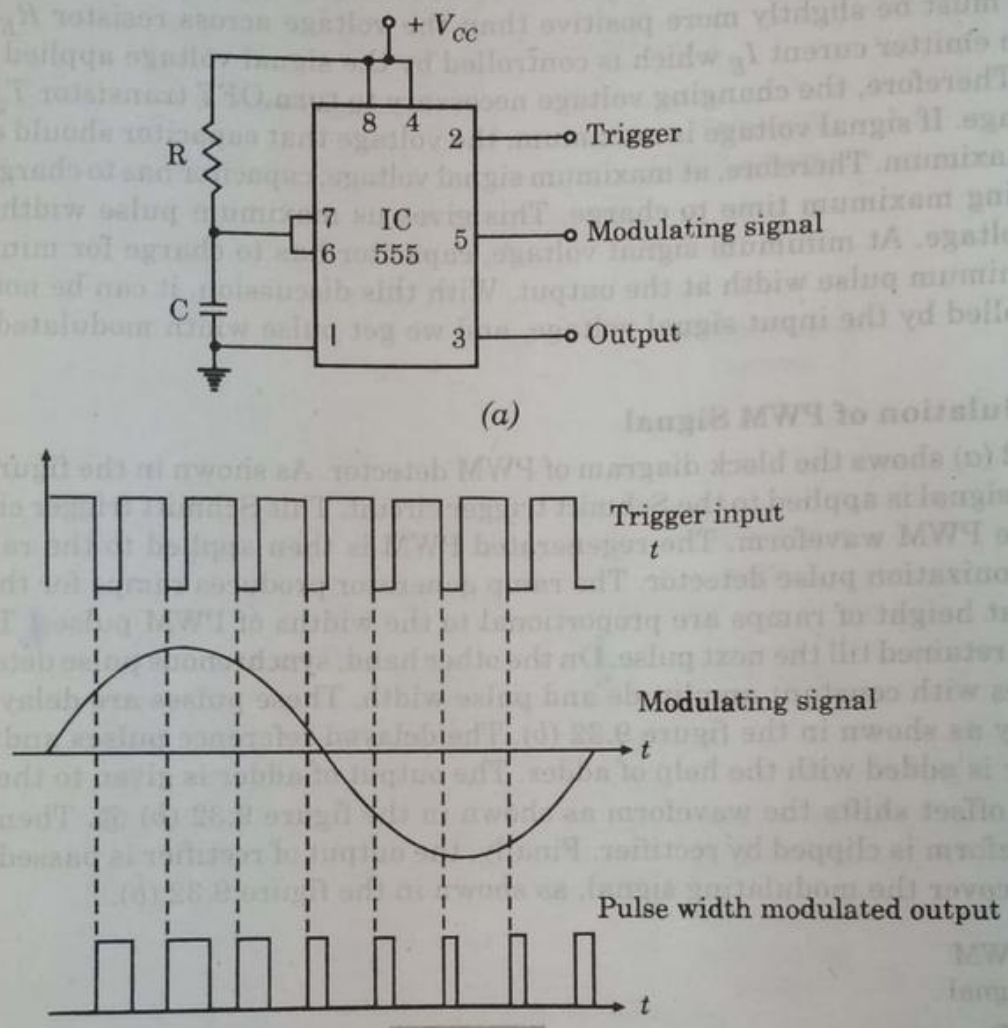


Fig. 9.30. *(b)*

Figure 9.31 illustrates another monostable multivibrator circuit to generate pulse width modulation (PWM).

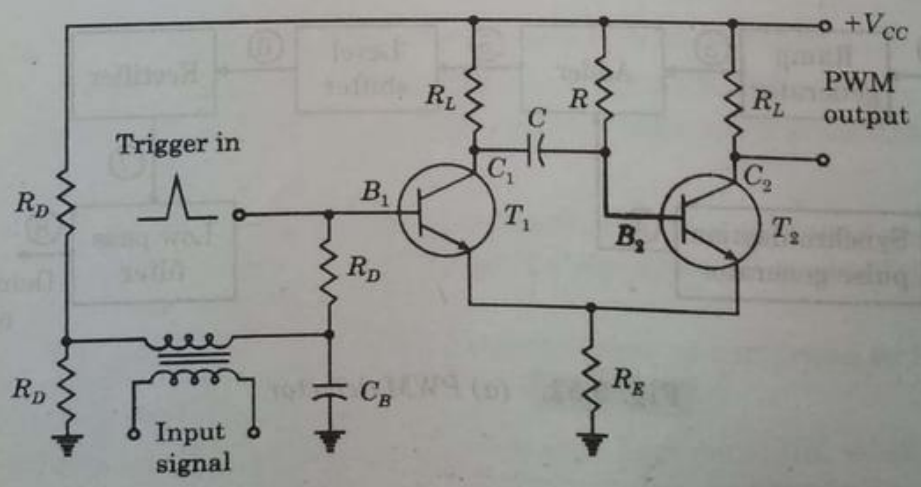


Fig. 9.31. *Monostable multivibrator generating pulse width modulation (PWM).*

Scanned by CamScanner

The stable state for above circuit is achieved when T_1 is OFF and T_2 is ON. The positive going trigger pulse at B_1 switches T_1 ON. Because of this, the voltage at C_1 falls as T_1 now begins to draw the collector current. As a result, voltage at B_2 also falls and T_2 is switched OFF, C begins to charge up to the collector supply voltage (V_{CC}) through resistor R. After a time determined by the supply voltage and the RC time constant of the charging network, the base of the T_2 becomes sufficiently positive to switch T_2 ON. The transistor T_1 is simultaneously switched OFF by regenerative action and stays OFF until the arrival of the next trigger pulse. To make T_2 ON, the base of the T_2 must be slightly more positive than the voltage across resistor R_E. This voltage depends on the emitter curent I_E which is controlled by the signal voltage applied at the base of transistor T_1. Therefore, the changing voltage necessary to turn OFF transistor T_2 is decided by the signal voltage. If signal voltage is maximum, the voltage that capacitor should charge to turn ON T_2 is also maximum. Therefore, at maximum signal voltage, capacitor has to charge to maximum voltage requiring maximum time to charge. This gives us maximum pulse width at maximum input signal voltage. At minimum signal voltage, capacitor has to charge for minimum voltage and we get minimum pulse width at the output. With this discussion, it can be noted that pulse width is controlled by the input signal voltage, and we get pulse width modulated waveform at the output.

9.14.4. Demodulation of PWM Signal

Figure 9.32 (*a*) shows the block diagram of PWM detector. As shown in the figure 9.32 (*a*), the received PWM signal is applied to the Schmitt trigger circuit. This Schmitt trigger circuit removes the noise in the PWM waveform. The regenerated PWM is then applied to the ramp generator and the synchronization pulse detector. The ramp generator produces ramps for the duration of pulses such that height of ramps are proportional to the widths of PWM pulses. The maximum ramp voltage is retained till the next pulse. On the other hand, synchronous pulse detector produces reference pulses with constant amplitude and pulse width. These pulses are delayed by specific amount of delay as shown in the figure 9.32 (*b*). The delayed reference pulses and the output of ramp generator is added with the help of adder. The output of adder is given to the level shifter. Here, negative offset shifts the waveform as shown in the figure 9.32 (*b*) ⑥. Then the negative part of the waveform is clipped by rectifier. Finally, the output of rectifier is passed through low-pass filter to recover the modulating signal, as shown in the figure 9.32 (*b*).

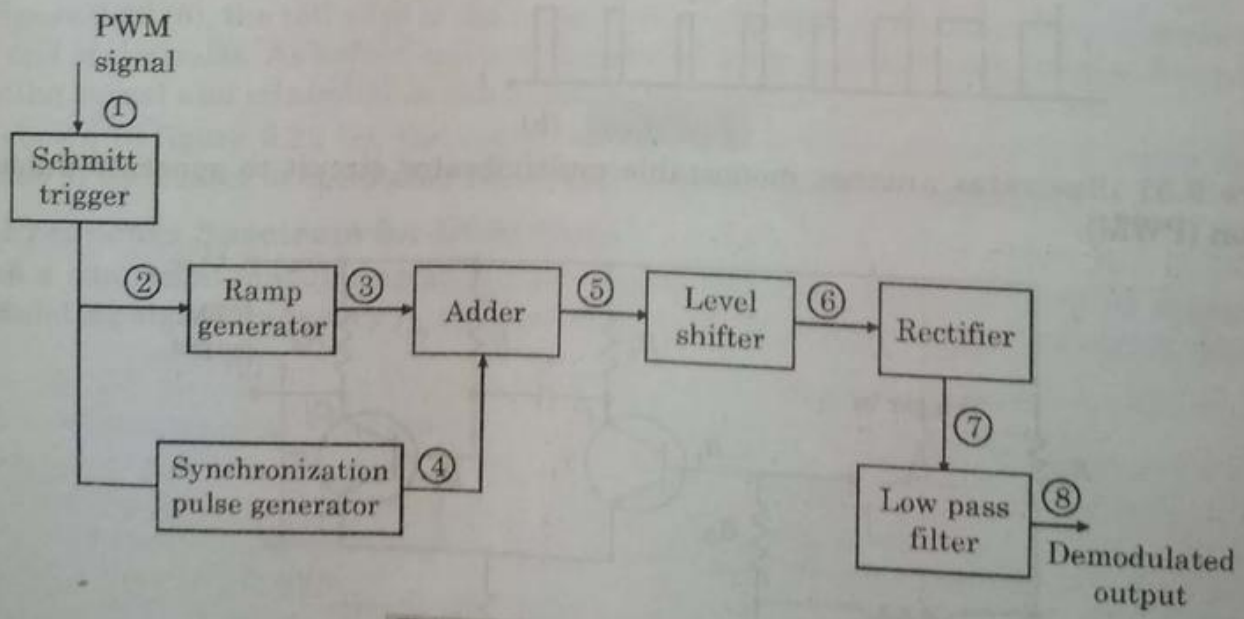


Fig. 9.32. *(a) PWM detector*

Scanned by CamScanner

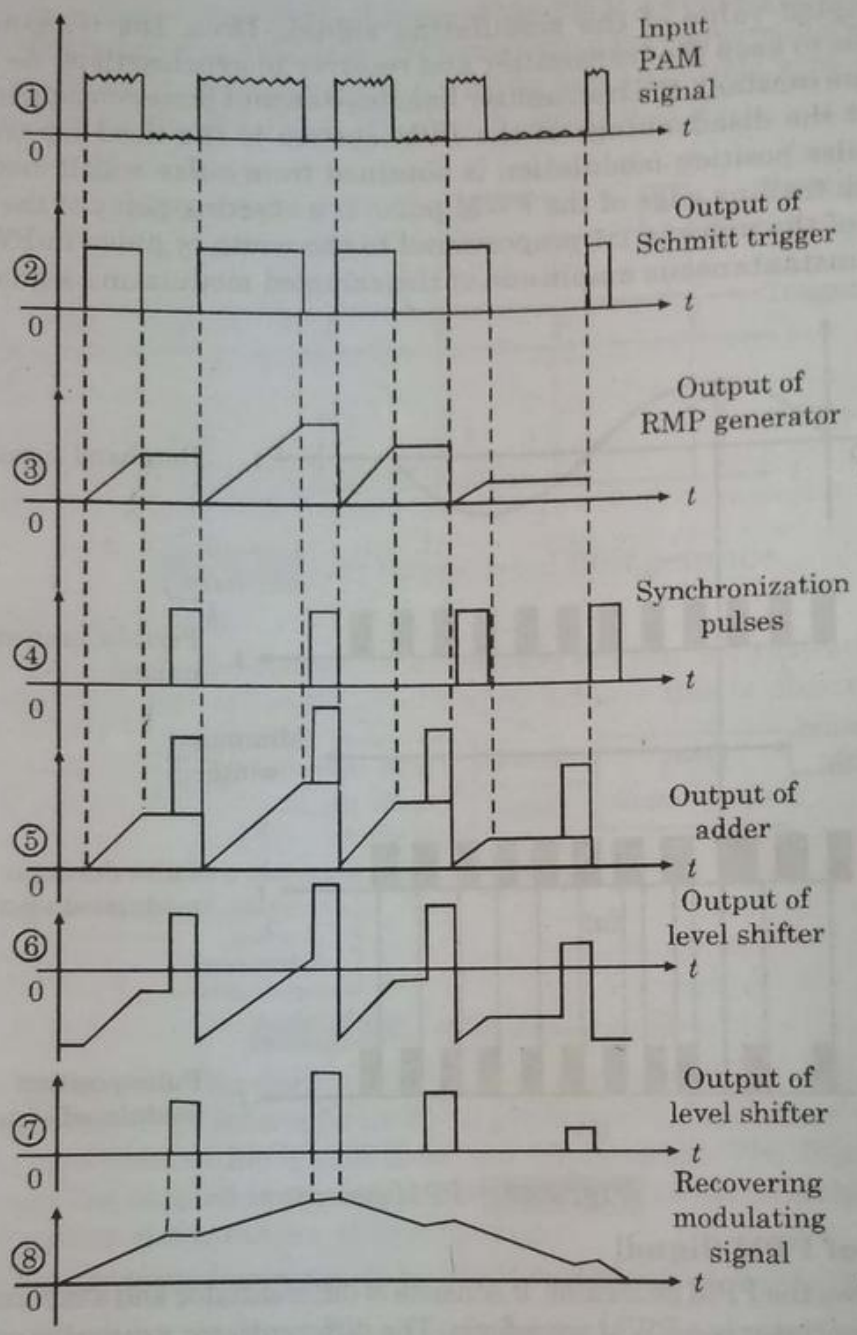


Fig. 9.32. *(b) Waveforms for PWM detection circuit.*

9.14.5 Advantages of PWM

(*i*) Unlike, PAM, noise is less, since in PWM, amplitude is held constant.

(*ii*) Signal and noise separation is very easy, as shown in figure 9.32 (*b*) ②.

(*iii*) PWM communication does not require synchronization between transmitter and receiver.

9.14.6. Disadvantages of PWM

(*i*) In PWM, pulses are varying in width and therefore their power contents are variable. This requires that the transmitter must be able to handle the power contents of the pulse having maximum pulse width.

(*ii*) Large bandwidth is required for the PWM communication as compared to PAM.

9.14.7. Pulse Position Modulation

In this system, the amplitude and width of the pulses are kept constant, while the position of each pulse, with reference to the position of a reference pulse, is changed according to the

Scanned by CamScanner

instantaneous sampled value of the modulating signal. Thus, the transmitter has to send synchronizing pulses to keep the transmitter and receiver in synchronism. As the amplitude and width of the pulses are constant, the transmitter handles constant power output, a definite advantage over the PWM. But the disadvantage of the PPM system is the need for transmitter-receiver synchronization. Pulse position modulation is obtained from pulse width modulation, shown in the figure 9.33. Each trailing edge of the PWM pulse is a starting point of the pulse in the PPM. Therefore, position of the pulse is 1:1 proportional to the width of pulse in PWM and hence it is proportional to the instantaneous amplitude of the sampled modulating signal.

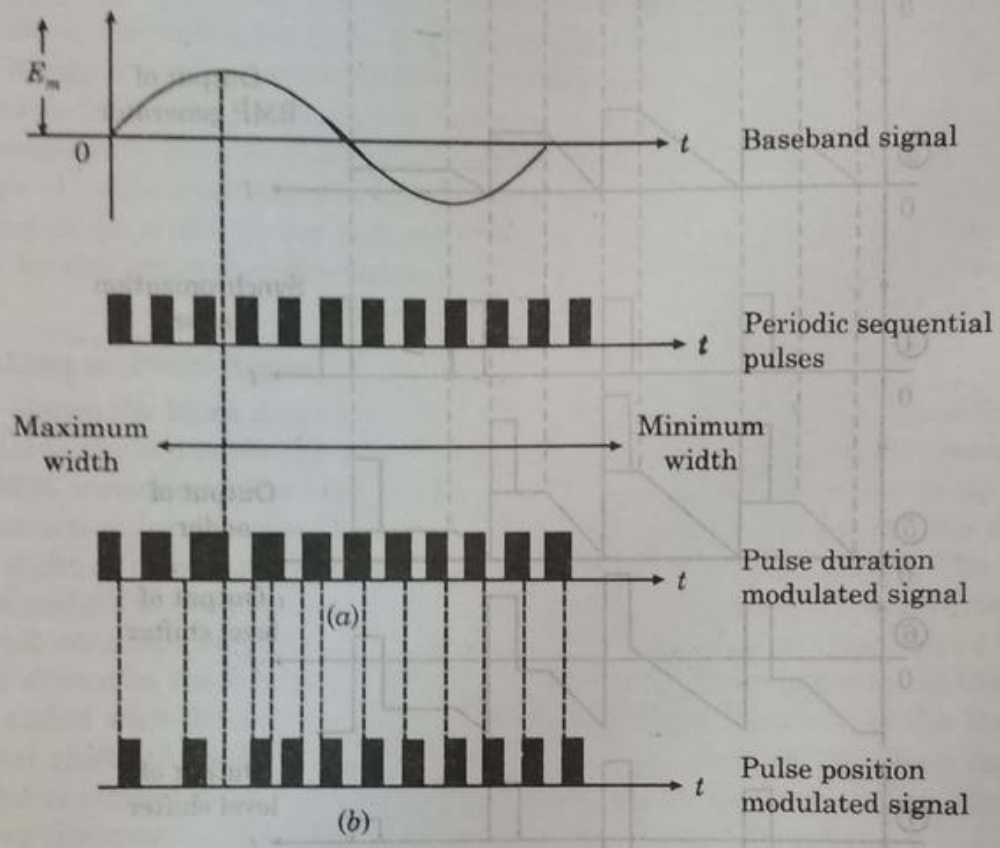


Fig. 9.33. *PPM generator*

9.14.8. Generation of PPM Signal

Figure 9.34 (*a*) shows the PPM generator. It consists of differentiator and a monostable multivibrator. The input to the differentiator is a PWM waveform. The differentiator generates positive and negative

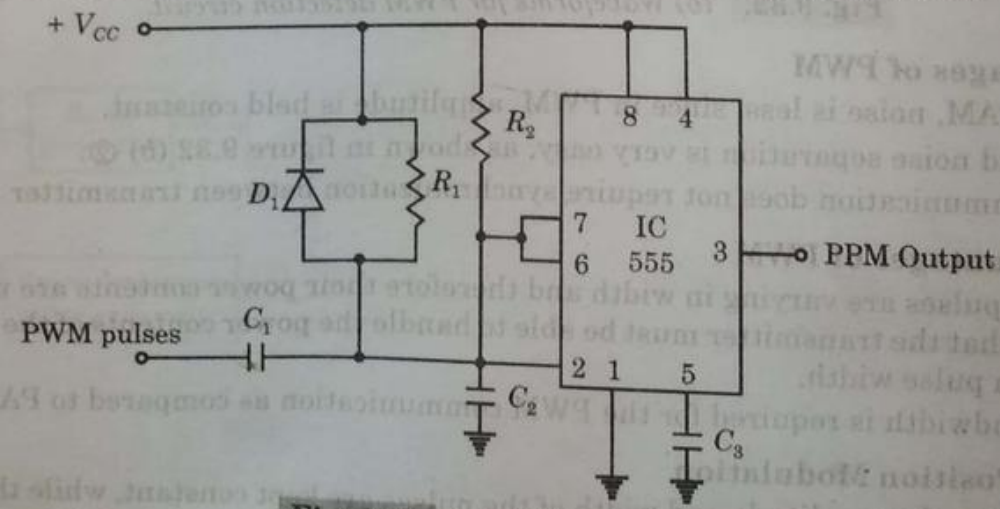


Fig. 9.34. *(a) PPM generator*

Scanned by CamScanner

spikes corresponding to leading and trailing edges of the PWM waveform. Diode D1 is used to bypass the positive spikes. The negative spikes are used to the trigger monostable multivibrator. The monostable multivibrator then generates the pulses of same width and amplitude with reference to trigger to give pulse position modulated waveform, as shown in figure 9.34 (*b*).

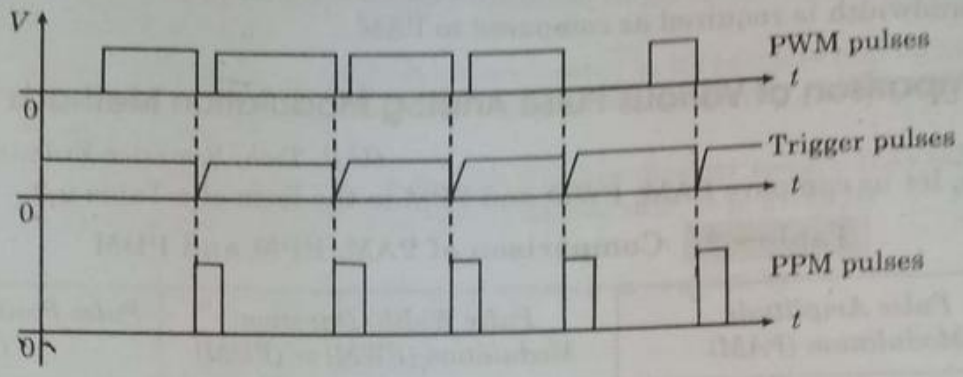


Fig. 9.34. *(b) Waveforms of PPM generator*

9.14.9. Demodulation of PPM

In case of pulse-position modulation, it is customary to convert the received pulses that vary in position to pulses that vary in length. One way to achieve this is illustrated in figure 9.35.

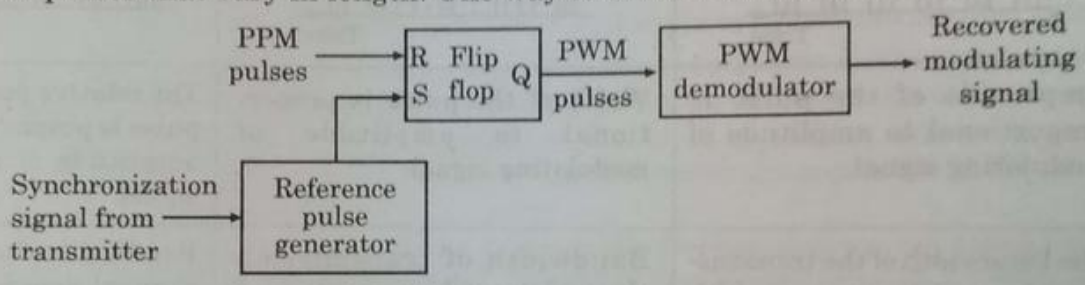


Fig. 9.35. *PPM demodulator*

As shown in figure 9.35, flip-flop circuit is set or turned 'ON' (giving high output) when the reference pulse arrives. This reference pulse is generated by reference pulse generator of the receiver with the synchronization signal from the transmitter. The flip-flop circuit is reset or turned 'OFF' (giving low output) at the leading edge of the position modulated pulse. This repeats and we get PWM pulses at the output of the flip-flop.

The PWM pulses are then demodulated by PWM demodulator to get orginal modulating signal.

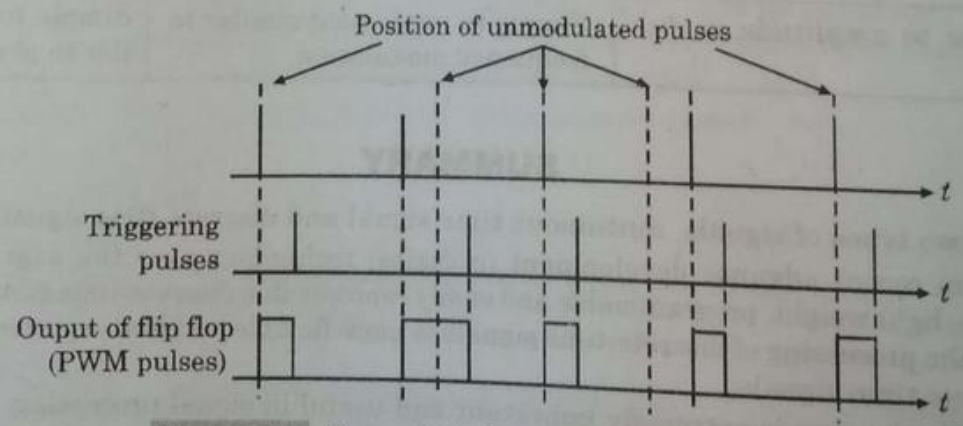


Fig. 9.36. *Demodulation waveform for PPM*

9.14.10. Advantages of PPM

(*i*) Like PWM, in PPM, amplitude is held constant thus less noise interference.

(*ii*) Like PPM, signal and noise separation is very easy.

Scanned by CamScanner

(*iii*) Because of constant pulse widths and amplitudes, transmission power for each pulse is same.

9.14.11. Disadvantages of PPM

(*i*) Synchronization between transmitter and receiver is required.

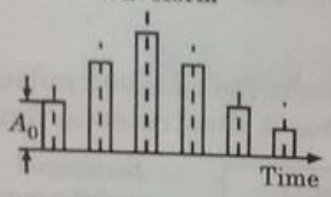
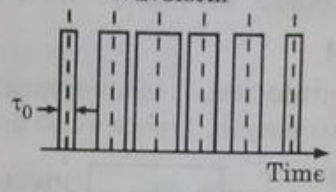
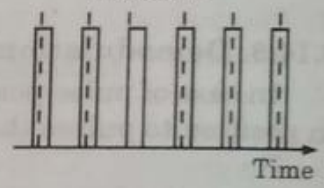
(*ii*) Large bandwidth is required as compared to PAM.

9.15. Comparison of Various Pulse Analog Modulation Methods

(U.P. Tech, Semester Examination, 2003-04)

In this section, let us compare PAM, PWM and PPM in the form of a Table 9.2.

Table 9.2. Comparison of PAM, PPM and PDM

S. No.	*Pulse Amplitude Modulation (PAM)*	*Pulse Width/Duration Modulation (PWM) or (PDM)*	*Pulse Position Modulation (PPM)*
1	Waveform 	Waveform 	Waveform 
2	Amplitude of the pulse is proportional to amplitude of modulating signal.	Width of the pulse is proportional to amplitude of modulating signal.	The relative position of the pulse is proprotional to the amplitude of modulating signal.
3	The bandwidth of the transmission channel depends on width of the pulse.	Bandwidth of transmission channel depends on rise time of the pulse.	Bandwidth of transmission channel depends on rising time of the pulse.
4	The instantaneous power of the transmitter varies.	The instantaneous power of the transmitter varies.	The instantaneous power of the transmitter remains constant.
5	Noise interference is high System is complex .	Noise, interference is minimum.	Noise, interference is minimum.
6	Similar to amplitude modulation.	Simple to implement similar to frequency modulation.	Simple to implement similar to phase modulation.

SUMMARY

1. There are two types of signals, continuous time signal and discrete-time signals.
2. Due to some recent advance development in digital technology over the past few decades, the inexpensive, light weight, programmable and easily reproducible discrete-time systems are available. Therefore, the processing of discrete-time signals is more flexible and is also preferable to processing of continuous-time signals.
3. The sampling theorem is extremely important and useful in signal processing.
4. With the help of sampling theorem, a continuous-time signal may be completely represented and recovered from the knowledge of samples taken uniformly.
5. The concept of sampling provides a widely used method for using discrete-time system technology to implement continuous-time systems and process the continuous-time signals.

Scanned by CamScanner

6. Sampling of the signals is the fundamental operation in signal-processing. A continuous time signal is first converted to discrete-time signal by sampling process.
7. The sufficient number of samples of the signal must be taken so that the original signal is represented in its samples completely. Also, it should be possible to recover or reconstruct the original signal completely from its samples. The number of samples to be taken depends on maximum signal frequency present in the signal.
8. A continuous-time signal may be completely represented in its samples and recovered back if the sampling frequency is $f_s \geq 2 f_m$. Here, f_s is the sampling frequency and f_m is the maximum frequency present in the 'signal.
9. When the sampling rate becomes exactly equal to $2 f_m$ samples per second, then it is called Nyquist rate. Nyquist rate is also called the *minimum* sampling rate. It is given by

$$f_s = 2 f_m$$

10. Maximum sampling interval is called *Nyquist interval*. It is given by

Nyquist Interval $\quad T_s = \dfrac{1}{2 f_m}$ seconds.

11. The low pass filter is used to recover original signal from its samples. This is also known as interpolation filter.
12. A low-pass filter is that type of filter which passes only low-frequencies upto a specified cut-off frequency and rejects all other frequenices above cut-off frequency.
13. The process of reconstructing a continuous-time signal $x(t)$ from its samples is called as interpolation.
14. Aliasing is the phenomenon in which a high frequency component in the frequency-spectrum of the signal takes identity of a lower-frequency component in the spectrum of the sampled signal.
15. Because of the overlap due to aliasing phenomenon, it is not possible to recover original signal $x(t)$ from sampled signal $g(t)$ by low-pass filtering since the spectral components in the overlap regions add and hence the signal is distorted.
16. Since any information signal contains a large number of frequencies, so, to decide a sampling frequency is always a problem. Therefore, a signal is first passed through a low-pass filter. This low-pass filter blocks all the frequencies which are above f_mHz. This process is known as band limiting of the original signal $x(t)$. This low-pass filter is called prelias filter because it is used to prevent aliasing effect.
17. To avoid aliasing, we must have:
    (*i*) Preaias filter must be used to limit band of frequencies of the signal to f_m Hz.
    (*ii*) Sampling frequency 'f_s' must be selected such that

$$f_s > 2 f_m$$

18. The bandpass signal $x(t)$ whose maximum bandwidth is $2f_m$ can be completely represented into and recovered from its samples if it is sampled at the minimum rate of twice the bandwidth. Here, f_m is the maximum frequency component present in the signal.
19. Basically, there are three types of sampling techniques as under:
    (*i*) Instantaneous sampling
    (*ii*) Natural sampling
    (*iii*) Flat top sampling.
20. Natural sampling is a practical method and will be discussed in this section.
21. Spectrum of Naturally Sampled signal:

$$G(f) = \frac{\tau A}{T_s} \cdot \sum_{n=-\infty}^{\infty} \text{sinc}\,(nf_s, \tau)\, X(f - nf_s)$$

22. Flot top sampling like natural sampling is also a practically possible sampling method. But natural sampling is little complex whereas it is quite easy to get flat top samples.

Scanned by CamScanner

CHAPTER 10

Waveform Coding Techniques

Inside this Chapter

10.1. Introduction 535
10.2. Discretization in Time and Amplitude 535
10.3. Concept of Quantization 536
10.4. Pulse Code Modulation (PCM) 536
10.5. A PCM Generator or Transmitter 538
10.6. PCM Transmission Path 539
10.7. PCM Receiver 540
10.8. Quantizer 541
10.9. Working Principles of Quantizer 542
10.10. A Uniform Quantizer with Incorrect Quantization Characteristics 543
10.11. Transmission Bandwidth in a PCM System 544
10.12. Quantization Noise/Error in PCM 545
10.13. Signal to Quantization Noise Ratio for Linear Quantization 548
10.14. Effect of Noise on the PCM System 549
10.15. Robust Quantization: Necessity of Nonuniform Quantization in a PCM System 558
10.16. Necessity of Nonuniform Quantization for Speech Signal 559
10.17. Nonuniform Quantization 561
10.18. Companding (*i.e.*, Companded PCM) 561
10.19. Compander Characteristic 562
10.20. Different Types of Compressor Characteristics 562
10.21. Applications of PCM 564
10.22. Advantages of PCM 564
10.23. Disadvantages of PCM 564
10.24. Delta Modulation 564
10.25. Adaptive Delta Modulation 567
10.26. Differential Pulse Code Modulation (DPCM) 569
10.27. Comparison of Digital Pulse Modulation Methods 571

Scanned by CamScanner

10.1. Introduction

To transport an information-bearing signal from one point to another point over a communication channel, we can use digital or analog techniques. As discussed earlier in chapter 1, the use of digital communication offers several important advantages as compared to analog communication. In particular, a digital communication system offers the following highly attractive features:

(*i*) ruggedness to channel noise and external interference unmatched by any analog communication system.

(*ii*) flexible operation of the system.

(*iii*) integration of diverse sources of information into a common format.

(*iv*) security of information in the course of its transmission from source to the destination.

In view of above reasons, digital communications have become the dominant form of communication technology in our society.

However, to handle the transmission of analog message signals (*i.e.*, voice and video signals)* by digital means, the signal has to undergo an analog-to-digital conversion. We already know about the pulse amplitude modulation (PAM) technique. We also know the disadvantages of using PAM technique. Using the waveform coding technique, we convert the analog PAM signal into a digital signal. This digital signal is in the form of a train or stream of binary digits 0 and 1. Thus, with waveform coding techniques, we enter into the world of digital communication. After sampling an analog signal, the next step in its digital transmission is the generation of the "coded version" (digital representation) of the signal. Pulse Code Modulation (PCM) provides one method to meet such a requirement.

In PCM, the message signal is sampled and amplitude of each sample is approximated (rounded off) to the nearest one of a finite set of discrete levels. This will enable us to represent both time and amplitude in discrete form. Hence, it is possible to transmit the message signal by means of a digital (coded) waveform. Conceptually PCM is simple to understand. It was the first method which was developed for the digital coding of the waveforms. PCM is the most applied of all the digital coding systems in use today. PCM is therefore widely accepted as the standard against which the other digital coder systems are calibrated. Here, we can note one more point that PCM belongs to a class of signal coders known as **waveform coders**. In this chapter, we shall present a detailed discussion of different digital pulse modulation techniques.

10.2. Discretization in Time and Amplitude

As discussed earlier, pulse modulation may be classified under two heads *i.e.*, pulse analog modulation and pulse digital modulation. In case of pulse analog modulation, only time is expressed in the digital form and any one of the pulse parameters (*i.e.*, pulse amplitude, duration or position) is varied in a continuous manner in accordance with the message signal. Pulse amplitude modulation (PAM), Pulse duration (width) modulation (PWM) and Pulse position modulation (PPM) are the examples of pulse analog modulation. In these modulation schemes, information transmission is accomplished in an analog form at discrete times. On the other hand, in the pulse digital modulation, the time and the pulse parameter (usually the amplitude) occur in discrete form and digital coded form respectively. Pulse digital modulation is therefore basically a scheme which converts the analog signal to its corresponding digital form. It is for this reason that the analog-to-digital conversion is sometimes known as **pulse digital modulation**.

The simplest form of pulse digital modulation is called pulse code modulation (PCM). In this system (*i.e.*, PCM), the message signal is first sampled and then amplitude of each sample is rounded off to the nearest one of a finite set of allowable values known as quantization levels, so

* Most of the signals in our suroundings are analog signals.

Scanned by CamScanner

that both time and amplitude are in the discrete form. This means that in pulse code modulation both parameters *i.e.*, time and amplitude are expressed in discrete form. This process is called **discretization** in time and amplitude.

10.3. Concept of Quantization

In communication systems, sometimes it happens that we are available with analog signal, however, we have to transmit a digital signal for a particular application. In such cases, we have to convert an analog signal into digital signal. This means that we have to convert a continuous time signal in the form of digits. To see how a signal can be converted from analog to digital form, let us consider an analog signal as shown in figure 10.1(*a*). First of all, we get samples of this signal according to sampling theorem. For this purpose, we mark the time-instants t_0, t_1, t_2 and so on, at equal time-intevals along the time axis. At each of these time-instants, the magnitude of the signal is measured and thus samples of the signal are taken. Figure 10.1(*b*) shows a representation of the signal of figure 10.1(*a*) in terms of its samples.

Now, we can say that the signal in figure 10.1(*b*) is defined only at the sampling instants. This means that it no longer is a continuous function of time, but rather, it is a discrete-time signal. However, since the magnitude of each smaple can take any value in a continuous range, the signal in figure 10.1(*b*) is still an analog signal.

This difficulty is neatly resolved by a process known as **quantization**. In quantization, the total amplitude range which the signal may occupy is divided into a number of standard levels.

As shown in figure 10.1(*c*), amplitudes of the signal $x(t)$ lie in the range $(-m_p, m_p)$ which is partitioned into L intervals, each of magnitude $\Delta v = \frac{2m_p}{L}$. Now, each smaple is approximated or rounded off to the nearest quantized level as shown in figure. Since each sample is now approximated to one of the L numbers therefore the information is digitized.

The quantized signal is an approximation of the original one. We can improve the accuracy of the quantized signal to any desired degree simply by increasing the number of levels L.

10.4. Pulse Code Modulation (PCM)

(U.P. Tech-Semester Exam. 2002-2003)

Pulse-code modulation is known as a **digital pulse modulation technique**. In fact, the pulse-code modulation (PCM) is quite complex compared to the analog pulse modulation techniques (*i.e.*, PAM, PWM and PPM) in the sense that the message signal is subjected to a great number of operations. Figure 10.2 shows the basic elements of a PCM system. It consists of three main parts *i.e.*, transmitter, transmission path and receiver. The essential operations in the transmitter of a PCM system are sampling, quantizing and encoding as shown in figure 10.2. As discussed earlier, sampling is the operation in which an analog (*i.e.*, continous-time) signal is sampled according to the sampling theorem resulting in a discrete-time signal. The quantizing and encoding operations are usually performed in the same circuit which is known as an **analog-to-digital converter (ADC).**

Also, the essential operations in the receiver are regeneration of impaired signals, decoding and demodulation of the train of quantized samples. These operations are usually performed in the same cirucit which is known as a digital-to-analog converter (DAC).

Further, at intermediate points, along the transmission route from the transmitter to the receiver, regenerative repeaters are used to reconstruct (*i.e.*, regenerate) the transmitted sequence of coded pulses in order to combat the accumulated effects of signal distortion and noise.

As discussed in article 10.3, the quantization refers to the use of a finite set of amplitude levels and the selection of a level nearest to a particular sample value of the message signal as the representation for it. In fact, this operation combined with sampling, permits the use of

Scanned by CamScanner

coded pulses for representing the message signal. *Thus, it is the combined use of quantizing and coding that distinguishes pulse code modulation form analog modulation techniques.*

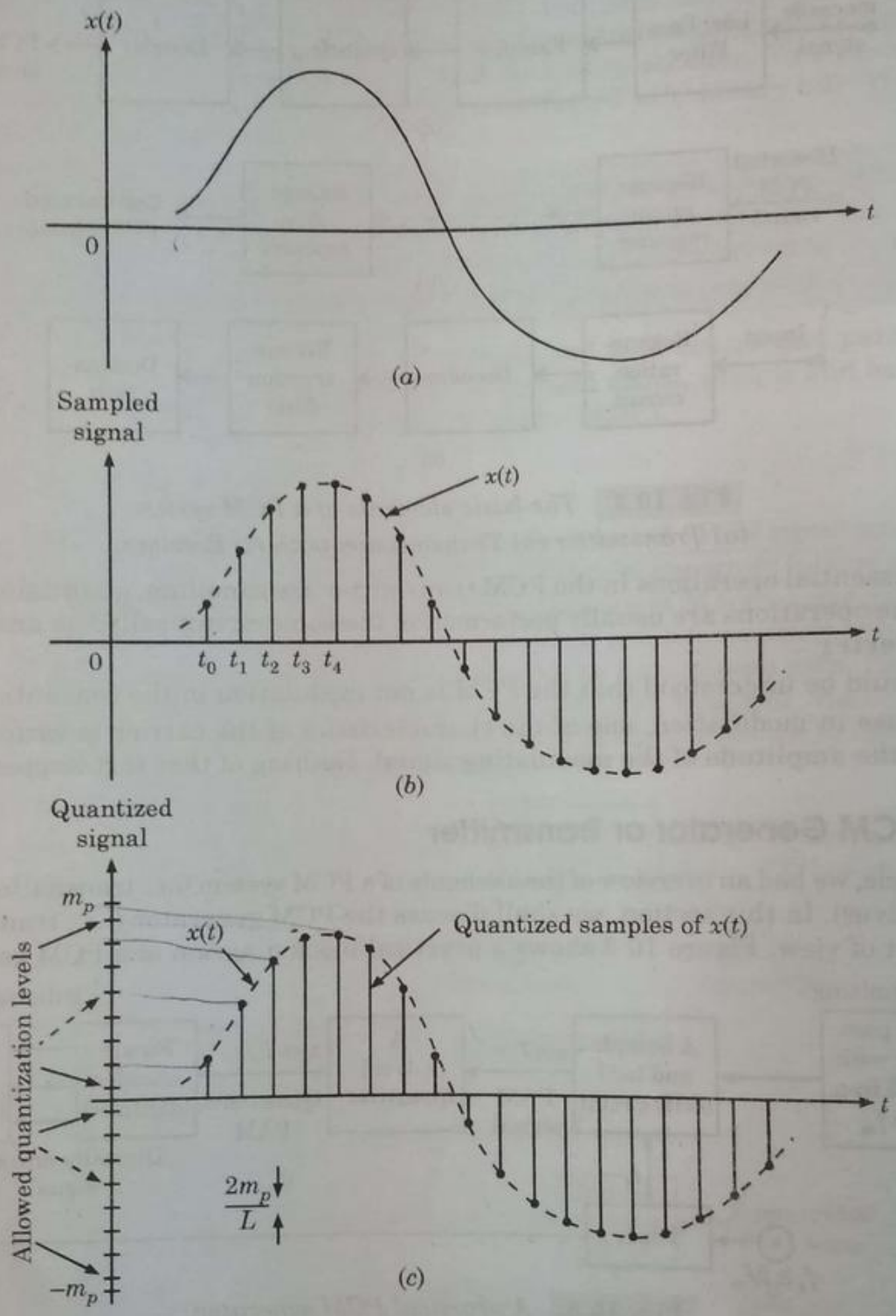


Fig. 10.1. *(a) An analog signal, (b) Samples of analog siagnal, (c) Quantization*

Now, let us summarize PCM in the form of few points as under:

(*i*) PCM is a type of pulse modulation like PAM, PWM or PPM but there is an important difference between them PAM, PWM or PPM are "analog" pulse modulation systems whereas PCM is a "digital" pulse modulation system.

(*ii*) This means that the PCM output is in the coded digital form. It is in the form of digital pulses of constant amplitude, width and position.

Scanned by CamScanner

(*iii*) The information is transmitted in the form of "code words". A PCM system consists of a PCM encoder (transmitter) and a PCM decoder (receiver).

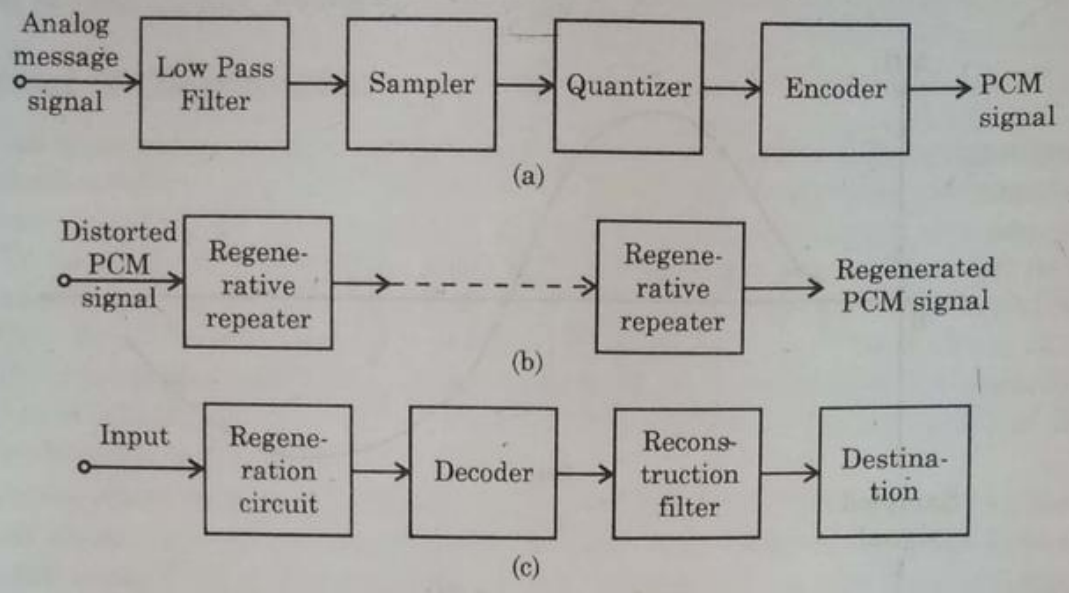


Fig. 10.2. *The basic elements of a PCM system*
(a) Transmitter (b) Transmission path (c) Receiver.

(*iv*) The essential operations in the PCM transmitter are sampling, quantizing and encoding.

(*v*) All the operations are usually performed in the same circuit called as **analog-to-digital converter**.

(*vi*) It should be understood that the PCM is not modulation in the conventional sense.

(*vii*) Because in modulation, one of the characteristics of the carrier is varied in proportion with the amplitude of the modulating signal. Nothing of that sort happen in PCM.

10.5. A PCM Generator or Transmitter

In the last article, we had an overview of the elements of a PCM system (*i.e.*, transmitter, transmission-path and receiver). In this section, we shall discuss the PCM generator (*i.e.*, transmitter) from a practical point of view. Figure 10.3 shows a practical block diagram of a PCM generator.

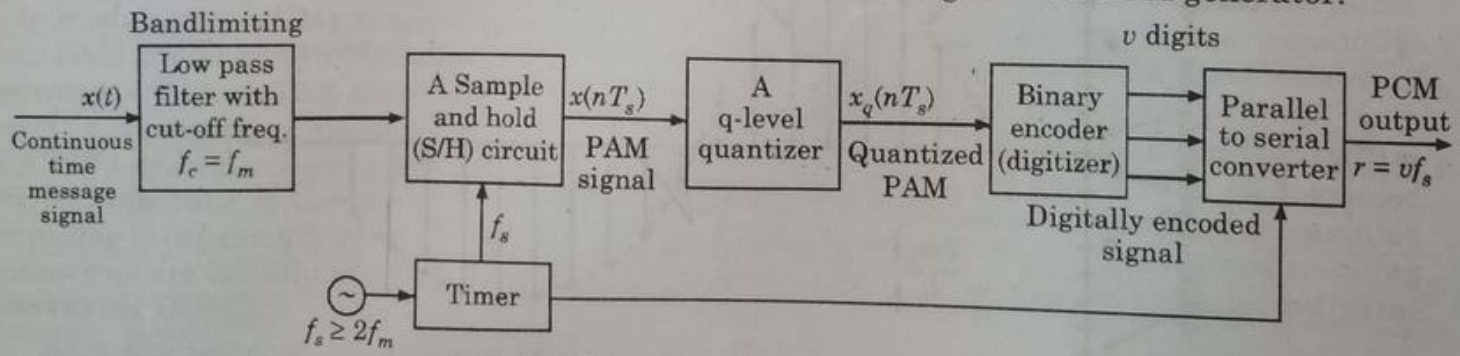


Fig. 10.3. *A practical PCM generator*

In PCM generator of figure 10.3, the signal $x(t)$ is first passed through the low-pass filter of cutoff frequency f_m Hz. This low-pass filter blocks all the frequency components which are lying above f_m Hz.*

This means that now the signal $x(t)$ is bandlimited to f_m Hz. The sample and hold circuit then samples this signal at the rate of f_s. Sampling frequency f_s is selected sufficiently above nyquist rate to aviod aliasing *i.e.*,

$$f_s \geq 2f_m$$

* Recall that this filter is used to avoid aliasing.

Scanned by CamScanner

In figure 10.3, the output of sample and hold circuit is denoted by $x(nT_s)$. This signal $x(nT_s)$ is discrete in time and continuous in amplitude. A q-level quantizer compares input $x(nT_s)$ with its fixed digital levels. It assigns any one of the digital level to $x(nT_s)$ with its fixed digital levels. It then assigns any one of the digital level to $x(nT_s)$ which results in minimum distortion or error. This error is called **quantization error**. Thus, output of quantizer is a digital level called $x_q(nT_s)$.

Now, the quantized signal level $x_q(nT_s)$ is given to binary encoder. This encoder converts input signal to 'v' digits binary word. Thus $x_q(nT_s)$ is converted to 'v' binary bits. This encoder is also known as digitizer.

Note: *It may be noted that it is not possible to transmit each bit of the binary word separately on transmission line. Therefore 'v' binary digits are converted to serial bit stream to generate single baseband signal. In a parallel to serial converter, usually a shift register does this job. The output of PCM generator is thus a single baseband signal of binary bits.*

Also, an oxcillator generates the clocks for sample and hold circuit and parallel to serial converter. In the pulse code modulation generator discussed above, sample and hold, quantizer and encoder combinely form an analog to digital converter (ADC).

10.6. PCM Transmission Path

The Path between the PCM transmitter and PCM receiver over which the PCM signal travel, is called as **PCM transmission path** and it is as shown in figure 10.4. The most important feature of PCM system lies in its ability to control the effects of distortion and noise when the PCM wave travels on the channel. PCM accomplishes this capacity by means of using a chain of regenerative repeaters as shown in figure 10.4. Such repeaters are spaced close enough to each other on the transmission path. The regenerative performs three basic operations namely equalization, timing and decision making. Hence, each repeater actually reproduces the clean noise free PCM signal from the PCM signal distorted by the channel noise. This improves the performance of PCM in presence of noise.

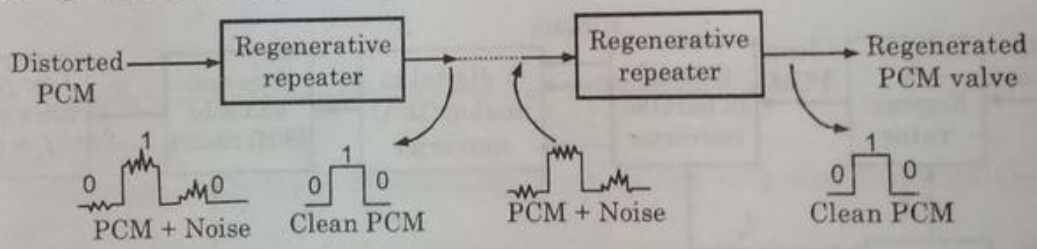


Fig. 10.4. *PCM transmission path.*

10.6.1. Block Diagram of a Repeater

Figure 10.5 shows the block diagram of a regenerative repeater.

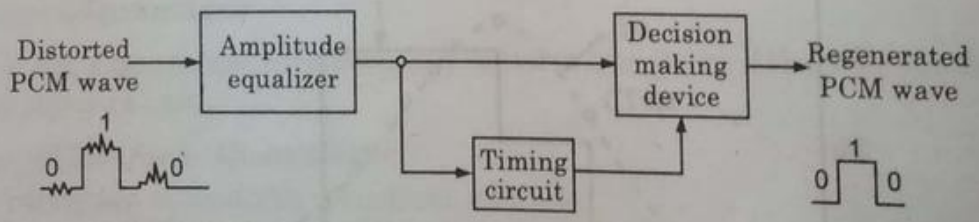


Fig. 10.5. *Block diagram of a regenerative repeater.*

The amplitude equalizer shapes the distorted PCM wave so as to compensate for the effects of amplitude and phase distortions. The timing circuit produces a periodic pulse train which is derived from the input PCM pulses. This pulse train is then applied to the decision making device. The decision making device uses this pulse train for sampling the equalized PCM pulses. The sampling is carried out at the instants where the signal to noise ratio is maximum.

Scanned by CamScanner

The decision device makes a decision about whether the equalized PCM wave at its input has a 0 value or 1 value at the instant of sampling. Such a decision is made by comparing equalized PCM with a reference level called **decision threshold** as illustrated in figure 10.6*a*. At the output of the decision device, we get a clean PCM signal without any trace of noise.

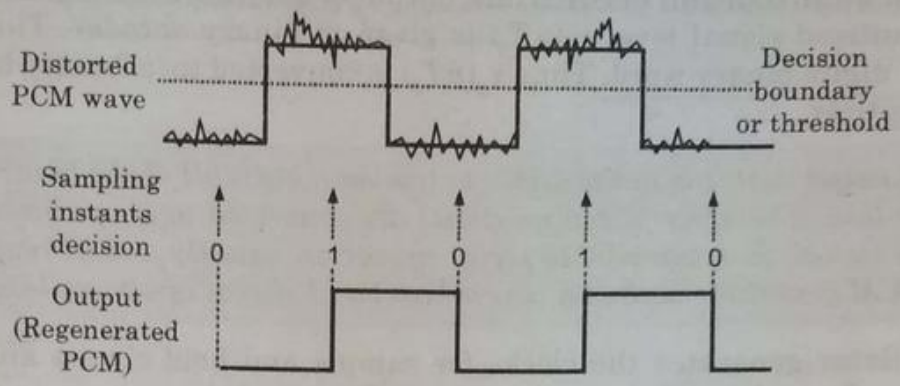


Fig. 10.6*a*. *Waveforms of a regenerative repeater.*

10.7. PCM Receiver

In this section, we shall discuss a PCM receiver from practical point of view. Figure 10.6(*a*) shows the block diagram of PCM receiver and figure 10.6 (*b*) shows the reconstructed signal. The regenerator at the start of PCM receiver reshapes the pulse and removes the noise. This signal is then converted to parallel digital words for each sample.

Now, the digital word is converted to its analog value denoted as $x_q(t)$ with the help of a sample and hold circuit. This signal, at the output of sample and hold circuit, is allowed to pass through a lowpass reconstruction filter to get the appropriate original message signal denoted as $y(t)$.

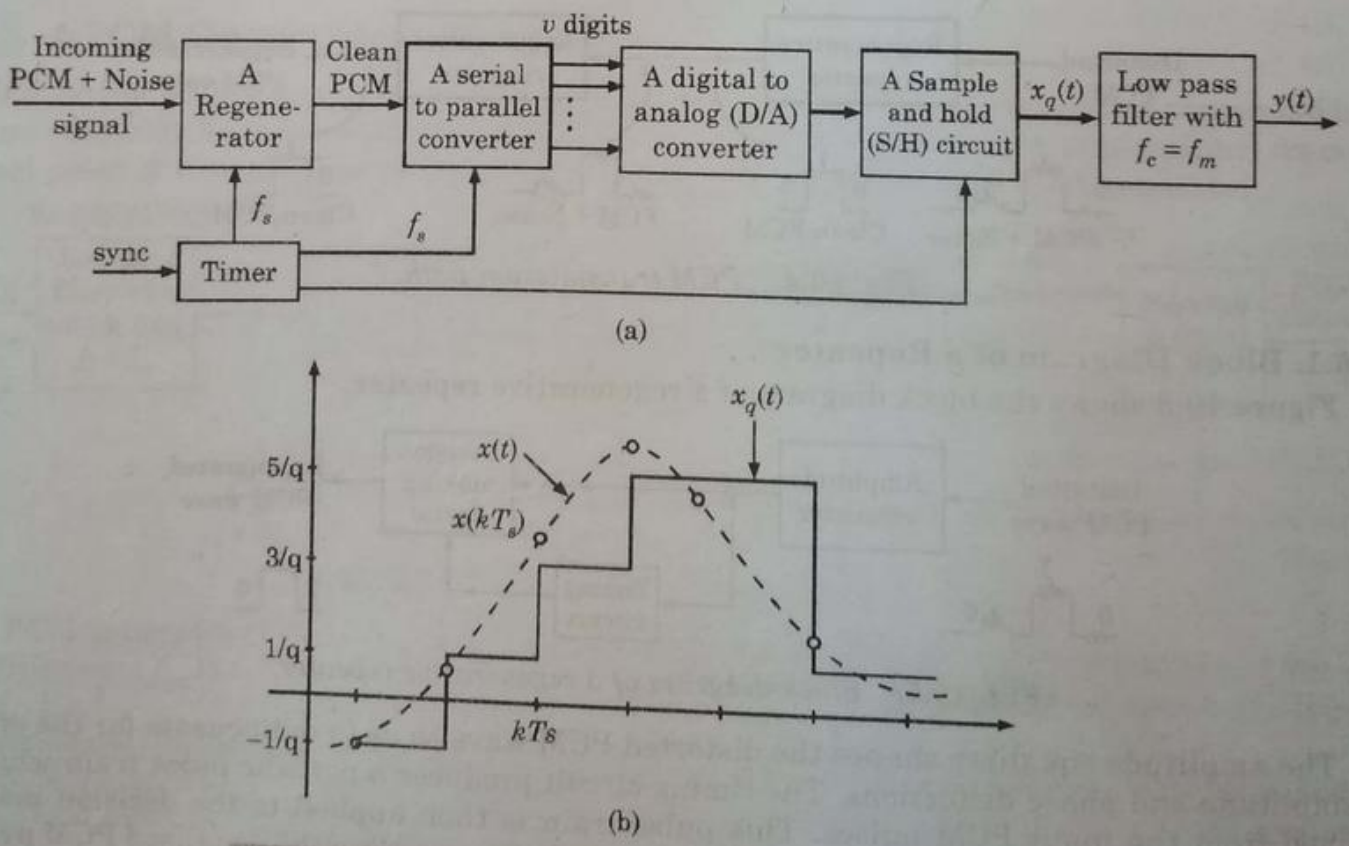


Fig. 10.6. *(a) PCM receiver (b) Reconstructed waveform.*

Scanned by CamScanner

Note: *As shown in reconstructed signal of figure 10.6 (b), it is impossible to reconstruct exact original signal x(t) because of permanent quantization error introduced during quantization at the transmitter. In fact, this quantization error can be reduced by increasing the binary levels. This is equivalent to increasing binary digits (bits) per sample. But increasing bits 'v' increases the singnaling rate as well as transmission bandwidth as we have observed in last article. Therefore the choice of these parameters is made, in such a manner that noise due to quantization error (i.e., also called as quantization noise) is in tolerable limits.*

10.8. Quantizer

As discussed in article 10.5, a q-level quantizer compares the discrete-time input $x(nT_s)$ with its fixed digital levels. It assigns any one of the digital level to $x(nT_s)$ with its fixed digital levels. It then assigns any one of the digital level to $x(nT_s)$ which results in minimum distortion or error. This error is called quantization error. Thus, the output of a quantizer is a digital level called $x_q(nT_s)$.

10.8.1. Classification of Quantization Process

Figure 10.7 shows the classification of quantization process.

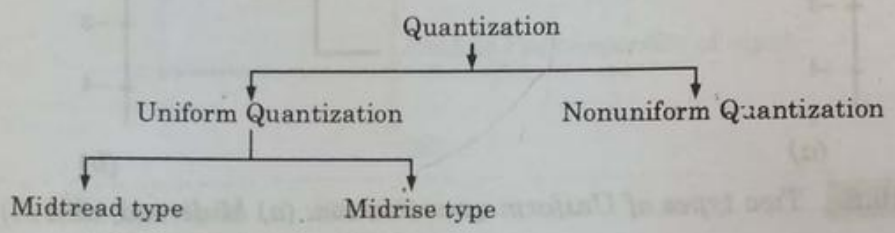


Fig. 10.7. *Classification of quantization process.*

The quantization process can be classified into two types as under:

(*i*) Uniform quantization

(*ii*) Non-uniform quantization.

This classification is based on the step size as defined earlier.

(*i*) Uniform Quantizer

A uniform quantizer is that type of quantizer in which the 'step size' remains same throughout the input range.

(*ii*) Nonuniform Quantizer

A non-uniform quantizer is that type of quantizer in which the 'step-size' varies according to the input signal values.

10.8.2. A Uniform Quantizer

As discussed earlier, a quantizer is called as an uniform quantizer if the step size remains constant throughout the input range.

10.8.2.1. Types of Uniform Quantizer

There are two types of uniform quantizer as under:

(*i*) Symmetric quantizer of the midtread type

(*ii*) Symmetric quantizer of the midrise type

Basically, quantizers can be of a **uniform** or **nonuniform type**. In a uniform quanitizer, the representation levels are uniformly spaced; otherwise, the quantizer is nonuniform. Now, let us consider only uniform quantizers, nonuniform quantizer shall be considered later on.

The quantizer characteristic can also be *midtread* or *midrise type*. Figure 10.8(*b*) shows the input-output characteristic of a uniform quantizer of the midtread type, which is so called because

Scanned by CamScanner

the origin lies in the middle of a tread of the staircaselike graph. Figure 10.8(*b*) shows the corresponding input-output characteristic of a uniform quantizer of the midrise type, in which the origin lies in the middle of a rising part of the staircaselike graph. It may be noted that both the midtread and midrise types of uniform quantizers illustrated in figure 10.8 are *symmetric* about the origin.

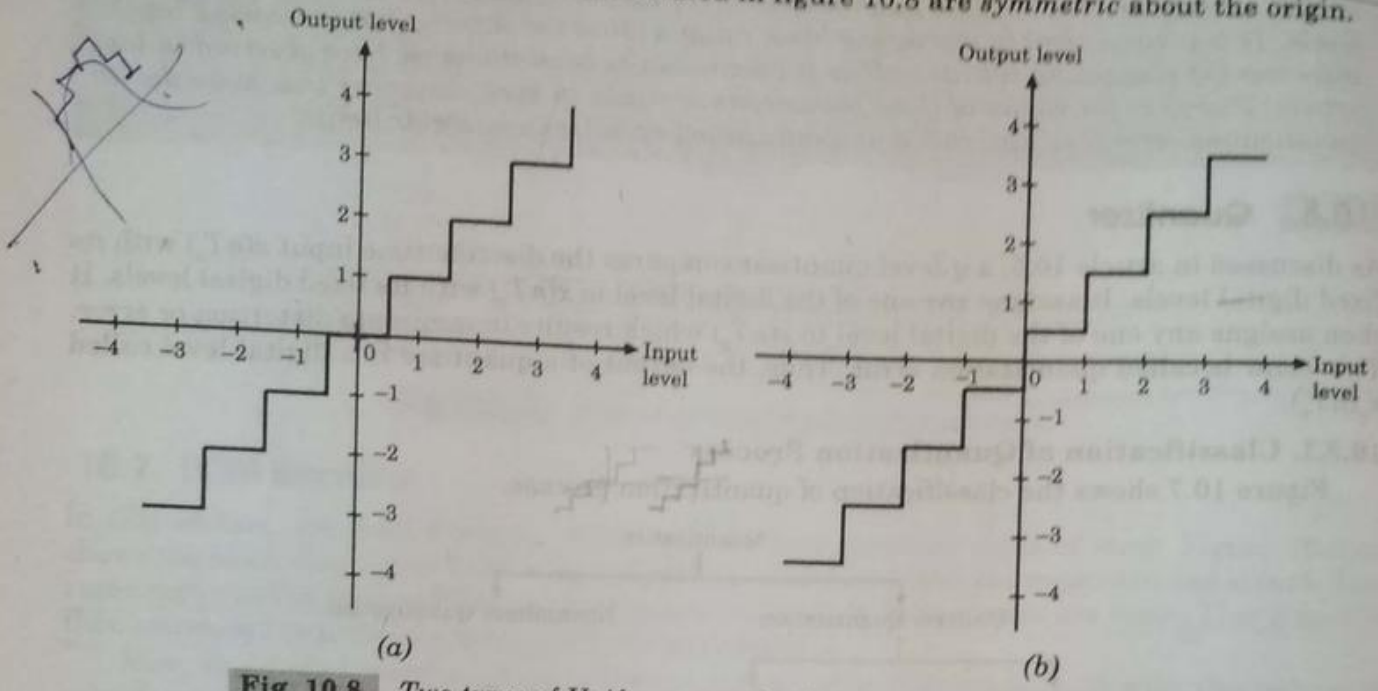


Fig. 10.8. *Two types of Uniform quantization: (a) Midtread, and (b) Midrise.*

10.9. Working Principle of Quantizer

In this section, let us see how uniform quantization takes place. For this purpose, we shall consider uniform quantizer of midrise type. Figure 10.9(*a*) shows the transfer characteristics of a uniform quantizer of midrise type. In figure 10.9(*a*), let us assume that the input to the quantizer $x(nT_s)$ varies from $-4\,\Delta$ to $+4\,\Delta$. This means that the peak to peak value of $x(nT_s)$ will be between -4Δ to $+4\Delta$. Here 'Δ' is the step size.

Thus, input $x(nT_s)$ can take any value between $-4\,\Delta$ to $+4\,\Delta$. Now, the fixed digital levels are avilable at $\pm\frac{\Delta}{2}, \pm\frac{3}{2}\Delta, \pm\frac{5}{2}\Delta$ and $\pm\frac{7}{2}\Delta$. These levels are available at quantizer because of its characteristics.

Hence, according to figure 10.9(*a*), we have

If $\quad x(nT_s) = 4\,\Delta$, then $x_q(nT_s) = \frac{7}{2}\Delta$

and if $\quad x(nT_s) = -4\,\Delta$, then $x_q(nT_s) = -\frac{7}{2}\Delta$

Thus, it may be observed from figure 10.9(*b*) that maximum quantization error would be $\pm\frac{\Delta}{2}$.

From above, we conclude that quantization error may be expressed as

$$\varepsilon = x_q(nT_s) - x(nT_s \qquad ...(10.1)$$

here 'ε' represents the quantization error

Now, when $x(nT_s) = 0$, quantizer will assign any one of the nearest binary levels *i.e.*, either $\Delta/2$ or $-\Delta/2$. If $\Delta/2$ is assigned, then quantization error will be,

Scanned by CamScanner

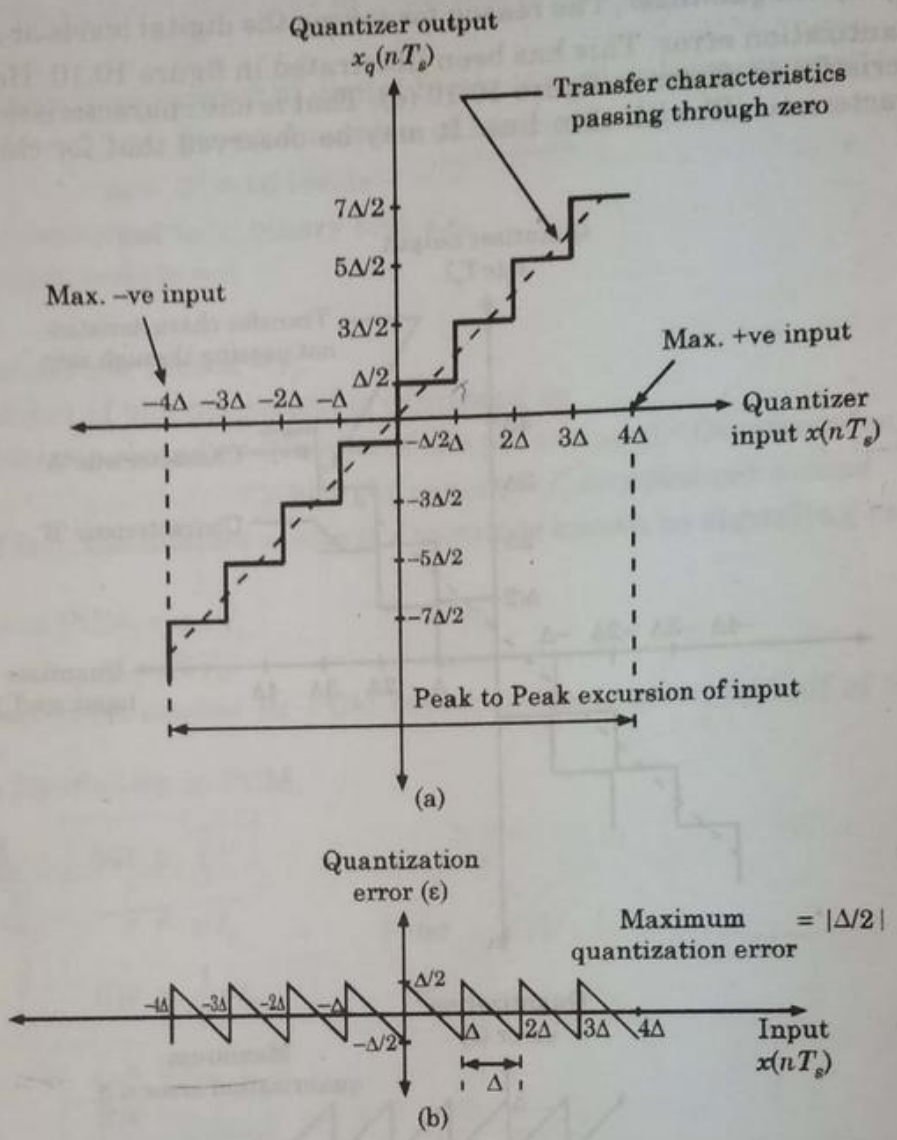


Fig. 10.9. *(a) Transfer characteristic of a quantizer*
(b) Variation of quantization error with input
$\varepsilon = x_q(nT_s) - x(nT_s) = \Delta/2 - 0 = \Delta/2$

From figure 10.9(*a*), it may also be observed that

for $\quad \Delta < x(nT_s) < 2\Delta, \; x_q(nT_s) = \frac{3}{2}\Delta$

or $\quad -\Delta < x(nT_s) < -2\Delta, \; x_q(nT_s) = -\frac{3}{2}\Delta$

This means that the maximum quantization error will be $\pm\,\Delta/2$.
In other words, maximum quantization error is given by

$$\varepsilon_{max} = \left|\frac{\Delta}{2}\right| \qquad \text{...(10.2)}$$

10.10. A Uniform Quantizer with Incorrect Quantization Characteristics

The quantizer discussed in last section is known as *uniform quantizer* since the step size remains same throughout the input range. Also, if step size varies according to the input, then quantizer

Scanned by CamScanner

is known as *non uniform quantizer*. The reason for taking the digital levels at $\Delta/2, \pm\frac{3}{2}\Delta$.... etc. is to reduce the quantization error. This has been illustrated in figure 10.10. Hence, there are two possible characteristics as shown in figure 10.10 (*a*). That is one characteristic 'A' with thick line and second characteristic 'B' with thin line. It may be observed that for characteristic 'A', we have

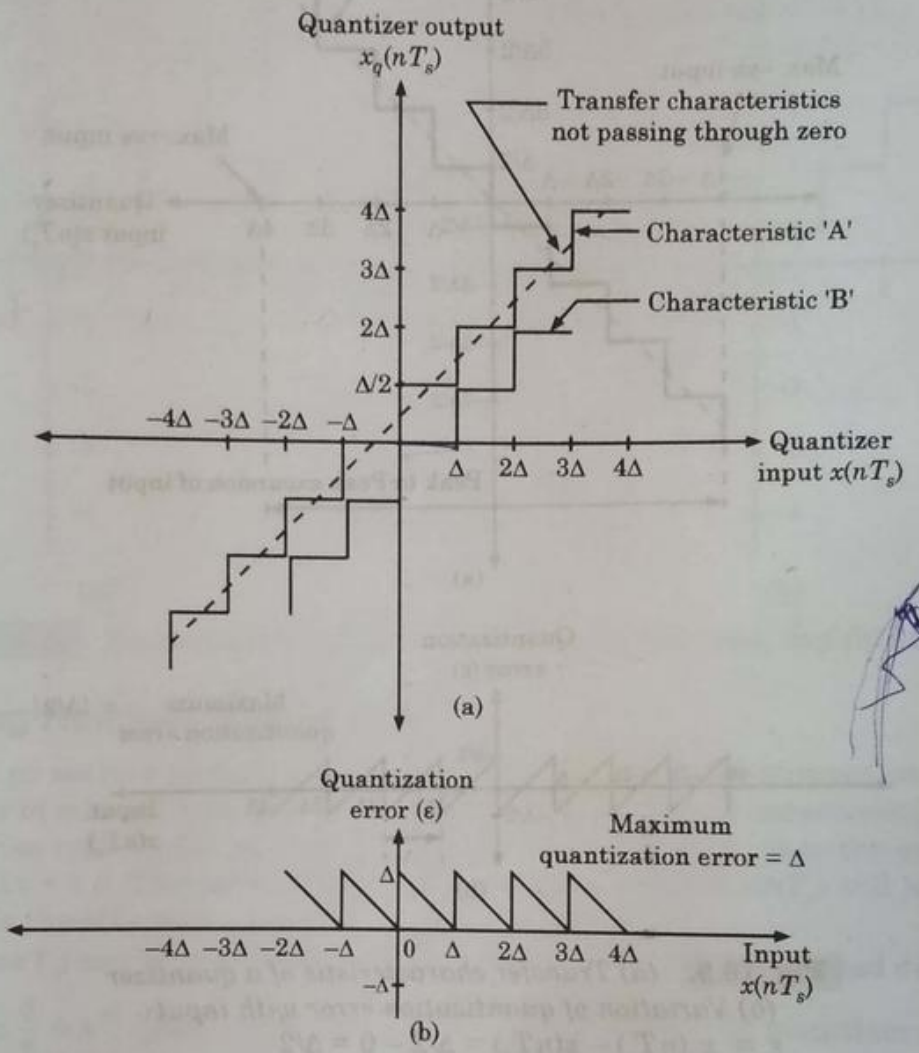


Fig. 10.10. *(a) Incorrect quantization characteristic (b) Increased quantization error.*

If $0 < x(nT_s) < \Delta$; then output $x_q(nT_s) = \Delta$

or $2\Delta < x(nT_s) < 3\Delta$; then output $x_q(nT_s) = 3\Delta$

Therefore, the maximum quantization error will be equal to Δ as shown in figure 10.10 (*b*). Similarly for characteristic '*B*' maximum quantization error is equal to $-\Delta$. The dotted line shows actual transfer characteristic which passes through origin for characteristic shown in figure 10.10(*a*). On the other hand, it does not pass through characteristic of figure 10.10(*a*). Hence, the digital levels are taken at $\pm\Delta/2, \pm\frac{3}{2}\Delta$ etc. It provides correct quantization characteristic sand reduces quantization error.

10.11. Transmission Bandwidth in a PCM System

In this section, we shall evaluate the transmission bandwidth for PCM system. Let us assume that the quantizer use '*v*' number of binary digits to represent each level.

Scanned by CamScanner

Then, the number of levels that may be represented by 'v' digits will be,

$$q = 2^v \quad ...(10.3)$$

Here 'q' represents total number of digital levels of a q-level quantizer.

For example, if $v = 4$ bits, the total number of levels will be,

$$q = 2^4 = 16 \text{ levels}$$

Each sample is converted to 'v' binary bits. *i.e.*,

Number of bits per sample = v.

We know that,

Number of samples per second = f_s

Therefore, Number of bits per second is expressed as

(Number of bits per second) = (Number of bits per samples) × (Number of samples per second)

$$= v \text{ bits per sample} \times f_s \text{ samples per second} \quad ...(10.4)$$

As a matter of fact, the number of bits per second is known as **signaling rate** of PCM and is denoted by 'r' *i.e.*,

Signaling rate in PCM, $r = v f_s$ ...(10.5)

where $f_s \geq 2 f_m$

Also, since bandwidth needed for PCM transmission is given by half of the signaling rate therefore,we have

Transmission Bandwidth in PCM,

$$BW \geq \frac{1}{2} r \quad ...(10.6)$$

But $r = v f_s$

Therefore, $BW \geq \frac{1}{2} v f_s$ ...(10.7)

Again, since $f_s \geq 2 f_m$

Hence, $BW \geq v f_m$ ...(10.8)

This is the required expression for bandwidth of a PCM system.

10.12. Quantization Noise/Error in PCM

In this section, we shall derive an expression for quantization noise (*i.e.*, error) in a PCM system for linear quantization or uniform quantization. Because of quantization, inherent errors are introduced in the signal. This error is called **quantization error**. As defined earlier, the quantization error is given as

$$\varepsilon = x_q(nT_s) - x(nT_s) \quad ...(10.9)$$

Let us assume that the input $x(nT_s)$ to a linear or uniform quantizer has continuous amplitude in the range $-x_{max}$ to $+x_{max}$.

From figure 10.9(*a*), it may be observed that the total excursion of input $x(nT_s)$ is mapped into 'q' levels on vertical axis. This means that when input is 4Δ, output is $\frac{7}{2}\Delta$ and when input is -4Δ, output is $-\frac{7}{2}\Delta$. Thus, $+x_{max}$ represents $\frac{7}{2}\Delta$ and $-x_{max}$ represents $-\frac{7}{2}\Delta$. Therefore, the total amplitude range becomes,

Total amplitude range = $x_{max} - (-x_{max}) = 2\,x_{max}$

Scanned by CamScanner

Now, if this total amplitude range is divided into 'q' levels of quantizer, then the step size 'Δ' will be,

$$\text{'step size'}\ \Delta = \frac{x_{max} - (-x_{max})}{q} = \frac{2x_{max}}{q} \qquad ...(10.10)$$

Again, now if signal $x(t)$ is normalized to minimum and maximum values equal to 1, then we have

$$x_{max} = 1$$

$$-x_{max} = -1$$

Therefore, step side would be,

$$\Delta = \frac{2}{q} \quad \text{(for normalized signal)} \qquad ...(10.11)$$

Now, if step size 'Δ' is considered as sufficiently small, then it may be assumed that the quantization error 'ε' will be an uniformly distributed random variable. We know that the maximum quantization error is given as,

$$\varepsilon_{max} = \left|\frac{\Delta}{2}\right| \qquad ...(10.12)$$

i.e.,

$$-\frac{\Delta}{2} \le \varepsilon_{max} \le \frac{\Delta}{2} \qquad ...(10.13)$$

Hence, over the interval $\left(-\frac{\Delta}{2}, \frac{\Delta}{2}\right)$ quantization error may be assumed as an uniformly distributed random variable. ...(10.14)

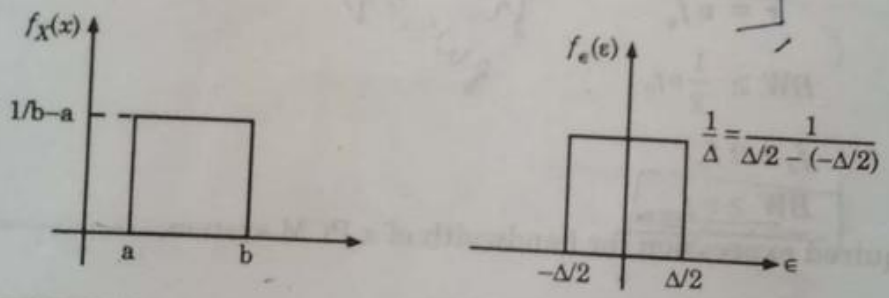


Fig. 10.11. *(a) A Uniform distribution (b) A Uniform distribution for quantization error*

Figure 10.11(*a*) shows an uniformly distrubuted random variable 'X' over an interval (a, b). Recall that the PDF of uniformly distributed random variable 'X' is given as

$$f_X(x) = \begin{cases} 0 & \text{for } x \le a \\ \dfrac{1}{b-a} & \text{for } a < x \le b \\ 0 & \text{for } x > b \end{cases} \qquad ...(10.15)$$

Thus, with the help of equation (10.15), the probability density function (PDF) for quantization error 'ε' may be defined as

$$f_\varepsilon(\varepsilon) = \begin{cases} 0 & \text{for } \varepsilon \le \dfrac{\Delta}{2} \\ \dfrac{1}{\Delta} & \text{for } -\dfrac{\Delta}{2} < \varepsilon \le \dfrac{\Delta}{2} \\ 0 & \text{for } \varepsilon > \dfrac{\Delta}{2} \end{cases} \qquad ...(10.16)$$

Scanned by CamScanner

Also, from figure 10.11(*b*), it may be observed that quantization error 'ε' has zero average value. In other words, the mean 'm_ε' of the quantization error is zero.

Further, we know that the signal to quantization noise ratio of the quantizer is defined as,

$$\frac{S}{N} = \frac{\text{Signal power (normalized)}}{\text{Noise power (normalized)}} \quad ...(10.17)$$

If type of signal at input *i.e.*, $x(t)$ is known, then it is possible to calculate signal power.

The noise power is expressed as,

$$\text{Noise power} = \frac{{}^{*}V^2_{\text{noise}}}{R} \quad ...(10.18)$$

Here, V^2_{noise} is taken as the mean square value of noise voltage. Since, here noise is defined by random varible 'ε', and *PDF* $f_\varepsilon(\varepsilon)$ therefore, its mean square value is given as,

$$\text{mean square value} = E[\varepsilon^2] = \overline{\varepsilon^2} = V^2_{\text{noise}} \quad ...(10.19)$$

We know that the mean square value of a random variable '*X*' is expressed as,

$$\overline{X}^2 = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx \quad ...(10.20)$$

Here
$$E[\varepsilon^2] = \int_{-\infty}^{\infty} \varepsilon^2 f_\varepsilon(\varepsilon)\, d\varepsilon \quad ...(10.21)$$

Using equation (10.16), above equation may be written as,

$$E[\varepsilon^2] = \int_{-\Delta/2}^{\Delta/2} \varepsilon^2 \times \frac{1}{\Delta}\, d\varepsilon = \frac{1}{\Delta}\left[\frac{\varepsilon^3}{3}\right]_{-\Delta/2}^{-\Delta/2}$$

or
$$E[\varepsilon^2] = \frac{1}{\Delta}\left[\frac{(\Delta/2)^3}{3} + \frac{(\Delta/2)^3}{3}\right] = \frac{1}{3\Delta}\left[\frac{\Delta^3}{8} + \frac{\Delta^3}{8}\right]$$

Simplifying, we get

or
$$E[\varepsilon^2] = \frac{\Delta^2}{12} \quad ...(10.22)$$

Now, using equation (10.19), the mean square value of noise voltage would be

$$V^2_{noise} = \text{mean square value} = \frac{\Delta^2}{12}$$

Also, if load resistance, $R = 1$ ohm, then the noise power is normalized *i.e.*,

$$\text{Noise power (normalized)} = \frac{V^2_{noise}}{1} \quad \text{(putting } R = 1 \text{ in equation (10.18)}$$

$$= \frac{\Delta^2/12}{1} = \frac{\Delta^2}{12}$$

* Here, we are assuming that $x(t)$ is a voltage signal.

Scanned by CamScanner

Hence, finally, we write

$$\left.\begin{array}{l}\text{Normalized noise power}\\ \text{or Quantization noise power}\\ \text{or Quantization error (in terms of power)}\end{array}\right\} = \frac{\Delta^2}{12}, \text{ for linear quantization.} \quad ...(10.23)$$

10.13. Signal to Quantization Noise Ratio for Linear Quantization

We know that in a PCM system for linear quantization the signal to quantization noise ratio is given as,

$$\frac{S}{N} = \frac{\text{Normalized signal power}}{\text{Normalized noise power}}$$

But, normalized noise power has been calculated as $\frac{\Delta^2}{12}$.

Therefore,
$$\frac{S}{N} = \frac{\text{Normalized signal power}}{(\Delta^2/12)} \quad ...(10.24)$$

We know that the number of bits 'v' and quantization levels are related as,

$$q = 2^v \quad ...(10.25)$$

Let us assume that input $x(nT_s)$ to a linear quantizer has continuous amplitude in the range $-x_{max}$ to $+x_{max}$.

Therefore, total amplitude range

$$= x_{max} - (-x_{max}) = 2x_{max}$$

Now, the step size will be

$$\Delta = \frac{2x_{max}}{q} \quad ...(10.26)$$

Here, substituting the value of q from equation (10.25) in equation (10.26), we get

$$\Delta = \frac{2x_{max}}{2^v}$$

Now substituting this value in equation (10.24) we get,

$$\frac{S}{N} = \frac{\text{Normalized signal power}}{\left(\frac{2x_{max}}{2^v}\right)^2 \cdot \frac{1}{12}}$$

Let normalized signal power be denoted as 'P'.

Then,
$$\frac{S}{N} = \frac{P}{\frac{4x_{max}^2}{2^{2v}} \times \frac{1}{12}} = \frac{3P}{x_{max}^2} \cdot 2^{2v}$$

This is the required relation for signal to quantization noise ratio for linear quantization in a PCM system.

Hence, signal to quantization noise ratio:

$$\frac{S}{N} = \frac{3P}{x_{max}^2} \cdot 2^{2v} \quad ...(10.27)$$

This expression shows that signal to noise power ratio of quantizer increases exponentially with increasing bits per sample.

Scanned by CamScanner

Now if we assume that inpur $x(t)$ is normalized, *i.e.*,

$$x_{max} = 1 \quad ...(10.28)$$

Then, signal to quantization noise ratio will be,

$$\frac{S}{N} = 3 \times 2^{2v} \times P \quad ...(10.29)$$

Also, if the destination signal power 'P' is normalized, *i.e.*,

$$P \leq 1 \quad ...(10.30)$$

Then the signal to noise ratio will be given as

$$\frac{S}{N} \leq 3 \times 2^{2v} \quad ...(10.31)$$

Because $x_{max} = 1$ and $P \leq 1$, the signal to noise ratio given by equation (10.31) is said to be normalized.

Expressing the signal to noise ratio in decibels, we get

$$\left(\frac{S}{N}\right) dB = 10 \log_{10}\left(\frac{S}{N}\right) dB \leq 10 \log_{10} [3 \times 2^{2v}]$$

or

$$\left(\frac{S}{N}\right) dB \leq (4.8 + 6\, v) \text{ dB}$$

Thus, Signal to Quantization noise ratio for normalized values of power 'P' and amplitude of input $x(t)$, will be

$$\left(\frac{S}{N}\right) dB \leq (4.8 + 6v)\, dB \quad ...(10.32)$$

10.14. Effect of Noise on the PCM System

To illustrate the effect of noise on the transmitted pulses, let us consider figure 10.12.

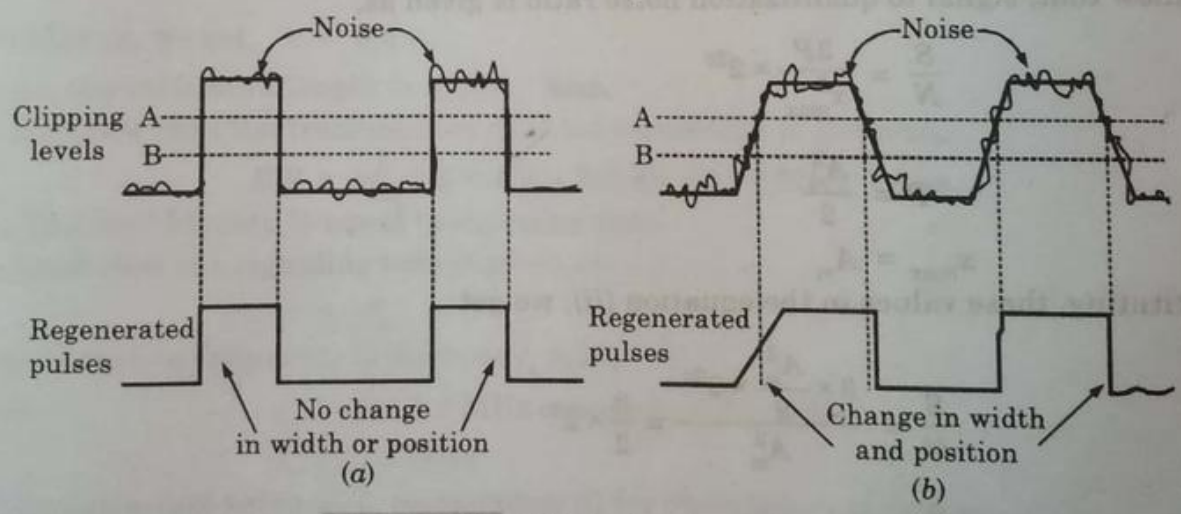


Fig. 10.12. *Effect of noise on PCM.*

First, let us look at figure 10.12(*a*). Due to the noise superimposed on the pulse, only the PAM system shall be affected. However, the PWM, PPM and PCM systems will remain unaffected. The regeneration of the pulses is achieved by using a clipper circuit with reference levels A and B. Now, let us observe figure 10.12 (*b*). Here, the sides of the transmitted pulse are not perfectly vertical. In practice, the transmitted pulses usually have slightly sloping sides (edges). As the noise is superimposed on them, the width and the position of the regenerated pulses is changed. Now, this is going to distort the information contents in the PWM and PPM signals.

Scanned by CamScanner

However, PCM is still unaffected as it does not contain any information in the width or the position of the pulses. Hence, PCM has much better noise immunity as compared to PAM, PWM and PPM systems.

Example 10.1. Derive an expressin for signal to quantization noise ratio for a PCM system which emplys linear (*i.e.*, uniform) quantization technique. Given that input to the PCM system is a sinusoidal signal. *(U.P.S.C.I.E.S. Examination-1999)*

or

A PCM system uses a uniform quantizer followed by a v bit encoder. Show that rms signal to quantization noise ratio is approximately given as (1.8 + 6v) dB.

(U.P. Tech-Semester Exam. 2002-2003)

Solution: Let us assume that the modulating signal is a sinusoidal voltage, having a peak amplitude equal to A_m. Also, let this signal cover the complete excursion of representation levels.

Then, the power of this signal will be,

$$P = \frac{V^2}{R}$$

Here, V = rms value

i.e.,
$$V = \left[A_m/\sqrt{2}\right]^2.$$

Therefore, we have

$$P = \frac{A_m^{\ 2}}{2} \cdot \frac{1}{R} \qquad ...(i)$$

In case when $R = 1$, the power P is normalized *i.e*,

Normalized power $P = \frac{A_m^2}{2}$ with $R = 1$ in equation (*i*).

We know that, signal to quantization noise ratio is given as,

$$\frac{S}{N} = \frac{3P}{x_{\max}^2} \times 2^{2v} \qquad ...(ii)$$

Here
$$P = \frac{A_m^2}{2}$$

and
$$x_{max} = A_m$$

Substituting, these values in the equation (*ii*), we get

$$\frac{S}{N} = \frac{3 \times \frac{A_m^2}{2} \times 2^{2v}}{A_m^2} = \frac{3}{2} \times 2^{2v}$$

or
$$\frac{S}{N} = 1.5 \times 2^{2v}$$

Expressing signal to noise power ratio in dB, we get

$$\left(\frac{S}{N}\right) dB = 10 \log_{10}\left(\frac{S}{N}\right)$$

or
$$\left(\frac{S}{N}\right) dB = 10 \log_{10} (1.5 \times 2^{2v})$$

Scanned by CamScanner

or $$\left(\frac{S}{N}\right) dB = 10 \log_{10} (1.5) + 10 \log_{10} (2^{2v})$$

or $$\left(\frac{S}{N}\right) dB = 1.76 + 2v \times 10 \times 0.3$$

Therefore, we have

$$\left(\frac{S}{N}\right) dB \text{ in PCM} = \left(\frac{S}{N}\right) dB = 1.8 + 6v \text{ (for sinusoidal signal)} \quad \textbf{Ans.}$$

Example 10.2. A Television signal having a bandwidth of 4.2 MHz is transmitted using binary PCM system. Given that the number of quantization levels is 512. Determine:

(*i*) Code word length

(*ii*) Transmission bandwidth

(*iii*) Final bit rate

(*iv*) Output signal to quantization noise ratio.

Solution: Givn that the bandwidth is 4.2 MHz. This means that highest frequency component will have frequency of 4.2 MHz *i.e.*,

$$f_m = 4.2 \text{ MHz}$$

Also, given that Quantization levels, $q = 512$

(*i*) We know that the number of bits and quantization levels are related in binary PCM as under:

$$q = 2^v$$

i.e., $$512 = 2^v$$

or $$\log_{10} 512 = v \log_{10} 2$$

or $$v = \frac{\log_{10} 512}{\log_{10} 2}$$

Simplifying, we get, $v = 9$ bits

Hence, the code word length is 9 bits. **Ans.**

(*ii*) We know that the transmission channel bandwidth is given as,

$$BW \geq v f_m \geq 9 \times 4.2 \times 10^6 \text{ Hz} \geq 37.8 \text{ MHz} \quad \textbf{Ans.}$$

(*iii*) The final bit rate is equal to signaling rate.

We know that the signaling rate is given as,

$$r = v f_s \qquad \text{...(i)}$$

Here, sampling frequency is given as $f_s \geq 2f_m$

Thus, $$f_s \geq 2 \times 4.2 \text{ MHz} \quad \text{since } f_m = 4.2 \text{ MHz}$$

or $$f_s \geq 8.4 \text{ MHz}$$

Substituting this value of 'f_s' in equation (*i*) for signaling rate, we get

or $$r = 9 \times 8.4 \times 10^6 \text{ bits/sec} = 75.6 \times 10^6 \text{ bits/sec} \quad \textbf{Ans.}$$

The transmission bandwidth may also be obtained as,

$$BW \geq \frac{1}{2} r \geq \frac{1}{2} \times 75.6 \times 10^6 \text{ bits/sec}$$

or $BW \geq 37.8$ MHz which is same as the value obtained earlier.

Scanned by CamScanner

(*iv*) The output signal to noise ratio is expressed as

$$\left(\frac{S}{N}\right) dB \leq (4.8 + 6v)\ dB$$

But $v = 9$

Therefore, $$\left(\frac{S}{N}\right) dB \leq 4.8 + 6 \times 9$$

or $$\left(\frac{S}{N}\right) dB \leq 58.8 \text{ dB}$$ **Ans.**

Example 10.3. The bandwidth of an input signal to the PCM is restricted to 4 kHz. The input signal varies in amplitude from – 3.8 V to + 3.8 V and has the average power of 30 mW. The required signal to noise ratio is given as 20 dB. The PCM modulator produces binary output. Assuming uniform quantization,

(*i*) Find the number of bits required per sample.

(*ii*) Outputs of 30 such PCM coders are time multiplexed. What would be the minimum required transmission bandwidth for this multiplexed signal?

(Punjab Technical University-1999)

Solution: The given value of signal to noise ratio is 20 dB.

This means that.

$$\left(\frac{S}{N}\right) dB = 10 \log_{10}\left(\frac{S}{N}\right) = 20\ dB$$

Hence, $$\frac{S}{N} = 100$$

(*i*) We know that the signal to quantization noise ratio is given as,

$$\frac{S}{N} = \frac{3P \cdot 2^{2v}}{x_{max}^2}$$

Here, we are given

$$x_{max} = 3.8\ V$$

$$P = 30 \text{ mW}$$

and $$\frac{S}{N} = 100$$

Therefore, $$100 = \frac{3 \times 30 \times 10^{-3} \times 2^{2v}}{(3.8)^2}$$

Solving, we get $v = 6.98$ bits = 7 bits **Ans.**

(*ii*) The maximum frequency is given as

$$f_m = 4 \text{ kHz}$$

We know that the transmission bandwidth is expressed as,

$$BW \geq v f_m$$

Since there are 30 PCM coders which are time multiplexed, the transmission bandwidth must be,

$$BW \geq 30 \times v \times f_m \geq 30 \times 7 \times 4 \text{ kHz} \geq 840 \text{ kHz}$$ **Ans.**

Scanned by CamScanner

We also know that the signaling rate is two times the transmission bandwidth, *i.e.*

Signaling rate, $r = 840 \times 2$ bits/sec = 1680 bits/sec. **Ans.**

Example 10.4. The information in an analog signal voltage waveform is to be transmitted over a PCM system with an accuracy of ± 0.1% (full scale). The analog voltage waveform has a bandwidth of 100 Hz and an amplitude range of – 10 to + 10 volts.

(*i*) Find the minimum sampling rate required.

(*ii*) Find the number of bits in each PCM word.

(*iii*) Find minimum bit rate required in the PCM signal.

(*iv*) Find the minimum absolute channel bandwidth required for the transmission of the PCM signal. *(West Bengal-1999)*

Solution: Here an accuracy is given as ± 0.1%. This means that the quantization error must be ± 0.1% or the maximum quantization error must be ± 0.1%.

Thus, $\varepsilon_{max} = \pm 0.1\% = \pm 0.001$

We know that the maximum quantization error for an uniform quantizer is expressed as,

$$\varepsilon_{max} = \left|\frac{\Delta}{2}\right|$$

or $$\left|\frac{\Delta}{2}\right| = 0.001$$

Therefore,

$$\text{Step size } \Delta = 2 \times 0.001 = 0.002$$

We know that the step size, number of quantization levels and maximum value of the signal are related as

$$\Delta = \frac{2\,x_{max}}{q} \qquad ...(i)$$

Here, given $|x_{max}| = 10$ volts

Substituting, values of Δ and x_{max} in equation (*i*), we get

$$0.002 = \frac{2 \times 10}{q}$$

or $$q = \frac{20}{0.002} = 10{,}000$$

Hence, the number of levels are 10,000.

(*i*) The maximum frequency in the signal is given as 100 Hz *i.e.*,

$$f_m = 100 \text{ Hz}$$

By sampling theorem minimum sampling frequency should be,

$$f_s \geq 2 f_m \geq 2 \times 100 \geq 200 \text{ Hz}$$ **Ans.**

(*ii*) We know that minimum 10,000 levels should be used to quantize the signal. If binary PCM is used, then number of bits for each samples may be calculated as under, *i.e.*

$$q = 2^v$$

Here, q = number of levels

v = bits in PCM,

Thus, $$10{,}000 = 2^v$$

$$\log_{10} 10{,}000 = v \log_{10} 2$$

or $$v = \frac{\log_{10} 10{,}000}{\log_{10} 2} = 13.288 = 14 \text{ bits}$$ **Ans.**

Scanned by CamScanner

(*iii*) The bit rate or signaling rate is expressed as,

$$r \geq v f_s \geq 14 \times 200 \geq 2800 \text{ bits per second.} \quad \textbf{Ans.}$$

(*iv*) The transmission bandwidth for PCM is expressed as,

$$BW \geq \frac{1}{2} r \geq \frac{1}{2} \times 2800 \geq 1400 \text{ Hz} \quad \textbf{Ans.}$$

Example 10.5. Twenty four voice signals are sampled uniformly and then have to be time division multiplexed. The highest frequency component for each voice signal is equal to 3.4 kHz. Now

(*i*) If the signals are pulse amplitude modulated using Nyquist rate sampling, what would be the minimum channel bandwidth required.

(*ii*) If the signals are pulse code modulated with an 8 bit encoder, what would be the sampling rate? The bit rate of system is given as 1.5×10^6 bits/sec.

Solution: (*i*) As a matter of fact, if N channels are time division multiplexed, then minimum transmission bandwidth is expressed as,

$$BW = N f_m$$

Here, f_m is the maximum frequency in the signals.

Given, $f_m = 3.4$ kHz

Therefore $BW = 24 \times 3.4 \text{ kHz} = 81.6 \text{ kHz}$ **Ans.**

(*ii*) The signaling rate of the system is given as,

$$r = 1.5 \times 10^6 \text{ bits/sec}$$

Since there are 24 channels, the bit rate of an individual channel is,

$$r \text{ (one channel)} = \frac{1.5 \times 10^6}{24} = 62500 \text{ bits/sec}$$

Further, since each sample is encoded using 8 bits, the samples per second will be,

$$\text{Sample/sec} = \frac{r \text{ (one channel) bits/sec}}{\text{bits/sample}}$$

Note that the samples per seconds is nothing but sampling frequecy.

Thus, we have $f_s = \dfrac{62500 \text{ bits/sec}}{8 \text{ bits/sample}}$

Solving, we get, $f_s = 7812.5$ Hz or samples per second **Ans.**

Example 10.6. A PCM system uses a uniform quantizer followed by a 7-bit binary encoder. The bit rate of the system is equal to 50×10^6 bits/sec.

(*i*) What is the maximum message signal bandwidth for which the system operates satisfactorily?

(*ii*) Calculate the output signal to quantization noise ratio when a full load sinusoidal modulating wave of frequency 1 MHz is applied to the input.

(U.P. Tech-Semester Exam. 2002-2003)

Solution: (*i*) Let us assume that the message bandwidth be f_m Hz. Therefore sampling frequency should be,

$$f_s \geq 2 f_m$$

The number of bits given as $v = 7$ bits

We know that the signaling rate is given as,

$$r \geq v \cdot f_s$$

Thus

$$r \geq 7 \times 2 f_m$$

Scanned by CamScanner

Substituting value for r, we get

$$50 \times 10^6 \geq 14 f_m$$

or $$f_m \leq 3.57 \text{ MHz} \quad \textbf{Ans.}$$

Thus the maximum message bandwidth is 3.57 MHz.

(*ii*) The modulating wave is sinusoidal. For such signal, the signal to quantization noise ratio is expressed as,

$$\left(\frac{S}{N}\right) dB = 1.8 + 6v$$

Substituting the value of v, we get

$$\left(\frac{S}{N}\right) dB = 1.8 + 6 \times 7 = 43.8 \text{ dB} \quad \textbf{Ans.}$$

Example 10.7. The information in an analog waveform with maximum frequency f_m = 3 kHz is to be transmitted over an M-level PCM system where the number of quantization levels is M = 16. The quantization distortion is specified not to exeed 1% of peak to peak analog signal.

(*i*) What would be the maximum number of bits per sample that should be used in this PCM system?

(*ii*) What is the minimum sampling rate and what is the resulting bit transmission rate?

Solution: (*i*) Since the number of quantization levels given here are $M = 16$,

$$q = M = 16$$

We know that the bits and levels in binary PCM are related as,

$$q = 2^v$$

Here, v = number of bits in a codeword

Thus, $$16 = 2^v$$

or $$v = 4 \text{ bits.} \quad \textbf{Ans.}$$

(*ii*) Again since $$f_m = 3 \text{ kHz}$$

By sampling theorem, we know that

$$f_s \geq 2 f_m$$

Thus, $$f_s \geq 2 \times 3 \text{ kHz} \geq 6 \text{ kHz} \quad \textbf{Ans.}$$

Hence, the minimum sampling rate is 6 kHz

Also, bit transmission rate or signaling rate is given as,

$$r \geq v f_s \geq 4 \times 6 \times 10^3$$

or $$r \geq 24 \times 10^3 \text{ bits per second} \quad \textbf{Ans.}$$

Example 10.8. A signal having bandwidth equal to 3.5 kHz is sampled, quantized and coded by a PCM system. The coded signal is then transmitted over a transmission channel of supporting a transmission rate of 50 k bits/sec. Determine the maximum signal to noise ratio that can be obtained by this system.

The input signal has peak to peak value of 4 volts and rms value of 0.2 V.

(Pune University-1998)

Solution: The maximum frequency of the signal is given as 3.5 kHz,

i.e., $$f_m = 3.5 \text{ kHz}$$

Therefore sampling frequency will be

$$f_s \geq 2 f_m \geq 2 \times 3.5 \text{ kHz} \geq 7 \text{ kHz}$$

We know that the signaling rate is given by

$$r \geq v f_s$$

Scanned by CamScanner

Substituting values of $r = 50 \times 10^3$ bits/sec and $f_s \geq 7 \times 10^3$ Hz in above equation, we get

$$50 \times 10^3 \geq v \cdot 7 \times 10^3$$

Simplifying, we get

$$v \leq 7.142 \text{ bits}$$

or $$v \cong 8 \text{ bits}$$

The rms value of the signal is 0.2 V. Therefore the normalized signal power will be,

Normalized signal power $P = \dfrac{(0.2)^2}{1}$ *

i.e., $$P = 0.04\ W$$

Further, the maximum signal to noise ratio is given by,

$$\frac{S}{N} = \frac{3P \cdot 2^{2v}}{x_{\max}^2}$$

Substituting the values of $P = 0.04$, $v = 8$ and $x_{max} = 2$ in above equation, we have

$$\frac{S}{N} = \frac{3 \times 0.04 \times 2^{2\times 8}}{4} = 1966.08 \cong 33 \text{ dB} \qquad \textbf{Ans.}$$

Example 10.9. A signal $x(t)$ is uniformly distributed in the range $\pm x_{max}$. Evaluate maximum signal to noise ratio for this signal.

Solution: Given that the signal is uniformly distributed in the range $\pm x_{max}$, therefore we can write its PDF (using the Standard Uniform Distribution) as under:,

$$f_X(x) = \begin{cases} 0 & \text{for } x < -x_{max} \\ \dfrac{1}{2x_{\max}} & \text{for } -x_{max} < x < x_{max} \\ 0 & \text{for } x > x_{max} \end{cases}$$

Figure 10.13 shows this PDF,

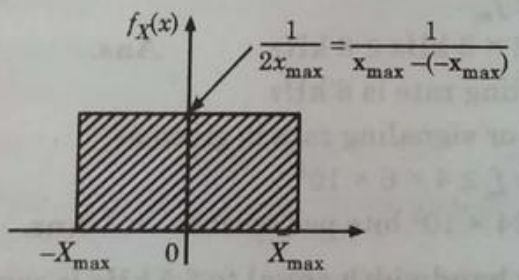


Fig 10.13. *PDF of a uniformly distributed random variable.*

The mean square value of a random variable X is given as,

$$\overline{X}^2 = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\, dx$$

Therefore, mean square value of $x(t)$ will be,

$$\overline{x^2(t)} = \int_{-x_{\max}}^{x_{\max}} x^2(t) \cdot \frac{1}{2x_{\max}}\, dx = \frac{1}{2x_{\max}} \cdot \left[\frac{x^3(t)}{3}\right]_{-x_{\max}}^{x_{\max}}$$

* $R = 1$ for normalized power.

Scanned by CamScanner

or $$\overline{x^2(t)} = \frac{x_{max}^2}{3} \quad ...(i)$$

The signal power $= \dfrac{\overline{x^2(t)}}{R}$

Normalized signal power $P = \dfrac{\overline{x^2(t)}}{1}$ [since $R = 1$]

Putting the value of $\overline{x^2(t)}$ from (*i*), we get

$$P = \frac{x_{max}^2}{q}$$

We know that the relation between step size, maximum amplitude of signal and number of levels is given as

Step size $$\Delta = \frac{2x_{max}}{q}$$

$$\therefore \quad x_{max} = \frac{\Delta q}{2}$$

$$\therefore \text{ Normalized signal power, } P = \frac{\left(\frac{\Delta q}{2}\right)^2}{3} = \frac{\Delta^2 q^2}{12}$$

We also know that

Normalized noise power $= \dfrac{\Delta^2}{12}$

Therefore, signal to noise power ratio $\dfrac{S}{N} = \dfrac{\text{Normalized signal power}}{\text{Normalized noise power}}$

$$\frac{S}{N} = \frac{\Delta^2 q^2/12}{\Delta^2/12} = q^2$$

Since $q = 2^v$, above equation will be,

$$\frac{S}{N} = 2^{2v}$$

or $$\left(\frac{S}{N}\right) dB = 10 \log_{10}(2^{2v}) \text{ dB} \approx 6v$$

This is required expression for maximum value of signal to noise ratio.

Example 10.10. Consider an audio signal consisting of the sinusoidal term given as

$$x(t) = 3 \cos (500 \pi t)$$

(*i*) Determine the signal to quantization noise ratio when this is quantized using 10 bit PCM.

(*ii*) How many bits of quantization are needed to achieve a signal to quantization noise ratio of atleast 40 dB?

Solution: Here given that $x(t) = 3 \cos (500 \pi t)$

This is sinusoidal signal applied to the quantizer.

(*i*) Let us assume that peak value of cosine wave defined by $x(t)$ covers the complete range of quantizer.

Scanned by CamScanner

i.e., $A_m = 3V$ covers complete range

In example 10.1, we have derived signal to noise ratio for a sinusoidal signal. It is expressed as

$$\left(\frac{S}{N}\right) dB = 1.8 + 6v$$

Since here 10 bit PCM is used *i.e.*,

$$v = 10$$

Thus, $$\left(\frac{S}{N}\right) dB = 1.8 + 6 \times 10 = 61.8 \text{ dB} \quad \textbf{Ans.}$$

(*ii*) For sinusoidal signal, again, let us use the same relation

i.e., $$\left(\frac{S}{N}\right) dB = 1.8 + 6v \text{ dB}$$

To get signal to noise ratio of at least 40 dB we can write above equation as,

$$1.8 + 6v \geq 40 \text{ dB}$$

Solving this, we get $v \geq 6.36$ bits ≈ 7 bits

Hence, at least 7 bits are required to get signal to noise ratio of 40 dB. **Ans.**

Example 10.11. A 7 bit PCM system employing uniform quantization has an overall signaling rate of 56 k bits per second. Calculate the signal to quantization noise that would result when its input is a sine wave with peak amplitude equal to 5 Volt. Find the dynamic range for the sine wave inputs in order that the signal to quantization noise ratio may be less than 30 dBs. What is the theoretical maximum frequency that this system can handle?

(Madras University-1999)

Solution: The number of bits in the PCM system are

$$v = 7 \text{ bits}$$

Assume that 5 V peak to peak voltage utilizes complete range of quantizer. Then, we can find the signal to quantization noise ratio as,

$$\left(\frac{S}{N}\right) dB = 1.8 + 6v \text{ dB} = 1.8 + 6 \times 7 = 43.8 \text{ dB}$$

We know that the signaling rate is given, as,

$$r = v f_s$$

Substituting $r = 56 \times 10^3$ bits/second and $v = 7$ bits in above equation, we get

$$56 \times 10^3 = 7 \cdot f_s$$

Simplifying, we get

Sampling frequency, $f_s = 8 \times 10^3$ Hz

Further, using sampling theorem we have,

$$f_s \geq 2 f_m$$

Thus, maximum frequency that can be handled is given as,

$$f_m \leq \frac{f_s}{2} \leq \frac{8000}{2} \leq 4000 \text{ Hz},$$

$$f_m \leq 4 \text{ kHz} \quad \textbf{Ans.}$$

10.15. Robust Quantization: Necessity of Nonuniform Quantization in a PCM System

(U.P. Tech., Sem. Examination, 2003-2004)

As discussed earlier, in uniform quantization, the quantizer has a linear characteristics. The step size also remains same throughout the range of quantizer. Therefore, over the complete range of inputs, the maximum quantization error also remains same.

Scanned by CamScanner

We know that the quantization error is given as under:

$$\text{Maximum quantization error} = \varepsilon_{max} = \left|\frac{\Delta}{2}\right| \quad ...(10.33)$$

Also, step size 'Δ' is expressed as,

$$\Delta = \frac{2x_{max}}{q}$$

If $x(t)$ is normalized, its maximum value *i.e*, $x_{max} = 1$.

$$\text{Therefore, we have,} \quad \Delta = \frac{2}{q} \quad ...(10.34)$$

Let us consider an example of PCM system in which we take $v = 4$ bits.
Then number of levels q will be,

$$q = 2^4 = 16 \text{ levels.}$$

Thus, using equation (10.34) the step size Δ will be,

$$\Delta = \frac{2}{q} = \frac{2}{16} = \frac{1}{8}$$

Hence, quantization error is given from equation (10.33) as,

$$\varepsilon_{max} = \left|\frac{\Delta}{2}\right| = \left|\frac{1}{2 \times 8}\right| = \frac{1}{16}$$

Hence, note that here the quantization error is $\frac{1}{16}$th part of the full voltage* range. For simplicity, we assume that full range voltage is 16 volts. Then maximum quantization error will be 1 volt. However, for the low signal amplitudes like 2 volts, 3 volts etc., the maximum quantization error of 1 volt is quite high *i.e.*, about 30 to 50%. This means that for signal amplitudes which are close to 15 volts, 16 volts etc., the maximum quantization error (which is same throughout the range) of 1 volt can be considered to be small. In fact, this problem arises because of uniform quantization. Therefore non-uniform quantization should be used in such cases.

In other words, we can say that it is desirable that SNR_q should remain essentially constant over a wide range of input power levels. A quantizer that satisfies all these requirements is known as a ***Robust Quantizer****. Infact, such a robust performance can be obtained by using a* ***nonuniform quantization****.*

10.16. Necessity of Nonuniform Quantization for Speech Signal

We know that speech and music signals are characterized by large crest factor. This means that for such signals the ratio of peak to rms value is quite high.

$$\text{Crest factor} = \frac{\text{Peak value}}{\text{rms value}} \quad ...(10.35)$$

$$= \text{Very high for speech and music.}$$

We know that the signal to noise ratio is given by,

$$\frac{S}{N} = (3 \times 2^{2v} \times P) \quad ...(10.36)$$

Expressing in decibles, the last expression becomes

$$\left(\frac{S}{N}\right) dB = 10 \log_{10} (3 \times 2^{2v} \times P)$$

* If we consider the input signal as a voltage signal.

Scanned by CamScanner

If we normalize the signal power *i.e.*, if $P = 1$, then above equation becomes,

$$\left(\frac{S}{N}\right) dB \geq 4.8 + 6v)\ dB \qquad ...(10.37)$$

Here, power P is defined as

$$P = \frac{V_{signal}^2}{R} = \frac{x^2(t)}{R}$$

$$V_{signal}^2 = \text{mean square value of signal voltage} = x^2(t)$$

Hence, normalized power will be,

$$P = \frac{x^2(t)}{1} \qquad [\text{with } R = 1]$$

$$P = x^2(t) \qquad ...(10.38)$$

From equation (10.35), crest factor is given as,

$$\text{Crest factor} = \frac{\text{Peak value}}{\text{rms value}} = \frac{x_{max}}{[x^2(t)]^{1/2}} \qquad ...(10.39)$$

or

$$\text{Crest factor} = \frac{x_{max}}{\sqrt{P}} \qquad \text{since } P = x^2(t) \qquad ...(10.40)$$

When we normalize the signal $x(t)$, then

$$x_{max} = 1 \qquad ...(10.41)$$

Substituting this value of x_{max} in equation (10.40), we get

$$\text{Crest factor} = \frac{1}{\sqrt{P}} \qquad ...(10.42)$$

For a large crest factor of voice (*i.e.*, speech) and music signals, P should be very very less than one in above equation.

i.e., $P << 1$ for large crest factor

Therefore, actual signal to noise ratio would be significantly less than the value which is given by equation (10.37) since in this equation $P = 1$.

Again, consider equation (10.36).

$$\left(\frac{S}{N}\right) = 3 \times 2^{2v} \times P$$

$$(3 \times 2^{2v} \times P|_{P<<1} << (3 \times 2^{2v} \times P)|_{P=1} \qquad ...(10.43)$$

This equation illustrates that the signal to noise ratio for large crest factor signal ($P << 1$) would be very very less than that of the calculated theoretical value. The theoretical value is obtained for normalized power ($P = 1$) by equation (10.37).

Therefore, such large crest factor signals (*i.e.*, speech and music) should use non-uniform quantization to overcome the problem just discussed. Signal to noise ratio reduces at low power levels ($P << 1$) just now we have observed by equation (10.43). This means that at low signal levels, signal to noise ratio reduces *i.e.*, noise increases. However, the quantization noise is directly related to step size. Therefore, at low signal levels ($P << 1$), noise can be kept low by keeping step size low. This means that at low signal levels, signal to noise ratio can be increased by decreasing step size 'Δ'. This means that step size 'Δ' should be varied according to the signal level to keep signal to noise ratio at the required value. This is nothing but **nonuniform quantization.** Now let's see how *nonuniform quantization* is achieved through companding in the section to follow.

Scanned by CamScanner

10.17. Nonuniform Quantization

If the quantizer characteristics is nonlinear and the step size is not constant instead if it is variable, dependent on the amplitude of input signal then the quantization is known as **nonuniform quantization**. In non-uniform quantization, the step size is reduced with the reduction in signal level. For weak signals ($P << 1$), the step size is small, therefore the quantization noise reduces, to improve the signal to quantization noise ratio for weak signals. The step size is thus varied according to the signal level to keep the signal to noise ratio adequately high. This is nonuniform quantization. The non-uniform quantization is practically achieved through a process called **companding**. We shall discuss companding in the next section.

10.18. Companding (i.e., Companded PCM)

As a matter of fact, companding is nonuniform quantization. It is required to be implemented to improve the signal to quantization noise ratio of weak signals. We know that the quantization noise is given by

$$N_q = \frac{\Delta^2}{12}$$

This shows that in the uniform quantization, once the step size is fixed, the quantization noise power remains constant. However, the signal power is not constant. It is proportional to the square of signal amplitude. Hence signal power will be small for weak signals, but quantization noise power is constant. Therefore, the signal to quantization noise for the weak signals is very poor. This will affect the quality of signal. The remedy is to use companding. Companding is a term derived from two words *i.e.*, compression and expansion as under:

Companding = Compressing + Expanding

In pratice, it is difficult to implement the non-uniform quantization because it is not known in advance about the changes in the signal level. Therefore, a particular method is used. The weak signals are amplified and strong signals are attenuated before applying them to a uniform quantizer. This process is called as **compression** and the block that provides it is called as a **compressor**.

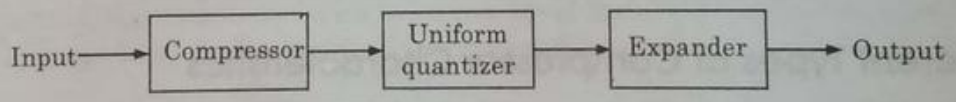


Fig. 10.14. *A companding model.*

At the receiver exactly opposite is followed which is called expansion. The circuit used for providing expansion is called as an **expander**. The compression of signal at the transmitter and expansion at the receiver is combined to be called **companding**. The process of companding has been shown in the form of a block diagram in figure 10.14.

10.18.1. Compressor Characteristic

Figure 10.15 shows the compressor characteristics. As shown in figure 10.15, the compressor provides a higher gain to the weak signals and smaller gain to the strong input signals. Thus weak signals are artificially boosted to improve the signal to quantization noise ratio. It may be noted that this compressor characteristics has been shown only for the positive input signal but we can draw it even for the negative input signals using the some principle. In fact, the compressor is included at the PCM transmitter.

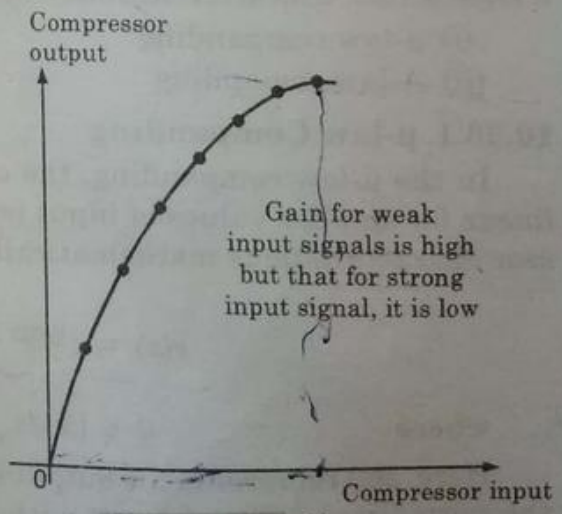


Fig. 10.15. *Compressor characteristics.*

Scanned by CamScanner

10.18.2. Expander Characteristics

Figure 10.16 shows the expander characteristics. This characteristics is exactly the inverse of the compressor characteristics. This ensures that all the artificially boosted signals by the compressor are brought back to their original amplitudes at the receiver end.

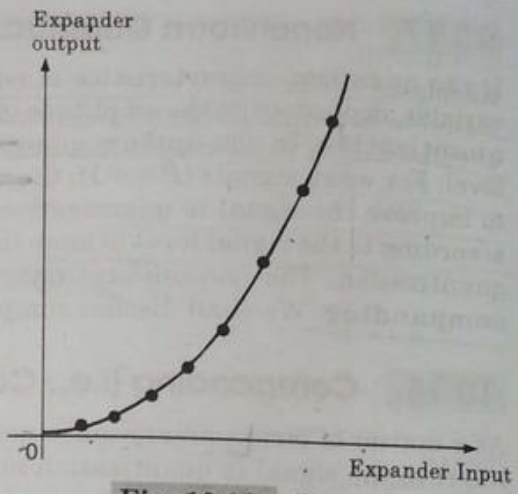


Fig. 10.16. *Expander characteristics*

10.19. Compander Characteristic

Figure 10.17 shows the compander characteristics which is the combination of the compressor and expander characteristics. Due to the inverse nature of compressor and expander, the overall characteristics of the compander is a straight line (dotted line in figure 10.17. This indicates that all the boosted signals are brought back to their original amplitudes.

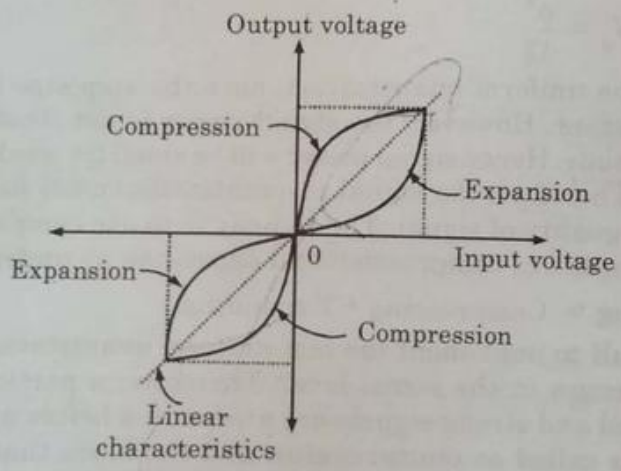


Fig. 10.17. *Companding curves for PCM system*

10.20. Different Types of Compressor Characteristics

Ideally, we need a linear compressor characteristics for small amplitudes of the input signal and a logarithmic characteristic elsewhere. In pratice, this is achieved by using following two methods:

(*i*) μ-law companding

(*ii*) A-law comapding

10.20.1. μ-law Companding

In the μ-law companding, the compressor characteristic is continuous. It is approximately linear for smaller values of input levels and logarithmic for high input levels. The μ-law compressor characteristic is mathematically expressed as under:

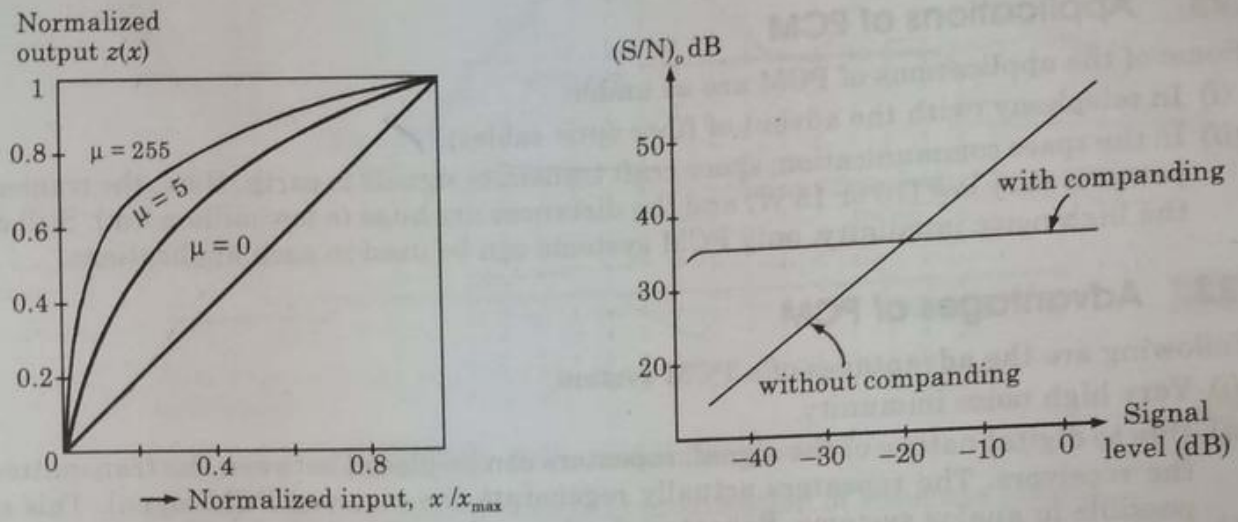
$$z(x) = (\text{sgn } x)\frac{\ln(1+\mu|x|/x_{max})}{\ln(1+\mu)} \quad ...(10.44)$$

where $0 \le |x|/x_{max} \le 1.$

Here, $z(x)$ represents the output and x is the input to the compressor. Also, $|x|/x_{max}$ represents the normalized value of input with respect to the maximum value x_{max}. Further, (sgn x) term represents ± 1 *i.e.*, positive and negative values of input and output. The μ-law compressor characteristics for different values of μ have been shown in figure 10.18 (*a*). The practically used value of μ is 255.

Scanned by CamScanner

It may be noted that the characteristic corresponding to $\mu = 0$ corresponds to the uniform quantization. The μ-law comanding is used for speech and music signals. It is used for PCM telephone systems in United States, Canada and Japan. Figure 10.18 (*b*) shows the variation of signal to quantization noise ratio with respect to signal level, with and without companding. It is obvious that SNR is almost constant at all the signal levels when companding is used.



(a) Compressor characteristic of a μ-law compressor

(b) PCM performance with μ-law companding

Fig. 10.18.

10.20.2. A-law Comanding

In the A-law companding, the compressor characteristic is piecewise, made up of a linear segment for low level inputs and a logarithmic segment for high level inputs. Figure 10.19 shows the A-law compressor characteristics for different values of A. Corresponding to $A = 1$, we observe that the characteristic is linear which corresponds to a uniform quantization. The practically used value of A is 87.56. The A-law companding is used for PCM telephone systems in Europe. The linear segment of the characteristics is for low level inputs whereas the logarithmic segments is for high level input. It is mathematically expressed as under:

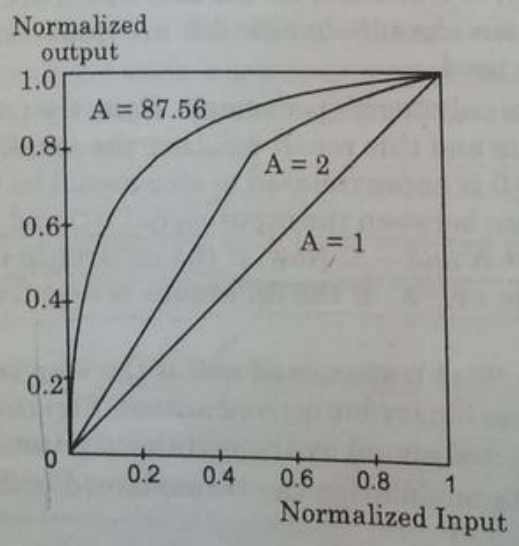


Fig. 10.19. *Compressor characteristic of A-law compressor.*

Scanned by CamScanner

$$\frac{z(x)}{x_{max}} = \begin{cases} \dfrac{A\,|x|/x_{max}}{1+\log_e A} & \text{for } 0 \le \dfrac{|x|}{x_{max}} \le 1 \\ \dfrac{1+\log_e [A\,|x|/x_{max}]}{1+\log_e A} & \text{for } \dfrac{1}{A} \le \dfrac{|x|}{x_{max}} \le 1 \end{cases} \quad ...(10.44a)$$

10.21. Applications of PCM

Some of the applications of PCM are as under:

(*i*) In telephony (with the advent of fibre optic cables).

(*ii*) In the space communication, space craft transmits signals to earth. Here, the transmitted power is very low (10 or 15 W) and the distances are huge (a few million km). Still due to the high noise immunity, only PCM systems can be used in such applications.

10.22. Advantages of PCM

Following are the advantages of a PCM system:

(*i*) Very high noise immunity.

(*ii*) Due to digital nature of the signal, repeaters can be placed between the transmitter and the receivers. The repeaters actually regenerate the received PCM signal. This is not possible in analog systems. Repeaters further reduce the effect of noise.

(*iii*) It is possible to store the PCM signal due to its digital nature.

(*iv*) It is possible to use various coding techniques so that only the desired person can decode the received signal.

10.23. Disadvantages of PCM

A PCM system has few drawbacks as under:

(*i*) The encoding, decoding and quantizing circuitry of PCM is complex.

(*ii*) PCM requires a large bandwidth as compared to the other systems.

10.24. Delta Modulation

(U.P. Tech-Semester Exam. 2002-2003)

We have observed in PCM that it transmits all the bits which are used to code a sample. Hence, signaling rate and transmission channel bandwidth are quite large in PCM. To overcome this problem, Delta Modulation is used.

Delta modulation transmits only one bit per sample. Here, the present sample value is compared with the previous sample value and this result whether the amplitude is increased or decreased is transmitted. Input signal $x(t)$ is approximated to step signal by the delta modulator. This step size is kept fixed. The difference between the input signal $x(t)$ and staircase approximated signal is confined to two levels, *i.e.*, $+\Delta$ and $-\Delta$. Now, if the difference is positive, then approximated signal is increased by one step, *i.e.*, 'Δ'. If the difference is negative, then approximated signl is reduced by 'Δ'.

When the step is reduced, '0' is transmitted and if the step is increased, '1' is transmitted. Hence, for each sample, only one binary bit is transmitted. Figure 10.20 shows the analog signal $x(t)$ and its staircase approximated signal by the delta modulator.

Thus, The principle of delta modulation can be explained with the help of few equations as under:

The error between the sampled value of $x(t)$ and last approximated sample is given as,

$$e(nT_s) = x(nT_s) - \hat{x}(nT_s) \quad ...(10.45)$$

where $e(nT_s)$ = error at present sample

Scanned by CamScanner

$x(nT_s)$ = sampled signal of $x(t)$

$\hat{x}(nT_s)$ = last sample approximation of the staircase waveform.

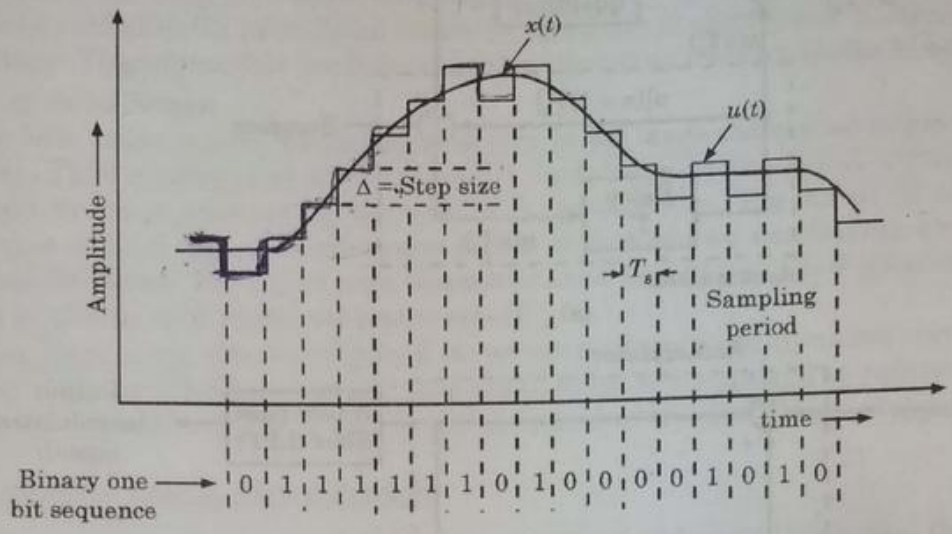


Fig 10.20. *Delta modulation waveform.*

If we assume $u(nT_s)$ as the present sample approximation of staircase output,

then, $$u[(n-1)T_s] = \hat{x}(nT_s) \quad ...(10.46)$$

= last sample approximation of staircase waveform

Let us define a quantity $b(nT_s)$ in such a way that,

$$b(nT_s) = \Delta\, sgn\, [e(nT_s)] \quad ...(10.47)$$

This means that depending on the sign of error $e(nT_s)$, the sign of step size Δ is decided. In other words, we can write

$$b(nT_s) = \begin{cases} +\Delta & \text{if } x(nT_s) \geq \hat{x}(nT_s) \\ -D & \text{if } x(nT_s) < \hat{x}(nT_s) \end{cases} \quad ...(10.48)$$

Also, If $b(nT_s) = +\Delta$ then a binary '1' is transmitted

and if $b(nT_s) = -\Delta$ then a binary '0' is transmitted.

Here, T_s = Sampling interval.

Figure 10.21(*a*) shows the transmitter (*i.e.*, generation of Delta Modulated signal).

The summer in the accumulator adds quantizer output ($\pm\Delta$) with the previous sample approximation. This gives present sample approximation. *i.e.*,

$$u(nT_s) = u(nT_s - T_s) + [\pm\Delta]$$

or $$u(nT_s) = u[(n-1)\,T_s] + b(nT_s) \quad ...(10.49)$$

The previous sample approximation $u[(n-1)\,T_s]$ is restored by delaying one sample period T_s. The sampled input signal $x(nT_s)$ and staircase approximated signal $\hat{x}(nT_s)$ are subtracted to get error signal $e(nT_s)$.

Thus, depending on the sign of $e(nT_s)$, one bit quantizer generates an output of $+\Delta$ or $-\Delta$. If the step size is $+\Delta$, then binary '1' is transmitted and if it is $-\Delta$, then binary '0' is transmitted.

At the receiver end, shown in figure 10.21(*b*), the accumulator and low-pass filter (LPF) are used. The accumulator generates the staircase approximated signal output and is delayed by one sampling period T_s. It is then added to the input signal. If input is binary '1' then it adds $+\Delta$ step to the previous output (which is delayed). If input is binary '0' then one step 'Δ' is subtracted from the delayed signal. Also, the low-pass filter has the cutoff frequency equal to highest frequency in $x(t)$. This low-pass filter smoothens the staircase signal to reconstruct orignal message signal $x(t)$.

Scanned by CamScanner

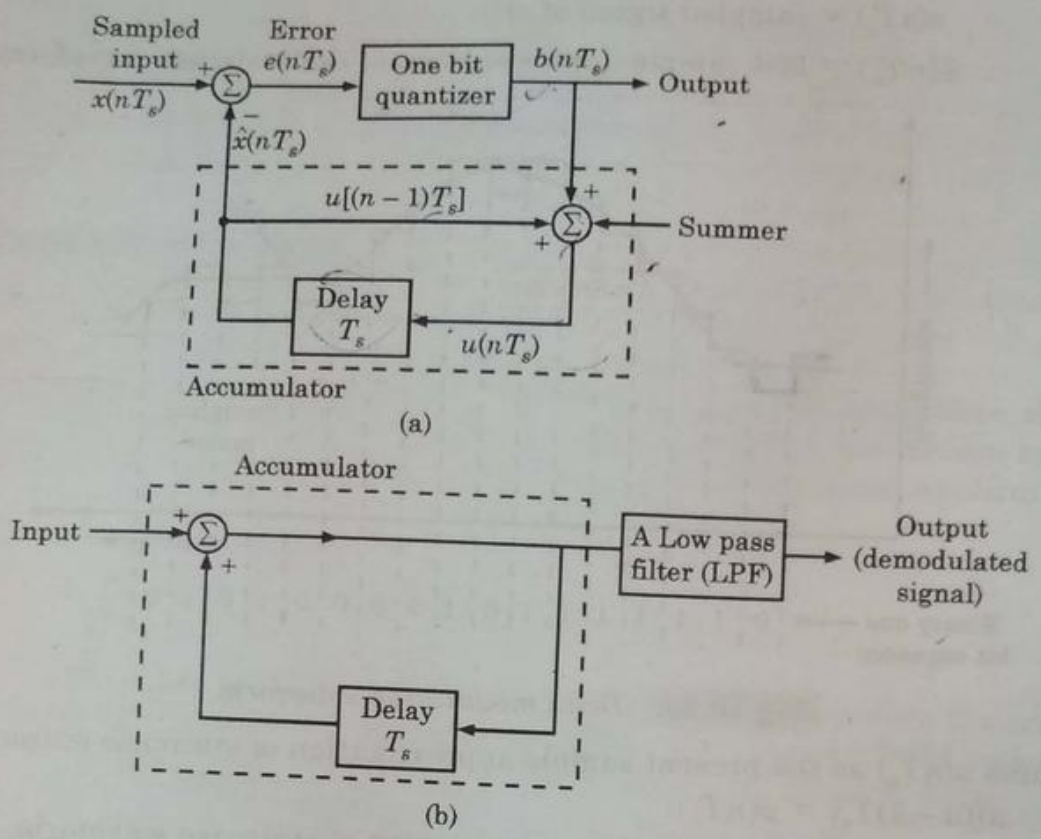


Fig. 10.21. *(a) A Delta modulation transmitter (b) A Delta modulation receiver*

10.24.1. Advantages of Delta Modulation

The delta modulation has certain advantages over PCM as under:

1. Since, the delta modulation transmits only one bit for one sample, therefore the signaling rate and transmission channel bandwidth is quite small for delta modulation compared to PCM.
2. The transmitter and receiver implementation is very much simple for delta modulation. There is no analog to digital converter required in delta modulation.

10.24.2. Disadvantges of Delta Modulation *(U.P. Tech., Sem. Examination, 2003-2004)*

The delta modulation has two major drawbacks as under:

(*i*) Slope overload distortion,

(*ii*) Granular or idle noise

Now, let us discuss these two drawbacks in detail.

(*i*) Slope Overload Distortion

This distortion arises because of large dynamic range of the input signal.

As can be observed from figure 10.22, the rate of rise of input signal $x(t)$ is so high that the staircase signal cannot approximate it, the steep size 'Δ' becomes too small for staircase signal

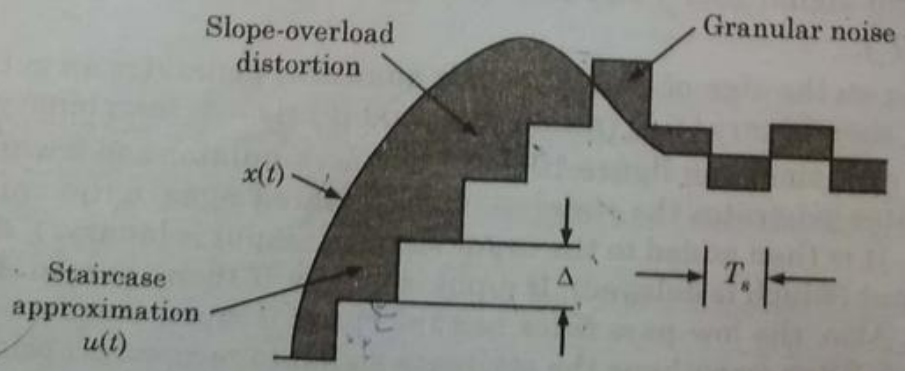


Fig. 10.22. *Quantization errors in delta modulation.*

Scanned by CamScanner

$u(t)$ to follow the step segment of $x(t)$. Hence, there is a large error between the staircase approximated signal and the original input signal $x(t)$. This error or noise is known as **slope overload distortion.** To reduce this error, the step size must be increased when slope of signl $x(t)$ is high.

Since the step size of delta modulator remains fixed, its maximum or minimum slopes occur along straight lines. Therefore, this modulator is also known as **Linear Delta Modulator (LDM).**

(*ii*) Granular or Idle Noise

Granular or Idle noise occurs when the step size is too large compared to small variations in the input signal. This means that for very small variations in the input signal, the staircase signal is changed by large amount (Δ) because of large step size. Figure 10.22 shows that when the input signal is almost flat, the staircase signal $u(t)$ keeps on oscillating by $\pm\,\Delta$ around the signal. The error between the input and approximated signal is called **granular noise.** The solution to this problem is to make step size small.

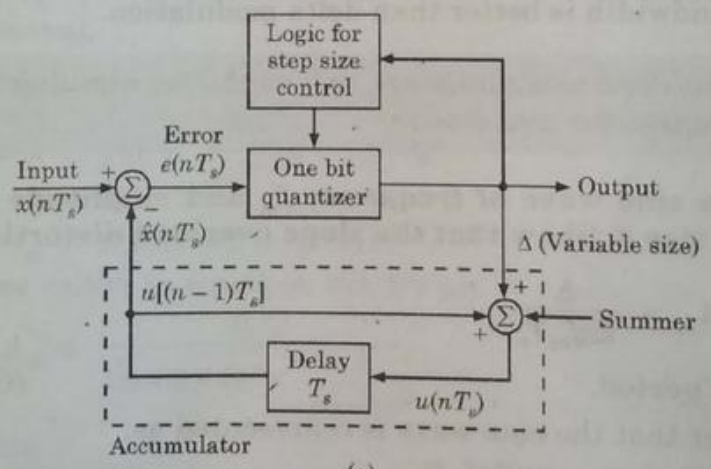
Therefore, a large step size is required to accommodate wide dynamic range of the input signal (to reduce slope overload distortion) and small steps are required to reduce granular noise. Infact, Adaptive delta modulation is the modification to overcome these errors.

10.25. Adaptive Delta Modulation

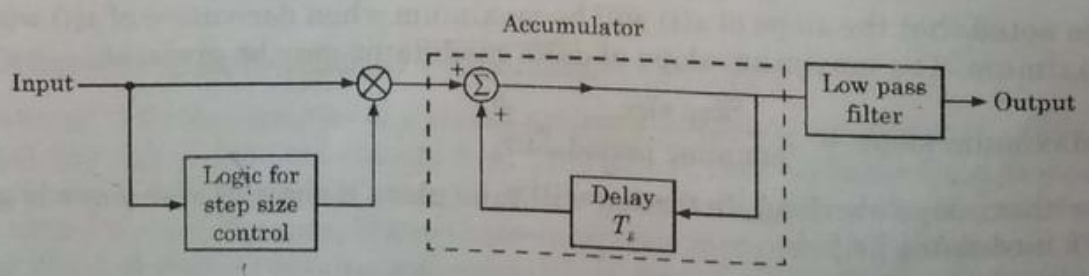
To overcome the quantization errors due to slope overload and granular noise, the step size (Δ) is made adaptive to variations in the input signal $x(t)$. Particularly in the steep segment of the signal $x(t)$, the step size is increased. Also, if the input is varying slowly, the step size is reduced. Then, this method is known as *Adaptive Delta Modulation (ADM).*

The adaptive delta modulators can take continuous changes in step size or discrete changes in step size.

Figure 10.23(*a*) shows the transmitter and 10.23(*b*) shows receiver of adaptive delta modulator. The logic for step size control is added in the diagram. The step size increases or decreases according to a specified rule depending on one bit quantizer output. As an example, if one bit quantizer output is high (*i.e.* 1), then step size may be doubled for next sample. If one bit quantizer output is low, then step size may be reduced by one step. Figure 10.24 shows the staircase waveforms of adaptive delta modultor and the sequence of bits to be transmitted.



(*a*)



(*b*)

Fig. 10.23. *Adaptive Delta Modulator (a) Transmitter (b) Receiver*

Scanned by CamScanner

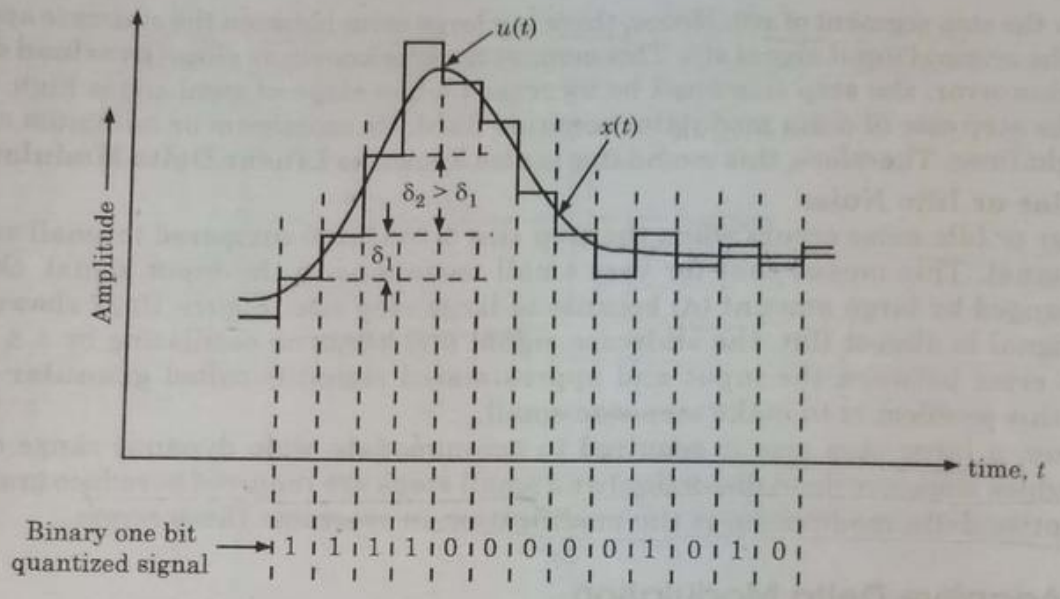


Fig. 10.24. *Waveforms for adaptive delta modulation*

In the receiver of adaptive delta modulator shown in figure 10.23(*b*), there are two portions. The first portion produces the step size from each incoming bit. Exactly the same process is followed as that in transmitter. The previous input and present input decides the step size. It is then applied to an accumulator which builds up staircase waveform. The low-pass filter then smoothens out the staircase waveform to reconstruct the original signal.

10.25.1. Advantages of Adaptive Delta Modulation

Adaptive delta modulation has certain advantages over delta modulation as under:

(*i*) The signal to noise ratio becomes better than ordinary delta modulation because of the reduction in slope overload distortion and idle noise.

(*ii*) Because of the variable step size, the dynamic range of ADM is wider than simple DM.

(*iii*) Utilization of bandwidth is better than delta modulation.

Note: *Other advantages of delta modulation are, only one bit per sample is required and simplicity of implementation of transmitter and receiver.*

Example 10.12. Given a sine wave of frequency f_m and amplitude A_m applied to a delta modulator having step size Δ. Show that the slope overload distortion will occur if

$$A_m > \frac{\Delta}{2\pi f_m T_s}$$

here T_s is the sampling period. *(GATE Examination-1999)*

Solution: Let us consider that the sine wave is represented as,

$$x(t) = A_m \sin(2\pi f_m t)$$

It may be noted that the slope of $x(t)$ will be maximum when derivative of $x(t)$ with respect to 't' will be maximum. The maximum slope of delta modulator may be given as,

$$\text{Maximum slope} = \frac{\text{Step size}}{\text{Sampling period}} = \frac{\Delta}{T_s} \quad ...(i)$$

We know that, slope overload distortion will take place if slope of sine wave is greater than slope of delta modulator *i.e.*,

$$\max\left|\frac{d}{dt}x(t)\right| > \frac{\Delta}{T_s}$$

Scanned by CamScanner

or $$\max\left|\frac{d}{dt} A_m \sin(2\pi f_m t\right| > \frac{\Delta}{T_s}$$

or $$\max|A_m 2\pi f_m \cos(2\pi f_m t)| > \frac{\Delta}{T_s}$$

or $$A_m 2\pi f_m > \frac{\Delta}{T_s}$$

or $$A_m > \frac{\Delta}{2\pi f_m T_s}$$ **Hence Proved.**

Example 10.13. A delta modulator system is designed to operate at five times the Nyquist rate for a signal having a bandwidth equal to 3 kHz bandwidth. Calculate the maximum amplitude of a 2 kHz input sinusoid for which the delta modulator does not have slope over load. Given that the quantizing step size is 250 mV. Also, derive the formula that you use.

(U.P.S.C. I.E.S. Engg. Examination-1999)

Solution: In last example, we have derived the relation for slope overload distortion which will occur if,

$$A_m > \frac{\Delta}{2\pi f_m T_s} \quad ...(i)$$

Thus, slope overload will not occur if,

$$A_m \le \frac{\Delta}{2\pi f_m T_s} \quad ...(ii)$$

The maximum frequency in the signal is,

$$f_m = 3 \text{ kHz}$$

Hence, Nyquist rate $= 2 f_m = 2 \times 3 \text{ kHz} = 6 \text{ kHz}$

Sampling frequency = 5 times Nyquist rate

or $f_s = 5 \times 6 \text{ kHz} = 30 \text{ kHz}$

Hence, Sampling interval,

$$T_s = \frac{1}{f_s} = \frac{1}{30 \times 10^3} \text{ seconds.}$$

Given that step size $\Delta = 250\ mV = 250 \times 10^{-3}$ V

or $\Delta = 0.25$ V

Again, Given that $f_m = 2 \text{ kHz} = 2 \times 10^3$ Hz

Substituting all these values in equation (*ii*), we get

$$A_m \le \frac{0.25}{2\pi \times 2 \times 10^3 \times \frac{1}{30 \times 10^3}}$$

Simplifying, we obtain

$$A_m \le 0.6 \text{ volts}$$ **Ans.**

10.26. Differential Pulse Code Modulation (DPCM)

It may be observed that the samples of a signal are highly correlated with each other. This is due to the fact that any signal does not change fast. This means that its value from present sample to next sample does not differ by large amount. The adjacent samples of the signal carry the same information with a little difference. When these samples are encoded by a standard PCM system, the resulting encoded signal contains some redundant information. Figure 10.25 illustrates this **redundant information.**

Scanned by CamScanner

Figure 10.25 shows a continuous time signal $x(t)$ by dotted line. This signal is sampled by flat top sampling at intervals T_s, $2T_s$, $3T_s$ ... nT_s. The sampling frequency is selected to be higher than nyquist rate. The samples are encoded by using 3 bit (7 levels) PCM. The sample is quantized to the nearest digital level as shown by small circles in the figure 10.25. The encoded binary value of each sample is written on the top of the samples. We can observe from figure 10.25 that the samples taken at $4T_s$, $5T_s$ and $6T_s$ are encoded to same value of (110). This information can be carried only by one sample. But three samples are carrying the same information means that it is redundant. Consider another example of samples taken at $9T_s$ and 10 T_s. The difference between these samples s only due to last bit and first two bits are redundant, since they do not change.

If this redundancy is reduced, then overall bit rate will decrease and number of bits required to transmit one sample will also be reduced. This type of digital pulse modulation scheme is known as **Differential Pulse Code Modulation (DPCM).**

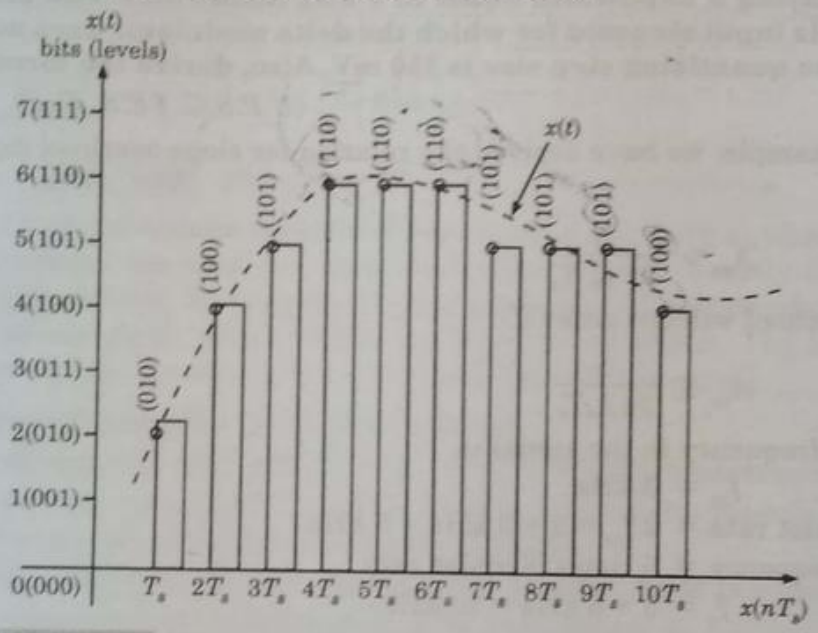


Fig 10.25. *Illustration of redundant information in PCM.*

In fact the differential pulse code modulation works on the principle of prediction. The value of the present sample is predicted from the past samples. The prediction may not be exact but it is very close to the actual sample vlaue. Figure 10.26 shows the transmitter of Differential Pulse Code Modulation (DPCM) system. The sampled signal is denoted by $x(nT_s)$ and the predicted signal is denoted by $\hat{x}(nT_s)$. The comparator finds out the difference between the actual sample value $x(nT_s)$ and predicated sample value $\hat{x}(nT_s)$. This is known as Prediction error and it is denoted by $e(nT_s)$. It can be defined as,

$$e(nT_s) = x(nT_s) - \hat{x}(nT_s) \qquad ...(10.50)$$

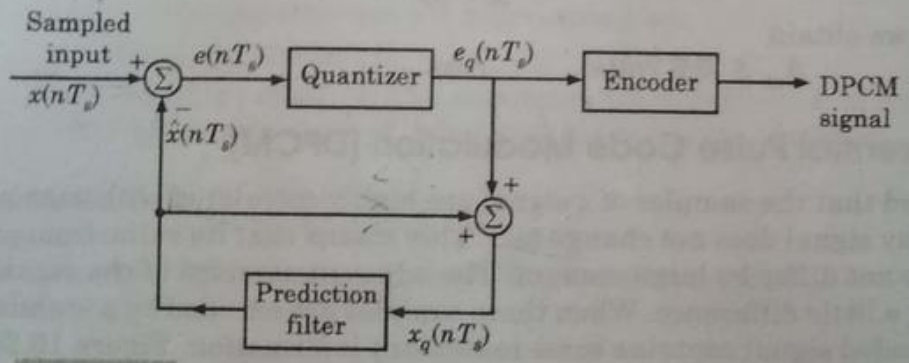


Fig. 10.26. *A Differential pulse code modulation transmitter.*

Scanned by CamScanner

Thus, error is the difference between unquantized input sample $x(nT_s)$ and prediction of it $\hat{x}(nT_s)$. The predicted value is produced by using a prediction filter. The quantizer output signal gap $e_q(nT_s)$ and previous prediction is added and given as input to the prediction filter. This signal is called $x_q(nT_s)$. This makes the prediction more and more close to the actual sampled signal. We can observe that the quantized error signal $e_q(nT_s)$ is very small and can be encoded by using small number of bits. Thus number of bits per sample are reduced in DPCM.

The quantizer output can be written as,

$$e_q(nT_s) = e(nT_s) + q(nT_s) \quad ...(10.51)$$

Here $q(nT_s)$ is the quantization error. As shown in figure 10.26, the prediction filter input $x_q(nT_s)$ is obtained by sum $\hat{x}(nT_s)$ and quantizer output *i.e.*,

$$x_q(nT_s) = \hat{x}(nT_s) + e_q(nT_s) \quad ...(10.52)$$

Substituting the value of $e_q(nT_s)$ from equation (10.51) in the above equation, we get,

$$x_q(nT_s) = \hat{x}(nT_s) + e(nT_s) + q(nT_s) \quad ...(10.53)$$

Equation (10.50) is written as,

$$e(nT_s) = x(nT_s) - \hat{x}(nT_s)$$

$$\therefore \quad e(nT_s) + \hat{x}(nT_s) = x(nT_s) \quad ...(10.54)$$

Therefore, the value of $e(nT_s) + \hat{x}(nT_s)$ from above equation into equation (10.53), we get,

$$x_q(nT_s) = x(nT_s) + q(nT_s) \quad ...(10.55)$$

Hence, the quantized version of the signal $x_q(nT_s)$ is the sum of original sample value and quantization error $q(nT_s)$. The quantization error can be positive or negative. Thus equation (10.55) does not depend on the prediction filter characteristics.

10.26.1. Reconstruction of DPCM Signal

Figure 10.27 shows the block diagram of DPCM receiver.

The decoder first reconstructs the quantized error signal from incoming binary signal. The prediction filter output and quantized error sigals are summed up to give the quantized version of the orginal signal. Thus the signal at the receiver differs from actual signal by quantization error $q(nT_s)$, which is introduced permanently in the reconstructed signal.

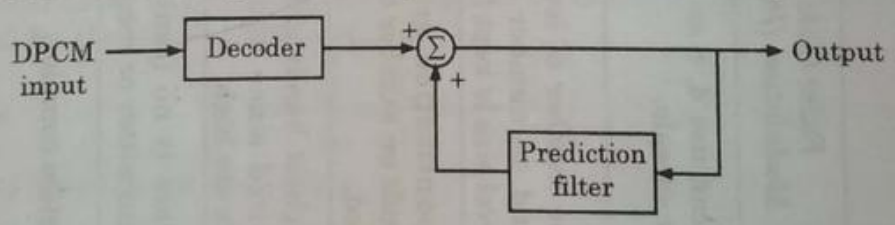


Fig. 10.27. *DPCM receiver*

10.27. Comparison of Digital Pulse Modulation Methods

After discussing all the digital pulse modulation methods in details, let us now compare all these methods from different aspects.

Table 10.1 shows the comparision of PCM, Differential PCM, Delta Modulation and Adaptive Delta Modulation. This comparison is carried out on the basis of various parameters like transmission bandwidth, quantization error, number of transmitter bits per sample etc.

Scanned by CamScanner

Table 10.1. Comparison between PCM, Delta Modulation, Adaptive Delta Modulation and Differential Pulse Code Modulation

S. No.	*Parameter of comparison*	*Pulse Code Modulation (PCM)*	*Delta modulation (DM)*	*Adaptive Delta Modulation (ADM)*	*Differential Pulse Code Modulation (DPCM)*
1.	Number of bits.	It can use 4, 8 or 16 bits per sample.	It uses only one bit for one sample.	Only one bit is used to encode one sample.	Bits can be more than one but are less than PCM.
2.	Levels and step size	The number of levels depend on number of bits. Level size is kept fixed.	Step size is kept fixed and cannot be varied.	According to the signal variation, step size varies (*i.e.* Adapted).	Here, Fixed number of levels are used.
3.	Quantization error and distortion	Quantization error depends on number of levels used.	Slope overload distor-tion and granular noise are present.	Quantization noise is present but other errors are absent.	Slope overload distor-tion and quantization noise is present.
4.	Transmission bandwidth	Highest bandwidth is required since number of bits are high	Lowest bandwidth is required.	Lowest bandwidth is required.	Bandwidth required is lower than PCM.
5.	Feedback	There is no feedback in transmitter or receiver.	Feedback exists in transmitter.	Feedback exists.	Here, Feedback exists.
6.	Complexity of implementation	System complex.	Simple.	Simple.	Simple

Scanned by CamScanner

MISCELLANEOUS SOLVED EXAMPLES

Example 10.14. A binary channel with bit rate r_b = 36000 bits per second (*b/s*) is available for PCM voice transmission. Evaluate the appropriate values of the sampling rate f_s, the quantizing level q, and the number of binary digits v. Assume f_m = 3.2 kHz.

Solution: Here, we require that

$$f_s \geq f_m = 6400$$

and

$$vf_s \leq r = 36000$$

Therefore, we have, $v \leq \frac{r}{f_s} \leq \frac{36000}{6400} = 5.6$

Hence, we have $v \cong 5$,

and also, $q = 2^v = 2^5 = 32$,

and $f_s = \frac{36000}{5} = 7200 \text{ Hz} = 7.2 \text{ kHz}$ **Ans.**

$f_s \geq 2 f_m$

$\frac{r}{n} \geq 2 f_m$ $\quad r = n f_s$

$f_s = \frac{r}{n}$

$\frac{r}{2 f_m} \geq n$

$\boxed{n \leq \frac{r}{2 f_m}}$

Example 10.15. An analog is quantized and is to be transmitted by using a PCM system. If each sample at the receiving end of the system must be known to witin ± 0.5 per cent of the peak-to-peak full-scale value, how many digits must sample contain?

Solution: Let us consider that $2m_p$ is the peak-to-peak value of the signal.

The peak error is then $0.005(2m_p) = 0.01\ m_p$, and the peak-to-peak error will be $(2(0.01\ m_p) = 0.02\ m_p$(the maximum step size Δ).

Thus, the required number of quantizing levels will be

$$q = \frac{2m_p}{\delta} = \frac{2m_p}{0.02m_p} = 100 \leq 2^v$$

Hence, the number of binary digits needed for each sample is $v = 7$. **Ans.**

Example 10.16. An analog signal is sampled at the Nyquist rate f_s and quantized into L levels. Find the time duration τ of 1 bit of the binary-encoded signal.

Solution: Let v be number of bits per sample. Then we have

$$v = [\log_2 q]$$

where $[\log_2 q]$ indicates the next higher integer to be taken if $\log_2 q$ is not integer value *i.e*, vf_s binary pulses must be transmitted per second.

Thus, we have $$\tau = \frac{1}{vf_s} = \frac{T_s}{v} = \frac{T}{[\log_2 q]}$$

where T_s is the Nyquist interval. **Ans**

Example 10.17. The output signal-to-quantizing-noise ratio $(SNR)_0$ in a PCM system is defined as the ratio of average signal power to average quantizing noise power. For a full-scale sinusoidal modulating signal with amplitude A, prove that

(U.P. Tech-Semester Exam. 2002-2003)

$$(SNR)_o = \left(\frac{S}{N_q}\right)_o = \frac{3}{2}q^2 \quad ...(i)$$

or

$$\left(\frac{S}{N_q}\right)_{odB} = 10\log\left(\frac{S}{N_q}\right)_o = 1.76 + 20 \log q \quad ...(ii)$$

where q is the number of quantizing levels.

Scanned by CamScanner

Solution: Since, here peak-to-peak excursion of the quatizer input is $2A$. Therefore, the quantizer step size will be

$$\Delta = \frac{2A}{q}$$

Then, the average quantizing noise power is

then, $$N_q = \langle q_e^2 \rangle = \frac{\delta^2}{12} = \frac{A^2}{3q^2}$$

The output signal-to-quantizing-noise ratio of a PCM system for a full scale test tone is, therefore,

$$(\text{SNR})_0 = \left(\frac{S}{N_q}\right)_o = \frac{A^2/2}{A^2/(3q^2)} = \frac{3}{2}q^2$$

Expressing this in decibels, we have

$$\left(\frac{S}{N_q}\right)_{0dB} = 10\log\left(\frac{S}{N_q}\right)_0 = 1.76 + 20\log q \quad \textbf{Hence Proved.}$$

Example 10.18. In a binary PCM system, the output signal-to-quantizing-noise ratio is to be held to a minimum value of 40 dB. Determine the number of required levels, and find the corresponding output signal-to-quantizing-noise ratio. *(GATE Examination-1997)*

Solution: In a binary PCM system, $q = 2^v$, where v is the number of binary digits. Then, we have

$$\left(\frac{S}{N_q}\right)_{0dB} = 1.76 + 20\log 2^v = 1.76 + 6.02\,v\ dB \qquad ...(i)$$

Now, since $$\left(\frac{S}{N_q}\right)_{0dB} = 40\ dB$$

Therefore, $$\left(\frac{S}{N_q}\right)_0 = 10{,}000$$

Thus, we have, $$q = \sqrt{\frac{2}{3}\left(\frac{S}{N_q}\right)_0} = \sqrt{\frac{2}{3}(10000)} = [81.6] \cong 82$$

and the number of binary digits v is

$$v = [\log_2 82] = [6.36] \cong 7$$

Then, the number of levels required is $q = 2^7 = 128$, and corresponding output signal-to-quantizing noise ratio will be

$$\left(\frac{S}{N_q}\right)_{0dB} = 1.76 + 6.02 \times 7 = 43.9\ \text{dB} \quad \textbf{Ans.}$$

Note: *Equation (i) indicates that each bit in the code word of a binary PCM system contributes 6 dB to the output signal-to-quantizing noise ratio. In fact, this is called the 6 dB rule.*

Example 10.19. A compact disk (CD) recording system samples each of two stereo signal with a 16-bit analog-to-digital converter (ADC) at 44.1 kb/s.

Scanned by CamScanner

(*i*) Determine the output signal-to-quantizing-noise ratio for a full-scale sinusoid signal.

(*ii*) The bit stream of digitized data is augmented by the addition of error-corresponding bits, clock extraction bits, and display and control bit fields. These additional bits represent 100 per cent overhead. Determine the output bit rate of the CD recording system.

(*iii*) The CD can record an hour's worth of music. Determine the number of bits recorded on a CD.

(*iv*) For a comparison, a high-grade collegiate dictionary may contain 1500 pages, 2 columns per page, 100 lines per column, 8 words pe line, 6 letters per word, and 7 bits per letter on an average. Determine the number of bits required to describe the dictionary, and estimate the number of comparable books that can be stored on a CD.

Solution: (*i*) We have

$$\left(\frac{S}{N_q}\right)_{0dB} = 1.76 + 6.02 \times 16 = 98.08 \text{ dB } \textbf{Ans.}$$

The very high SNR of the disk has the effect increasing the dynamic range of recording, resulting in the excellent clarity of sound from a CD.

(*ii*) The input bit rate will be

$$2(44.1)(10^3)(16) = 14.11(10^6) \text{ b/s} = 1.411 \text{ M b/s}$$

Including the additional 100 per cent overhead, the output bit rate will be

$$2(1.411)(10^6)\text{b/s} = 2.822 \text{ Mb/s } \textbf{Ans.}$$

(*iii*) The number of bits recorded on a CD will be

$$2.822(10^6)\,(3600) = 10.16(10^9) \text{ bits} = 10.16 \text{ gigabits (GB) } \textbf{Ans.}$$

(*iv*) The number of bits required to describe the dictionary will be

$$1500(2)(100)(8)(6)(7) = 100.8(10^6) \text{ bits} = 100.8 \text{ Mb } \textbf{Ans.}$$

Including the additional 100 per cent overhead, we have

$$\frac{10.16(10^9)}{2(100.8)(10^6)} = 50.4 \textbf{ Ans.}$$

Thus, a disk contains the equivalent of about 50 comparable-books storage capacity.

Example 10.20.(*i*) Plot the μ law compression characteristic for μ = 255.

(*ii*) If m_p = 20 V and 256 quantizing levels are employed, what is the voltage between levels when there is no compression? For μ = 255, what is the smallest and what is the largest effective separatioin between levels?

Solution: For μ = 255, we have $y = \pm\dfrac{In(1+255|x|)}{In256}$ for $|x| < 1$

where $x = m/m_p$. The plot of μ-law compression characteristic for μ = 255 has been shown in figure 10.28.

(*ii*) With no compression (that is, a uniform quantizing) the step size Δ is

$$\Delta = \frac{2m_p}{q} = \frac{40}{256} = 0.156 \text{ V}$$

With compression (that is, a nonuniform quantizing), the smallest effective separation between levels will be the one closed to the origin, and the largest affective separation levels will be the one closest $|x| = 1$.

Scanned by CamScanner

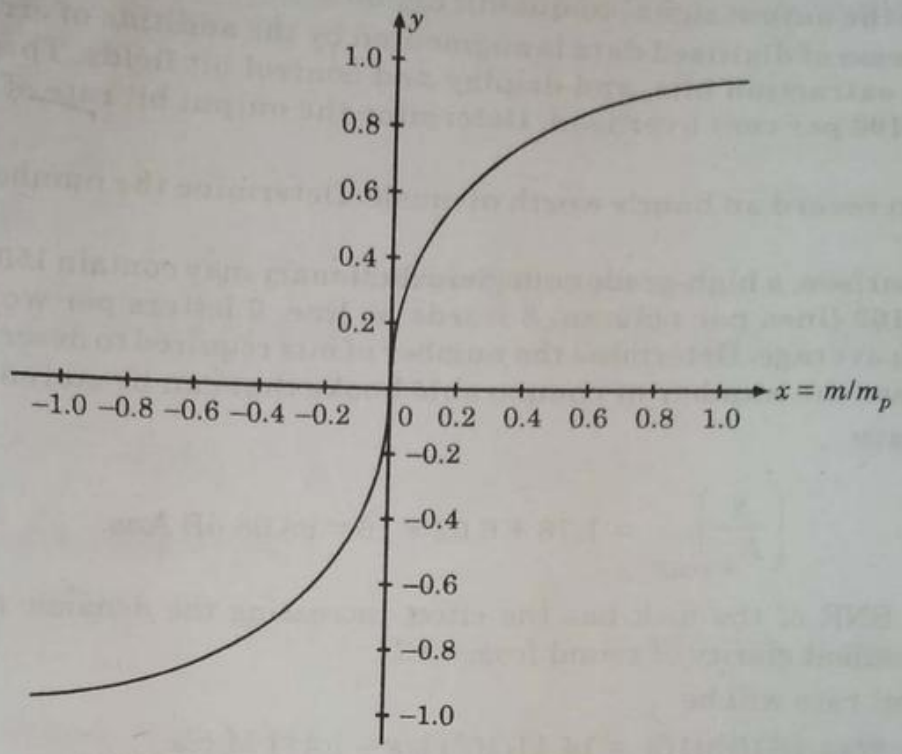


Fig. 10.28. *The μ law characteristic for μ = 255.*

Let x_1 be the value of x corresponding to $y = 1/127$, that is,

$$\frac{\text{In}(1 + 255\,|\,x\,|)}{In256} = \frac{1}{127}$$

Solving for $|\,x_1|$, we obtain

$$|\,x_1| = 1.758(10^{-4})$$

Thus, the smallest effective separation between levels is given by

$$\Delta_{\min} = m_p\,|\,x_1| = 20(1.75)(10^{-4}) = 3.5(10^{-3})\text{ V} = 3.5\text{ mV}$$

Next, let x_{127} be the value of x corresponding to $y = 1 - 1/27$, *i.e.*,

$$\frac{\text{In}(1 + 255\,|\,x_{127}\,|)}{\text{In } 256} = \frac{126}{127}$$

Solving for $|\,x_{127}|$, we get

$$|\,x_{127}| = 0.957$$

Thus, the largest effective separation between levels is given by $\Delta_{\max}$, *i.e.*

$$\Delta_{\max} = m_p(1 - |\,x_{127}|) = 20(1 - 0.957) = 0.86\text{ V}$$ **Ans.**

Example 10.21. Consider an audio signal with spectral components limited to the frequency band of 300 to 3300 Hz. A PCM signal is generated with a sampling rate of 8000 samples/s. The required output signal-to-quantizing-noise ratio is 30 dB.

(*i*) What is the minimum number of uniform quantizing levels needed, and what is the minimum number of bits per sample needed?

(*ii*) Calculate the minimum system bandwidth required.

(*iii*) Repeat parts (*i*) and (*ii*) when a μ-law compander is used with μ = 255.

(Karnataka University-1997)

Solution: (*i*) Here, we have

$$\left(\frac{S}{N_q}\right)_{0dB} = 1.76 + 20\log q \geq 30$$

Scanned by CamScanner

$$\log q \geq \frac{1}{20}(30 - 1.76) = 1.412$$

or $$q \geq 25.82$$

Thus, the minimum number of uniform quatizing levels required is 26.

$$v = [\log_2 q] = [\log_2 26] = [4.7] \cong 5 \text{ bits per sample}$$

The minimum number of bits per sample is 5.

(*ii*) The minimum required system bandwidth will be

$$f_{PCM} = \frac{v}{2} f_s = \frac{5}{2}(8000)$$

$$f_{PCM} = 20{,}000 \text{ Hz} = 20 \text{ kHz}$$

we have $$\left(\frac{S}{N_q}\right)_{0dB} = 20 \log q - 10.1 \geq 30$$

$$\log q \geq \frac{1}{20}(30 + 10.1) = 20.005$$

or $$q \geq 101.2$$

Thus, the minimum number of quantizing levels needed is 102.

Also, $$v = [\log_2 q] = [6.67] \cong 7$$

The minimum number of bits per sample is 7.

The minimum bandwidth required for this case will be

$$f_{PCM} = \frac{v}{2} f_s = \frac{7}{2}(8000) = 28000 \text{ Hz} = 28 \text{ kHz}$$ **Ans.**

Example 10.22. Consider a sinusoidal signal $m(t) = A \cos \omega_m t$ applied to a delta modulator with step size Δ. Show that the slope overload distortion will occur if

$$A > \frac{\Delta}{\omega_m T_s} = \frac{\Delta}{2\pi}\left(\frac{f_s}{f_m}\right) \quad \text{...(i)}$$

where $f_s = 1/T_s$ is the sampling frequency.

Solution: We have

$$m(t) = A \cos \omega_m t$$

so that $$\frac{dm(t)}{dt} = -A\omega_m \sin \omega_m t$$

To avoid the slope overload, we require that

$$\frac{\Delta}{T_s} \geq \left|\frac{dm(t)}{dt}\right|_{max} = A\omega_m$$

or $$A \leq \frac{\Delta}{\omega_m T_s}$$

Thus, if $A > \Delta/(\omega_m T_s)$, slope overload distortion will occur. **Hence Proved.**

Example 10.23. For a sinusoidal modulating signal

$$m(t) = A \cos \omega_m t$$

$$\omega_m = 2\pi f_m$$

Prove that the maximum output signal-to-quantizing-noise ration in a DM system under the assumption of no slope overload is given by

Scanned by CamScanner

$$(\text{SNR})_0 = \left(\frac{S}{N_q}\right)_0 = \frac{3f_s^3}{8\pi f_m^2 f_M} \quad ...(i)$$

where $f_s = 1/T_s$ is the sampling rate and f_M is the cutoff frequency of a low-pass filter at the output end of the reciver.

Solution: For no-slope-overload, we must have

$$A < \frac{\Delta}{\omega_m T_s} = \frac{\Delta}{2\pi}\left(\frac{f_s}{f_m}\right)$$

Thus, the maximum permissible value of the output signal power equals

$$P_{\max} = \frac{A^2}{2} = \frac{\Delta^2 f_s^2}{8\pi^2 f_m^2} \quad ...(ii)$$

We know that the mean quantizing error, or the quantizing noise power, $\langle q_e^2 \rangle = \Delta^2/3$. Let the bandwidth of a postreconstruction low-pass filter at the output end of the reciever be $f_M \geq f_m$ and $f_M << f_s$. Then, assuming that the quantizing noise power P_q is uniformly distributed over the frequency band up to f_s, the output quantizing noise power within the bandwidth f_M is

$$N_q = \left(\frac{\Delta^2}{3}\right)\frac{f_M}{f_s} \quad ...(iii)$$

Combining equation (*ii*) and (*iii*), we see that the maximum output signal-to-quantizing-noise ratio is

$$\left(\frac{S}{N_q}\right)_0 = \frac{P_{\max}}{N_q} = \frac{3f_s^3}{8\pi f_m^2 f_M} \quad \textbf{Hence Proved.}$$

Example 10.24. A DM system is designed to operate at 3 times the Nyquist rate for a signal with a 3 kHz bandwidth. The quantizing step size is 250 mV.

(*i*) Determine the maximum amplitude of a 1-kHz input siinusoid for which the delta modulator does not show slope overload.

(*ii*) Determine the posfiltered output signal-to-quantizing-noise ratio for the signal of part (*i*)

Solution: We have

$$m(t) = A\cos\omega_m t = A\cos 2\pi(10^3)t$$

$$\left|\frac{dm(t)}{dt}\right|_{\max} = A(2\pi)(10^3)$$

The maximum allowable amplitude of the input sinusoid is

$$A_{\max} = \frac{\Delta}{\omega_m T_s} = \frac{\Delta}{\omega_m}$$

$$f_s = \frac{250}{2\pi(10^3)}3(2)(3)(10^3) = 71.2 \text{ mV} \quad \textbf{Ans.}$$

(*ii*) Assuming that the cutoff frequency of the low-pass filter is f_m, we have

$$(\text{SNR})_0 = \left(\frac{S}{N_q}\right)_0 = \frac{3[(3)(6)(10^3)]^3}{8\pi^2(10^3)^3} = 221.6 = 23.5\ dB \quad \textbf{Ans.}$$

Scanned by CamScanner

Example 10.25. The pulse rate in a DM system is 56,000 per sec. The input signal is 5 cos (2π 1000 t) + 2 cos (2π 2000 t) V, with t in sec. Find the minimum value of step size which will avoid slope overload distortion. What would be the disadvantages of choosing a value of larger than the minimum? *(GATE Examination-1998)*

Solution: Input signal,

$$m(t) = 5\cos(2000\,\pi t) + 2\cos(4000\,\pi t) = m_1(t) + m_2(t)$$

To avoid slope overloading, we have

$$\left|\frac{dm_1(t)}{dt}\right|_{max} \le \Delta_1 f_s,$$

where Δ is step size and f_s is sampling rate.

or $$\Delta_1 f_s \ge \left|\frac{dm_1(t)}{dt}\right|_{max}$$

$$\Delta_{1min} \text{ (minimum value of step size)} = \left|\frac{dm_1(t)}{dt}\right|_{max} = \frac{5.(2000\pi)}{56000}$$

$$= \frac{10\pi}{56} = 0.56 \text{ V}$$

Also, $$\Delta_2, f_s \ge \left|\frac{dm_2(t)}{dt}\right|_{max}$$

or $$\Delta_2, \min = \frac{8000\pi}{56000} = \frac{\pi}{7} = 0.45 \text{ V}$$

Hence, larger step size out of two will be the required step size. *i.e.*, = 0.56 V.

If a value larger than the minimum will be choosen, then granular noise will occur.

Example 10.26. Bandwidth of the input to pulse code modulator is restricted to 4 kHz. The input varies from -3.8V to 3.8 V and has the average power of 30 mW, the required signal to quantization noise power ratio is 20 dB. The modulator produces binary output. Assume uniform quantization Calculate the number of bits required per sample.

Solution: Given, $$\left(\frac{S}{N_q}\right)_{odB} = 20 \text{ dB}$$

or $$10\log\left(\frac{S}{N_q}\right)_o = 20 \text{ dB}$$

or $$\left(\frac{S}{N_q}\right)_o = 100$$

Quantizer step size, $$\Delta = \frac{2A}{L}$$

where $L = 2^n$, n is the number of bianry digits.

then, average quantizing power is,

$$N_q = \langle q_e^2 \rangle = \frac{\Delta^2}{12} = \frac{A^2}{3L^2}$$

Scanned by CamScanner

then, $$\left(\frac{S}{N_q}\right)_o = \frac{\text{Average signal power}}{\text{Average quantizing power}}$$

or $$100 = \frac{30\times10^{-3}}{A^2/3L^2}$$

or $$L = \sqrt{\frac{30\times10^{-3}}{3\times100\times(3.8)^2}} = 126.67$$

or $$2^n = 128$$

Hence, $n = 7$ = number of bits required per sample.

SUMMARY

1. The use of digital communications offers several important advantages as compared to analog communications.
2. Digital communications have become the dominant form of communication technology in our society.
3. To handle the transmission of analog message signals (*i.e.*, voice and video signals) by digital means, the signal has to undergo an analog-to-digital conversion.
4. The simplest form of pulse digital modulation is called pulse code modulation (PCM).
5. In this system (PCM) the message signal is first sampled and then amplitude of each sample is rounded off to the nearest one of a finite set of allowable values known as quantization levels, so that both time and amplitude are in the discrete form. This means that in pulse code modulation both parameters *i.e.*, time and amplitude are expressed in discrete form. This process is called discretization in time and amplitude.
6. We can improve the accuracy of the quantized signal to any desired degree simply by increasing the number of levels L.
7. Pulse-code modulation is known as a digital pulse modulation techniques. The pulse-code modulation (PCM) is quite complex compared to the Analog pulse modulation techniques (*i.e.*, PAM, PWM and PPM) in the sense that the message signal is subjected to a great number of operations.
8. The essential operations in the receiver are regeneration of impaired signals, decoding and demodulation of the train of quantized samples.
9. Quantization refers to the use of a finite set of amplitude levels and the selection of a level nearest to a particular sample value of the message signal as the representation for it.
10. This operation combined with sampling, permits the use of coded pulses for representing the message signal.
11. It is the combined use of quantizing and coding that distinguishes pulse code modulation form analog modulation techniques.
12. Basically, the quantizers are of two types:
    (*i*) Uniform quantizer
    (*ii*) Non-uniform quantizer.
13. A uniform quantizer is that type of quantizer in which the 'step size' remains same through out the input range.
14. A non-uniform quantizer is that type of quantizer in which the 'step-wise' varies according to the input values.
15. Transmission bandwidth in PCM is given by

$$BW \geq \frac{1}{2}r$$

But $$r = v f_s$$

Scanned by CamScanner

CHAPTER 11

Digital Multiplexers

Inside this Chapter

11.1. Introduction 587
11.2. Multiplexing 587
11.3. A PAM/TDM System 588
11.4. Introduction to Digital Multiplexers 595
11.5. Classification of Digital Multiplexers 596
11.6. Multiplexing Hierarchy for Digital Communication 596
11.7. North American Hierarchy 600
11.8. T Lines 600
11.9. A PCM-TDM System (T1 Carrier System) 601
11.10. E Lines 603

11.1. Introduction

In last two chapters, we have discussed the pulse analog modulation and pulse digital modulation methods. In this chapter, we shall discuss an important process in communication system known as multiplexing. In the articles to follow, we shall discuss what is multiplexing and what are the types of multiplexing and then proceed for a detailed aspects of multiplexing.

11.2. Multiplexing

Multiplexing may be defined as a technique which allows many users to share a common communication channel simultaneously. There are two major types of multiplexing techniques. They are as under:

(*i*) Frequency division multiplexing (FDM),

(*ii*) Time division multiplexing (TDM).

11.2.1. Frequency Division Multiplexing (FDM)

This technique permits a fixed frequency band to every user in the complete channel bandwidth. Such frequency slot is allotted continuously to that user. As an example consider that the channel bandwidth is 1 MHz. Let there be ten users, each requiring upto 100 kHz bandwidth. Then the complete channel bandwidth of 1 MHz can be divided into ten frequency bands, *i.e.* each of 100 kHz and every user can be allotted one independent frequency band. This technique is known as **Frequency Division Multiplexing** (FDM). It is mainly used for modulated signal. This is due to the fact that a modulated signal can be placed in any frequency band by just changing the

Scanned by CamScanner

carrier frequency. However, at the receiver, these frequency multiplexed signals can be separated by the use of tuned circuits (*i.e.*, bandpass filters) of their respective frequency band. And for every band, there are independent tuned circuits and demodulators.

11.2.2. Time Division Multiplexing (TDM)

As discussed earlier, in PAM, PPM and PDM the pulse is present for a short duration and for most of the time between the two pulses, no signal is present. This free space between the pulses can be occupied by pulses from other channels. This is known as Time Division Multiplexing (TDM). Thus, time division multiplexing (TDM) makes maximum utilization of the transmission channel.

Hence, we can say that in FDM, all the signals are transmitted simultaneously over the same communication medium, and the signals occupy frequency slots. However, in TDM, the signals to be multiplexed are transmitted sequentially one after the other. Each signal occupies a short time slot as shown in figure 11.1. Thus, the signals are isolated from each other in the time domain, but all of them occupy the same slot in the frequency spectrum. Therefore, in TDM, the complete bandwidth of the communication channel is available to each signal being transmitted.

Figure 11.1 shows the concept of TDM.

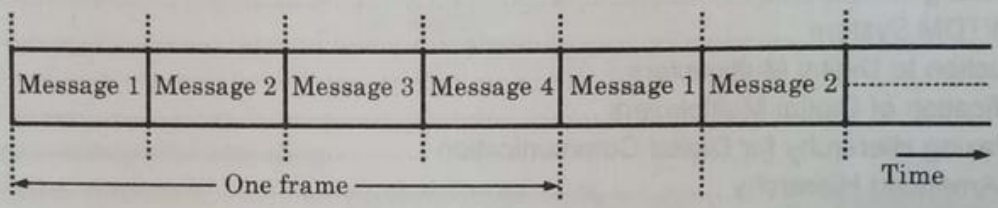


Fig. 11.1. *Illustration of TDM concept*

At this stage, it may be noted that in context of TDM, we define one important term *i.e.*, frame. One frame corresponds to the time period required to transmit all the signals once on the transmission channel. This has been shown in figure 11.1. Here, we have total four message signals to be transmitted. Hence, one frame will correspond to the time period required to transmit all the four signals once on the channel.

The TDM system can be used to multiplex analog or digital signals, however it is more suitable for the digital signal multiplexing.

11.3. A PAM/TDM System

Now, let us discuss a PAM/TDM system. Infact, this system combines the concepts of PAM and TDM both as shown in figure 11.2.

Working Principle

Here, the multiplexer is a single pole rotating switch or commutator. This switch can be a mechanical switch or an electronic switch and it rotates at f_s rotations per second. As the switch arm rotates, it is going to make contact with the position 1, 2, 3 or N for a short time. There are N analog signals, to be multiplexed, which are connected to these contracts. Hence, the switch arm will connect these N input signals one by one to the communication channel.

The waveform of a TDM signal which is being transmitted has been shwon in figure 11.3. It shows that the rotary switch samples each message during each of its rotations.

Since, each rotation corresponds to one frame, therefore, one frame is completed in T_s seconds where $T_s = 1/f_s$.

Scanned by CamScanner

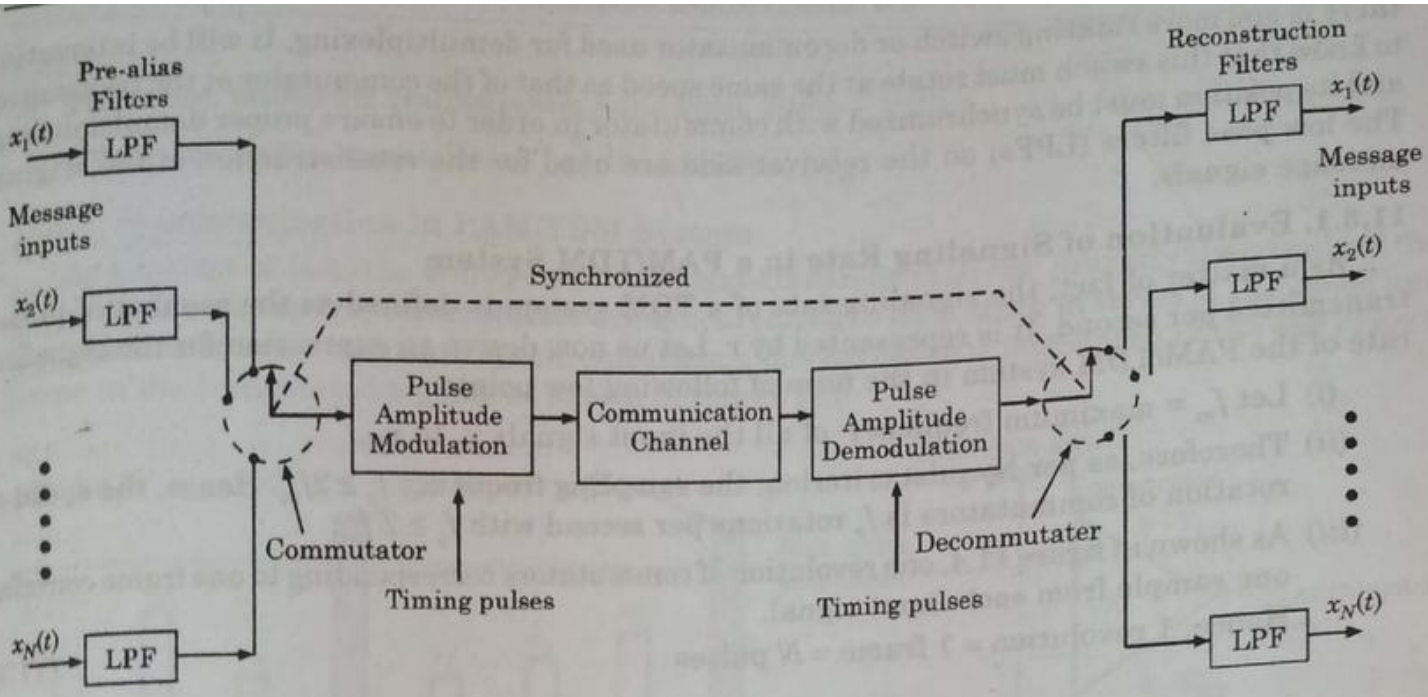


Fig. 11.2. *Block diagram of a PAM/TDM system*

Hence, the function of the commutator is two fold as under:

(*i*) To take narrow sample of each input message at a rate f_s which is higher than $2f_m$.

(*ii*) To sequentially interleave the N samples inside the interval $T_s = 1/f_s$.

Now, the multiplexed signal at the output of the commutator is applied to a pulse amplitude modulator. It converts the PAM pulses into a form suitable for transmission over the communication channel. The input message signals are passed through low pass filters before applying them to the commutator. These filters are actually the antialiasing filters which avoid the aliasing. The cutoff frequency of each low pass filter (LPF) is f_m Hz.

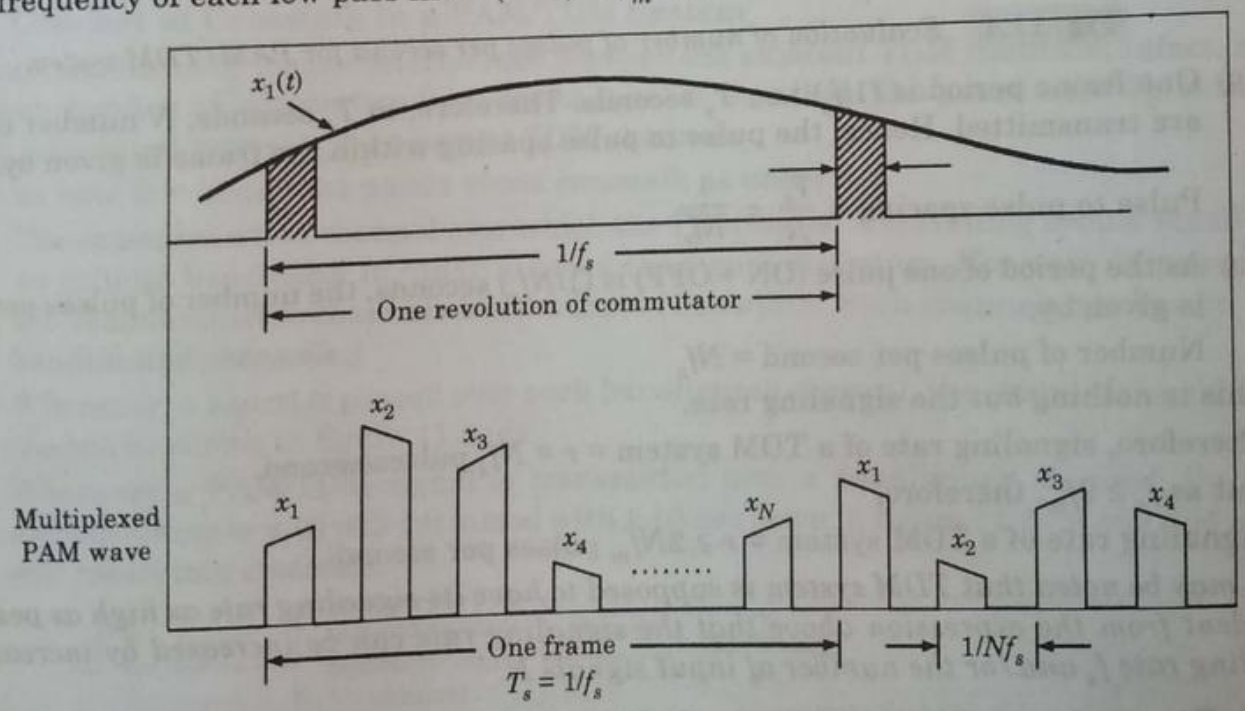


Fig. 11.3. *(a) Sampling of the first input (b) Multiplexed PAM signal transmitted on the transmission channel*

At the receiving end of PAM/TDM system, the received signal is applied to a pulse amplitude demodulator which performs the reverse operation of pulse amplitude modulator. At the receiver,

Scanned by CamScanner

there is one more rotating switch or decommutator used for demultiplexing. It will be interesting to know that this switch must rotate at the same speed as that of the commutator at the transmitter and its position must be synchronized with commutator in order to ensure proper demultiplexing. The low pass filters (LPFs) on the receiver side are used for the reconstruction of the original message signals.

11.3.1. Evaluation of Signaling Rate in a PAM/TDM System

As a matter of fact, the signaling rate of a TDM system is defined as the number of pulses transmitted per second. It is represented by r. Let us now derive an expression for the signaling rate of the PAM/TDM system in the form of following few points:

(*i*) Let f_m = maximum frequency of all the input signals x_1 to x_N.

(*ii*) Therefore, as per Nyquist criterion, the sampling frequency $f_s \geq 2f_m$. Hence, the speed of rotation of commutators is f_s rotations per second with $f_s \geq 2\,f_m$.

(*iii*) As shown in figure 11.4, one revolution of commutators corresponding to one frame contains one sample from each input signal.

Hence, 1 revolution = 1 frame = N pulses ...(11.1)

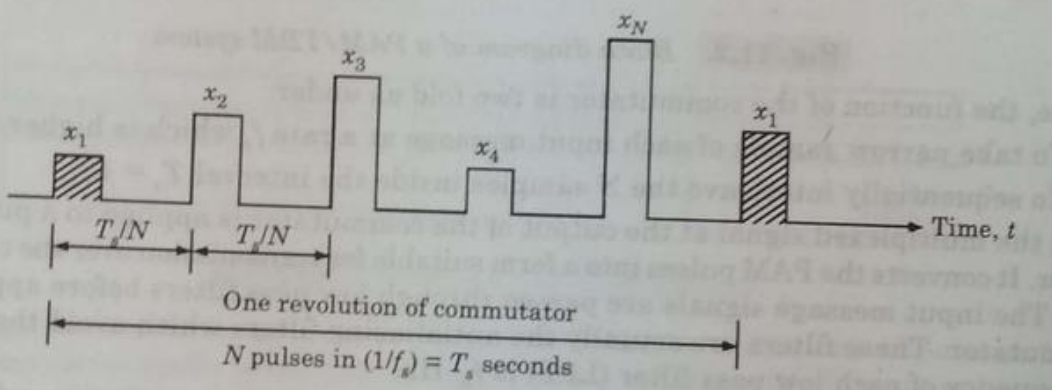


Fig. 11.4. *Evaluation of number of pulses per second for PAM/TDM system.*

(*iv*) One frame period is $(1/f_s)$ i.e., T_s seconds. Therefore, in T_s seconds, N number of pulses are transmitted. Hence, the pulse to pulse spacing within the frame is given by,

$$\text{Pulse to pulse spacing} = \frac{T_s}{N} = \frac{1}{Nf_s} \qquad ...(11.2)$$

(*v*) As the period of one pulse (ON + OFF) is $(1/Nf_s)$ seconds, the number of pulses per second is given by,

Number of pulses per second = Nf_s

This is nothing but the signaling rate.

Therefore, signaling rate of a TDM system = $r = Nf_s$ pulses/second.

But as $f_s \geq 2f_m$, therefore,

Signaling rate of a TDM system = $r \geq 2Nf_m$ pulses per second.

It may be noted that TDM system is supposed to have its signaling rate as high as possible. It is evident from the expression above that the signaling rate can be increased by increasing the sampling rate f_s and/or the number of input signals N.

11.3.2. Transmission Bandwidth of a PAM/TDM Channel

The minimum transmission bandwidth of a PAM-TDM channel is given by

$$BW = \frac{1}{2}\,(\text{signaling rate})$$

Scanned by CamScanner

Therefore, minimum transmission bandwidth $BW \geq \frac{1}{2} \times 2N\, f_m$

Hence, minimum transmission bandwidth $BW = Nf_m$.

11.3.3. Synchronization in PAM/TDM System

As a matter of fact, the multiplexed PAM signals can be received properly if and only if the transmitter and receiver commutators are synchronized to each other in terms of the speed and the position. In order to ensure synchronization, a marker pulse is introduced at the end of each frame in the transmitted signals as shown in figure 11.5.

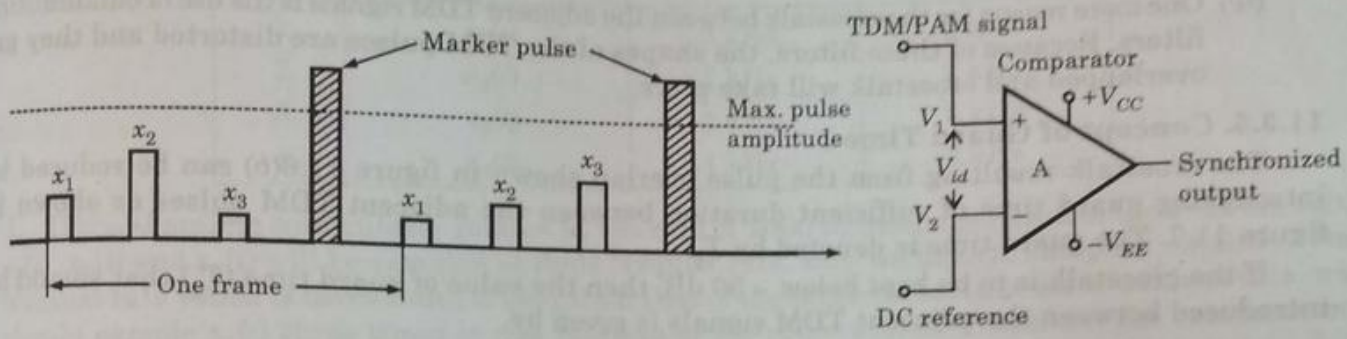


Fig. 11.5. *Illustration of frame synchronization and detection.*

The amplitude of this pulse is kept higher than the maximum permissible amplitude of the multiplexed channels. At the receiver end, the received signal is compared with a DC reference level. The comparator responds to only the marker pulse to produce output. Thus, the marker pulse is separated from the remaining multiplexed channels. Due to the introduction of synchronizing pulse, only three signals instead of four can now be transmitted.

11.3.4. Concept of Crosstalk in a PAM/TDM System

Crosstalk basically means interference between the adjacent TDM channels. Infact, it is an unwanted coupling of information from one channel to the other. The guard time 'T_g' is the time spacing introduced between the adjacent TDM channels.

Let us note few important points about crosstalk as under:

(*i*) The communication channel over which the TDM signal is travelling should ideally have an infinite bandwidth in order to avoid the signal distortion. However, in practice, all the communication channels have a finite bandwidth. Such channels are known as the bandlimited channels.

(*ii*) Whenever, a signal is passed over such bandlimited channel, the shape of the signal will change as shown in figure 11.6 (*a*).

(*iii*) Whenever a PAM/TDM signal is transmitted over a bandlimited channel, the signal corresponding to $x_1(t)$ will get mixed with $x_2(t)$ as shown in figure 11.6 (*b*) and this overlap will result into crosstalk.

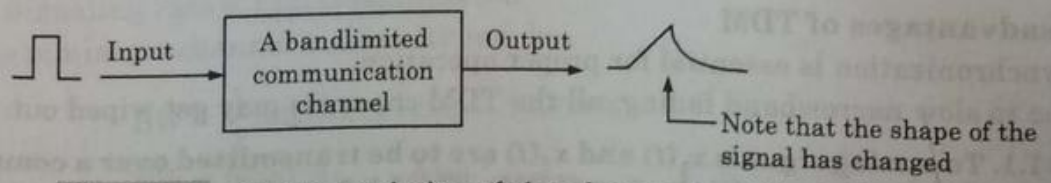


Fig. 11.6. *(a) Transmission of signal over a bandlimited channel*

Scanned by CamScanner

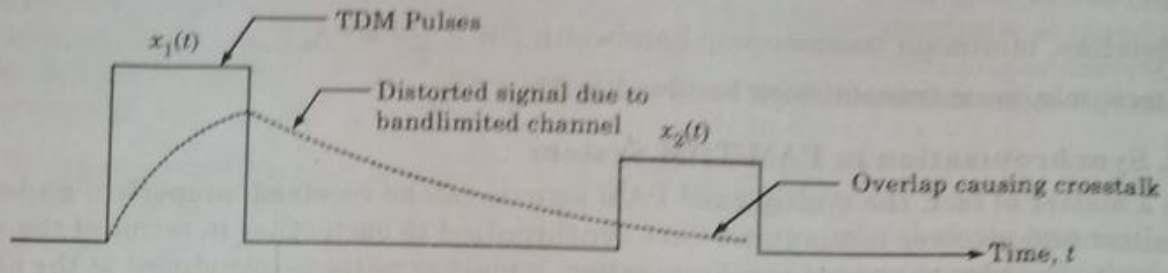


Fig. 11.6. *(b) Illustration of crosstalk in TDM*

(*iv*) One more reason for the crosstalk between the adjacent TDM signals is the use of bandlimiting filters. Because of these filters, the shapes of the TDM pulses are distorted and they get overlapped and crosstalk will take place.

11.3.5. Concept of Guard Time

The crosstalk resulting from the pulse overlap shown in figure 11.6(*b*) can be reduced by introducing guard time of sufficient duration between the adjacent TDM pulses as shown in figure 11.7. The guard time is denoted by T_g.

If the crosstalk is to be kept below – 30 dB, then the value of guard time (T_g) that should be introduced between the adjacent TDM signals is given by,

$$\text{Guard time} \qquad T_g > \frac{0.55}{BW} \qquad \text{...(11.3)}$$

where BW = Bandwidth of the channel.

The equation (11.3) shows that the guard time required to avoid the crosstalk will decrease with increase in the channel bandwidth. As T_g reduces, we can increase the signaling rate of the PAM/TDM system.

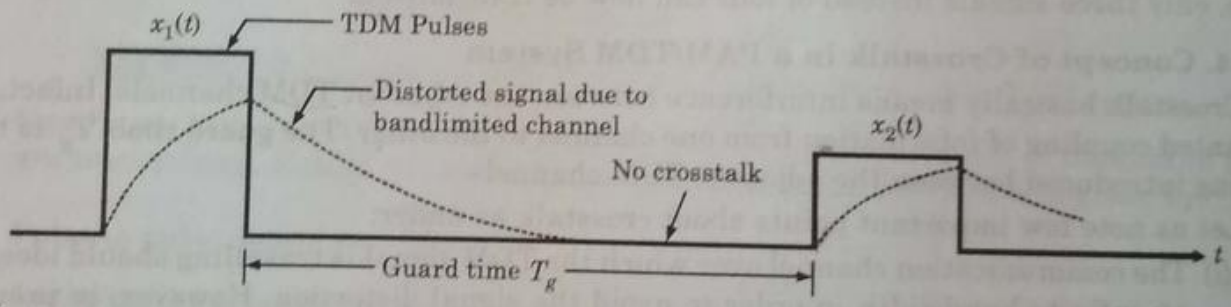


Fig. 11.7. *Elimination of crosstalk due to guard time.*

11.3.6. Advantages of TDM

(*i*) Full available channel bandwidth can be utilized for each channel.
(*ii*) Intermodulation distortion is absent.
(*iii*) TDM circuitry is not very complex.
(*iv*) The problem of crosstalk is not severe.

11.3.7. Disadvantages of TDM

(*i*) Synchronization is essential for proper operation.
(*ii*) Due to slow narrowband fading, all the TDM channels may get wiped out.

Example 11.1. To analog signals $x_1(t)$ and $x_2(t)$ are to be transmitted over a common channel by means of time division multiplexing (TDM). The highest frequency of $x_1(t)$ is 4 kHz and that of $x_2(t)$ is 4.5 kHz. What will be the minimum value of permissible sampling rate?

Solution: The highest frequency component of the composite signal consisting of $x_1(t)$ and $x_2(t)$ is 4.5 kHz. Therefore, the minimum value of permissible sampling rate will be,

$$f_{s(\min)} = 2 \times 4.5 \text{ kHz} = 9 \text{ kHz} \qquad \textbf{Ans.}$$

Scanned by CamScanner

CHAPTER 12

Digital Baseband Transmission

Inside this Chapter

12.1. Introduction 609
12.2. A Baseband Digital Communication System 609
12.3. Introduction to Discrete PAM Signals (Digital Data Formats) 611
12.4. Line Coding and its Properties 612
12.5. Various PAM Formats or Line Codes 613
12.6. Unipolar RZ and NRZ 614
12.7. Polar RZ and NRZ 615
12.8. Bipolar NRZ (Alternate Mark Inversion (AMI)] 616
12.9. Split Phase Manchester Format 616
12.10. Polar Quaternary NRZ Format 616
12.11. High Density Bipolar (HDB) Signalling 617
12.12. B8ZS Line Code 618
12.13. Power Spectra of Discrete PAM Signals (Various Line Codes) 618
12.14. Power Spectral Density (psd) of NRZ Unipolar Format 619
12.15. Power Spectral Density (psd) of NRZ Polar Format 621
12.16. Power Spectral Density (psd) of NRZ Bipolar Format 621
12.17. Power Spectral Density (psd) of the Manchester Format 622
12.18. Comparison of Various Discrete PAM Formats on the Basis of Power Spectra 625
12.19. Difference between Source Coding and Line Coding 627
12.20. Introduction to Matched Filter and Intersymbol Interference (ISI) 627
12.21. Integrate and Dump Filter (i.e., Receiver) 628
12.22. The Optimum Filter (i.e., Optimum Receiver) 638
12.23. Matched Filter 643
12.24. The Correlator 653
12.25. Intersymbol Interference (ISI) 655
12.26. Cause of Intersymbol Interference (ISI) 656
12.27. Nyquist's Criterion for Distortionless Baseband Binary Transmission 657
12.28. Ideal Solution 659
12.29. Concept of Eye Pattern 660

Scanned by CamScanner

12.1. Introduction

Although a significant portion of communication, these days, is in analog form, it is now being replaced rapidly by digital communication. This chapter deals with the problems of transmitting digital data over a channel. Hence, the starting messages are assumed to be digital. Infact, to begin with, we shall consider the binary case where the data consists of only two symbols *i.e.* 1 and 0. We assign a distinct waveform (*i.e.* pulse) to each of these two symbols. The resulting sequence of these pulses is transmitted over a channel. At the receiver end, these pulses are detected and are converted back to binary data (*i.e.* 1 s and 0 s).

As a matter of fact, the pulse code Modulation (PCM), Delta modulation (DM), Adaptive delta modulation (ADM) etc. are used to convert an analog signal to digital data. This digital data is the sequence of binary symbols. This digital data needs to be transmitted over the channel. In digital data stream, the occurrence of 1's and 0's is not exactly equal. Therefore, the signal contains frequencies right from very low frequencies to high frequencies. This means that the baseband transmission of digital data requires a low-pass channel. In other words, this channel should transmit very low frequencies as well highest frequency that can be represented by the signal. Thus, the low-pass channel used to transmit digital data can be approximated by the low-pass filter (LPF). When the digital data is transmitted over a channel, each received pulse is affected somewhat by adjacent pulses. This means that a signal received at a particular time is due to the transmitted pulse and some interference due to the adjacent pulses. This interference due to adjacent pulses is known as **Intersymbol Interference (ISI)**. Because of this interference (ISI), the received signal is distorted from the expected value. Therefore, errors are introduced in the detection of the symbols. To control the effect of ISI, care is taken so that the interference between the transmitted symbols is minimum. This needs pulse shaping.

Further, we know that any transmission channel adds noise to the signal. This noise is known as the channel noise. The channel noise is normally additive white noise. Thus, a noisy signal is received at the receiver. In the presence of such noise, the receiver should decide which of the possible waveform was transmitted. When this decision is made, then the symbol (1 or 0) is recovered exactly without any noise. Therefore, the channel noise can introduce errors in the signal and decision for a particular symbol can be wrong. *This means that the ISI and channel noise simultaneously affect the received signal and create errors in the signal. Special cares must be taken to reduce the noise and increase the signal to noise ratio.*

In this chapter, first, we shall discuss the digital representation of signals. Then, we shall discuss ISI and method to minimize it.

Also, it may be noted that the optimum detection of such a pulse signal involves the use of a linear time invariant (LTI) filter known as a **matched filter.**

12.2. A Baseband Digital Communication System

A baseband digital communication system is made up of several elements as described under:

1. Source

The input to a digitial system is in the form of sequence of digits. The input can be the output from such sources as a data set, a computer, a digitized voice signal (PCM or DM), a digital facsimile or television or telemetry equipment. Most of the discussion in this chapter is restricted to the binary case (*i.e.* communication schemes using only two symbols).

2. Multiplexer

Generally speaking, the capacity of a practical channel of transmitting data is much larger than the data rate of individual sources. To utilize this capacity effectively, we combine several sources through a digital multiplexer using the process of interleaving. Thus, a channel is time-shared by several messages simultaneously.

Scanned by CamScanner

3. Line Coder

The output of a multiplexer is coded into electrical pulses or waveforms for the purpose of transmission over the channel. This process is known as **line coding or transmission coding**. There are several possible ways of assigning waveforms (*i.e.* pulses) to the digital data. In the binary case (two symbols), for example, conceptually the simplest line code is **on-off**, where a 1 is transmitted by a pulse $p(t)$ and a 0 is transmitted by no pulse (*i.e.* zero signal), as shown in figure 12.1(*a*). Another commonly used code is **polar**, where 1 is transmitted by a pulse $p(t)$ and 0 is transmitted by a pulse $-p(t)$ as shown in figure 12.1(*b*). The polar scheme is the most power

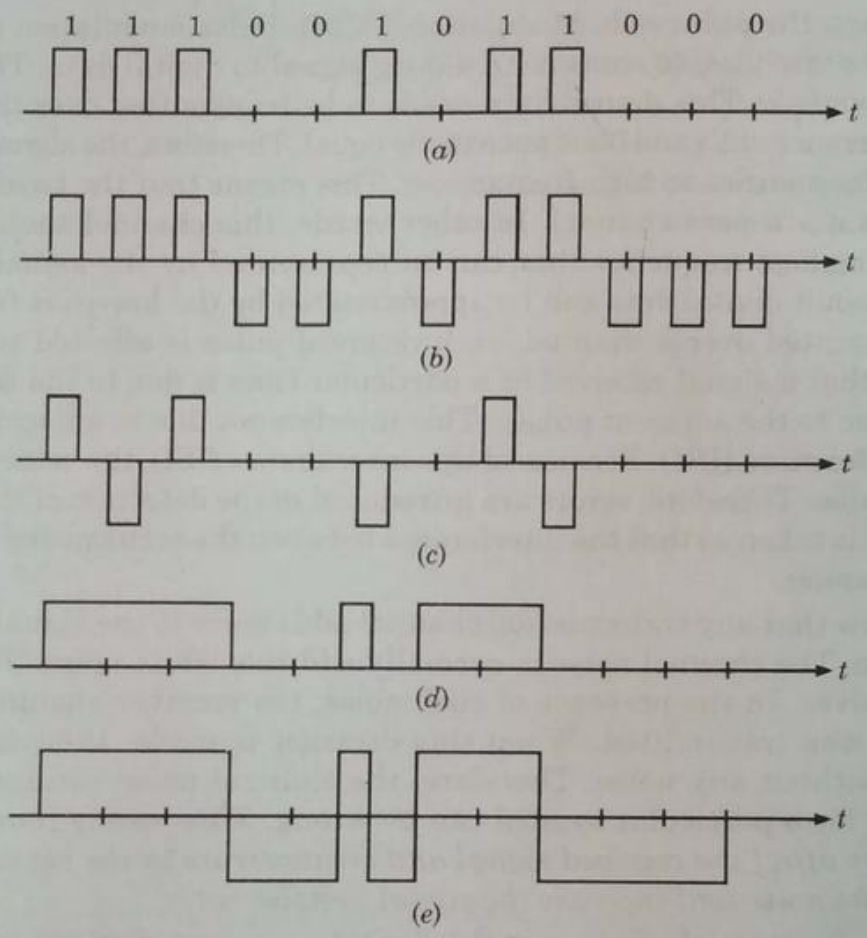


Fig. 12.1. *Some line codes (a) On-off (RZ). (b) Polar (RZ), (c) Bipolar (RZ), (d) On-off NRZ), (e) Polar (NRZ).*

efficient code, since for a given noise immunity (*i.e.* error probability) this code needs the least power. Another popular code in PCM is **bipolar**, also known as **pseudoternary** or **alternate mark inversion (AMI)**, where 0 is encoded by no pulse and 1 is encoded by a pulse $p(t)$ or $-p(t)$, depending on whether the previous 1 is encoded by $-p(t)$ or $p(t)$. In short, pulses representing consecutive 1's alternate in sign, as shown in figure 12.1(*c*). This code has the advantage that if an error is made in the detection of pulses, the received pulse sequence will violate the bipolar rule and the error is immediately detected (although not corrected).

Another line code that in the past appeared promising is the duobinary (and modified duobinary) proposed by Lender. Although this code is better than the bipolar in terms of bandwidth efficiency, it has lost its appeal due to some practical problems and will not be discussed here.

In our discussion so far, we have used half-width pulses just for the sake of illustration. We can select other widths. However, full-width pulses are often used in some applications. Whenever full-width pulses are used, the pulse amplitude is held to a constant value throughout the pulse interval (it does not have a chance to go to zero before the next pulse begins). For this reason these schemes are known as **nonreturn-to-zero** (NRZ) schemes in contrast to **return-to-zero** (RZ) schemes as shown in figure 12.1(*a*), (*b*), (*c*). Figure 12.1(*d*) shows an on-off NRZ signal, whereas figure 12.1(*e*) illustrates a polar NRZ signal.

Scanned by CamScanner

4. Regenerative Repeater

Regenerative repeaters are used at regularly spaced intervals along a digital transmission line to detect the incoming digital signal and regenerate new clean pulses for further transmission along the line. This process periodically eliminates, and thereby combats, the accumulation of noise and signal distortion along the transmission path. If the pulses are transmitted at a rate of R_b pulses per second, we require the periodic timing information—the clock signal at R_b Hz—to sample the incoming pulses at a repeater. This timing information can be extracted from the received signal itself if the line code is chosen properly. The polar signal in figure 12.1(*b*), for example, when rectified, results in a periodic signal of clock frequency R_b Hz which contains the desired periodic timing signal of frequency R_b Hz. When this signal is applied to a resonant circuit tuned to frequency R_b, the output, which is a sinusoid of frequency R_b Hz, can be used for timing. The on-off signal can be expressed as the sum of a periodic signal (of clock frequency) and a polar signal, as shown in figure 12.2. Because of the presence of the periodic component, we can extract the timing information from this signal using a resonant circuit tuned to the clock frequency. Thus, a bipolar signal, when rectified, becomes an on-off signal. Hence, its timing information can be extracted the same way as that for an on-off signal.

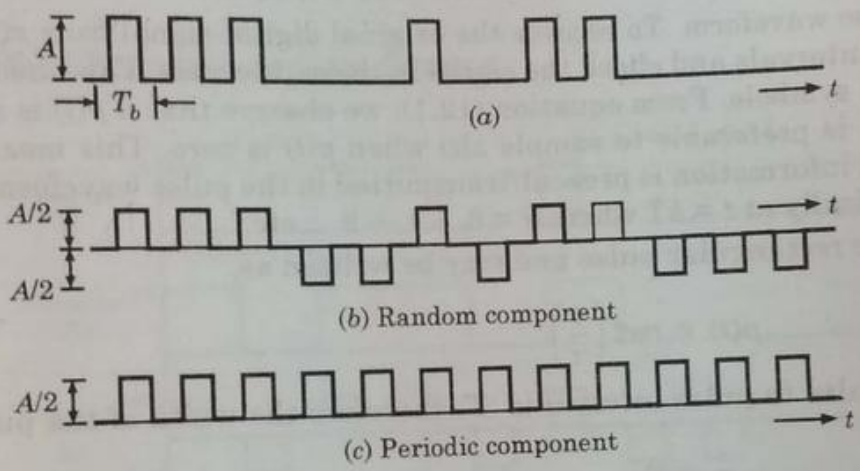


Fig. 12.2. *An on-off signal is the sum of a polar signal and a clock frequency periodic signal.*

The timing signal (*i.e.*, the resonant circuit output) is sensitive to the incoming bit pattern. In the on-off or bipolar case, a '0' is transmitted by "no pulse". Hence, if there are too many 0's in a sequence (no pulses), there is no signal at the input of the resonant circuit and the sinusoidal output of the resonant circuit starts decaying, thus, causing error in the timing information. We shall discuss later the ways of overcoming this problem. A line code in which the bit pattern does not affect the accuracy of the timing information is said to be a **transparent** line code. The polar scheme (where each bit is transmitted by some pulse) is transparent, whereas, the on-off and bipolar schemes are non-transparent.

12.3. Introduction to Discrete PAM Signals (Digital Data Formats)

There are various techniques used to convert the analog signal to digital signal. But, the other way of obtaining digital data is from the source such as computers. The information from such a source is inherently discrete in nature. If such a discrete signal is transmitted over a bandlimited channel, then the signal gets dispersed. This causes the pulses to overlap and cause distortion. Such a distortion is called as intersymbol interference (ISI). In order to avoid this, we should not transmit the discrete signal as it is on the transmission medium. This data is first converted into a PAM suitable format and then transmitted over a communication channel. The various formats used are also called as **line codes**.

Scanned by CamScanner

The use of an appropriate waveform for baseband representation of digital data is basic to its transmission from a source to a destination. This means that digital pulse modulation can be used for transmitting the output of a digital source.

We know that the signal symbols can amplitude modulate some carrier to generate amplitude modulated pulse train. Therefore, such signal may be represented as,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, p(t-kT) \qquad ...(12.1)$$

Here, a_k is the amplitude of kth symbol in the message sequence.

$p(t)$ is the pulsed carrier signal, *i.e.* is a basic pulse shape. It's pulses are modulated by a_k.

T is the maximum duration (*i.e.*, time period) allowed for the carrier pulse. The unmodulated carrier pulse $p(t)$ is the rectangular pulse and it can take variable duty cycle. It can be represented as,

$$p(t) = \begin{cases} 1 & \text{for} \quad t = 0 \\ 0 & \text{for} \quad t = \pm T, \pm 2T, ... \end{cases} \qquad ...(12.2)$$

$x(t)$ is the pulse waveform. To recover the original digital signal from $x(t)$ we have to sample $x(t)$ at some fixed intervals and check the signal in these intervals. This checking is the detection of the transmitted symbols. From equation (12.1), we observe that if $p(t)$ is zero, then $x(t)$ is also zero. Therefore, it is preferable to sample $x(t)$ when $p(t)$ is zero. This means that at this time $[p(t) = 0]$ no digital information is present/transmitted in the pulse waveform. Therefore, $x(t)$ can be sampled periodically at $t = kT$ where $k = 0, \pm 1, \pm 2, ...$etc.

Also, $p(t)$ is the rectangular pulse and may be written as,

$$p(t) = rect\left(\frac{t}{\tau}\right) \qquad ...(12.3)$$

Because, the pulse to pulse interval is 'T', therefore the width of the pulse τ should be less than or equal to T.

i.e., $\tau \leq T$

The signalling rate is given as,

$$r = \frac{1}{T} \qquad ...(12.4)$$

If 'T' represents the duration of one bit, then $T = T_b$ and signalling rate will be,

$$r = \frac{1}{T_b} \qquad ...(12.5)$$

12.4. Line Coding and its Properties

The digital data can be transmitted by various transmission or line codes such as on-off, polar, bipolar and so on. This is called **line-coding**. Each type of line-code has its advantages and disadvantages.

Thus, among other desirable properties, a line code must have the following properties:

1. Transmission bandwidth

For a line-code, the transmission bandwidth must be as small as possible.

2. Power efficiency

For a given bandwidth and a specified detection error probability, the transmitted power for a line code should be as small as possible.

Scanned by CamScanner

3. Error detection and correction capability

It must be possible to detect and preferably correct detection errors. For example, in a bipolar case, a signal error will cause bipolar violation and thus can easily be detected.

4. Favourable power spectral density

It is desirable to have zero power spectral density (PSD) at $w = 0$ (*i.e.*, dc) since ac coupling and transformers are used at the repeaters. Significant power in low-frequency components causes dc wander in the pulse stream when ac coupling is used. The a.c. coupling is required since the dc paths provided by the cable pairs between the repeater sites are used to transmit the power required to operate the repeaters.

5. Adequate timing content

It must be possible to extract timing or clock information from the signal.

6. Transparency

It must be possible to transmit a digital signal correctly regardles of the pattern of 1's and 0's.

12.5. Various PAM Formats or Line Codes

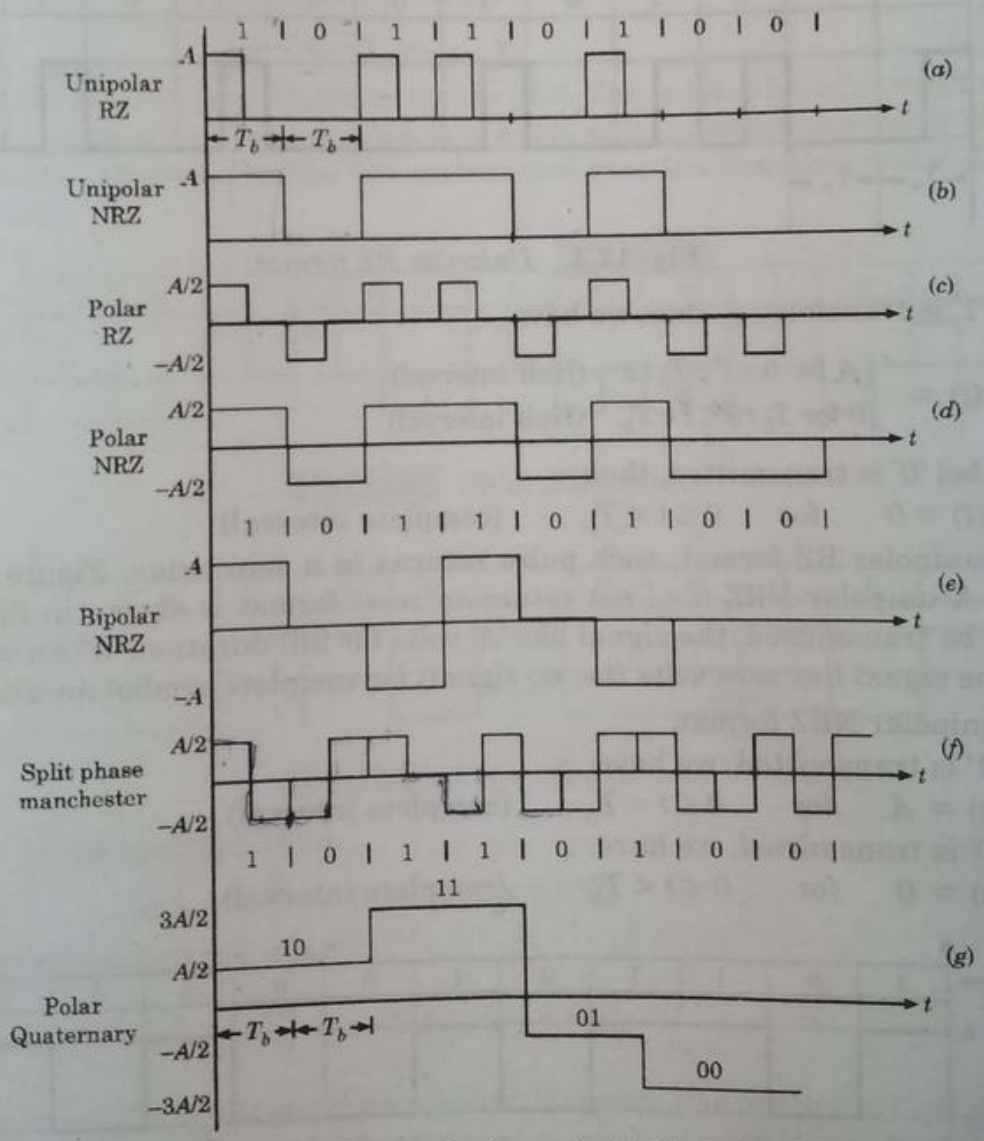


Fig. 12.3. *Various digital PAM signals formats (a) Unipolar RZ (b) Unipolar NRZ (c) Polar RZ (d) Polar NRZ (e) Bipolar NRZ (f) Split phase Manchester (g) Polar quaternary NRZ.*

Scanned by CamScanner

Some of the important PAM formats are as under:

(*i*) Non-return to zero (NRZ) and return to zero (RZ) unipolar format.

(*ii*) NRZ and RZ polar format.

(*iii*) Non-return to zero bipolar format.

(*iv*) Manchester format.

(*v*) Polar quaternary NRZ format.

All the formats have been shown for a binary message 10110100. Figure 12.3 shows various PAM formats or line codes.

12.6. Unipolar RZ and NRZ

In unipolar format, the waveform has a single polarity. The waveform can have +5 or +12 volts when high. The waveform is simple on-off. In the unipolar RZ form, the waveform has zero value when symbol '0' is transmitted and waveform has '*A*' volts when '1' is transmitted. In RZ form, the '*A*' volts is present for $T_b/2$ period if symbol '1' is transmitted and for remaining $T_b/2$, waveform returns to zero value, *i.e.*, for unipolar RZ form, we have

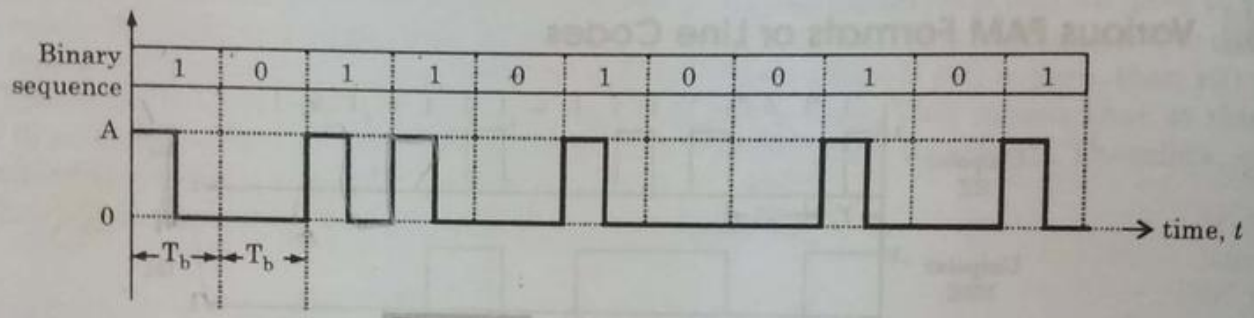


Fig. 12.4. *Unipolar RZ format.*

If symbol '1' is transmitted, then we have

$$x(t) = \begin{cases} A \text{ for } 0 \le t < T_b/2 & \text{(Half interval)} \\ 0 \text{ for } T_b/2 \le t < T_b & \text{(Half interval)} \end{cases} \quad ....(12.6)$$

and if symbol '0' is transmitted, then

$$x(t) = 0 \quad \text{for} \quad 0 \le t < T_b \quad \text{(complete interval)} \quad ...(12.7)$$

Hence, in unipolar RZ format, each pulse returns to a zero value. Figure 12.4 shows this signal format. A unipolar NRZ (*i.e.*, not return to zero) format is shown in figure 12.5. When symbol '1' is to be transmitted, the signal has '*A*' volts for full duration. When symbol '0' is to be transmitted, the signal has zero volts (*i.e.* no signal) for complete symbol duration.

Thus, for unipolar NRZ format,

If symbol '1' is transmitted, we have

$$x(t) = A \quad \text{for} \quad 0 \le t < T_b \quad \text{(complete interval)} \quad ...(12.8)$$

If symbol '0' is transmitted, we have

$$x(t) = 0 \quad \text{for} \quad 0 \le t < T_b \quad \text{(complete interval)} \quad ...(12.9)$$

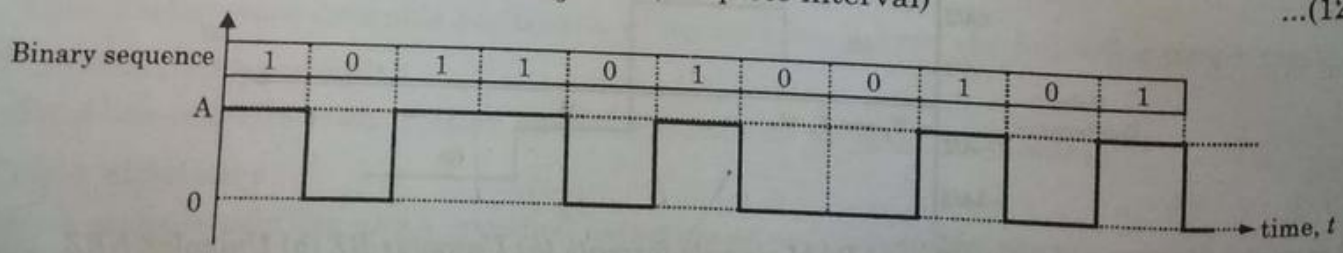


Fig. 12.5. *Unipolar NRZ format.*

Scanned by CamScanner

For NRZ format, it may be observed that the pulse does not return to zero on its own. If symbol '0' is to be transmitted, then pulse becomes zero. *Internal computer waveforms are usually of unipolar NRZ type.*

Because, there is no separation between the pulses, therefore, the receiver needs synchronization to detect unipolar NRZ pulse. As compared to RZ format, NRZ pulse width (pulse to pulse interval is same) is more. Thus, energy of the pulse is more. However, unipolar format has some average DC value. This DC value does not carry any information.

12.7. Polar RZ and NRZ

In the polar RZ format, symbol '1' is represented by positive voltage polarity whereas symbol '0' is represented by negative voltage polarity. Because this is RZ format, the pulse is transmitted only for half duration. Thus, for polar RZ, if symbol '1' is transmitted, then

$$x(t) = \begin{cases} +\dfrac{A}{2} & \text{for } 0 \le t < T_b/2 \\ 0 & \text{for } T_b/2 \le t < T_b \end{cases} \qquad \text{...(12.10)}$$

and if symbol '0' is transmitted, then

$$x(t) = \begin{cases} -\dfrac{A}{2} & \text{for } 0 \le t < T_b/2 \\ 0 & \text{for } T_b/2 \le t < T_b \end{cases} \qquad \text{...(12.11)}$$

Polar RZ waveform has been shown in figure 12.6. The polar NRZ is shown in figure 12.7. In polar NRZ format, symbol '1' is represented by positive polarity whereas symbol '0' is represented by negative polarity. These polarities are maintained over the complete pulse duration *i.e.*, for polar NRZ, we have

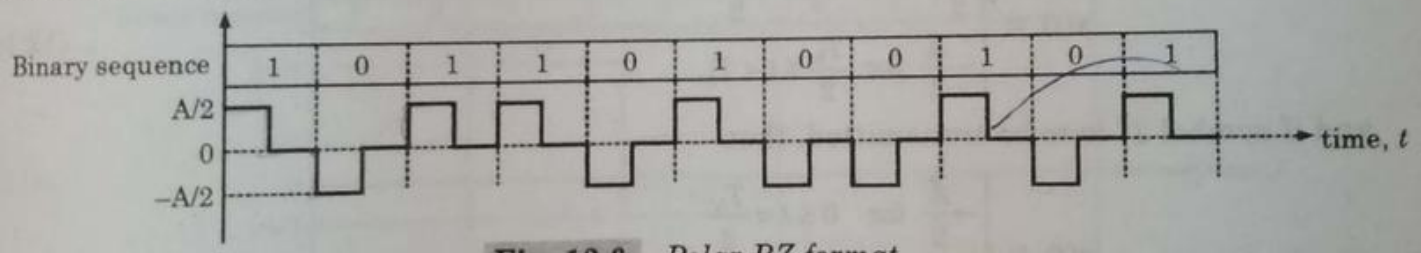


Fig. 12.6. *Polar RZ format.*

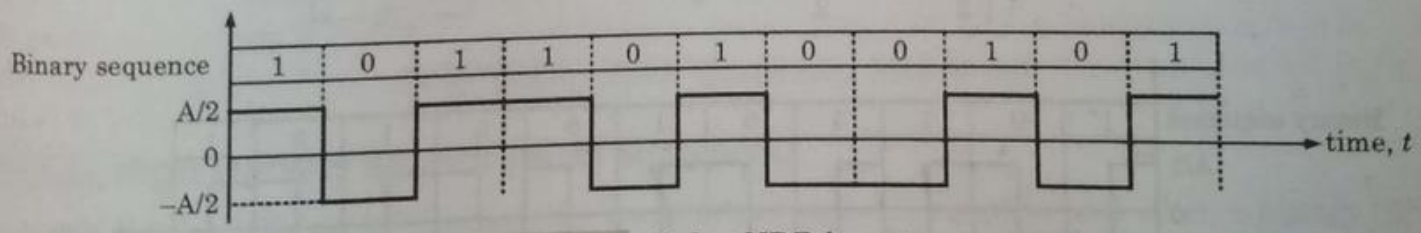


Fig. 12.7. *Polar NRZ format.*

If symbol '1' is transmitted, then

$$x(t) = +\frac{A}{2} \quad \text{for} \quad 0 \le t < T_b \qquad \text{...(12.12)}$$

and if symbol '0' is transmitted, then

$$x(t) = -\frac{A}{2} \quad \text{for} \quad 0 \le t < T_b \qquad \text{...(12.13)}$$

Since polar RZ and NRZ formats are bipolar, therefore, the average DC value is minimum in these waveforms. If probabilities of occurrence of symbols '1' and '0' are same, then average DC components of the waveform would be zero.

Scanned by CamScanner

12.8. Bipolar NRZ [Alternate Mark Inversion (AMI)]

In this format, the successive '1's are represented by pulses with altenate polarity and '0's are represented by no pulses. Figure 12.8 illustrates the Bipolar NRZ or AMI waveform. If there are even number of 1's, the DC component of the waveform would be zero. The advantage of this format is that the ambiguities due to transmission sign inversion are eliminated.

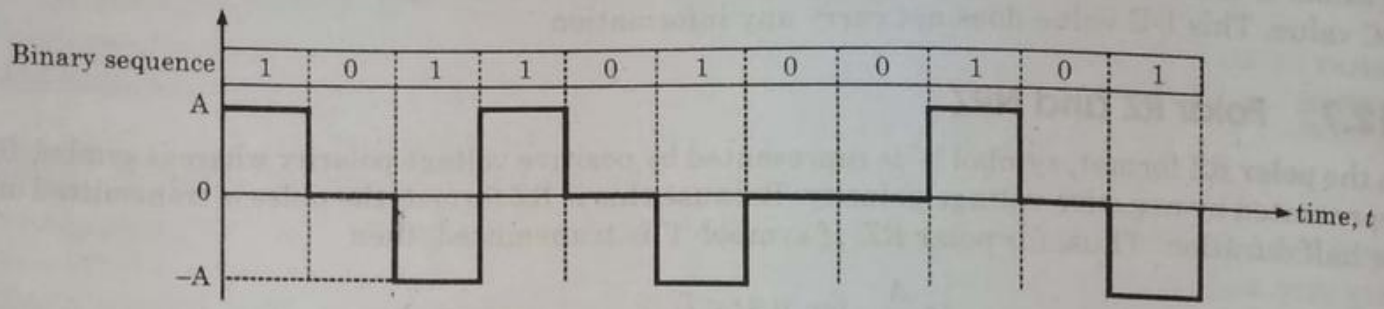


Fig. 12.8. *Bipolar NRZ format (AMI).*

12.9. Split Phase Manchester Format

This type of waveform is shown in figure 12.9. In this case, if symbol '1' is to be transmitted, then a positive half interval pulse is followed by a negative half interval pulse. If symbol '0' is to be transmitted, then a negative half interval pulse is followed by a positive half interval pulse. Hence, for any symbol the pulse takes positive as well as negative value *i.e.*,

If symbol '1' is to be transmitted, then

$$x(t) = \begin{cases} \dfrac{A}{2} & \text{for } 0 \le t < \dfrac{T_b}{2} \\ -\dfrac{A}{2} & \text{for } \dfrac{T_b}{2} \le t < T_b \end{cases} \qquad \text{...(12.14)}$$

and if symbol '0' is to be transmitted, then

$$x(t) = \begin{cases} -\dfrac{A}{2} & \text{for } 0 \le t < \dfrac{T_b}{2} \\ \dfrac{A}{2} & \text{for } \dfrac{T_b}{2} \le t < T_b \end{cases} \qquad \text{...(12.15)}$$

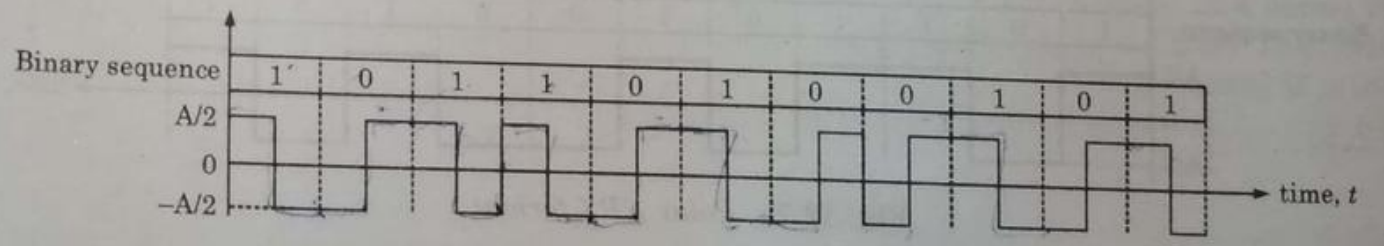


Fig. 12.9. *Split phase manchester format.*

The primary advantage of this format is that irrespective of the probability of occurrence of symbol '1' and '0' the waveform has zero average value. Therefore by this mode, the power saving is quite more.

However, the drawback of this format is that it requires absolute sense of polarity at the receiver end.

12.10. Polar Quaternary NRZ Format

Figure 12.10 shows the waveform of this format. This format is derived to reduce the signalling rate '*r*'. The message bits are grouped in the blocks of two. Therefore there are four possible

Scanned by CamScanner

combinations 00, 01, 10 and 11. To these four combinations, four amplitude levels are assigned. The Table 12.1 shows how this can be achieved

Table 12.1. Polar quaternary NRZ Format: Combinations of bits

Message combination	$x(t) = a_n$
00	$-\frac{3A}{2}$
01	$-\frac{A}{2}$
10	$\frac{A}{2}$
11	$\frac{3A}{2}$

In the waveform of figure 12.10, the first combination of two bits is 10. Thus, from Table 12.1, we may observe that the level should be $\frac{A}{2}$. The second combination in figure 12.10 is 11, hence from Table 12.1, the level taken is $\frac{3A}{2}$. Similarly other levels are selected. Hence, for two message bits only one pulse is transmitted with duration $2T_b$, *i.e.*,

$$T_s = 2T_b$$

and signalling rate is given as,

$$r = \frac{r_b}{2} = \frac{1}{2T_b} \quad ...(12.16)$$

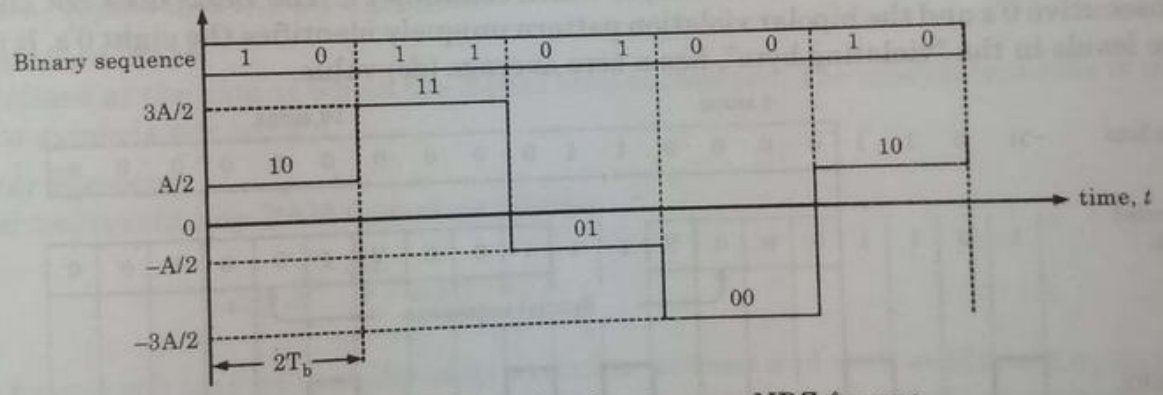


Fig. 12.10. *Polar Quaternary NRZ format.*

12.11. High Density Bipolar (HDB) Signalling

In case of bipolar NRZ or AMI signal, the transmitted signal is equal to zero when a binary "0" is to be transmitted. This is true even for the unipolar RZ and unipolar NRZ signals. The absence of transmitted signal can cause problems in synchronization at the receiver, if long sequence of binary "0"s are being transmitted. This problem can be solved by adding (transmitting) pulses when long strings of 0's exceeding a number n are being transmitted. This type of coding is called as **High Density Bipolar coding**. It is denoted by **HDBN**. Here, $N = 1, 2, 3, ...$ The most widely used HDB format is with $N = 3$ *i.e.*, HDB3.

In the string of message bits, when $(N + 1)$ or mm number of zeros occur, they are replaced by special binary sequences of $(N + 1)$ length. As shown in figure 12.11, these sequences contain some binary 1's which are necessary for synchronization at the receiver end. The $(N + 1)$ long special sequences for the HDB3 coding are 000V and B00V where B and V both are considered to be binary 1's. When the number of consecutive zeros exceed $(N + 1)$ *i.e.*, 4 in case of HDB3, the abovementioned special sequences are inserted as shown in figure 12.11.

Scanned by CamScanner

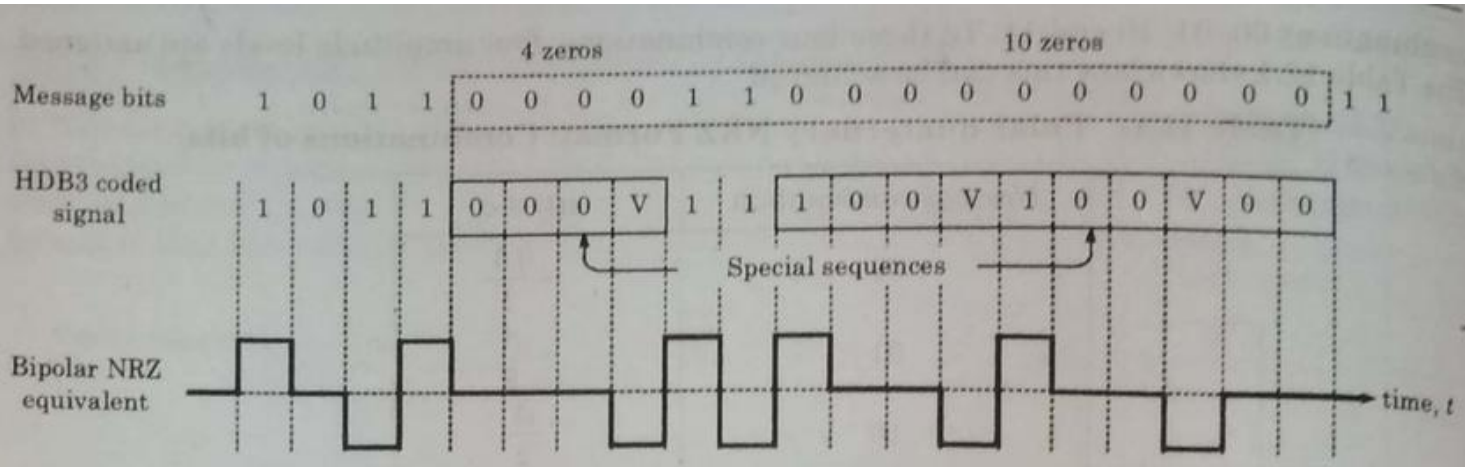


Fig. 12.11.

12.12. B8ZS Line Code

We have discussed about the codes in this chapter. We know that in order to have synchronization between the transmitter and receiver, the line code needs to cross the zero line frequently. As per U.S. T_1 standard, not more than 15 0's can be sent in succession to ensure proper synchronization. In order to solve the problems related to synchronization a new line code called B8Zs (Binary 8-zeros suppression) was developed. Whenever eight successive 0's are detected, the implementation of this line code will automatically insert a special 8 bit sequence containing a bipolar violation. This can bed easily detected and corrected by the CSU/DSU (channel service unit/digital service unit). Figure 12.12 gives a clear idea about the B8ZS line code. The violations (BPV in figure 12.12), will distinguish a byte substituted for all 0's from a normal byte which contains 1's. The B8ZS does not allow more than 8, consecutive 0's and the bipolar violation pattern uniquely identifies the eight 0's. It may that the voltage levels in the "violating byte", has a zero average (dc) value.

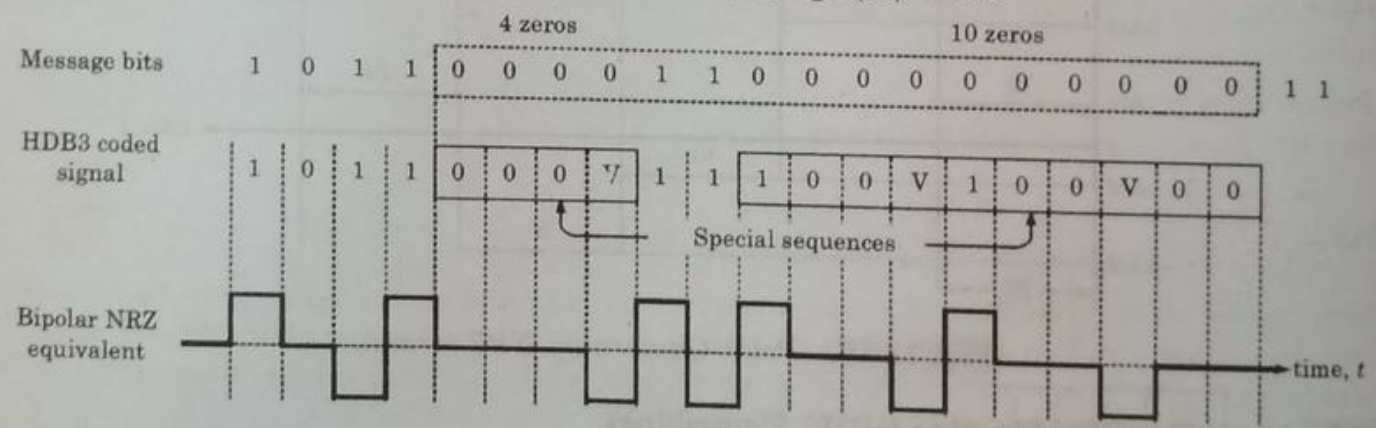


Fig. 12.12.

12.13. Power Spectra of Discrete PAM Signals (Various Line Codes)

As discussed earlier, we can represent all the discrete PAM signals with the help of a single expression as under:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k p(t-kT) \qquad \text{...(12.17)}$$

where a_k = coefficient
$p(t)$ = the basic pulse shape
T = symbol duration

Now, let us assume that the basic pulse $p(t)$ is centered at the origin ($t = 0$) and normalized such that $p(0) = 1$.

Scanned by CamScanner

Table 12.2 lists the value of coefficients a_k for different PAM formats.

Table 12.2. Values of a_k for various data formats.

S. No.	NRZ formats	Coefficient a_k		Basic pulse $p(t)$
		for symbol 0	for symbol 1	
1.	Unipolar NRZ	$a_k = 0$	$ak = A$	Basic pulse $p(t)$ is a rectangular
2.	Polar NRZ	$a_k = -A$	$a_k = A$	pulse of unit amplitude and
3.	Bipolar NRZ	$a_k = 0$	$a_k = A$ or $-A$ alternately for 1s.	duration T_b.
4.	Manchester	$a_k = -A$	$a_k = +A$	$p(t)$ consists of double pulses of amplitude ± 1 and duration T_b.
5.	Polar Quaternary	$a_k = -3A/2$ for 00	$-A/2$ for 01 and	$p(t)$ is a rectangular pulse of unit amplitude and duration $2\,T_b$.

Now, let us discuss few important terms as under:

(i) Data Signalling Rate

It is defined as the number of bits of data transmitted per second. It is measured in bits/second.

The data signalling rate is also called as data rate and it is defined as follows:

$$R_b = 1/T_b \qquad ...(12.18)$$

where T_b represents the bit duration.

(ii) Modulation Rate

It is defined as the rate at which the signal level is changed. The modulation rate is measured in bauds or symbols per second.

(iii) Power Spectra

We can represent any PAM format as follows:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k p(t-kT)$$

Each format $x(t)$ may be considered as a random process and each coefficient a_k as a random variable. Let these coefficients be generated by a discrete stationary random source which is characterized by the following relation between autocorrelation and ensemble average *i.e.*,

$$R_\tau(n) = E[a_k\, a_{k-n}] \qquad ...(12.19)$$

where $R_\tau(n)$ = Autocorrelation function

and E = Expectation operator

Also, power spectral density (psd) of PAM signal $x(t)$ is given by

$$S(f) = \frac{1}{T}|P(f)|^2 \sum_{n=-\infty}^{\infty} R_\tau(n)\, e^{-j2\pi nfT} \qquad ...(12.20)$$

where $P(f)$ = Fourier transform of the basic pulse $p(t)$.

It may be noted that the values of $P(f)$ and $R_\tau(n)$ depends upon the type of PAM format.

12.14. Power Spectral Density (psd) of NRZ Unipolar Format

For simplicity, let us assume that the 0s and 1s are equally likely to happen or have equal probability.

$$P(a_k = 0) = P(a_k = A) = 1/2$$

Scanned by CamScanner

We know that a discrete stationary random source is characterized by the following autocorrelation:

$$R_\tau(n) = [a_k\, a_{k-n}]$$

Hence, for $n = 0$, above expression becomes

$$R_\tau(n) = E[a_k\, a_k] = E[a_k{}^2] \qquad ...(12.21)$$

$$E[a_k]^2 = 0^2 P(a_k = 0) + (A)^2\, P(a_k = A)$$

or

$$E[a_k]^2 = A^2 \times \frac{1}{2} = \frac{A^2}{2} \qquad ...(12.22)$$

Now, let us obtain the product $a_k\, a_{k-n}$ for $n \neq 0$. The four possible values of this product are 0, 0, 0 and A^2. With the assumption that the successive symbols in a binary sequence are equiprobable (with a probability equal to 1/2), the probability of the four values 0, 0, 0 and a^2 will be equal to 1/4 each.

Therefore, for $n \neq 0$, we have

$$E[a_k\, a_{k-n}] = (3 \times 0 \times 1/4) + (A^2 \times 1/4)$$

or

$$E[a_k\, a_{k-n}] = \frac{A^2}{4} \text{ for } n \neq 0 \qquad ...(12.23)$$

Thus, the autocorrelation function $R_\tau(n)$ can be expressed as under:

$$R_\tau(n) = \begin{cases} A^2/2 \text{ for } n = 0 & ...(12.24) \\ A^2/4 \text{ for } n \neq 0 & ...(12.25) \end{cases}$$

The basic pulse $p(t)$ is a rectangular pulse of unity amplitude and duration T_b. Hence, its Fourier transform is given by

$$P(f) = T_b \operatorname{sinc}(fT_b) \qquad ...(12.26)$$

where the sinc function is expressed as,

$$\operatorname{sinc} x = \frac{\sin(\pi x)}{\pi x}$$

Now, substituting the expression for $P(f)$ and $R_\tau(n)$ in equation (12.20) and substituting $T = T_b$ to get power spectral density (psd) as under:

$$S(f) = \frac{1}{T_b} \times [T_b \operatorname{sinc}(fT_b)]^2 \left[\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{A^2}{4} e^{-j2\pi nfT_b} + \sum_{n=0} \frac{A^2}{2} e^{-j2\pi nfT_b} \right] \qquad ...(12.27)$$

or

$$S(f) = T_b \operatorname{sinc}^2(fT_b) \left[\frac{A^2}{2} e^0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{A^2}{4} e^{-j2\pi nfT_b} \right]$$

or

$$S(f) = \frac{A^2 T_b}{2} \operatorname{sinc}^2(fT_b) + \frac{A^2 T_b}{4} \operatorname{sinc}^2(fT_b) \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} e^{-j2\pi nfT_b} \qquad ...(12.28)$$

According to Poisson's expression, we can write,

$$\sum_{n=-\infty}^{\infty} e^{-j2\pi nfT_b} = \frac{1}{T_b} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T_b}\right) \qquad ...(12.29)$$

where $\delta(f)$ is the unit impulse at $f = 0$ in the frequency domain.

Substituting equation (12.29) into equation (12.28) and noting that the sinc function passes through zero at $f = \pm 1/T_b, \pm 2/T_b ...$, we can write equation (12.28) in the simplified form as under:

Scanned by CamScanner

$$S(f) = \frac{A^2 T_b}{4} \sin c^2(fT_b) + \frac{A^2}{4}\delta(f) \quad ...(12.30)$$

This expression can be plotted as shown in figure 12.13 (curve *a*).

12.15. Power Spectral Density (psd) of NRZ Polar Format

Here, let us follow a similar procedure as followed in the previous section to obtain the psd of NRZ polar format.

For the NRZ polar format, we have

$$R_\tau(n) = \begin{cases} a^2 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases} \quad ...(12.31)$$

We know the basic pulse $p(t)$ is same for all the PAM formats, therefore, its Fourier transform will also remain same.

Hence, $P(f) = T_b \text{ sinc } (fT_b)$

Thus, the power spectral density (psd) is given by,

$$S(f) = A^2 T_b \text{ sinc}^2 (fT_b) \quad ...(12.32)$$

The normalized form of this expression has been plotted in figure 12.13 as curve *b*.*

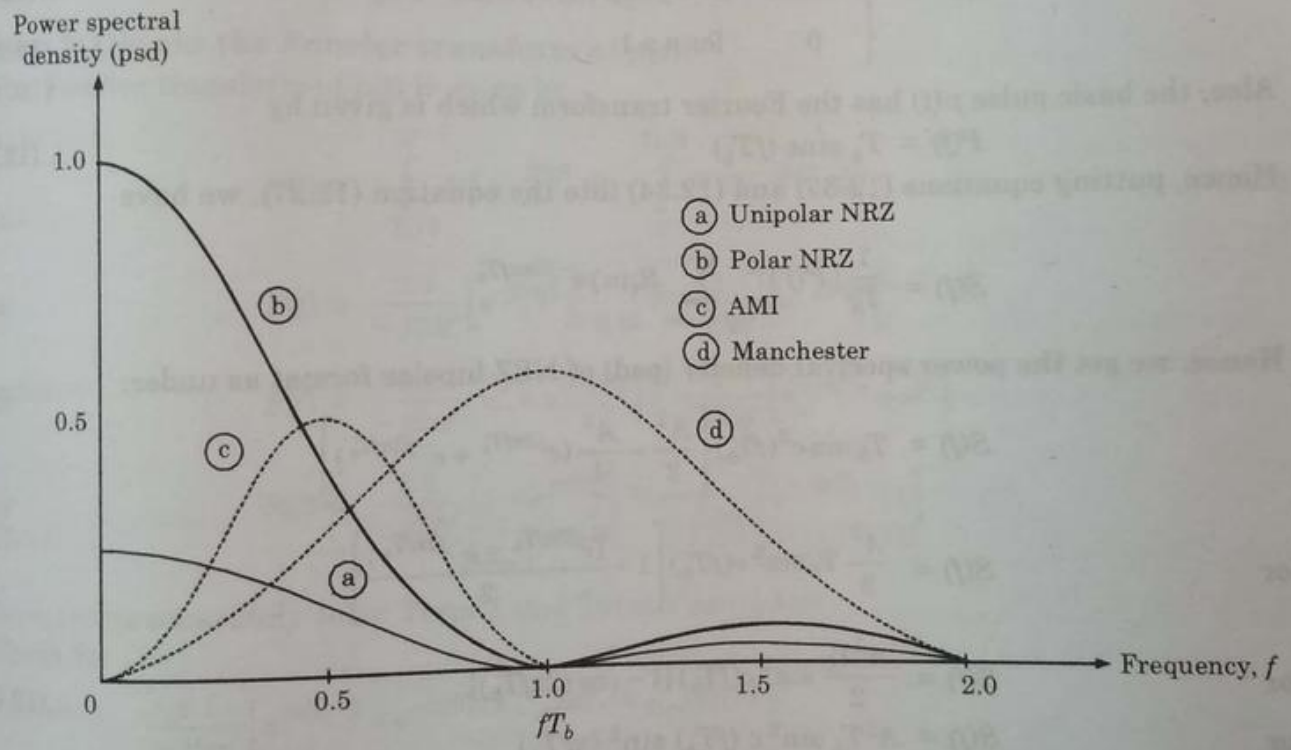


Fig. 12.13. *Power spectra of different line codes.*

Similarly, it is possible to obtain the expressions for the power spectral densities (psds) of other PAM formats.

12.16. Power Spectral Density (psd) of NRZ Bipolar Format

We know that the bipolar NRZ format has three levels A, 0 and $-A$. If we assume that the 1s and 0s in the binary input data occur with equal probability then the probabilities of occurrence of these levels will be given by

* Curve *b* of figure 12.13 shows that most of the power of polar NRZ format lies inside the main lobe of the sinc shaped curve extending upto the bit rate $1/T_b$.

Scanned by CamScanner

$$P(a_k = A) = 1/4$$
$$P(a_k = 0) = 1/2$$
$$P(a_k = -A) = 1/4$$

Therefore, for $n = 0$, we can have

$$E\left[a_k^2\right] = A^2 P(a_k = A) + (0)^2 P(a_k = 0) + (-A)^2 P(a_k = -A) = A^2/2$$

Similarly, for $n = 1$, the input sequence can assume four possible formats as (0, 0), (0, 1), (1, 0) and (1, 1). The corresponding values of the product $a_k\, a_{k-1}$ are 0, 0, 0 and 1 respectively.

Now, assuming that the 0s and 1s in the binary sequence have equal probability, we can write

$$E[a_k\, a_{k-1}] = 3(0)\,(1/4) + (-A^2)\,(1/4) = -A^2/4$$

Further, for $n > 1$ we find that

$$E[a_k\, a_{k-1}] = 0$$

Therefore, for the bipolar NRZ format, we have

$$R_\tau(n) = \begin{cases} A^2/2 & \text{for } n = 0 \\ -A^2/4 & \text{for } n = \pm 1 \\ 0 & \text{for } n > 1 \end{cases} \qquad \text{...(12.33)}$$

Also, the basic pulse $p(t)$ has the Fourier transform which is given by

$$P(f) = T_b \operatorname{sinc}(fT_b) \qquad \text{...(12.34)}$$

Hence, putting equations (12.33) and (12.34) into the equation (12.27), we have

$$S(f) = \frac{1}{T_b} |P(f)|^2 \sum_{n=-\infty}^{\infty} R_\tau(n)\, e^{-j2\pi nfT_b}$$

Hence, we get the power spectral density (psd) of NRZ bipolar format as under:

$$S(f) = T_b \operatorname{sinc}^2(fT_b)\left[\frac{A^2}{2} - \frac{A^2}{4}\left(e^{j2\pi fT_b} + e^{-j2\pi fT_b}\right)\right]$$

or
$$S(f) = \frac{A^2}{2} T_b \sin^2 c(fT_b)\left[1 - \frac{\left(e^{j2\pi fT_b} + e^{-j2\pi fT_b}\right)}{2}\right]$$

or
$$S(f) = \frac{A^2 T_b}{2} \sin^2 c(fT_b)\,[1 - \cos(2\pi fT_b)] \qquad \text{...(12.35)}$$

or
$$S(f) = A^2 T_b \sin^2 c\,(fT_b) \sin^2(\pi fT_b)$$

The normalized form of this equation is shown plotted in '*c*' in figure 12.13.

12.17. Power Spectral Density (psd) of the Manchester Format

Here, let us assume that the input binary data consists of 0s and 1*s* of equal probability (i.e., $\frac{1}{2}$). Then the autocorrelation function of this format is same as that for the NRZ polar format *i.e.*,

$$R_\tau(n) = \begin{cases} A^2 & \text{for } n = 0 \\ 0 & \text{for } n = 1 \end{cases} \qquad \text{...(12.36)}$$

Scanned by CamScanner

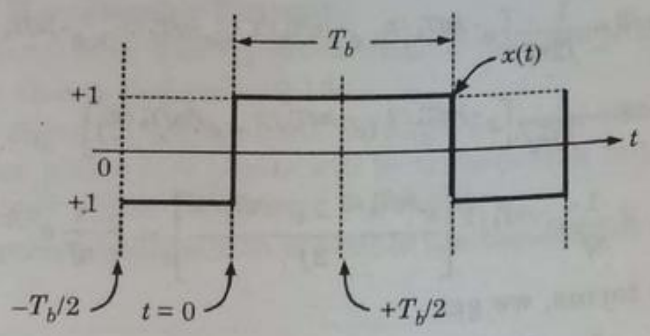


Fig. 12.14. *Manchester signal.*

The basic pulse $p(t)$ of the Manchester format is as under:

Step 1: Express the Manchester signal $x(t)$ mathematically in time domain:

The basic pulse $p(t)$ of the Manchester signal is as shown in figure 12.14. We can express the Manchester signal as under:

$$p(t) = \begin{cases} \pm 1 & \text{for } -T_b/2 \le t \le 0 \\ \mp 1 & \text{for } 0 \le t \le +T_b/2 \end{cases}$$

Step 2: Obtain the Fourier transform of $p(t)$:

The Fourier transform of $p(t)$ is given by

$$P(f) = \int_{-T_b/2}^{0} \pm 1\, e^{-j2\pi ft}\, dt - \int_{0}^{T_b/2} \mp 1\, e^{-j2\pi ft}\, dt \qquad \text{...(12.37)}$$

or
$$P(f) = \frac{\pm 1}{-j2\pi f}\left[e^{-j2\pi ft}\right]_{-T_b/2}^{0} \mp \frac{1}{-j2\pi f}\left[e^{-j2\pi ft}\right]_{0}^{T_b/2}$$

or
$$P(f) = \frac{\pm 1}{-j2\pi f}\left[e^{0} - e^{-j\pi fT_b}\right] \mp \frac{1}{-j2\pi f}\left[e^{-j\pi fT_b} - e^{0}\right]$$

or
$$P(f) = \underbrace{\frac{\pm 1}{-j2\pi f}\left[1 - e^{j\pi fT_b}\right]}_{\text{Term 1}} \mp \underbrace{\frac{1}{-j2\pi f}\left[e^{-j\pi fT_b} - 1\right]}_{\text{Term 2}}$$

Now, let us separately solve Term 1 and Term 2 as under:

Term 1:

$$\frac{\pm 1}{-j2\pi f}\left[e^{j\pi fT_b/2} \times e^{-j\pi fT_b/2} - e^{j\pi fT_b/2} \times e^{j\pi fT_b/2}\right]$$

$$= \frac{\pm 1}{-j2\pi f}\left[e^{j\pi fT_b/2} \times (e^{-j\pi fT_b/2} - e^{j\pi fT_b/2})\right]$$

$$= \frac{\pm 1}{-j2\pi f}\, e^{j\pi fT_b/2}\left(\frac{e^{j\pi fT_b/2} - e^{-j\pi fT_b/2}}{2j}\right)$$

$$= \pm \frac{1}{\pi f}\, e^{j\pi fT_b/2} \sin \pi fT_b/2$$

Term 2:

$$\mp \frac{1}{-j2\pi f}\left[e^{-j\pi fT_b} - 1\right]$$

Scanned by CamScanner

$$= \mp \frac{1}{-j2\pi f}\left[e^{-j\pi f T_b/2} \cdot e^{-j\pi f T_b/2} - e^{j\pi f T_b/2} \times e^{-j\pi f T_b/2}\right]$$

$$= \mp \frac{1}{-j2\pi f}\left[e^{-j\pi f T_b/2}\,(e^{-j\pi f T_b/2} - e^{j2\pi f T_b/2})\right]$$

$$= \mp \frac{1}{\pi f} e^{-j\pi T_b/2}\left[\frac{e^{j\pi f T_b/2} - e^{-j\pi f T_b/2}}{2j}\right] = \mp \frac{1}{\pi f} e^{-j\pi T_b/2} \sin(\pi f T_b/2) \quad \text{...(12.38)}$$

Now, adding both the terms, we get

$$P(f) = \pm \frac{1}{\pi f} e^{j\pi f T_b/2} \sin(\pi f T_b/2) \mp \frac{1}{\pi f} e^{-j\pi f T_b/2} \sin(\pi f T_b/2)$$

or $$P(f) = \pm \frac{1}{\pi f} \sin(\pi f T_b/2)\left[e^{j\pi f T_b/2} - e^{-j\pi f T_b/2}\right]$$

or $$P(f) = \frac{\pm 2j \cdot 1}{\pi f} \sin(\pi f T_b/2)\left[\frac{e^{j\pi f T_b/2} - e^{-j\pi f T_b/2}}{2j}\right]$$

or $$P(f) = \pm \frac{2j \cdot 1}{\pi f} \sin(\pi f T_b/2)(\pi f T_b/2) = \pm \frac{2j\,1}{\pi f} \sin^2(\pi f T_b/2)$$

Further, multiplying and dividing by $T_b/2$, we get,

$$P(f) = \pm \frac{2j\,(T_b/2)}{\pi f\, T_b/2} \sin^2(\pi f T_b/2)$$

or $$P(f) = \pm j(T_b)\left[\frac{\sin^2(\pi f T_b/2)}{\pi f T_b/2}\right]$$

Rearranging terms, we get

$$P(f) = \pm j(T_b) \frac{\sin(\pi f T_b/2)}{(\pi f T_b/2)} \sin(\pi f T_b/2)$$

or $$P(f) = \pm (T_b) \sin c \frac{(fT_b)}{2} \sin \frac{(\pi f T_b)}{2}$$

This is the required Fourier transform

Step 3: Obtain the expression for $S(f)$:

We know that the power spectral density (psd) $S(f)$ is given by

$$S(f) = \frac{1}{T_b} |P(f)|^2 \sum_{n=-\infty}^{\infty} R_\tau(n) \cdot e^{-j2\pi n f T_b}$$

Substituting the expressions for $P(f)$, and $R_\tau(n)$, we obtain

$$S(f) = T_b \sin c^2 \frac{(fT_b)}{2} \sin^2 \frac{(\pi f T_b)}{2} \cdot A^2$$

or $$S(f) = A^2 T_b \sin c^2 \frac{(fT_b)}{2} \sin^2 \frac{(\pi f T_b)}{2} \quad \text{...(12.38)}$$

This is the required expression for the power spectral density (psd). The normalized form of this power spectral density (psd) is plotted as curve "d" in figure 12.13. Most of the power lies inside the bandwidth equal to $2/T_b$ which is twice as large as that of unipolar, polar and bipolar formats of NRZ type.

Scanned by CamScanner

Features of the psd of a Manchester Format

(i) The main lobes extend from 0 to $2f_b$ rather than 0 to f_b (as NRZ).

(ii) At $f = 0$, $S(f) = 0$ as shown in figure 12.15.

(iii) If the Manchester signal is transmitted through an ideal low pass filter with a cut-off frequency of $2f_b$ then about 95% power will be transmitted but if the cut-off frequency is reduced to f_b then only about 70% power is passed.

(iv) The bandwidth required is $2f_b$ which is double the bandwidth of the NRZ signal.

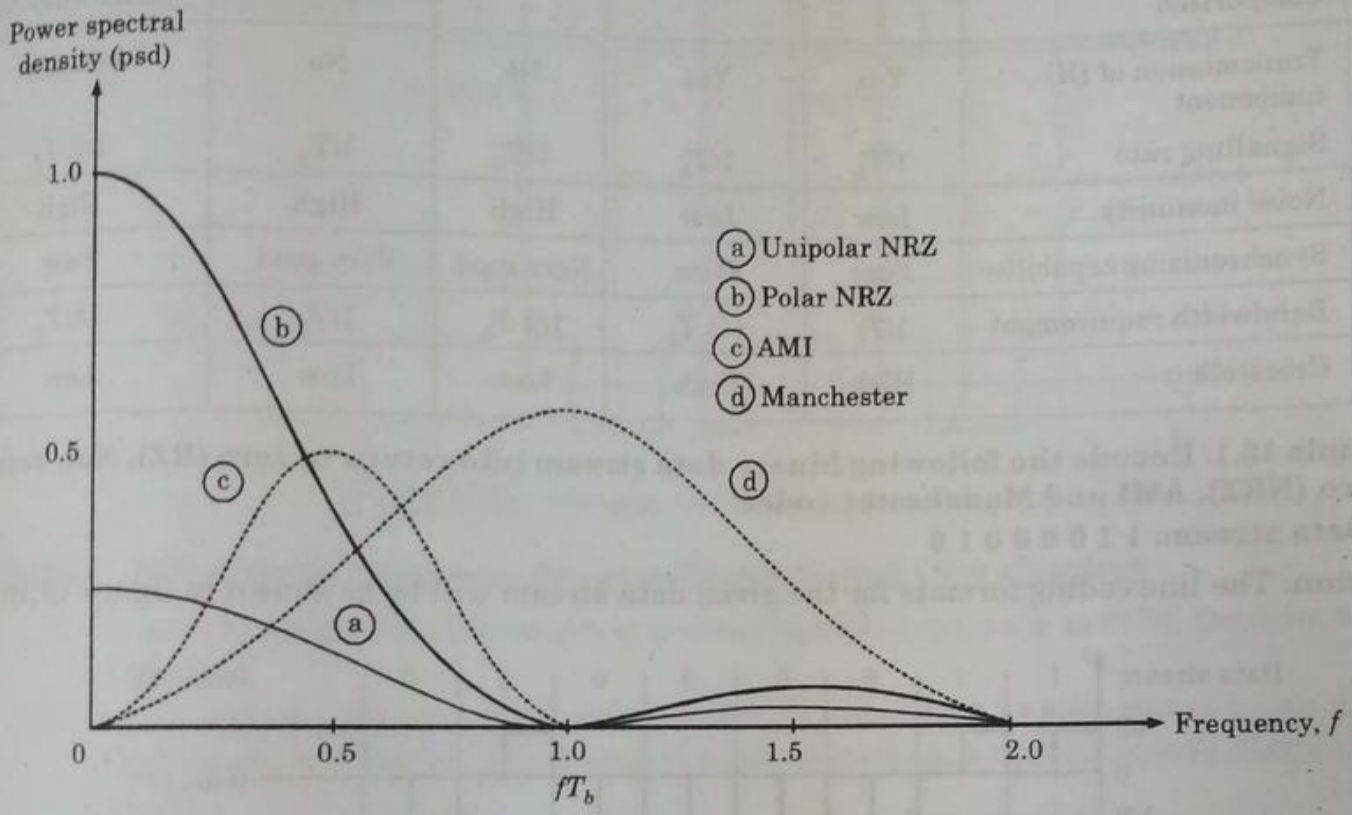


Fig. 12.15. *Power spectra of different line codes.*

12.18. Comparison of Various Discrete PAM Formats on the Basis of Power Spectra

As a matter of fact, power spectra is a graph power spectral density (psd) plotted against frequency. It gives the distribution of power of different frequency components. The power spectra for four important line codes have been shown in figure 12.15.

(i) Power Spectra of NRZ Unipolar

Curve *a* of figure 12.15 shows a normalized graph. Both the power spectral density (psd) and frequency are normalized. Most of the power of the NRZ unipolar format is concentrated between dc and the bit rate of input data.

(ii) Power Spectra of NRZ Polar Format

Curve *b* in figure 12.15 shows a normalized power spectra of NRZ polar format. Here, also like unipolar NRZ format, most of the power lies in main lobe of the sinc shaped curve. This main lobe extends upto a bit rate of $1/T_b$.

(iii) NRZ Bipolar (AMI) Format

Curve *c* in figure 12.15 shows the normalized power spectra for the AMI format. It can be seen that almost all the power lies inside the bandwidth equal to $1/T_b$. Similar to the two NRZ formats discussed earlier, however the spectral content of the NRZ bipolar format is relatively small around the zero frequency.

Scanned by CamScanner

(*iv*) Manchester Format

Curve *d* in figure 12.15 shows the normalized power spectra of Manchester format. Here most of power lies inside the bandwidth equal to $2/T_b$ which is twice the bandwidth of unipolar, polar and bipolar NRZ formats.

Table 12.3. Comparison of Various Line Codes

S. No.	Parameter of Comparison	Polar RZ	Polar NRZ	AMI	Manchester	Polar Quaternary NRZ
1.	Transmission of DC component	Yes	Yes	No	No	Possible
2.	Signalling rate	$1/T_b$	$1/T_b$	$1/T_b$	$1/T_b$	$1/2\ T_b$
3.	Noise immunity	Low	Low	High	High	High
4.	Synchronizing capability	Poor	Poor	Very good	Very good	Poor
5.	Bandwidth requirement	$1/T_b$	$1/2\ T_b$	$1/2\ T_b$	$1/T_b$	$1/2T_b$
6.	Crosstalk	High	High	Low	Low	Low

Example 12.1. Encode the following binary data stream into return to zero (RZ), Non-return to zero (NRZ), AMI and Manchester codes.

Data stream: 1 1 0 0 0 0 1 0

Solution: The line coding formats for the given data stream will be as shown in figure 12.16.

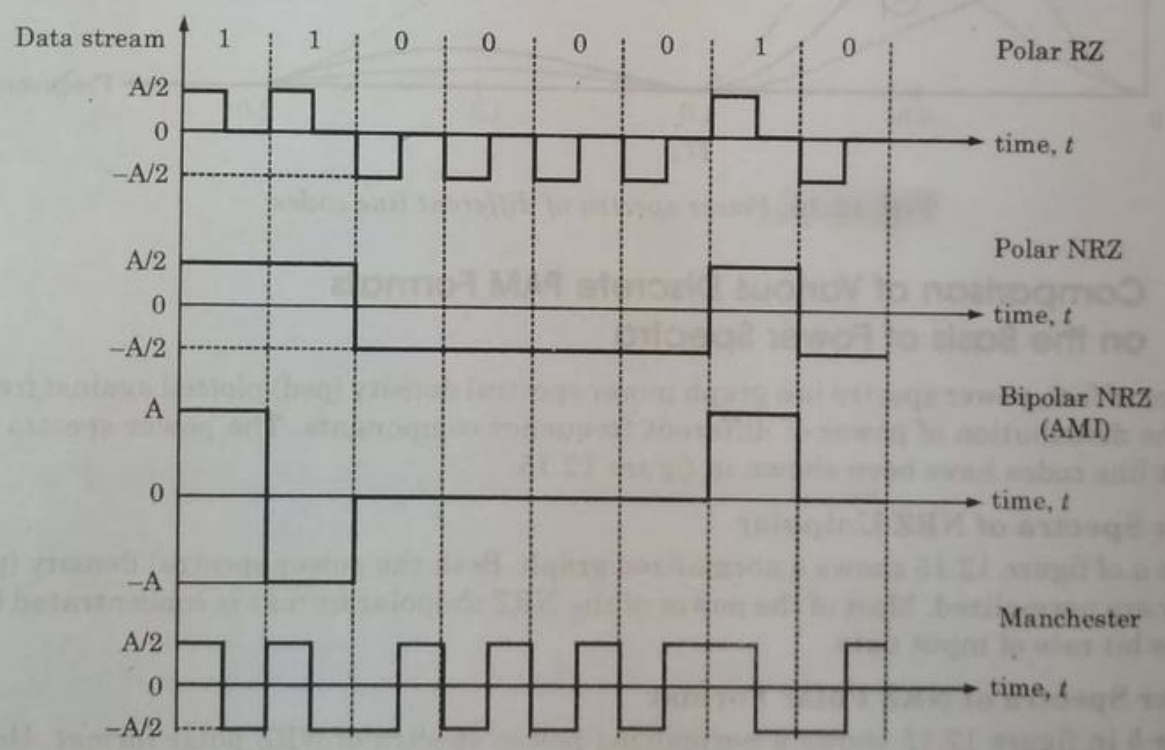


Fig. 12.16.

Example 12.2. Draw the following data formats for the bit stream 1 1 0 0 1 1 0:

(*i*) Polar NRZ

(*ii*) Unipolar RZ

(*iii*) AMI

(*iv*) Manchester

Scanned by CamScanner

Solution: The data formats will as shown in figure 12.17.

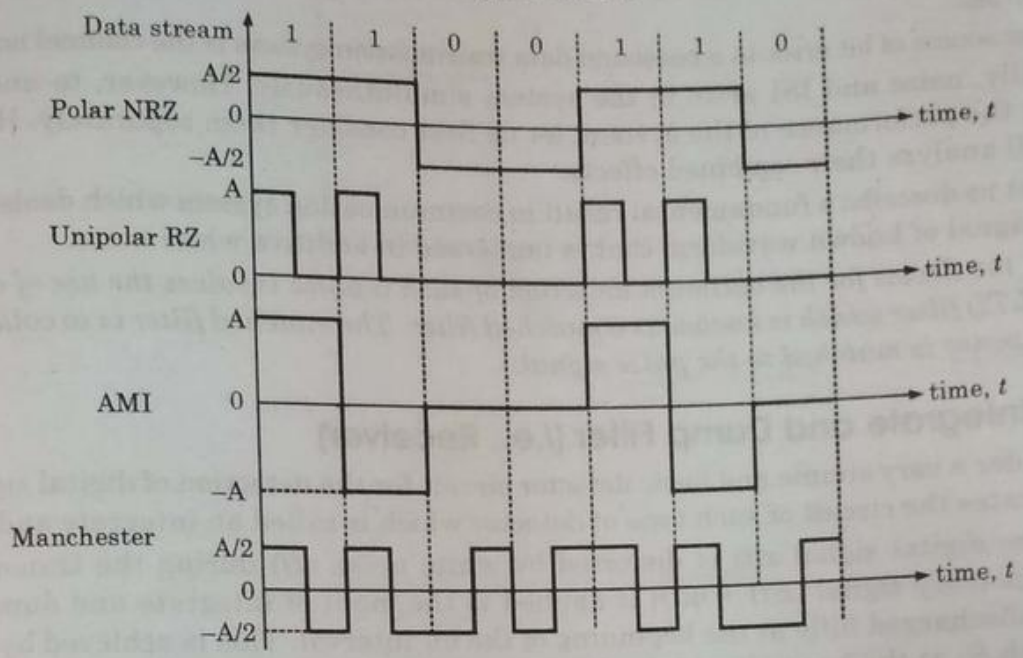


Fig. 12.17. *Various line codes for example 12.2.*

12.19. Difference between Source Coding and Line Coding

(*i*) We have already discussed various source coding techniques such as PCM, Delta modulation ADM etc.

(*ii*) Source coding technique are used in order to convert an analog signal to its digitally coded equivalent signal. The output of a source coder is a train of binary digits i.e., "0s" and "1s".

(*iii*) The line coding techniques convert the stream of binary digits into a format or code which is more suitable for transmission over a cable or any other medium.

(*iv*) This is because the data transmission must satisfy certain requirements such as synchronization, elimination of dc component, low cross talk etc.

(*v*) The line code formats should be such that the bandwidth requirements be as low as possible. Examples of line codes are RZ, NRZ, AMI, Manchester and other formats.

12.20. Introduction to Matched Filter and Intersymbol Interference (ISI)

As discussed earlier, there are various techniques for converting an analog information-bearing signal into digital form. There is another way in which digital data can rise in practice that is the data may represent the output of a source of information which is inherently discrete in nature (e.g., a digital computer).

As a matter of fact, digital data have a broad spectrum with a significant low frequency content. Therefore, baseband transmission of digital data requires the use of a low-pass channel with a bandwidth large enough to accommodate the essential frequency content of the data stream. However, usually the channel is dispersive in that its frequency response deviates from that of an ideal low-pass filter (LPF).

The result of data transmission over such a channel is that each received pulse is affected somewhat by adjacent pulses and hence giving rise to a common form of interference called **intersymbol interference (ISI)**.

Intersymbol interference (ISI) is a major source of bit errors in the reconstructed data stream

Scanned by CamScanner

at the receiver output. To correct for it, control has to be exercised over the pulse shape in the overall system.

Another source of bit error in a baseband data transmission system is the channel noise.

Basically, noise and ISI arise in the system simultaneously. However, to understand how they affect the performance of the system, let us first consider them separately. However, later on, we shall analyze their combined effects.

Now, let us describe a fundamental result in communication system which deals the detection of a pulse signal of known waveform that is immersed in additive white noise.

In fact, the device for the optimum detection of such a pulse involves the use of a linear-time-invariant (LTI) filter which is known as a matched filter. The matched filter is so called because its impulse response is matched to the pulse signal.

12.21. Integrate and Dump Filter (*i.e.*, Receiver)

Let us consider a very simple and basic detector circuit for the detection of digital signals. Figure 12.18 illustrates the circuit of such type of detector which is called an integrate and dump filter.

Here, the digital signal $x(t)$ is distorted by white noise $n(t)$ during the transmission over channel. This noisy signal $[x(t) + n(t)]$ is applied to the input of integrate and dump filter. The capacitor is discharged fully at the beginning of the bit interval. This is achieved by temporarily closing switch S_2 at the beginning of the bit interval. The integrator then integrates noisy input signal over one bit period. This integrated signal is shown as $y(t)$ in the figure. For the square pulse input, the output of the integrator would be a triangular pulse as shown in figure 12.19.

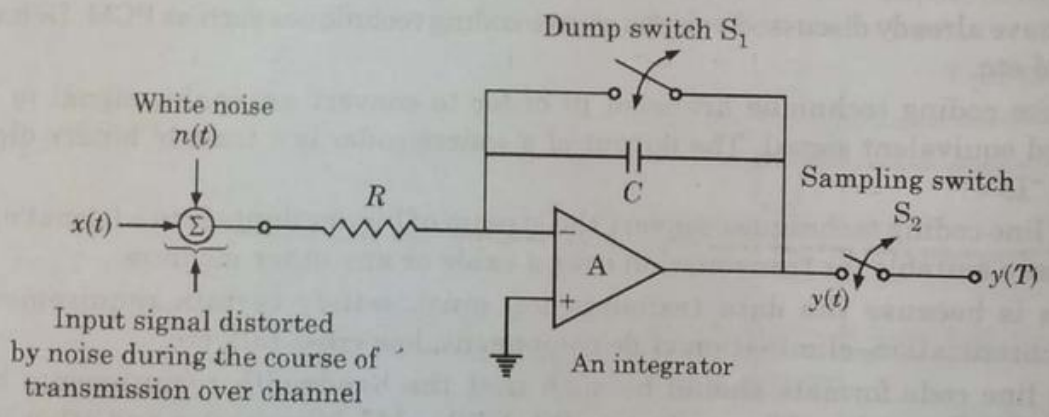


Fig. 12.18. *Integrate and dump filter.*

Figure 12.19(*b*) illustrates the waveform of $y(t)$. At the end of bit period *i.e.*, at $t = T$, the magnitude of $y(t)$ attains its maximum amplitude. Hence, the value of $y(t)$ is sampled at the end of bit period. It can be further proved that the signal to noise ratio is maximum at the end of bit period (*i.e.*, at $t = T$). Depending upon the value of $y(T)$, the decision is taken. The dump switch S_2 is then closed momentarily to discharge the capacitor to receive the next bit. Therefore, the integrator integrates (*i.e.* generates output) which is independent of the value of previous bit. This reveals the fact that the detection in integrate and dump filter is unaffected by the values of previous bits. In figure 12.19(*b*), the output of integrator would decrease after $t > T$.

Scanned by CamScanner

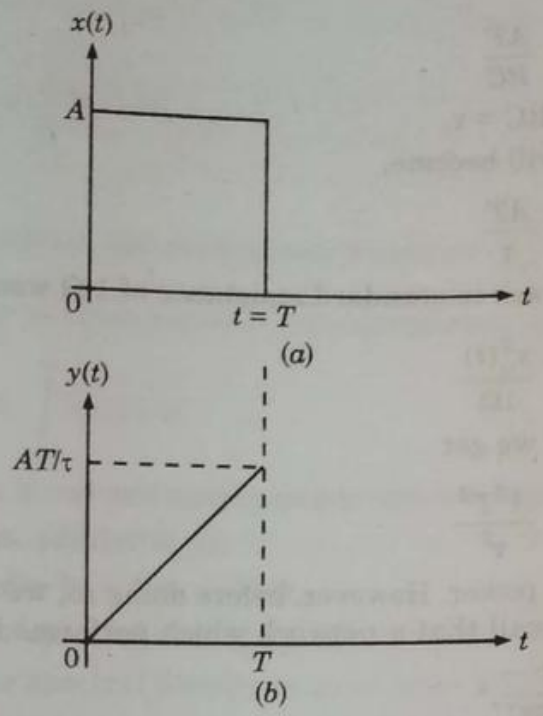


Fig. 12.19. *(a) Input pulse to the integrator (Here, we have assumed that the noise is absent). This pulse represents binary '1'. The width of pulse is 'T' (b) Output of integrator. The initial output is zero. At t = T, the output of integrator is y(t) = AT.*

12.21.1. Calculation of Signal to Noise Ratio for the Integrate and Dump Filter

We know that the output of the integrator may be written as,

$$y(t) = \frac{1}{RC}\int_0^T [x(t) + n(t)]\, dt \qquad ...(12.39)$$

Here, the integration is performed over one bit period *i.e.*, from 0 to T. The noisy signal $[x(t) + n(t)]$ is the input to the integrator. Equation (12.39) may be modified as

$$y(t) = \frac{1}{RC}\int_0^T x(t)\, dt + \frac{1}{RC}\int_0^T n(t)\, dt \qquad ...(12.40)$$

or $$y(t) = x_0(t) + n_0(t)$$

where $x_0(t)$ is the output signal voltage and $n_0(t)$ is the output noise voltage.

Now, let us consider output signal voltage which is

$$x_0(t) = \frac{1}{RC}\int_0^T x(t)\, dt \qquad ...(12.41)$$

Because the value of $x(t) = A$ form 0 to T, the above equation can be written as,

$$x_0(t) = \frac{1}{RC}\int_0^T A\, dt = \frac{A}{RC}\int_0^T 1 \cdot dt$$

Integrating, we get

$$x(t) = \frac{A}{RC}[t]_0^T$$

Scanned by CamScanner

or $$x_0(t) = \frac{AT}{RC} \quad ...(12.42)$$

Let the time constant be RC = τ.

Hence, equation (12.42) will become,

$$x_0(t) = \frac{AT}{\tau} \quad ...(12.43)$$

The normalized signal power in standard resistance of 1 Ω would be,

$$\text{Output signal power} = \frac{x_0^2(t)}{1\Omega}$$

Putting the value of $x_0(t)$, we get

$$\text{Output signal power} = \frac{A^2T^2}{\tau^2} \quad ...(12.44)$$

Now, let us find the noise power. However, before doing so, we have to calculate the transfer function of the integrator. Recall that a network which performs integration operation has the transfer function equal to $\frac{1}{j\omega RC}$.*

Now, since a delay of $t = T$ in time domain is equivalent to $e^{-j\omega T}$ in frequency domain, therefore, the network performing integration over the period of T can be represented by the transfer function as under:

$$H(f) = \frac{1-e^{-j\omega T}}{j\omega RC} \quad ...(12.45)$$

For $\omega = 2\pi f$ and $RC = \tau$, equation (12.45) takes the form

$$H(f) = \frac{1-e^{-j2\pi fT}}{j2\pi f\tau}$$

Simplifying, we get

or $$H(f) = \frac{1-[\cos(2\pi fT) - j\sin(2\pi fT)]^{**}}{j2\pi f\tau}$$

Separating the real and imaginary parts, we have

$$H(f) = \frac{\sin(2\pi fT)}{2\pi f\tau} - j\frac{1-\cos(2\pi fT)}{2\pi f\tau}$$

Then, the magnitude of above transfer function will become

$$|H(f)|^2 = \frac{\sin^2(2\pi fT) + 1 - 2\cos(2\pi fT) + \cos^2(2\pi fT)}{(2\pi f\tau)^2}$$

Simplifying the last equation, we obtain,

$$|H(f)|^2 = \frac{\sin^2(\pi fT)}{(\pi f\tau)^2} \quad ...(12.46)$$

Now, the average power of the output noise signal $n_0(t)$ may be obtained by integrating its power density spectrum.

---

* In fact, this can be very easily verified by taking the example of an RC circuit. The reason is that an RC low pass filter also basically performs integration operation.

** Using Euler's identity $e^{j\theta} = \cos\theta + j\sin\theta$.

Scanned by CamScanner

This means that

Power, $$P = \int_{-\infty}^{\infty} S(f)\, df$$

For a standard 1 Ω resistance, the noise power would be $\frac{\overline{n_0^2(t)}}{1\,\Omega} = \overline{n_0^2(t)}$.

Here, mean square value of noise signal is considered since it is a random signal. *i.e.*,

Noise power, $$\overline{n_0^2(t)} = \int_{-\infty}^{\infty} S_{no}(f)\, df \qquad ...(12.47)$$

Again, we know that the input and output power spectral densities are related as,

$$S_{no}(f) = |H(f)|^2 S_{ni}(f) \qquad ...(12.48)$$

where $H(f)$ = transfer function of filter,

$S_{no}(f)$ = power spectral density (psd) of output noise and

$S_{ni}(f)$ *i* = power spectral density (psd) of input noise.

Again, let us assume that white noise is present, therefore, the power spectral density (psd) of this noise would be,

$$S_{ni}(f) = \frac{N_0}{2} \qquad ...(12.49)$$

Substituting this value in equation (12.48), we get

$$S_{no}(f) = |H(f)|^2 \cdot \frac{N_0}{2}$$

Substituting this value in equation (12.49), we have

$$\overline{n_0^2(t)} = \int_{-\infty}^{\infty} |H(f)|^2 \cdot \frac{N_0}{2}\, df$$

Substituting the value of $|H(f)|^2$ from equation (16.48) in above equation, we obtain

$$\overline{n_0^2(t)} = \int_{-\infty}^{\infty} \frac{\sin^2(\pi f T)}{(\pi f \tau)^2} \cdot \frac{N_0}{2}\, df = \frac{N_0}{2} \int_{-\infty}^{\infty} \frac{\sin^2(\pi f T)}{(\pi f \tau)^2}\, df \qquad ...(12.50)$$

Now, let us substitute $\pi f \tau = x$

so that $$dx = \pi\tau\, df$$

or $$df = \frac{1}{\pi\tau}\, dx$$

Further, since $$f = \frac{x}{\pi\tau}$$

therefore, $$\pi f T = \frac{xT}{\tau}$$

With all these substitutions, equation (12.51) becomes,

$$\overline{n_0^2(t)} = \frac{N_0}{2} \int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{xT}{\tau}\right)}{x^2} \cdot \frac{1}{\pi\tau}\, dx$$

Scanned by CamScanner

Rearranging above equation, we get

$$\overline{n_0^2(t)} = \frac{N_0}{2}\int_{-\infty}^{\infty}\frac{\sin^2\left(\frac{xT}{\tau}\right)}{\left(\frac{xT}{\tau}\right)}\cdot\left(\frac{T}{\tau}\right)^2\cdot\frac{1}{\pi\tau}\,dx$$

$$= \frac{N_0}{2}\cdot\frac{T^2}{\pi\tau^3}\int_{-\infty}^{\infty}\frac{\sin^2\left(\frac{xT}{\tau}\right)}{\left(\frac{xT}{\tau}\right)}\cdot dx$$

Now, let $\quad \frac{xT}{\tau} = u$

so that $dx = \frac{\tau}{T}\,du$ and limits will remain unchanged, therefore above equation will become,

$$\overline{n_0^2(t)} = \frac{N_0}{2}\cdot\frac{T^2}{\pi\tau^3}\int_{-\infty}^{\infty}\frac{\sin^2 u}{u^2}\cdot\frac{\tau}{T}\,du = \frac{N_0T}{2\pi\tau^2}\int_{-\infty}^{\infty}\left(\frac{\sin u}{u}\right)^2 du$$

Because, here the function $\frac{\sin u}{u}$ is squared, the above equation can be written as,

$$\overline{n_0^2(t)} = \frac{N_0T}{2\pi\tau^2}\cdot 2\int_0^{\infty}\left(\frac{\sin u}{u}\right)^2 du = \frac{N_0T}{2\pi\tau^2}\cdot 2\cdot\frac{\pi}{2} = \frac{N_0T}{2\tau^2} \qquad ...(12.51)$$

This relation describes the noise power at the output. The signal to noise power ratio at output of integrator can be obtained as,

Signal to noise ratio, $\left(\frac{S}{N}\right)_0 = \frac{\text{Signal power}}{\text{Noise power}}$

Substituting the values of signal power from equation (12.44) and noise power from equation (12.51), we get

$$\left(\frac{S}{N}\right)_0 = \frac{\frac{A^2T^2}{\tau^2}}{\frac{N_0T}{2\tau^2}} = \frac{2A^2T}{N_0} = \frac{A^2T}{N_0/2}$$

Hence, we conclude

Signal to Noise ratio of Integrate and Dump receiver:

$$\left(\frac{S}{N}\right)_0 = \frac{A^2T}{N_0/2} \qquad ...(12.52)$$

This signal to noise ratio is also known as **figure of Merit.**

Few points about Signal to noise ratio:

(*i*) The result in equation (12.52) shows that signal to noise ratio improves in proportion to sampling period T. It is also increased as signal amplitude 'A' is increased.

(*ii*) Because noise has gaussian distribution and zero mean value at any time, therefore, the output of integrator also increases by a very small margin at the end of bit interval.

Figure 12.20 shows the waveform of $x_0(t)$ and $n_0(t)$ for the input signal $x(t)$.

Figure 12.20 shows the waveforms of integrate and dump filter receiver for the rectangular pulses input.

Scanned by CamScanner

It may observed that the output signal voltage reaches to the value of $\pm \frac{AT}{\tau}$ at the sampling instant. This is the maximum signal voltage. But the noise voltage $n_0(t)$ does not increase in same proportion. This is due to the fact that the noise has zero average value and Gaussian distribution.

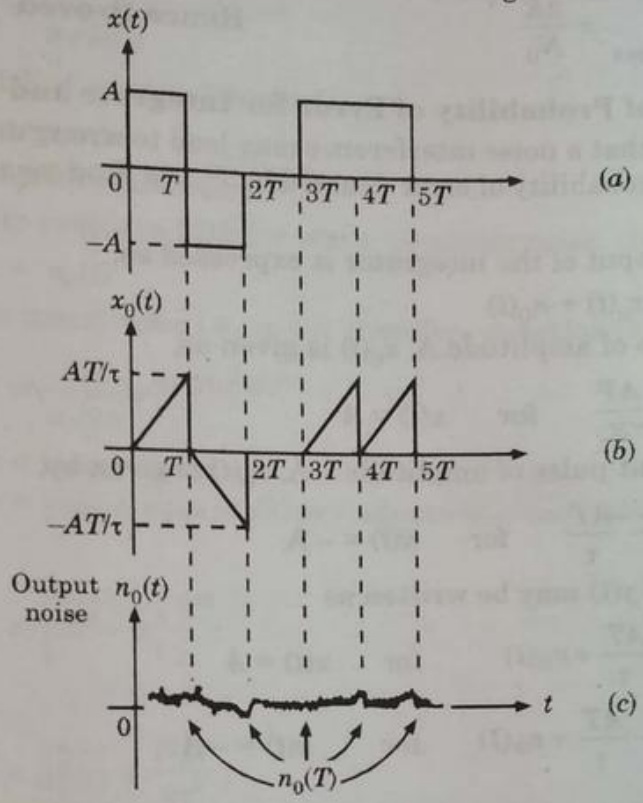


Fig. 12.20. *(a) Input signal to the integrate and dump filter (b) Output signal of the integrate and dump filter (c) Output noise of the integrate and dump filter.*

Example 12.3. Show that for an integrate and dump filter receiver, the maximum signal to noise ratio is expressed as,

$$\left(\frac{S}{N}\right)_0 = \frac{2E}{N_0}$$

Given that the input signal $x(t)$ is rectangular pulses of amplitudes $\pm A$ and duration T.

Solution: Recall that the energy of the signal $x(t)$ is given by the relation

$$E = \int_{-\infty}^{\infty} x^2(t)\, dt \qquad ...(i)$$

Given that $x(t)$ is a rectangular pulse of amplitude $\pm A$ and duration, T_1. Therefore, equation (*i*) can be written as

$$E = \int_0^T (\pm A)^2\, dt = A^2 \int_0^T dt = A^2 [t]_0^T = A^2\, T \qquad ...(ii)$$

We know that the signal to noise ratio of integrate and dump filter receiver is given as,

$$\left(\frac{S}{N}\right)_0 = \frac{A^2 T}{N_0 / 2} \qquad ...(iii)$$

Substituting, $A^2T = E$ from equation (*ii*) in equation (*iii*), we get

Scanned by CamScanner

$$\left(\frac{S}{N}\right)_0 = \frac{E}{N_0/2} = \frac{2E}{N_0}$$

Since output is maximum at sampling instant, T, therefore, the above value is maximum *i.e.*,

$$\left(\frac{S}{N}\right)_{0\,\max} = \frac{2E}{N_0}$$ **Hence Proved**

12.21.2. Computation of Probability of Error for Integrate and Dump Filter Receiver

As discussed earlier that a noise interference may lead to wrong decision at the receiver end. As a matter of fact, the probability of error denoted by P_e is a good measure of performance of the detector.

We know that the output of the integrator is expressed as,

$$y(t) = x_0(t) + n_0(t)$$

For the positive pulse of amplitude A, $x_0(t)$ is given as,

$$x_0(t) = \frac{AT}{\tau} \quad \text{for} \quad x(t) = A$$

Similarly, for the input pulse of amplitude – A, $x_0(t)$ is given by,

$$x_0(t) = -\frac{AT}{\tau} \quad \text{for} \quad x(t) = -A$$

Therefore, the output $y(t)$ may be written as

$$y(t) = \frac{AT}{\tau} + n_0(t) \quad \text{for} \quad x(t) = A \qquad \text{...(12.53)}$$

Similarly, $$y(t) = -\frac{AT}{\tau} + n_0(t) \quad \text{for} \quad x(t) = -A \qquad \text{...(12.54)}$$

Let us consider that $x(t) = -A$. Further, if noise $n_0(t)$ is greater than $\frac{AT}{\tau}$, then the output $y(t)$ would be positive according to equation (12.54). After that the receiver will decide in favour of symbol + A, which is wrong decision. This means that an error is introduced.

Similarly, let us consider that $x(t) = +A$. If noise $n_0(t) > -\frac{AT}{\tau}$ then, the output $y(t)$ will be negative according to equation (12.53). This leads to decision in favour of $-A$, which is erroneous. Based on the above discussion, we can make conclusions about probability of error in the form of a Table 12.4.

Table 12.4. Probability of error in integrate and dump filter receiver

S. No.	*Input x(t)*	*Value of $n_0(t)$ for error in the output*	*The probability of error P_e*
1.	$-A$	An error will be introduced if $n_0(t) > \frac{AT}{\tau}$	In this case, the probability of error may be obtained by calculating probability that $n_0(t) > \frac{AT}{\tau}$
2.	$+A$	An error will be introduced if $n_0(t) < -\frac{AT}{\tau}$	In this case, the probability of error may be obtained by calculating the probability that, $n_0(t) < -\frac{AT}{\tau}$

Scanned by CamScanner

As shown in Table 12.4, the error will be introduced depending upon probability that $n_0(t)$ takes a particular value. These probabilities may be obtained from PDF of $n_0(t)$. Recall that the probability density function (PDF) of the Gaussian distributed function is expressed by standard relation as under:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2} \quad ...(12.55)$$

where, $f_X(x)$ = the PDF of random function x.

m = the mean value

and σ = the standard deviation.

Here, because we have to evaluate PDF for white Gaussian noise, therefore, we have

$$x = n_0(t)$$

Since this noise has zero mean value *i.e.*, $m = 0$, therefore equation (12.55) may be written as,

$$f_X\{n_0(t)\} = \frac{1}{\sigma\sqrt{2\pi}} e^{-[n_0(t)]^2/2\sigma^2} \quad ...(12.56)$$

The standard deviation σ is expressed as,

$$\sigma = [\text{mean square value} - \text{square of mean value}]^{1/2}$$

This means that

$$\sigma_x = \left[\overline{X^2} - m_x^2\right]^{1/2}$$

Further, we have

mean square value $\overline{X^2} = \overline{n_0^2(t)} = \dfrac{N_0 T}{2\tau^2}$

Also, mean value $m_x = 0$ for this noise.

Therefore, we have

$$\sigma = \left[\overline{n_0^2(t)}\right]^{1/2} = \sqrt{\frac{N_0 T}{2\tau^2}} \quad ...(12.57)$$

Hence equation (16.56), can be written as,

$$f_X\{n_0(t)\} = \frac{1}{\sqrt{\dfrac{N_0 T}{2\tau^2}}\sqrt{2\pi}} \cdot e^{-[n_0(t)]^2 \Big/ 2\left(\frac{N_0 T}{2\tau^2}\right)}$$

At this stage, it may be noted that $n_0(t)$ is the function like 'x'. It is a random variable and we are evaluating its PDF.

On simplifying above equation, we get

$$f_X\{n_0(t)\} = \frac{\tau}{\sqrt{\pi N_0 T}} e^{-[n_0(t)]^2 \Big/ \left(\frac{N_0 T}{\tau^2}\right)} \quad ...(12.58)$$

This equation describes PDF of white Gaussian noise.

Figure 12.21 shows the graphical representation of this PDF.

From the property of PDF, we know that,

$$P\left[n_0(t) > \frac{AT}{\tau}\right] = \int_{\frac{AT}{\tau}}^{\infty} f_X[n_0(t)]\, d[n_0(t)] \quad ...(12.59)$$

Scanned by CamScanner

This equation gives the probability that $n_0(t)$ takes values greater than $\frac{AT}{\tau}$.

The above integration gives the area under the curve from $\frac{AT}{\tau}$ onwards. It has been illustrated by shaded region in figure. Similarly, the probability that $n_0(t)$ attains value less than $\frac{AT}{\tau}$ is given by area under the curve from $-\frac{AT}{\tau}$ onwards on left side. This portion if has also been shown shaded in figure 12.23.

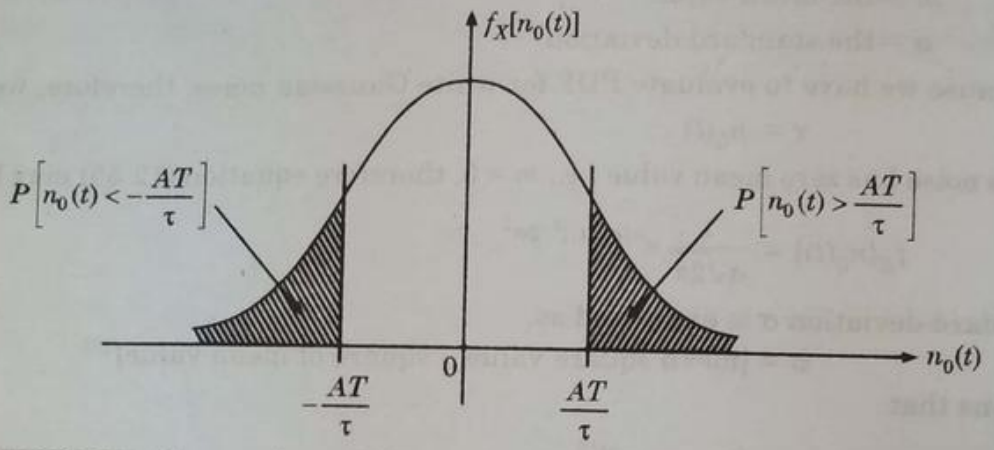


Fig. 12.21. *Illustration of PDF of white Gaussian noise having zero mean.*

Since the PDF curve is symmetric, therefore, we can write.

$$P\left[n_0(t) > \frac{AT}{\tau}\right] = P\left[n_0(t) < -\frac{AT}{\tau}\right] \quad ...(12.60)$$

We know that the above probabilities represent error probability. Because, occurrence of $-A$ or $+A$ is mutually exclusive, therefore, the probability of error is given by either of the two in above equation *i.e.*,

$$P_e = P\left[n_0(t) > \frac{AT}{\tau}\right] = P\left[n_0(t) < -\frac{AT}{\tau}\right]$$

Substituting value of $P\left[n_0(t) > \frac{AT}{\tau}\right]$ in above equation from equation (12.59), we get

$$P_e = P\left[n_0(t) > \frac{AT}{\tau}\right] = \int_{\frac{AT}{\tau}}^{\infty} f_X[n_0(t)]\, d\,[n_0(t)]$$

Again, substituting value of $f_X[n_0(t)]$ from equation (12.58), in above equation, we get

$$P_e = \int_{\frac{AT}{\tau}}^{\infty} \frac{\tau}{\sqrt{\pi N_0 T}} e^{-[n_0(t)]^2} \Big/ \left(\frac{N_0 T}{\tau^2}\right) d[n_0(t)] \quad ...(12.61)$$

Now, let us put

$$\frac{[n_0(t)]^2}{\frac{N_0 T}{\tau^2}} = y^2$$

So that $$\frac{n_0(t)}{\sqrt{N_0 T / \tau}} = y$$

Scanned by CamScanner

Thus, $$d[n_0(t)] = \frac{\sqrt{N_0 T}}{\tau} dy$$

when $$n_0(t) \to \infty, \text{ then } y \to \infty$$

when $$n_0(t) = \frac{AT}{\tau}$$

then $$y = \frac{AT/\tau}{\sqrt{N_0 T}/\tau} = \sqrt{\frac{A^2 T}{N_0}}$$

With all these substitutions, equation (16.61) takes the form

$$P_e = \int_{\sqrt{\frac{A^2T}{N_0}}}^{\infty} \frac{\tau}{\sqrt{\pi N_0 T}} e^{-y^2 \frac{\sqrt{N_0 T}}{\tau}} dy = \frac{1}{\sqrt{\pi}} \int_{\sqrt{\frac{A^2T}{N_0}}}^{\infty} e^{-y^2} dy$$

This equation can be rearranged as under:

$$P_e = \frac{1}{2} \cdot \left[\frac{2}{\sqrt{\pi}} \int_{\sqrt{\frac{A^2T}{N_0}}}^{\infty} e^{-y^2} dy \right] \quad \text{...(12.62)}$$

The integration inside brackets may be evaluated with the help of **complementary error function** *i.e.*,

$$\frac{2}{\sqrt{\pi}} \int_u^{\infty} e^{-y^2} dy = erfc\,(u)$$

Recall that this is a standard result and generally evaluated using numerical methods. Thus, with the help of this definition, equation (12.62) becomes,

$$P_e = \frac{1}{2} erfc \sqrt{\frac{A^2 T}{N_0}} \quad \text{...(12.63)}$$

This equation describes the probability of error P_e of the integrate and dump filter receiver.

Now, since $A^2T = E$, *i.e.*, energy of the bit, therefore, we have

$$P_e = \frac{1}{2} erfc \sqrt{\frac{E}{N_0}} \quad \text{...(12.64)}$$

Note: *It may be noted that basically 'erfc' is the monotonically decreasing function. Therefore P_e falls rapidly as the ratio $\frac{E}{N_0}$ increases. Hence, the maximum value of P_e is $\frac{1}{2}$ when $\frac{E}{N_0}$ is very very small. This means that even if the signal is lost entirely in the noise even then, the probability of error P_e will be $\frac{1}{2}$. Thus, the receiver would make incorrect decisions for half number of times.*

Scanned by CamScanner

12.22. The Optimum Filter (*i.e.* Optimum Receiver)

In the previous article, we discussed Integrate and Dump receiver. But, at this point, let us think whether Integrate and dump filter is an optimum* filter for the purpose of minimizing the probability of errors P_e. For this purpose, we shall discuss a generalized filter to receive binary coded signals. It is known as **optimum filter**. Therefore, let us consider the generalized Gaussian noise which is having zero mean. In the last section, we assumed 'white' Gaussian noise of zero mean.

Let the received signal be a binary waveform. Let us again assume that the polar NRZ signal is used to represent binary 1's and 0's. *i.e.*,

for binary '1',

$$x_1(t) = +A \quad \text{for one bit period } T$$

and for binary '0',

$$x_2(t) = -A \quad \text{for one bit period T}$$

Hence, the input signal $x(t)$ will be either $x_1(t)$ or $x_2(t)$ depending upon the polarity of the NRZ signal. Figure 12.22 shows the block diagram of a receiver for such a binary coded signal.

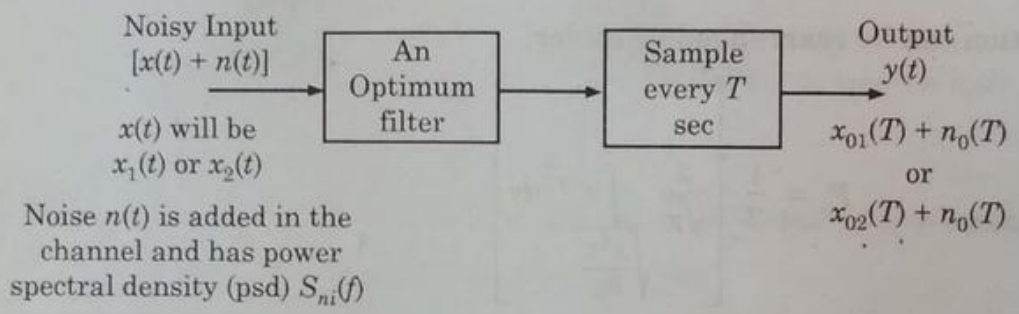


Fig. 12.22. *Block diagram of receiver for binary coded signal.*

As shown in this figure, the noise $n(t)$ is added to the signal $x(t)$ in the channel during the transmission.

Thus, input to the optimum filter is $[x(t) + n(t)]$ *i.e.*,

Input to the receiver $= x(t) + n(t)$

and Output from the receiver $= x_{01}(T) + n_0(T)$ or $x_{02}(T) + n_0(T)$

Also, in the absence of the noise $n(t)$, the output of the receiver will be,

$$y(T) = x_{01}(T) \quad \text{if} \quad x(t) = x_1(t)$$

and

$$y(T) = x_{02}(T) \quad \text{if} \quad x(t) = x_2(t)$$

Hence, in the absence of noise, decisions are taken clearly. However, if noise is present then, we select $x_1(t)$ if $y(T)$ is closer to $x_{01}(T)$ than $x_{02}(T)$ and we select $x_2(t)$ if $y(T)$ is closer to $x_{02}(T)$ than $x_{01}(T)$.

Therefore, the decision boundary will be in the middle of $x_{01}(T)$ and $x_{02}(T)$.

It is expressed as,

$$\text{Decision boundary} = \frac{x_{01}(T) + x_{02}(T)}{2} \qquad \text{...(12.65)}$$

12.22.1. Calculation of Probability of Error (P_e) for Optimum Filter

The probability of error (P_e) may be evaluated on the same basis as in the last section. Here, we shall consider generalized Gaussian noise. Let us consider that $x_2(t)$ was transmitted, but $x_{01}(T)$ is greater than $x_{02}(T)$. If noise $n_0(T)$ is positive and larger in magnitude compared to the voltage difference $\frac{1}{2}[x_{01}(T) + x_{02}(T)] - x_{02}(T)$, then, in this case, the incorrect decision will be taken. This means that the error will be generated if,

Scanned by CamScanner

$$n_0(T) \geq \frac{x_{01}(T) + x_{02}(T)}{2} - x_{02}(T) \geq \frac{x_{01}(T) - x_{02}(T)}{2} \quad \text{...(12.66)}$$

We have obtained the probability density function (PDF) for $n_0(t)$ in the previous section. It is given as

$$f_X[n_0(t)] = \frac{1}{\sigma\sqrt{2\pi}} e^{-[n_0(t)]^2/2\sigma^2} \quad \text{....(12.67)}$$

Here, $n_0(t)$ is the random function whose PDF is given by above equation. Also, σ is its standard deviation and the function has zero mean value.

Hence, to evaluate the probability of error, we must integrate the area under the PDF curve from $n_0(T) \geq \frac{x_{01}(T) - x_{02}(T)}{2}$. This portion of the curve has been shown shaded in figure 12.23.

Therefore, we have

$$P_e = P\left[n_0(T) \geq \frac{x_{01}(T) - x_{02}(T)}{2}\right] = \int_{\frac{x_{01}(T) - x_{02}(T)}{2}}^{\infty} f_X[n_0(t)]\, d[n_0(t)]$$

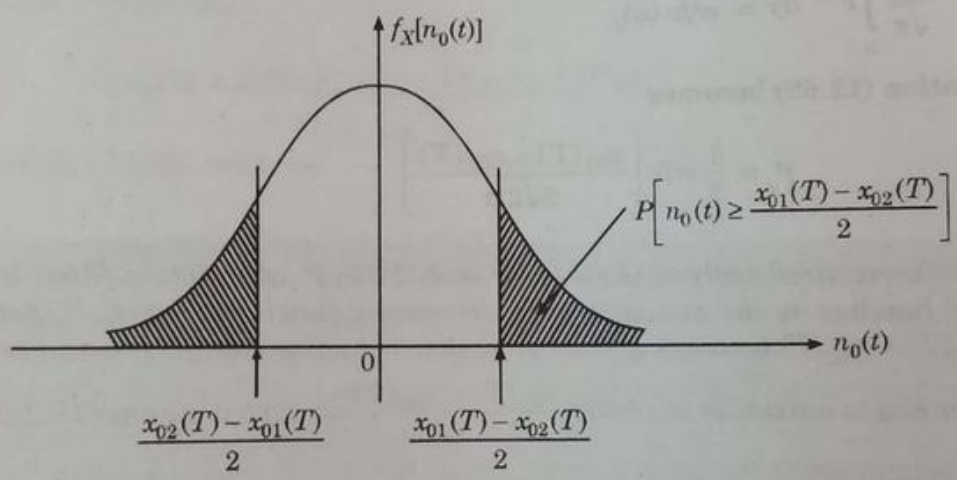


Fig. 12.23. *Evaluation of P_e for optimum filter receiver.*

Using equation (12.67), the last equation becomes,

$$P_e = \int_{\frac{x_{01}(T) - x_{02}(T)}{2}}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-[n_0(t)]^2/2\sigma^2} \quad \text{...(12.68)}$$

Let us substitute $\frac{[n_0(t)]^2}{2\sigma^2} = y^2$

so that $n_0(t) = \sigma\sqrt{2}\, y$

or $d\,[n_0(t)] = \sigma\sqrt{2}\, dy$

when $n_0(t) = \infty$, then $y = \infty$

and when $n_o(t) = \frac{x_{01}(T) - x_{02}(T)}{2}$

Scanned by CamScanner

then, we have $$y = \frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}$$

Substituting all these values in equation (12.30), we obtain,

$$P_e = \int_{\frac{x_{01}(T)-x_{02}(T)}{2\sqrt{2}\,\sigma}}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-y^2} \cdot \sigma\sqrt{2}\,dy = \frac{1}{\sqrt{\pi}} \int_{\frac{x_{01}(T)-x_{02}(T)}{2\sqrt{2}\,\sigma}}^{\infty} e^{-y^2} dy$$

Let us rearrange this equation as under:

$$P_e = \frac{1}{2}\left[\frac{2}{\sqrt{\pi}} \int_{\frac{x_{01}(T)-x_{02}(T)}{2\sqrt{2}\,\sigma}}^{\infty} e^{-y^2} dy\right] \quad \text{...(12.69)}$$

To solve this integration, let us use the following standard result.

$$\frac{2}{\sqrt{\pi}} \int_{u}^{\infty} e^{-y^2} dy = erfc\,(u)$$

Then, equation (12.69) becomes

$$P_e = \frac{1}{2}\, erfc\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}\right] \quad \text{...(12.70)}$$

Note: *This is the required expression for error probability P_e of optimum filter. It may be noted that the 'erfc' function is the monotonically decreasing function. Hence, P_e decreases as the difference $x_{01}(T) - x_{02}(T)$ becomes greater and the rms noise voltage σ becomes smaller. The optimum filter has to maximize the ratio $\frac{x_{01}(T) - x_{02}(T)}{\sigma}$ in such a manner that the probability of error P_e is maximum.*

12.22.2. Evaluation of Transfer Function for the Optimum Filter

Now, let us find the transfer function of the optimum filter in such a way that it will maximize the ratio $\frac{x_{01}(T) - x_{02}(T)}{\sigma}$.

For this, let us represent the difference signal $x_{01}(T) - x_{02}(T)$ by $x_0(T)$.
Thus, we have

$$x_0(T) = x_{01}(T) - x_{02}(T) \quad \text{...(12.71)}$$

This means that the optimum filter has to maximize the ratio $\frac{x_0(T)}{\sigma}$.

Now, let us derive the transfer function of the optimum filter such that the square of the ratio $\frac{x_0(T)}{\sigma}$ is maximized.

This square of the ratio is,

$$\left(\frac{S}{N}\right)_0 = \frac{x_0^2(T)}{\sigma^2}$$

Scanned by CamScanner

where $\left(\frac{S}{N}\right)_0$ is known as the signal to noise power ratio. Thus, we are maximizing square of $\left[\frac{x_0(T)}{\sigma}\right]$ for mathematical convenience. In this equation, $x_0^2(T)$ is the normalized signal power in 1 Ω load.

Further, $\sigma^2 = \overline{n_0^2(T)} = E[n_0{}^2(T)]$ is normalized noise power because mean value of noise is zero.*

Thus, the optimum filter has to maximize the ratio

$$\left(\frac{S}{N}\right)_0 = \frac{x_0^2(T)}{\overline{n_0^2(T)}} \text{ or } \frac{x_0^2(T)}{E[n_0^2(T)]} \text{ or } \frac{x_0^2(T)}{\sigma^2} \quad ...(12.72)$$

Again, let $X_0(f)$ be Furrier transform of $x_0(t)$. If $X(f)$ is the Furrier transform of input difference signal $x(t)$ [*i.e.*, $x(t) = x_1(t) - x_2(t)$], then we have,

$$X_0(f) = H(f)\, X(f) \quad ...(12.73)$$

Here, $H(f)$ is the transfer function of optimum filter. Also, $x_0(T)$ is found by taking inverse Fourier transform of $X_0(f)$ *i.e.*,

$$x_0(T) = IFT\,[X_0(f)] = \int_{-\infty}^{\infty} X_0(f)\, e^{j2\pi fT} df$$

Using equation (12.35), we have

$$x_0(T) = \int_{-\infty}^{\infty} H(f)\, X(f)\, e^{j2\pi fT}\, df \quad ...(12.74)$$

Also, the input noise to the optimum filter is $n(t)$. Let, its power spectral density (psd) be $S_{ni}(f)$. The output noise of optimum filter is $n_0(t)$. Let its power spectral density be $S_{n0}(f)$.

We know that the input and output power spectral densities of noise are related as,

$$S_{n0}(f) = |H(f)|^2\, S_{ni}(f) \quad ...(12.75)$$

The normalized noise power can be obtained by integrating the power spectral density. Thus, we have

$$\sigma^2 = \int_{-\infty}^{\infty} S_{n0}(f)\, df * = \int_{-\infty}^{\infty} |H(f)|^2\, S_{ni}(f)\, df \quad ...(12.76)$$

Substituting for $x_0(T)$ from equation (12.73) and σ^2 from equation (12.75) in equation (12.72) we get,

$$\left(\frac{S}{N}\right)_0 = \frac{x_0^2(T)}{\sigma^2} = \frac{\left|\int_{-\infty}^{\infty} H(f)\, X(f)\, e^{j2\pi fT} df\right|^2}{\int_{-\infty}^{\infty} |H(f)|^2\, S_{ni}(f)\, df} \quad ...(12.77)$$

Here, let us use the well known Schwarz's inequality which states that

$$\left|\int_{-\infty}^{\infty} \theta_1(x)\, \theta_2(x)\, dx\right|^2 \le \int_{-\infty}^{\infty} |\theta_1(x)|^2\, dx \int_{-\infty}^{\infty} |\theta_2(x)|^2\, dx \quad ...(12.78)$$

* $\because \sigma^2 = \overline{n_0^2(T)} - (\text{mean value})^2$.

Scanned by CamScanner

Let $\theta_1(f) = \sqrt{S_{ni}(f)}\, H(f)$

and $\theta_2(f) = \dfrac{1}{\sqrt{S_{ni}(f)}} X(f) e^{j2\pi fT}$

With all these substitutions, the equation (12.76) becomes,

$$\left(\frac{S}{N}\right)_0 = \frac{\left|\int_{-\infty}^{\infty} \theta_1(f) \cdot \theta_2(f)\, df\right|^2}{\int_{-\infty}^{\infty} |\theta_1(f)|^2\, df}$$

Applying Schwarz's inequality of equation (12.77) to the numerator of above equation, we have

$$\left(\frac{S}{N}\right)_0 \le \frac{\int_{-\infty}^{\infty} |\theta_1(f)|^2\, df \int_{-\infty}^{\infty} |\theta_2(f)|^2\, df}{\int_{-\infty}^{\infty} |\theta(f)|^2\, df}$$

$$\le \int_{-\infty}^{\infty} |\theta_2(f)|^2\, df \le \int_{-\infty}^{\infty} \frac{1}{S_{ni}(f)} |X(f) e^{j2\pi fT}|^2\, df$$

Now, substituting value of $\theta_2(f)$ in the above equation, we have

$$\left|X(f) e^{j2\pi fT}\right|^2 = |X(f)|^2 \qquad [\because |e^{j2\pi fT}| = 1]$$

Thus, we have

$$\left(\frac{S}{N}\right)_0 \le \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)}\, df \qquad \text{...(12.78a)}$$

The signal to noise power ratio expressed by equation (12.78) will be maximum when we consider equality.

In other words, we must have

$$\left(\frac{S}{N}\right)_{0\,\max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)}\, df$$

This is possible only when equality applies in Schwarz's inequality *i.e.*, in equation (12.77),

$$\left|\int_{-\infty}^{\infty} \theta_1(x)\, \theta_2(x)\, dx\right|^2 = \int_{-\infty}^{\infty} |\theta_1(x)|^2\, dx \int_{-\infty}^{\infty} |\theta_2(x)|^2\, dx$$

This equality is possible only if we have $\theta_1(f) = k\theta^*_2(f)$

Substituting the values of $\theta_1(f)$ and $\theta_2(f)$, we obtain,

$$\sqrt{S_{ni}(f)}\, H(f) = k \cdot \frac{1}{\sqrt{S_{ni}(f)}} X^*(f)\, e^{-j2\pi fT}$$

* By property of psd.

Scanned by CamScanner

Therefore, when this equation is satisfied, the signal to noise ratio $\left(\frac{S}{N}\right)_0$ will be maximum. In this case, the transfer function $H(f)$ can be obtained from above equation as under:

$$H(f) = k \cdot \frac{X^*(f)}{S_{ni}(f)} e^{-j2\pi fT}$$

It may be noted that at the beginning of this derivation, we stated that

$$\left(\frac{S}{N}\right)_0 = \frac{x_0^2(T)}{\sigma^2} = \left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2$$

Now we can have the following conclusion:

The optimum filter which minimizes the probability of error (P_e) has to maximize the ratio $\left[\frac{x_{01}(T) - x_{02}(T)}{s}\right]^2$. *To maximize this ratio, the filter has the transfer function given by following expression,*

$$H(f) = k \cdot \frac{X^*(f)}{S_{ni}(f)} e^{-j2\pi fT} \quad ...(12.79)$$

and the maximized ratio is expressed as,

$$\left(\frac{S}{N}\right)_{0\max} = \left[\frac{x_0^2(T)}{\sigma^2}\right]_{\max} = \left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{\max}$$

or

$$\left(\frac{S}{N}\right)_{0\max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)} df \quad ...(12.80)$$

12.23. Matched Filter

(U.P. Tech-Semester Exam. 2002-2003)

In previous section, we have discussed optimum filter. For this filter, we have considered generalized Gaussian noise. When this noise is white Gaussian noise, then the optimum filter is known as ***matched filter***. Recall that for the white Gaussian noise, the power spectral density is given as,

$$S_{ni}(f) = \frac{N_0}{2} \quad ...(12.81)$$

12.23.1. Calculation of Impulse Response for the Matched Filter

The transfer function of the optimum filter is expressed as,

$$H(f) = k \cdot \frac{X^*(f)}{S_{ni}(f)} e^{-j2\pi fT} \quad ...(12.82)$$

In this equation, if we substitute $S_{ni}(f) = \frac{N_0}{2}$ *i.e.*, psd of white noise, then $H(f)$ becomes transfer function of matched filter.

This means that

Transfer function of matched filter, $H(f) = k \cdot \frac{X^*(f)}{\frac{N_0}{2}} e^{-j2\pi fT}$

or

$$H(f) = \frac{2k}{N_0} X^*(f) e^{-j2\pi fT} \quad ...(12.83)$$

Scanned by CamScanner

From the property of Fourier transform, we know that

$$X^*(f) = X(-f) \qquad ...(12.84)$$

Using this property, we can write equation (12.83) as under:

$$H(f) = \frac{2k}{N_0} X(-f)\, e^{-j2\pi fT}$$

The impulse response of a matched filter can be evaluated by taking inverse Fourier transform of above equation.

Thus, we have

$$h(t) = IFT\,[H(f)] = IFT\left[\frac{2k}{N_0} X(-f)\, e^{-j2\pi fT}\right] \qquad ...(12.85)$$

The inverse Fourier transform of $X(-f)$ is $x(-t)$ and $e^{-j2\pi fT}$ represents time shift of T seconds. Hence, we have

$$FT\,[x(-t)] = X(-f)$$

and $$FT\,[x(T-t)] = X(-f)\, e^{-j2\pi fT}$$

With the help of all these properties of Fourier transform, the equation (12.86) takes the form

$$h(t) = \frac{2k}{N_0} x(T-t) \qquad ...(12.86)$$

Note that we have considered $x(t) = x_1(t) - x_2(t)$, therefore, equation (12.49) will become,

$$\boldsymbol{h(t) = \frac{2k}{N_0}[x_1(T-t) - x_2(T-t)]} \qquad ...(12.87)$$

Thus, two equations (12.86) and (12.87) give the required impulse response of the matched filter.

Example 12.4. A polar NRZ waveform has to be received with the help of a matched filter. Here, binary 1 is represented by a rectangular positive pulse. Also, binary zero is represented by a rectangular negative pulse. Determine the impulse response of the matched filter. Also, sketch it.

Solution: Let $x_1(t)$ represent the positive rectangular pulse with duration T as shown in figure 12.24(*a*).

Let $x_2(t)$ represent a negative rectangular pulse with duration T as shown in figure 12.24(*b*)

Thus, we have

$$x_1(t) = +A \quad \text{for} \quad 0 \le t \le T$$

and $$x_2(t) = -A \quad \text{for} \quad 0 \le t \le T \qquad ...(i)$$

Let us calculate the difference signal $x(t)$ *i.e.*,

$$x(t) = x_1(t) - x_2(t) \quad \text{for} \quad 0 \le t \le T$$

or $$x(t) = A - (-A)$$

or $$x(t) = 2A \quad \text{for} \quad 0 \le t \le T \qquad ...(ii)$$

This difference signal has been shown in figure 12.24(*c*). The time reversed version of $x(t)$ will be,

$$x(-t) = 2A \quad \text{for} \quad -T \le t \le 0 \qquad ...(iii)$$

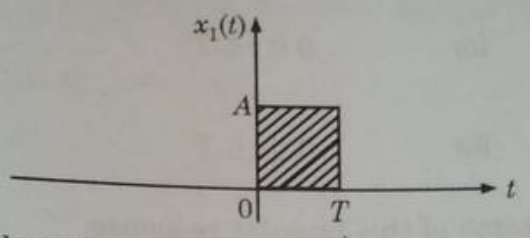
It may be noted that that in the above equation, the time reference is inverted. This signal has been shown in figure 12.24(*d*).

Now let us delay $x(-t)$ by T seconds.

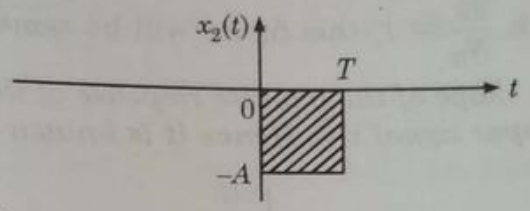
i.e., $$x(T-t) = 2A \quad \text{for} \quad 0 \le t \le T \qquad ...(iv)$$

The signal has been shown in figure 12.24(*e*)

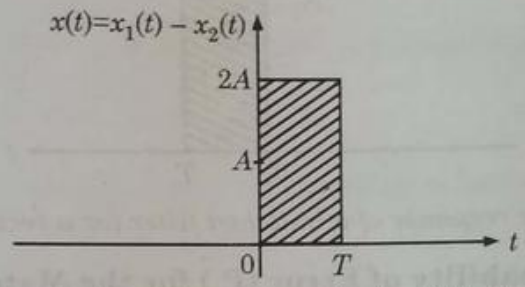
Scanned by CamScanner



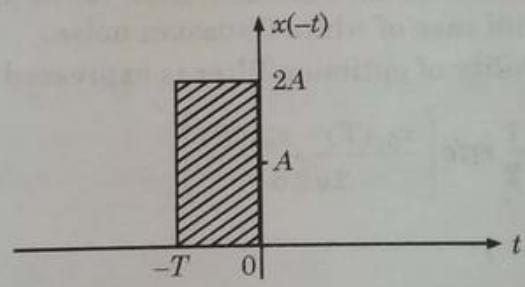
(*a*) The signal pulse $x_1(t)$ representing binary '1'. It is given by equation (*i*)



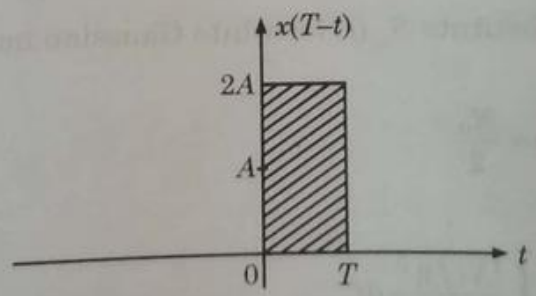
(*b*) The signal pulse $x_2(t)$ representing binary '0'. It is given by equation (*i*)



(*c*) The difference of $x_1(t)$ and $x_2(t)$. It has the amplitude of 2A. It is given by equation (*ii*)



(*d*) The difference signal which has been folded in time. It also has amplitude equal to 2A. It is given by equation (*iii*)



(*e*) The difference signal $x(-t)$ which has been delayed by T. It is same as the input signal $x(t)$ in figure 9.7(*c*). This is impulse response of the matched filter.

Fig. 12.24.

Impulse response of a matched filter is expressed as,

$$h(t) = \frac{2k}{N_0} x(T-t)$$

Substituting $x(T-t)$ from equation (*iv*), we get

Scanned by CamScanner

$$h(t) = \frac{2k}{N_0} \cdot 2A \quad \text{for} \quad 0 \le t \le T$$

or $$h(t) = \frac{4Ak}{N_0} \quad \text{for} \quad 0 \le t \le T$$

Figure 12.25 shows the sketch of this impulse response.

It may be noted that with $\frac{2k}{N_0} = 1$, this figure will be same as figure 12.24(*e*) and figure 12.24(*c*). *This proves that the shape of the impulse response of the matched filter is similar (or matched) to the shape of the input signal x(t).* ***Hence it is known as matched filter.***

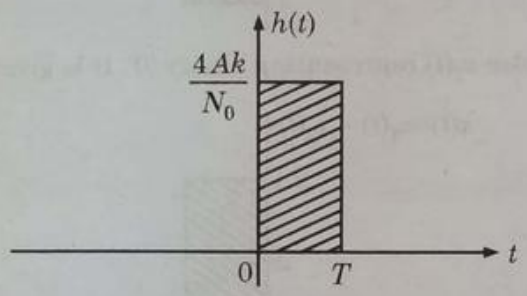


Fig. 12.25. *Impulse response of a matched filter for a rectangular pulse input.*

12.23.2. Calculation of Probability of Error (P_e) for the Matched Filter

To evaluate the probability of error for matched filter, let us again start with optimum filter and we shall consider the special case of white Gaussian noise.

We know that error probability of optimum filter is expressed as,

$$P_e = \frac{1}{2} erfc \left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma} \right] \quad ...(12.88)$$

In this equation, we have

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma} \right]^2_{\max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)} df$$

In this equation, let us substitute $S_{ni}(f)$ for white Gaussian noise.

Thus, we have

$$\text{psd of white noise} = S_{ni}(f) = \frac{N_0}{2}$$

Hence,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma} \right]^2_{\max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{\frac{N_0}{2}} df$$

$$= \frac{2}{N_0} \int_{-\infty}^{\infty} |X(f)|^2 df \quad ...(12.89)$$

Also, Parseval's power theorem states that,

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} x^2(t) dt = \int_0^T x^2(t) dt \quad ...(12.90)$$

Scanned by CamScanner

In the last integral we have taken limits from 0 to T because $x(t)$ exists from 0 to T only. We know that $x(t) = x_1(t) - x_2(t)$.

Hence above equation becomes,

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_0^T [x_1(t) - x_2(t)]^2 dt$$

or

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_0^T [x_1^2(t) + x_2^2(t) - 2x_1(t)\, x_2(t)]\, dt$$

$$= \int_0^T x_1^2(t)\, dt + \int_0^T x_2^2(t)\, dt - 2\int_0^T x_1(t)\, x_2(t)\, dt \qquad ...(12.91)$$

where, $\int_0^T x_1^2(t)\, dt = E_1$ *i.e.*, energy of $x_1(t)$ by standard relations.

and $\int_0^T x_2^2(t)\, dt = E_2$ *i.e.*, energy of $x_2(t)$ by standard relations.

and $\int_0^T x_1(t)\, x_2(t) = E_{12}$ represents energy due to autocorrelation between $x_1(t)$ and $x_2(t)$.

Now, if we choose $x_1(t) = -x_2(t)$, then these energies will be equal *i.e.*,

$$E_1 = E_2 = -E_{12} = E \qquad ...(12.92)$$

Substituting all these values in equation (12.92), we get

$$\int_{-\infty}^{\infty} |X(f)|^2 df = [E + E - 2(-E)] = 4E \qquad ...(12.93)$$

Substituting this value of $\int_{-\infty}^{\infty} |X(f)|^2 df$ in equation (12.89), we get

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{\max} = \frac{2}{N_0} \cdot 4E = \frac{8E}{N_0} \qquad ...(12.94)$$

Therefore,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]_{\max} = 2\sqrt{2} \cdot \sqrt{\frac{E}{N_0}}$$

Substituting this value of $\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]$ in equation (12.88), we get probability of error of matched filter as under:

$$P_e = \frac{1}{2}\, erfc \sqrt{\frac{E}{N_0}} \qquad ...(12.95)$$

Scanned by CamScanner

Note: *Here, erfc is the monotonically decreasing function. Therefore, when we substitute* $\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]_{max}$ *as the argument of erfc, we obtain minimum value of probability of error* P_e. *This means that the above equation really provides minimum probability of error of matched filter.*

Thus, **Minimum error probability of matched filter:**

$$P_e = \frac{1}{2} erfc \sqrt{\frac{E}{N_0}} \quad ...(12.96)$$

Few Points about Error Probability of Error, P_e of Matched Filter

From the equation (12.96), we can draw the following important conclusions:

(*i*) It is obvious from equation (12.96) that the *error probability depends only upon the signal energy E. It does not depend on the shape of the signal.*

(*ii*) We have obtained equation (12.96) by considering that $x_1(t) = -x_2(t)$. Also, in previous sections we have derived error probability for Intergrate and Dump filter where we have taken $x_1(t) = +A$ and $x_2(t) = -A$.

In other words, $x_1(t) = -x_2(t) = A$ and we obtained error probability of integrator as,

$$P_e = \frac{1}{2} erfc \sqrt{\frac{E}{N_0}} \quad ...(12.97)$$

This result is same as error probability of matched filter given be equation (12.97).

Hence, *Integrator is equivalent to matched filter when* $x_1(t) = -x_2(t) = A$ *or in other words, we can say that for a rectangular bipolar pulse input, the integrate and dump filter is same as a matched filter.*

Example 12.5. Prove that the maximum signal to noise ratio for the matched filter is found to be,

$$\left(\frac{S}{N}\right)_{0\,max} = \frac{2E}{N_0}$$

Solution: To derive this equation, let us consider the signal to noise power ratio of optimum filter which is given as,

$$\left(\frac{S}{N}\right)_{0\,max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)} df \quad ...(i)$$

We know that when white noise is present, then optimum filter is known as matched filter.

psd of white noise is $S_{ni}(f) = \frac{N_0}{2}$. Substituting this value in equation (*i*), we get

$$\left(\frac{S}{N}\right)_{0\,max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{\frac{N_0}{2}} df = \frac{2}{N_0} \int_{-\infty}^{\infty} |X(f)|^2 df \quad ...(ii)$$

Also, Rayleigh's energy theorem states that,

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} x^2(t)\, dt = E \text{ (energy)} \quad ...(iii)$$

Scanned by CamScanner

With this result, equation (*ii*) becomes

$$\left(\frac{S}{N}\right)_{0\max} = \frac{2}{N_0} \cdot E$$

Hence, maximum signal to noise power ratio of matched filter will be

$$\left(\frac{S}{N}\right)_{0\max} = \frac{2E}{N_0} \qquad \text{...(iv)}$$

Now, this equation can be rearranged as under:

$$\left(\frac{S}{N}\right)_{0\max} = \frac{E}{\frac{N_0}{2}}$$

Here, E is energy of signal $x(t)$ and $\frac{N_0}{2}$ is power spectral density (psd) of white noise.

Thus, we have

$$\left(\frac{S}{N}\right)_{0\max} = \frac{\text{Energy of the signal } x(t)}{\text{psd of white noise}} \qquad \textbf{Hence Proved}$$

Example 12.6. Prove that for a matched filter, the maximum signal component occurs at $t = T$ (*i.e.*, sampling instant) and has magnitude equal to E, *i.e.*, energy of the signal $x(t)$.

Solution: We know that the Fourier transform of the output signal $x_0(t)$ is expressed as,

$$X_0(f) = X(f)\, H(f) \qquad \text{...(i)}$$

where $X(f)$ = Fourier transform of input signal $x(t)$

and $H(f)$ = is transfer function of matched filter

Substituting value of $H(f)$ from equation (12.83), we get

$$X_0(f) = X(f) \cdot \frac{2k}{N_0} X^*(f)\, e^{-j2\pi fT} = \frac{2k}{N_0} X(f)\, X^*(f)\, e^{-j2\pi fT} \qquad \text{...(ii)}$$

Now, since $X(f)\, X^*(f) = |X(f)|^2$, therefore, equation (*ii*) becomes,

$$X_0(f) = \frac{2k}{N_0} |X(f)|^2\, e^{-j2\pi fT} \qquad \text{...(iii)}$$

From this expression, we can obtain $x_0(t)$ by taking inverse Fourier transform.

Thus, we have

$$x_0(t) = \int_{-\infty}^{\infty} X_0(f)\, e^{j2\pi ft} df^* = \int_{-\infty}^{\infty} \frac{2k}{N_0} |X(f)|^2\, e^{-j2\pi fT} \cdot e^{j2\pi ft} df$$

or

$$x_0(t) = \frac{2k}{N_0} \int_{-\infty}^{\infty} |X(f)|^2\, e^{j2\pi f(t-T)} df$$

At $t = T$, $x_0(t)$ will become,

$$x_0(T) = \frac{2k}{N_0} \int_{-\infty}^{\infty} |X(f)|^2\, e^{j2\pi f(T-T)} df$$

or

$$x_0(T) = \frac{2k}{N_0} \int_{-\infty}^{\infty} |X(f)|^2\, df \qquad [\because e^{j2\pi f(T-T)} = e^0 = 1] \qquad \text{...(iv)}$$

* By definition of IFT.

Scanned by CamScanner

By Rayleigh's energy theorem, we can write

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} x^2(t)\, dt = E,$$

Hence, equation (*iv*) becomes

$$x_0(T) = \frac{2k}{N_0} \cdot E$$

Also, maximum value of $x_0(T)$ will result when $\frac{2k}{N_0} = 1$.

Thus, $\quad x_0(T) = E \quad$ when $\quad t = T \quad$ **Hence Proved**

Example 12.7. Prove that the output signal of a matched filter is proportional to a shifted version of the autocorrelation function of the input signal to which the filter is matched.

Solution: For a matched filter, the output signal $x_0(t)$ may be obtained from its spectrum $X_0(f)$ by taking inverse Fourier transform.

This means that

$$x_0(t) = \int_{-\infty}^{\infty} X_0(f)\, e^{j2\pi ft} df$$

Substituting $X_0(f)$ from equation (*iii*) [last example], we get

$$x_0(t) = \int_{-\infty}^{\infty} \frac{2k}{N_0} |X(f)|^2 e^{-j2\pi fT} \cdot e^{j2\pi ft}\, df$$

or

$$x_0(T) = \frac{2k}{N_0} \int_{-\infty}^{\infty} |X(f)|^2 e^{j2\pi f(t-T)} df \qquad ...(i)$$

Recall that the energy spectral density (esd) is given as

$$\psi(f) = |X(f)|^{2*}$$

Hence, equation (*i*) becomes,

$$x_0(t) = \frac{2k}{N_0} \int_{-\infty}^{\infty} \psi(f) e^{j2\pi f(t-T)}\, df \qquad ...(ii)$$

We know that one of the property of esd states that the autocorrelation function $R(\tau)$ and energy spectral density $\psi(f)$ form a Fourier transform pair.

This means that

$$R(\tau) \leftrightarrow \psi(f) \qquad ...(iii)$$

Thus, we have

$$R(\tau) = \int_{-\infty}^{\infty} \psi(f)\, e^{j2\pi f\tau}\, df \qquad ...(iv)$$

and

$$\psi(f) = \int_{-\infty}^{\infty} R(\tau)\, e^{-j2\pi f\tau}\, d\tau \qquad ...(v)$$

* According to the Standard definition of esd.

Scanned by CamScanner

Applying equation (*iv*) to RHS of equation (*ii*), we get

$$x_0(t) = \frac{2k}{N_0} R(t-T)$$

This proves that $x_0(t)$ is proportional to autocorrelation function of $x(t)$ shifted to right by T. Here, $\frac{2k}{N_0}$ is a constant term and it depends upon psd of noise. **Hence Proved**

Example 12.8. In a binary transmission, one of the messages is represented by a rectangular pulse $x(t)$. An other message is transmitted by the absence of the pulse. Evaluate the signal to noise ratio at $t = T$. Assuming white noise with psd equal to $\frac{N_0}{2}$. Also sketch the impulse response of the matched filter and output of the matched filter.

Solution: Evaluation of Impulse response:

The rectangular pulse can be represented as

$$x(t) = \begin{cases} A & \text{for } 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \quad ...(i)$$

Further, we know that impulse response of the matched filter is given by,

$$h(t) = \frac{2k}{N_0} x(T-t) \quad ...(ii)$$

From equation (*i*), we can write,

$$x(-t) = \begin{cases} A & \text{for } -T \le t \le 0 \\ 0 & \text{otherwise} \end{cases} \quad ...(iii)$$

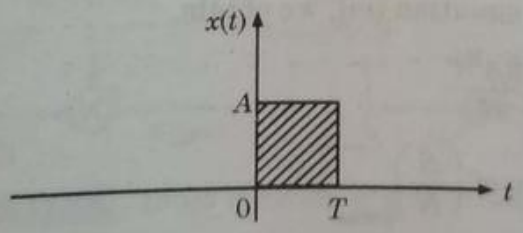
Let us delay this signal by T to get

$$x(T-t) = \begin{cases} A & \text{for } 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \quad ...(iv)$$

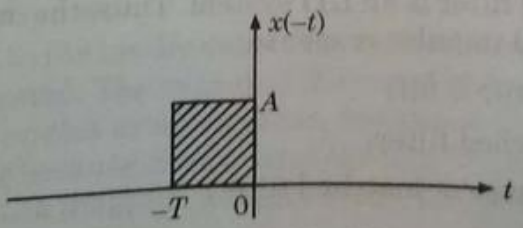
Substituting this value in equation (*ii*), we obtain

$$h(t) = \begin{cases} \frac{2k}{N_0} A & \text{for } 0 \le t \le T \\ 0 & \text{otherwise} \end{cases} \quad ...(v)$$

This equation gives impulse response of the matched filter. Figure 12.26 illustrates the sketch of this response.

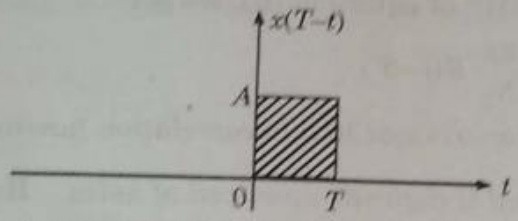


(*a*) Input signal $x(t)$. It is rectangular pulse given by equation (*i*)

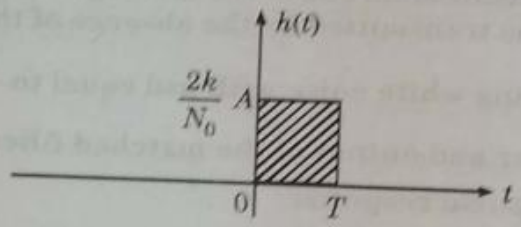


(*b*) Input signal pulse folded in time. This signal is represented by equation (*iii*)

Scanned by CamScanner



(c) Folded signal of Fig. (b) is delayed by T. This signal is represented by equation (iv)



(d) Impulse response of the matched filter to the input signal pulse in figure (a).

Fig. 12.26.

Calculation of $\left(\frac{S}{N}\right)_{0max}$:

We know that the maximum signal to noise ratio of the matched filter $\left(\frac{S}{N}\right)_{0max}$ is given by,

$$\left(\frac{S}{N}\right)_{0\max} = \frac{2E}{N_0} = \frac{E}{N_0/2} \qquad ...(vi)$$

Here, $\frac{N_0}{2}$ is given as psd of white noise.

Further, we can evaluate energy of signal $x(t)$ by,

$$E = \int_{-\infty}^{\infty} x^2(t)\,dt$$

Substituting, $x(t) = A$ and changing the limits from 0 to T as described in equation (i), the last equation becomes,

$$E = \int_0^T A^2\,dt = A^2[t]_0^T = A^2T$$

Substituting this value in equation (vi), we obtain,

$$\left(\frac{S}{N}\right)_{0\max} = \frac{2A^2T}{N_0}$$

which is the required value of $\left(\frac{S}{N}\right)_{0max}$

Output of the Matched Filter:

We know that the matched filter is an LTI system. Thus, the output of the matched filter will be the convolution of input and impulse response.

Thus, we have $\quad y(t) = x(t) \otimes h(t)$

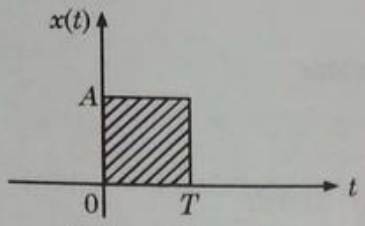
where $y(t)$ = output of matched filter,

$h(t)$ = impulse response of matched filter,

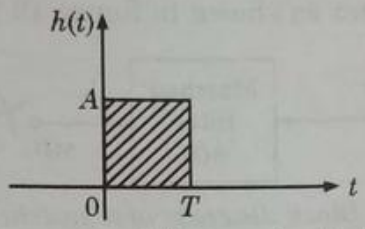
and $\quad x(t)$ = input signal.

Scanned by CamScanner

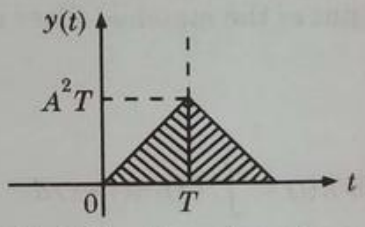
Without going into mathematical details, we can state that the convolution of two similar rectangular pulses is a triangular pulse. Here, $x(t)$ and $h(t)$ are two rectangular pulses. Therefore, $y(t)$ will be a triangular pulse. This has been shown in figure 12.27.



(a) Input signal $x(t)$



(b) Impulse response $h(t)$ assuming that $2k/N_0 = 1$



(c) Output of the matched filter for rectangular pulse input. It may be observed that output is maximum at $t = T$

Fig. 12.27.

As figure 12.27 shows the matched filter maximizes its output at $t = T$. Hence, $y(t)$ has a value of A^2T at $= T$.

12.24. The Correlator

In this article, we shall discuss a little different type of receiver which is known as **corrector**. Figure 12.28 shows the block diagram of this correlator.

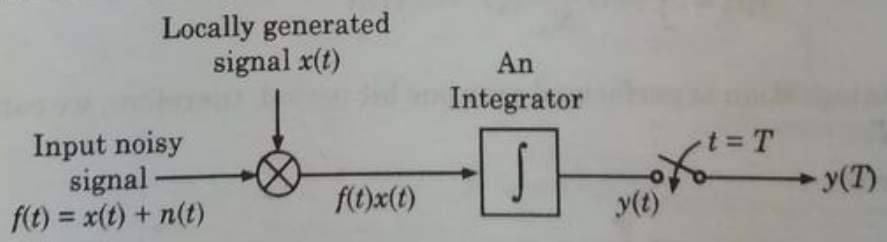


Fig. 12.28. *Block diagram of a correlator*

In figure 12.28, $f(t)$ represents input noisy signal,

i.e., $$f(t) = x(t) + n(t).$$

The signal $f(t)$ is multiplied to the locally generated replica of input signal $x(t)$. Then, result of multiplication $f(t) \cdot x(t)$ is integrated. The output of the integrator is sampled at $t = T$ (*i.e.*, end of one symbol period). Then based on this sampled value, decision is made. This is how the correlator works. It is known as correlator because *it correlates the received signal f(t) with a stored replica of the know signal x(t)*. In the block diagram of figure 12.29, the product $f(t)\,x(t)$ is integrated over one symbol period, *i.e.*, T.

Scanned by CamScanner

Thus, output $y(t)$ will be,

$$y(t) = \int_0^T f(t)\, x(t)\, dt$$

At $t = T$, this equation will become,
output of correlator,

$$y(T) = \int_0^T f(t)\, x(t)\, dt \qquad \text{...(12.98)}$$

Now consider the matched filter as shown in figure 12.29 below:

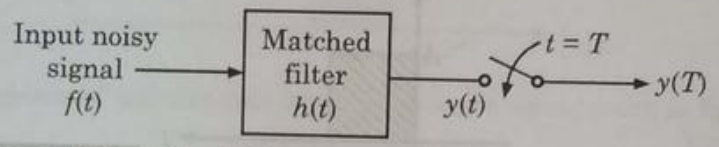


Fig. 12.29. *Block diagram of a matched filter receiver.*

In this block diagram, it may be observed that the matched filter does not need locally generated replica of input signal $x(t)$. The output of the matched filter is obtained by convolution of input $f(t)$ and its impulse response $h(t)$.

This means that

$$y(t) = f(t) \otimes h(t) = \int_{-\infty}^{\infty} f(\tau) \cdot h(t-\tau)\, d\tau \qquad \text{...(12.99)}$$

We know that the impulse response $h(t)$ of the matched filter is given by,

$$h(t) = \frac{2k}{N_0}\, x(T-t) \qquad \text{...(12.100)}$$

Therefore, $$h(t-\tau) = \frac{2k}{N_0}\, x(T-t+\tau)$$

Substituting this value of $h(t-\tau)$ in equation (12.99), we get

$$y(t) = \int_{-\infty}^{\infty} f(\tau)\, \frac{2k}{N_0}\, x(T-t+\tau)\, d\tau$$

Because the integration is performed over one bit period, therefore, we can change integration limits from 0 to T.

Hence, we write

$$y(t) = \frac{2k}{N_0} \int_0^T f(\tau)\, x(T-t+\tau)\, d\tau$$

At $t = T$, the last equation becomes,

$$y(T) = \frac{2k}{N_0} \int_0^T f(\tau)\, x(T-T+\tau)\, d\tau = \frac{2k}{N_0} \int_0^T f(\tau)\, x(\tau)\, d\tau$$

Now, let us substitute $\tau = t$ just for convenience for notation, then we have output of matched filter

Scanned by CamScanner

$$y(T) = \frac{2k}{N_0} \int_0^T f(t)\, x(t)\, dt \qquad ...(12.101)$$

Note: *This equation (i.e. 12.101) gives the output of matched filter. It may be observed that this equation and equation (12.98) (which gives output of correlator) are identical. In equation (12.101) the constant* $\frac{2k}{N_0}$ *is present which can be normalized to 1. The similarity between equation (12.98) and equation (12.101) shows that the matched filter and correlator provides same output.*

Therefore we can state as under:

The matched filter and correlator are two distinct, independent techniques which yield same result. Infact, these two techniques are used to synthesize the optimum filter.

12.25. Intersymbol Interference (ISI)

In a communication system, when the data is being transmitted in the form of pulses (*i.e.*, bits), the output produced at the receiver due to other bits or symbols interferes with the output produced by the desired bit. This is known as **intersymbol interference (ISI)**. The intersymbol interference will introduce errors in the detected signal. Let us consider figure 12.30 which shows the elements binary PAM system. The input signal consists of a binary data sequence $\{b_k\}$ with a bit duration of T_b seconds.

This sequence is applied to a pulse generator to produce a discrete PAM signal which given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, v(t - kT_b) \qquad ...(12.102)$$

where $v(t)$ denotes the basic pulse, normalized such that $v(0) = 1$. The first block of the system *i.e.*, pulse amplitude modulator converts this input sequence into polar form as under:

if $\quad b_k = 1$ then $a_k = 1$

and $\quad b_k = 0$ then $a_k = -1$

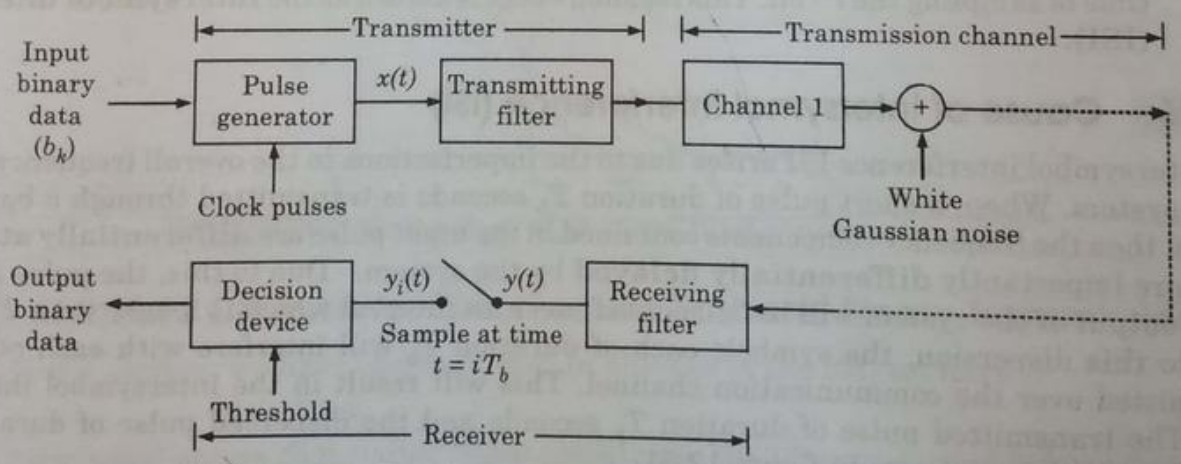


Fig. 12.30. *Baseband binary data transmission system.*

(*i*) The PAM signal $x(t)$ is then passed through a transmitting filter. The output of the transmitting filter is then transmitted over transmission channel. Let the impulse response of this channel be $h(t)$.

(*ii*) A random noise is then added to the transmitted signal when it travels over the transmission channel. Thus, the signal received at the receiving end is contaminated with noise.

Scanned by CamScanner

(iii) The channel output is passed through a receiving filter. This filter output is sampled synchronously with the transmitter. The sampling instants are determined by a clock or timing signal which is extracted from the receiving filter output.

(iv) The sequence of samples obtained at the output of receiving filter is used to reconstruct the original data sequence with the help of a decision making device.

(v) Each sample is compared to a threshold level in the decision making device. If the amplitude of the sample is higher than the threshold level then it is decided that a symbol 1 is received. On the other hand, if the signal has an amplitude lower than the threshold, then the decision is that a 0 is received.

The receiving filter output can be written as under:

$$y(t) = \mu \sum_{k=-\infty}^{\infty} a_k \, p(t - kT_b) + n(t) \qquad ...(12.102a)$$

where μ is a scaling factor and the noise $n(t)$ is the noise at the output of the receiving filter due to the added noise. The term $p(t - kT_b)$ represents the combined impulse response of the receiving filer.

The receiving filter output $y(t)$ is sampled at time $t_i = iT_b$ with i taking on integer values to provide following expression:

$$y(t_i) = \mu \sum_{k=-\infty}^{\infty} a_k \, p(iT_b - kT_b) + n(t_i) \qquad ...(12.103)$$

or

$$y(t_i) = \mu a_i + \mu \sum_{\substack{k=-\infty \\ k \neq i}}^{\infty} a_k \, p(iT_b - kT_b) + n(t_i) \qquad ...(12.104)$$

This is the receiver output $y(t)$ at instant $t = t_i$.

Now, it may be noted that equation (12.104) has two terms:

(i) The first term μa_i is produced by the ith transmitted bit. Theoretically, only this term should be present, however, practically it is not so.

(ii) The second term represents the residual effect of all the transmitted bits, obtained at the time of sampling the ith bit. This residual effect is known as the **intersymbol interference (ISI).**

12.26. Cause of Intersymol Interference (ISI)

The intersymbol interference ISI arises due to the imperfections in the overall frequency response of the system. When, a short pulse of duration T_b seconds is transmitted through a bandlimited system, then the frequency components contained in the input pulse are **differentially attenuated** and more importantly **differentially delayed** by the system. Due to this, the pulse appearing at the output of the system will be "dispersed" over an interval which is longer than T_b seconds. Due to this dispersion, the symbols each of duration T_b will interfere with each other when transmitted over the communication channel. This will result in the intersymbol interference (ISI). The transmitted pulse of duration T_b seconds and the dispersed pulse of duration more than T_b seconds are shown in figure 12.31.

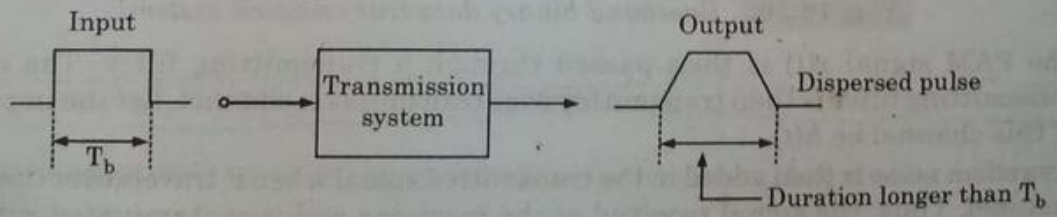


Fig. 12.31. *Cause of Intersymbol Interference (ISI).*

Scanned by CamScanner

Effect of Intersymbol Interference (ISI)

Following are the effects of ISI:

(*i*) In the absence of ISI and noise, the transmitted bit can be decoded correctly at the receiver.

(*ii*) The presence of ISI will introduce errors in the decision at the receiver output.

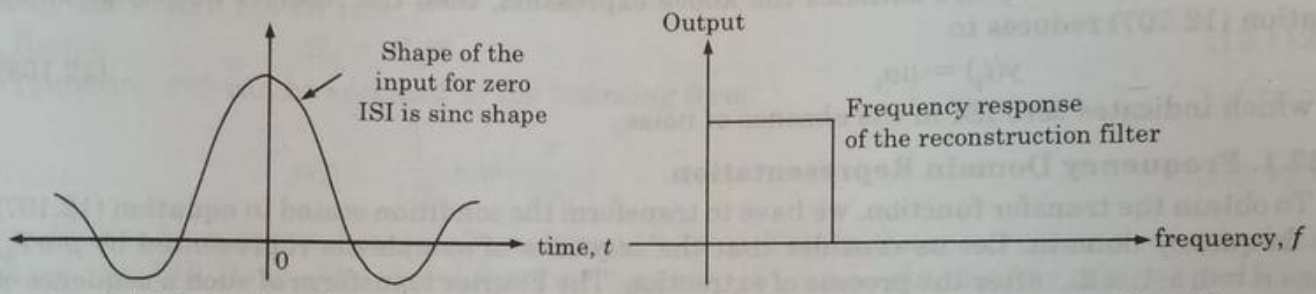
(*iii*) Hence, the receiver can make an error in deciding whether it has received a logic 1 or a logic 0.

Remedy to Reduce ISI

(*i*) It has been proved that the function which produces a zero intersymbol interference is a "sinc function". Hence, instead of a rectangular pulse if we transmit a sinc pulse then the ISI can be reduced to zero.

(*ii*) This is known as *Nyquist Pulse Shaping*. The sinc pulse transmitted to have a zero ISI has been shown in figure 12.32 (*a*).

(*iii*) Further, we know that Fourier transform of a sinc pulse is a rectangular function. Hence, to preserve all the frequency components, the frequency response of the filter must be exactly flat in the pass band and zero in the attenuation band as shown in figure 12.32 (*b*).



(a) Ideal pulse shape for zero ISI *(b) Frequency response of the filter*

Fig. 12.32.

- However, type of filter is practically not present or not possible. Hence, in practice, the frequency response of the filter is modified as shown in figure 12.33 with different roll off factors α to obtain the achievable filter response curves.

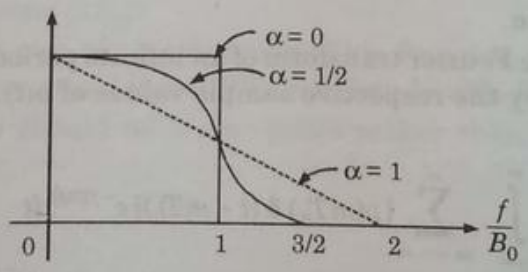


Fig. 12.33. *Illustration of practical filter characteristics.*

12.27. Nyquist's Criterion for Distortionless Basband Binary Transmission

In the previous section, we have observed that in absence of the ISI, we have

$$y(t_i) = \mu a_i \qquad \text{...(12.105)}$$

This expression shows that under these conditions, the ith transmitted bit can be decoded correctly. In order to minimize the effects of ISI, we have to design the transmitting and receiving filters properly. The transfer function of the channel and the shape of transmitted pulse are generally specified. Therefore, it becomes the first step towards design of filters. From this information we have to determine the transfer functions of the transmitting and receiving filters, to reconstruct the transmitted data sequence $\{b_k\}$. This is achieved by first extracting and then decoding the corresponding sequence of weights from the output $y(t)$.

Scanned by CamScanner

As expressed by following equation

$$y(t) = \mu \sum_{k=-\infty}^{\infty} a_k \, p(t - kT_b) \qquad \text{...(12.106)}$$

This shows that output $y(t)$ is dependent on a_k, the received pulse $p(t)$ and the sealing factor μ.

Extraction

Extraction is basically the process of sampling. The signal $y(t)$ is sampled at $t = iT_b$.

Decoding

The decoding should be such that the contribution of the weighted pulse *i.e.*, $a_k \, p(iT_b - kT_b)$ for $i = k$ be free from ISI. This can be stated mathematically as under:

$$p(iT_b - kT_b) = \begin{cases} 1 & \text{for } i = k \\ 0 & \text{for } i \neq k \end{cases} \qquad \text{...(12.107)}$$

where $p(0) = 1$ due to normalizing.

If $p(t)$ *i.e.*, received pulse satisfies the above expression, then the receiver output given by equation (12.107) reduces to

$$y(t_i) = \mu a_i \qquad \text{...(12.108)}$$

which indicates zero ISI in the absence of noise.

12.27.1. Frequency Domain Representation

To obtain the transfer function, we have to transform the condition stated in equation (12.107) into frequency domain. Let us consider that the sequence of samples is represented by $p(nT_b)$ where $n = 0, \pm 1, \pm 2...$ after the process of extraction. The Fourier transform of such a sequence of samples can be obtained, as we did for the sampling process of low pass signal. Therefore, the frequency domain representation of $p(nT_b)$ will be as under:

$$P_\delta(f) = F[p(nT_b)] = R_b \sum_{n=-\infty}^{\infty} P(f - nR_b) \qquad \text{...(12.109)}$$

where $R_b = 1/T_b$ *i.e.*, bit rate.

But, $P_\delta(f)$ also represents the Fourier transform of an infinite periodic sequence of unit impulses whose strengths are weighted by the respective sample values of $p(t)$. Hence, $P_\delta(f)$ can be written as under:

$$P_\delta(f) = \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} [p(mT_b)\, \delta(t - mT_b)]\, e^{-j2\pi ft} dt \qquad \text{...(12.110)}$$

where $\sum_{m=-\infty}^{\infty} p(mT_b)\, \delta(t - mT_b)$ represents the sequence of unit impulses weighted by the respective sample values.

Now, let $m = i - k$, therefore, if $i = k$ then $m = 0$ and if $i \neq k$ then $m \neq 0$.

Hence, let us apply condition of equation (12.107) to equation (12.110) to get, for $m = 0$.

$$P_\delta(f) = \int_{-\infty}^{\infty} p(0)\, \delta(t)\, e^{-j2\pi ft}\, dt$$

Using the sifting property of delta function, we have

$$P_\delta(f) = p(0)$$

But, $p(0) = 1$ due to normalization, hence $P_\delta(f) = 1$.

Scanned by CamScanner

Substituting this into equation (12.109), we obtain

$$1 = R_b \sum_{n=-\infty}^{\infty} P(f - nR_b)$$

Therefore, we have

$$\sum_{n=-\infty}^{\infty} P(f - nR_b) = 1/R_b = T_b \qquad ...(12.111)$$

This expression is called as the *Nyquist criterion for distortionless baseband transmission* in the absence of noise.

12.28. Ideal Solution

Note that the L.H.S. of equation (12.111) represents a series of shifted spectrums. For $n = 0$, the LHS corresponds to $P(f)$ and it represents a frequency function with the narrowest band which satisfies equation (12.111). The range of frequencies for $P(f)$ will extend from $-B_0$ to B_0 where B_0 corresponds to half the bit rate.

Hence, $$B_0 = R_b/2 \qquad ...(12.112)$$

Therefore, $P(f)$ can be specified in the following form:

$$P(f) = \frac{1}{2B_0} \text{rect}\left(\frac{f}{2B_0}\right) \qquad ...(12.113)$$

and it has been shown graphically in figure 12.34(*a*). This is the spectrum of a signal which produces zero ISI. Hence, the signal that produces zero ISI can be obtained by taking the IFT of $P(f)$.

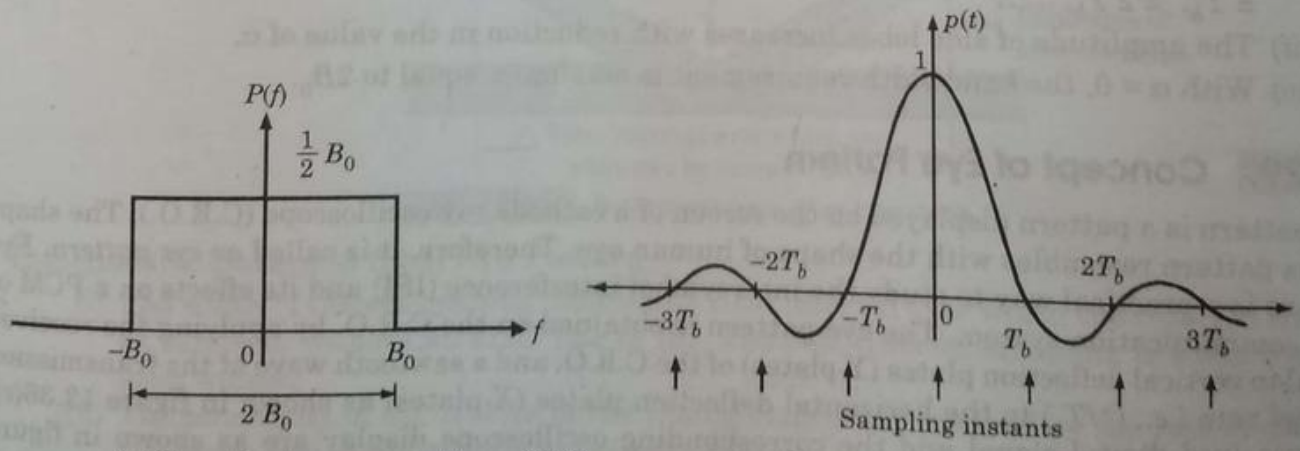
This means that we have

$$p(t) = F^{-1}[P(f)] = F^{-1}\left[\frac{1}{2B_0} \text{rect}\left(\frac{f}{2B_0}\right)\right]$$

or $$p(t) = \text{sinc}\,(2B_0 t) \qquad ...(12.114)$$

Figure 12.34(*b*) shows the plot of this function. This function $p(t)$ can be regarded as the impulse response of an ideal low pass filter (LPF) with bandwidth B_0.

Thus the shape of a pulse should be a sinc pulse rather than being a rectangular one to eliminate the ISI.



(a) Graphical representation of P(f). *(b) Time-domain representation.*

Fig. 12.34.

Scanned by CamScanner

Advantages of Using the Sinc Pulse

(*i*) Bandwidth requirement (of the channel) is reduced.

(*ii*) ISI is reduced to zero.

Possible Difficulties

(*i*) It is necessary that the amplitude characteristics of $P(f)$ should be flat from $-B_0$ to B_0 and zero outside this band. But, abrupt transition at $\pm B_0$ is not physically realizable.

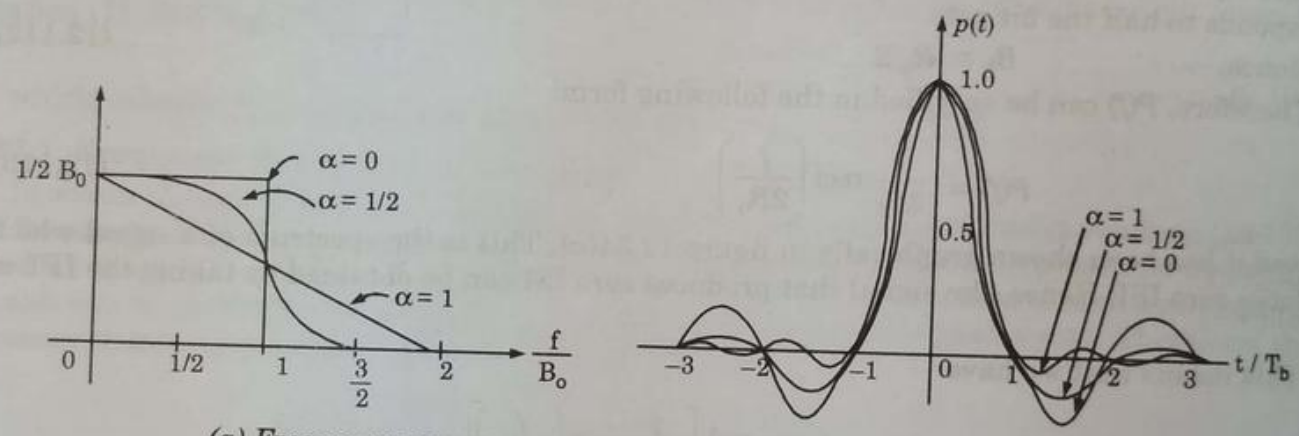
(*ii*) Due to discontinuity of $P(f)$ at $\pm B_0$, there is practically no margin of error in sampling times at the receiver end.

Remedy

(*i*) Due to these problems, it is practically not possible to use the sinc function as it is.

(*ii*) But, the practical difficulties can be overcome by increasing the bandwidth from $B_0 = R_b/2$ to an adjustable value between B_0 and $2B_0$.

(*iii*) The frequency response for different roll off factors (α) and the corresponding time response have been shown in figure 12.35(*a*) and (*b*) respectively.



(*a*) *Frequency response* (*b*) *Time response*

Fig. 12.35. *Responses for different roll-off factors* α.

Observations

(*i*) For $\alpha = 0.5$ and 1, the characteristics of $P(f)$ changes gradually with respect to frequency. Hence it is easier to realize this characteristics practically.

(*ii*) The time response has a sinc shape and all the sinc functions pass through zero at $t = \pm T_b, \pm 2T_b, \ldots\ldots$

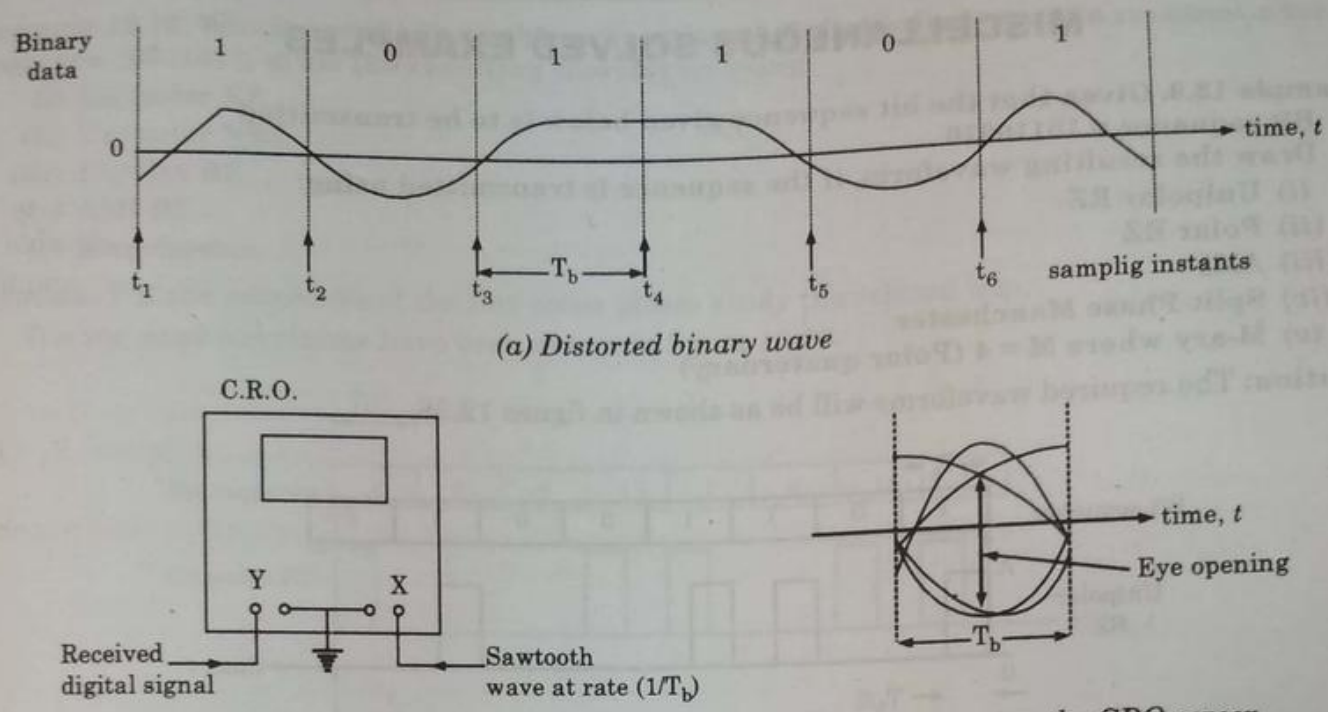
(*iii*) The amplitude of side lobes increases with reduction in the value of α.

(*iv*) With $\alpha = 0$, the bandwidth requirement is maximum equal to $2B_0$.

12.29. Concept of Eye Pattern

Eye pattern is a pattern displayed on the screen of a cathode ray oscilloscope (C.R.O.). The shape of this pattern resembles with the shape of human eye. Therefore, it is called as *eye pattern*. Eye pattern is a practical way to study the intersymbol interference (ISI) and its effects on a PCM or data communication system. The eye pattern is obtained on the C.R.O. by applying the received signal to vertical deflection plates (*Y*-plates) of the C.R.O. and a sawtooth wave at the transmission symbol rate *i.e.*, $(1/T_b)$ to the horizontal deflection plates (*X*-plates) as shown in figure 12.36(*c*). The received digital signal and the corresponding oscilloscope display are as shown in figure 12.36(*a*) and (*c*) respectively. The resulting oscilloscope display shown in figure 12.36(*c*) is called as the *eye pattern*. This is due to its resemblance to the human eye.

Scanned by CamScanner



(a) Distorted binary wave

(b) Oscilloscope connections. *(c) Eye pattern seen on the CRO screen.*

Fig. 12.36. *Obtaining eye pattern.*

The interior region of the eye pattern is called as the eye opening. The eye pattern provides a great deal of information about the performance of the system. The information obtainable is as follows (Figure 12.37)

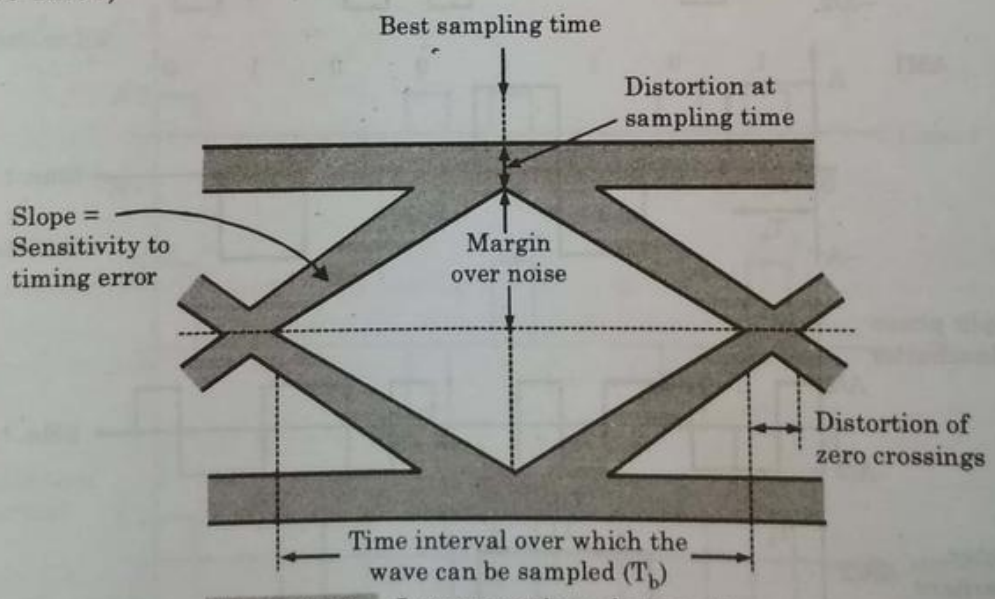


Fig. 12.37. *Interpretation of eye pattern.*

Information Obtained from Eye Pattern

(*i*) The width of the eye opening defines the time interval over which the received wave can be sampled, without an error due to ISI. The best time for sampling is when the eye is open widest.

(*ii*) The sensitivity of the system to the timing error is determined by the rate of closure of the eye as the sampling rate is varied.

(*iii*) The height of eye opening at a specified sampling time defines the margin over noise.

(*iv*) When the effect of ISI is severe, the eye is completely closed and it is impossible to avoid errors due to the combined presence of ISI and noise in the system.

Scanned by CamScanner

MISCELLANEOUS SOLVED EXAMPLES

Example 12.9. Given that the bit sequence given below is to be transmitted
Bit sequence = 10110010
Draw the resulting waveform, if the sequence is transmitted using:
(*i*) Unipolar RZ
(*ii*) Polar RZ
(*iii*) AMI
(*iv*) Split Phase Manchester
(*v*) M-ary where M = 4 (Polar quaternary)

Solution: The required waveforms will be as shown in figure 12.38.

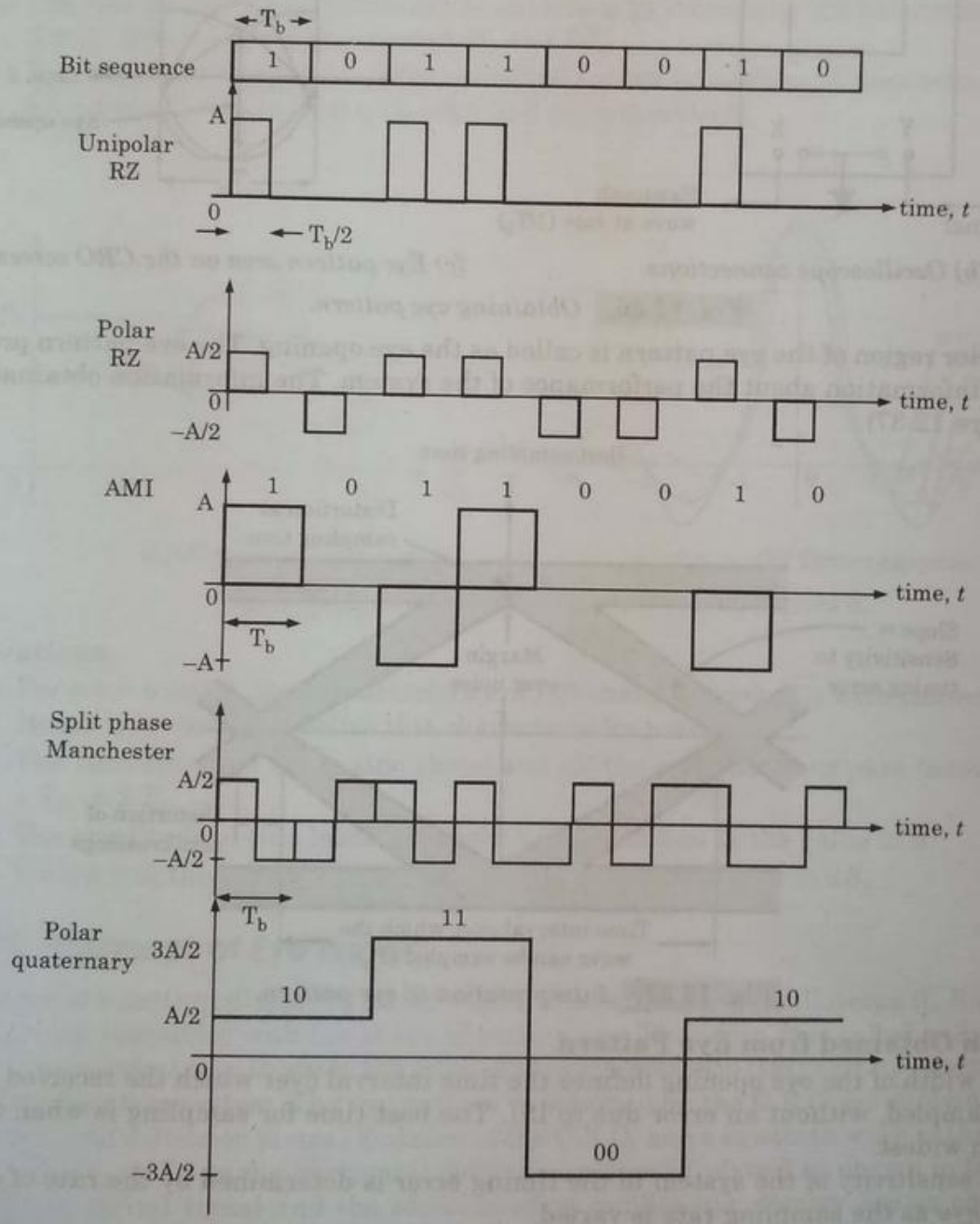


Fig. 12.38. *Waveforms for example 12.9.*

Scanned by CamScanner

Example 12.10. Which are the desirable properties of digital waveform ? To transmit a bit sequence 10011011, draw the resulting waveforms using:

(*i*) Unipolar RZ
(*ii*) Unipolar NRZ
(*iii*) Bipolar RZ
(*iv*) AMI RZ
(*v*) Manchester.

Solution: For the properties of the line codes please study the related text.

The required waveforms have been drawn in figure 12.39.

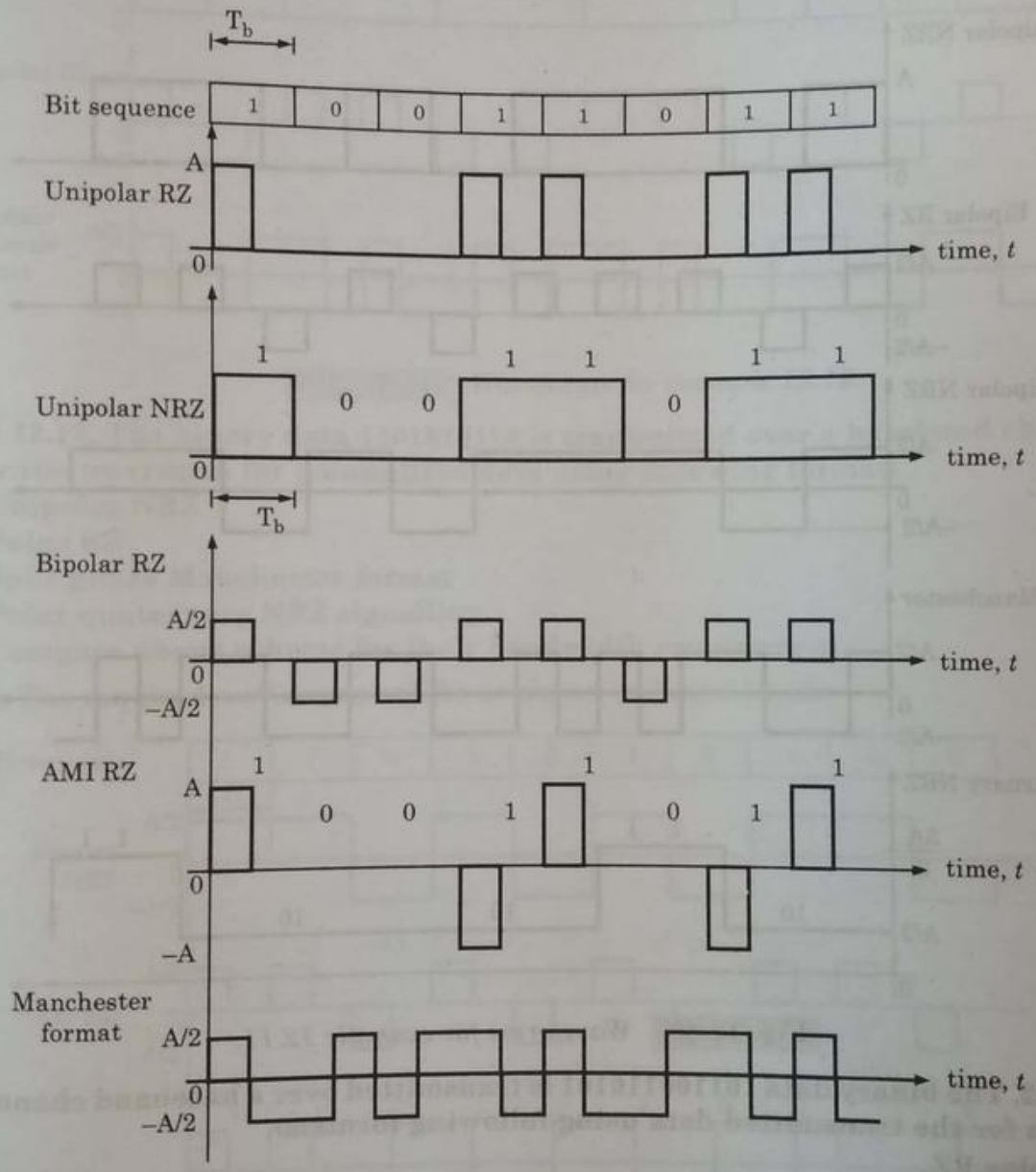


Fig. 12.39. *Waveforms for example 12.10.*

Example 12.11. The bit sequence 1 0 1 1 1 0 1 0 1 1 is to be transmitted using following formats:

(*i*) Unipolar RZ and NRZ
(*ii*) Bipolar RZ and NRZ
(*iii*) Split-phase Manchester
(*iv*) Polar quaternary NRZ.
Draw all the waveforms.

Scanned by CamScanner

Solution: The required waveforms will be as shown in figure 12.40.

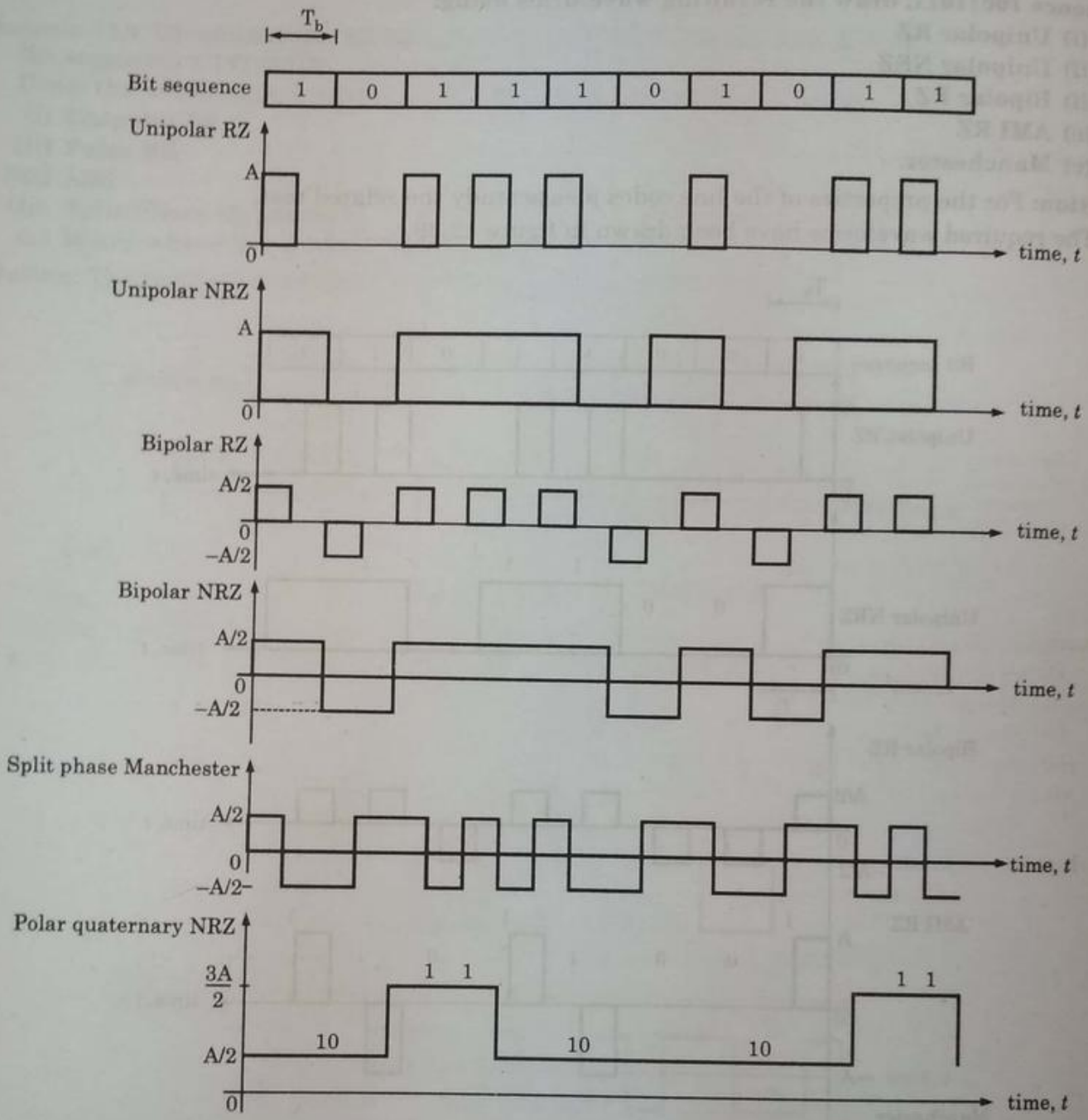


Fig. 12.40. *Waveforms for example 12.11.*

Example 12.12. The binary data 101100110101 is transmitted over a baseband channel. Draw the waveform for the transmitted data using following formats:

(*i*) Unipolar RZ
(*ii*) Unipolar NRZ
(*iii*) Bipolar RZ
(*iv*) Split-phase Manchester

Compare above schemes for their BW requirements.

Solution: The required waveforms will as shown in figure 12.41.

Scanned by CamScanner

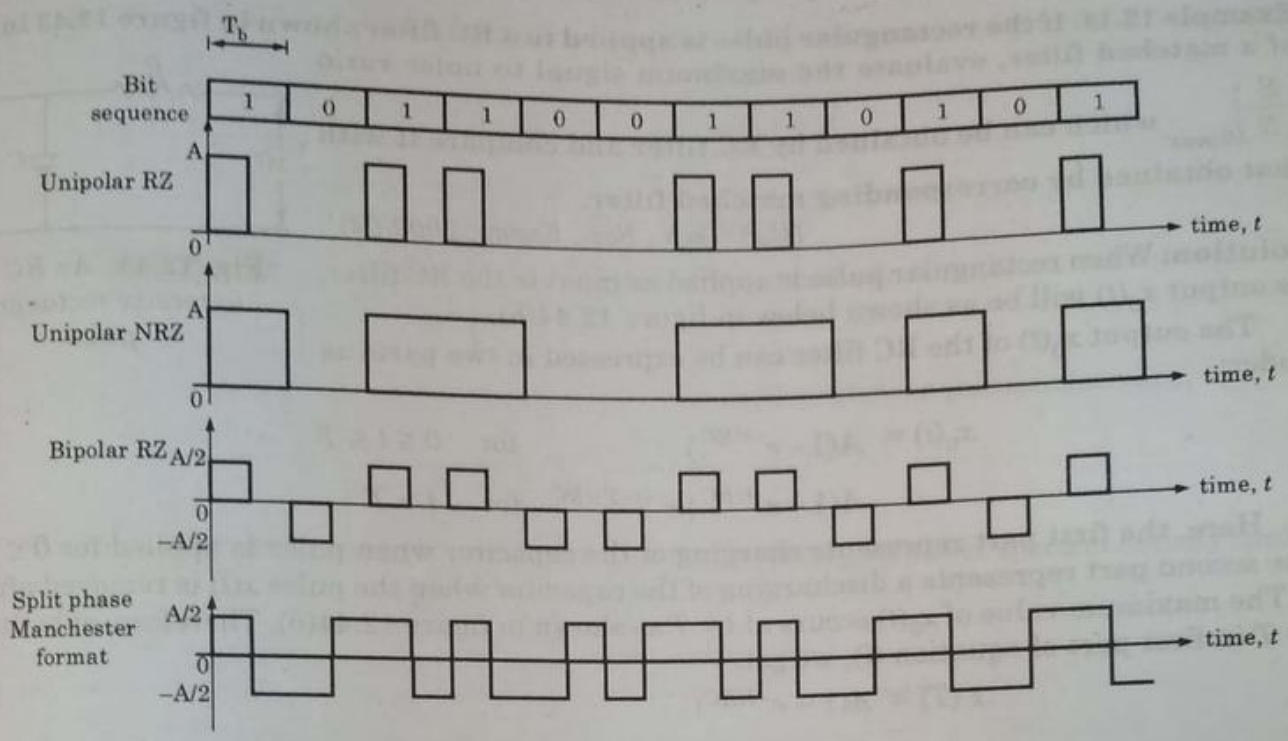


Fig. 12.41. *Waveforms for example 12.12.*

Example 12.13. The binary data 1101010110 is transmitted over a baseband channel. Draw the waveform for transmitted data using following format:

(*i*) Unipolar NRZ
(*ii*) Polar RZ
(*iii*) Split phase Manchester format
(*iv*) Polar quaternary NRZ signalling.
Compare above scheme for their bandwidth requirement.

Solution: The required waveforms will be as shown in figure 12.42.

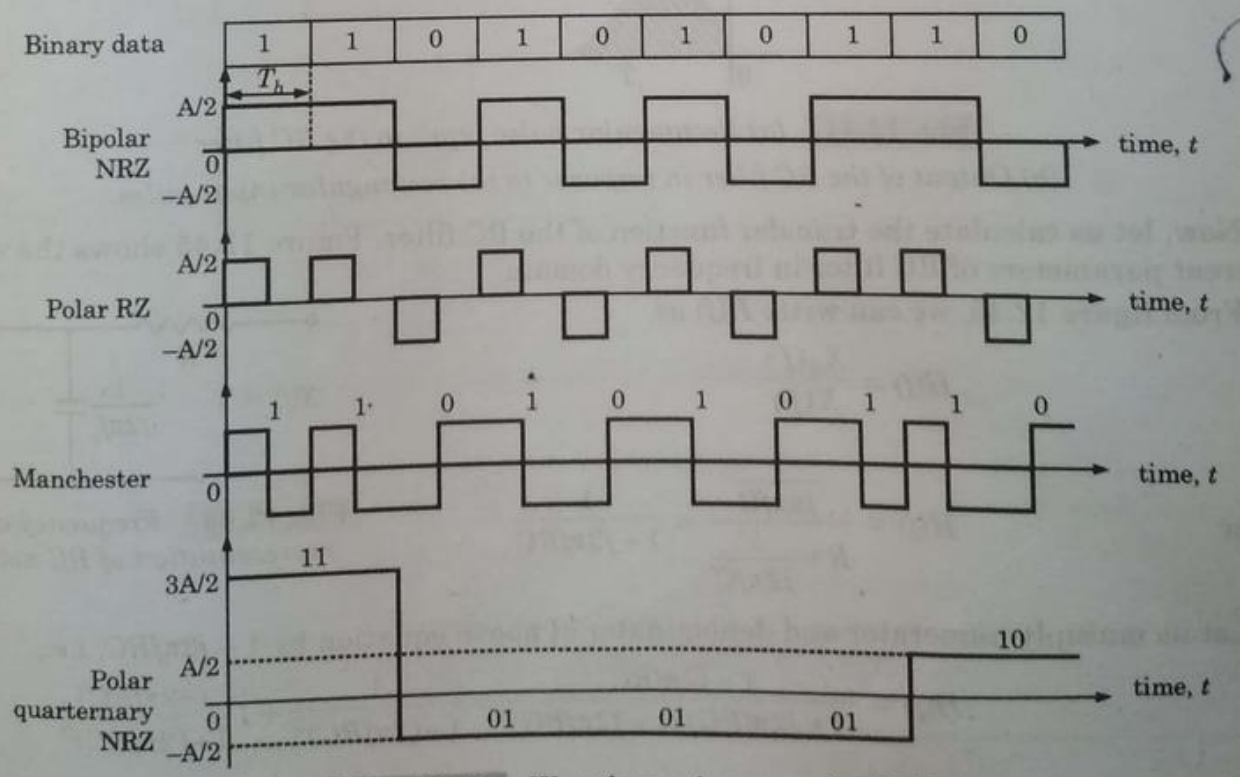


Fig. 12.42. *Waveforms for example 12.13.*

Scanned by CamScanner

Example 12.14. If the rectangular pulse is applied to a RC filter shown in figure 12.43 instead of a matched filter, evaluate the maximum signal to noise ratio $\left(\frac{S}{N}\right)_{0\,max}$ which can be obtained by RC filter and compare it with that obtained by corresponding matched filter.

(U.P. Tech., Sem. Exam., 2003-04)

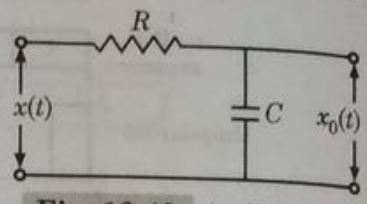


Fig. 12.43. *An RC filter to receive rectangular pulse.*

Solution: When rectangular pulse is applied as input to the RC filter, its output $x_0(t)$ will be as shown below in figure 12.44(*b*).

The output $x_0(t)$ of the RC filter can be expressed in two parts as under:

$$x_0(t) = \begin{cases} A(1-e^{-t/RC}) & \text{for} \quad 0 \le t \le T \\ A(1-e^{-t/RC})\,e^{-(t-T)/RC} & \text{for} \quad t > T \end{cases} \qquad ...(i)$$

Here, the first part represents charging of the capacitor when pulse is applied for $0 \le t \le T$. The second part represents a discharging of the capacitor when the pulse $x(t)$ is removed after $t > T$. The maximum value of $x_0(t)$ occurs at $t = T$ as shown in figure 12.44(*b*). Therefore, substituting $t = T$ in first part of equation (*i*), we get

$$x(T) = A(1 - e^{-T/RC}) \qquad ...(ii)$$

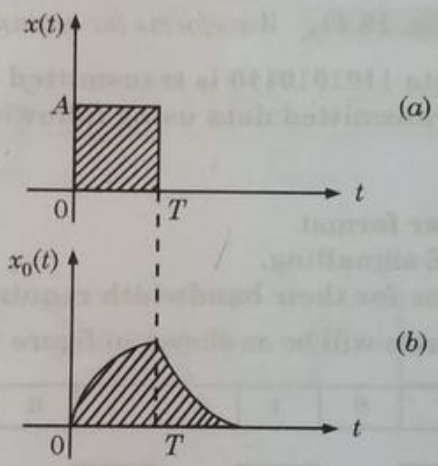


Fig. 12.44. *(a) Rectangular pulse input to the RC filter*
(b) Output of the RC filter in response to the rectangular input pulse.

Now, let us calculate the transfer function of the RC filter. Figure 12.45 shows the values of different parameters of RC filter in frequency domain.

From figure 12.45, we can write $H(f)$ as,

$$H(f) = \frac{X_0(f)}{X(f)}$$

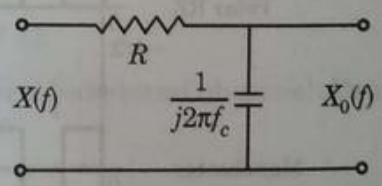


Fig. 12.45. *Frequency domain representation of RC network.*

or
$$H(f) = \frac{\frac{1}{j2\pi fC}}{R + \frac{1}{j2\pi fC}} = \frac{1}{1 + j2\pi fRC}$$

Let us multiply numerator and denominator of above equation by $1 - j2\pi fRC$, *i.e.*,

$$H(f) = \frac{1 - j2\pi fRC}{(1 + j2\pi fRC)(1 - j2\pi fRC)} = \frac{1}{1 + (2\pi fRC)^2} + j\frac{(-2\pi fRC)}{1 + (2\pi fRC)^2}$$

Scanned by CamScanner

Magnitude of $H(f)$ will be,

$$|H(f)|^2 = \frac{1}{[1+(2\pi fRC)^2]^2} + \frac{(2\pi fRC)^2}{[1+(2\pi fRC)^2]^2}$$

or
$$|H(f)|^2 = \frac{1+(2\pi fRC)^2}{[1+(2\pi fRC)^2]^2} = \frac{1}{1+(2\pi fRC)^2} \quad ...(iii)$$

We know that psds of input and output noise are related as,

$$S_{no}(f) = |H(f)|^2 S_{ni}(f)$$

The psd of white Gaussian noise is, $S_{ni}(f) = \frac{N_0}{2}$; hence, above equation becomes,

$$S_{no}(f) = |H(f)|^2 \cdot \frac{N_0}{2}$$

The normalized noise power $\overline{n_0^2(t)}$ is obtained by integrating its power spectral density (psd), *i.e.*,

$$\overline{n_0^2(t)} = \int_{-\infty}^{\infty} |H(f)|^2 \cdot \frac{N_0}{2}\, df^* = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2\, df \quad ...(iv)$$

Substituting $|H(f)|^2$ from equation (*iii*) in above equation,

$$\overline{n_0^2(t)} = \frac{N_0}{2} \int_{-\infty}^{\infty} \frac{1}{1+(2\pi fRC)^2}\, df$$

Since 'f is squared in above equation, therefore, we can change integration limits from $(-\infty, \infty)$ to $(0, \infty)$, *i.e.*,

$$\overline{n_0^2(t)} = \frac{N_0}{2} \cdot 2\int_0^{\infty} \frac{1}{1+(2\pi fRC)^2}\, df = N_0 \int_0^{\infty} \frac{1}{1+(2\pi fRC)^2}\, df \quad ...(v)$$

Let $\quad 2\pi fRC = y$

so that $\quad 2\pi RC\, df = dy$

or $\quad df = \frac{dy}{2\pi RC}$

Therefore, equation (*v*) becomes,

$$\overline{n_0^2(t)} = \frac{N_0}{2\pi RC} \int_0^{\infty} \frac{dy}{1+y^2} \quad ...(vi)$$

Here, let us use the following standard result of integration:

$$\int_0^{\infty} \frac{x^{m-1}}{1+x^n}\, dx = \frac{\pi/n}{\sin\left(\frac{m\pi}{n}\right)} \qquad n > m > 0 \quad ...(vii)$$

Now, we can rearrange equation (*vi*) as,

$$\overline{n_0^2(t)} = \frac{N_0}{2\pi RC} \int_0^{\infty} \frac{y^{1-1}}{1+y^2}\, dy \quad \text{since} \quad y^{1-1} = y^0 = 1$$

* By property of psd.

Scanned by CamScanner

Comparing this equation with equation (*vii*), we get

$$m = 1 \text{ and } n = 2$$

Thus,
$$n_0^2(t) = \frac{N_0}{2\pi RC} \cdot \frac{\pi/2}{\sin(\pi/2)} = \frac{N_0}{4RC} \quad \text{...(viii)}$$

The signal to noise power ratio is given as

$$\left(\frac{S}{N}\right)_0 = \frac{x_0^2(t)}{n_0^2(t)}$$

Substituting $x_0{}^2(t)$ from equation (*ii*) and $\overline{n_0^2(t)}$ from equation (*viii*), we get

$$\left(\frac{S}{N}\right)_0 = \frac{A^2(1-e^{-T/RC})^2}{\frac{N_0}{4RC}} = \frac{4RC\,A^2(1-e^{-T/RC})^2}{N_0}$$

Again, let $v = \frac{T}{RC}$, then above equation becomes,

$$\left(\frac{S}{N}\right)_0 = \frac{4 \cdot \frac{T}{v} A^2(1-e^{-v})^2}{N_0} = \frac{4A^2T}{N_0} \cdot \frac{(1-e^{-v})^2}{v} \quad \text{...(ix)}$$

For maximum value of $\left(\frac{S}{N}\right)_0$, $\frac{d\left(\frac{S}{N}\right)_0}{dv}$ should be equal to zero, *i.e.*,

$$\left(\frac{S}{N}\right)_{0\max} = \frac{d\left(\frac{S}{N}\right)_0}{dv} = \frac{4A^2T}{N_0} \cdot \frac{2ve^{-v}(1-e^{-v}) - (1-e^{-v})^2}{v^2} = 0$$

$\therefore$ $2v\,e^{-v}(1-e^{-v}) = (1-e^{-v})^2$

or $2v\,e^{-v} = 1 - e^{-v}$

or $2v\,e^{-v} + e^{-v} = 1$

or $e^{-v}(1+2v) = 1$

or $e^{v} = 1 + 2v$

Solving, we get

$$v = 1.26$$

Since $v = \frac{T}{RC}$, we can write,

$$\frac{T}{RC} = 1.26$$

Maximum value of $\left(\frac{S}{N}\right)_0$ will be obtained by substituting, $v = \frac{T}{RC} = 1.26$ in equation (*ix*)

Thus,
$$\left(\frac{S}{N}\right)_{0\max} = \frac{4A^2T}{N_0} \frac{(1-e^{-1.26})^2}{1.26} = 0.814 \cdot \frac{2A^2T}{N_0} \quad \text{...(x)}$$

In a last example we have obtained $\left(\frac{S}{N}\right)_{0\max}$ for a rectangular pulse input to the matched filter.

Scanned by CamScanner

Thus, we have

$$\left(\frac{S}{N}\right)_{0\max} \text{(matched filter)} = \frac{2A^2T}{N_0}$$

With this result we can write equation (x) as under:

$$\left[\left(\frac{S}{N}\right)_{0\max} \text{(RC filter)}\right] = 0.814 \times \left[\left(\frac{S}{N}\right)_{0\max} \text{matched filter}\right]$$

Thus, because of RC filter, the maximum signal to noise ratio obtained is reduced to 0.814 of that of matched filter. **Hence Proved**

Example 12.15. Find the impulse response of the matched filter for a Gaussian signal pulse given by,

$$x(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2\sigma^2}$$

The noise on the channel is a white noise with power density spectrum of $\frac{N}{2}$.

Calculate the maximum signal to noise ratio $\left(\frac{S}{N}\right)_{0\max}$ achieved by this filter.

Solution: The given gaussian pulse is,

$$x(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2\sigma^2} \quad ...(i)$$

We know that the impulse response of the matched filter is given as,

$$h(t) = \frac{2k}{N_0} x(T-t)$$

Let $\frac{2k}{N_0} = 1$, then above equation will be,

$$h(t) = x(T-t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(T-t)^2/2\sigma^2} \quad ...(ii)$$

This equation gives the required impulse response of the matched filter for Gaussian signal pulse.

We know that the maximum signal to noise ratio of matched filter is given as,

$$\left(\frac{S}{N}\right)_{0\max} = \frac{2E}{N_0} = \frac{2E}{N} \quad \left(\text{since } \frac{N_0}{2} = \frac{N}{2} \text{ given}\right) \quad ...(iii)$$

Here, E is the energy of the signal $x(t)$. It can be calculated as,

$$E = \int_{-\infty}^{\infty} x^2(t)\, dt$$

Substituting value of $x(t)$ from equation (i), we get

$$E = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{-t^2/\sigma^2} dt = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-t^2/\sigma^2}\, dt \quad ...(iv)$$

Now, substituting

$$\frac{t^2}{\sigma^2} = y^2$$

Scanned by CamScanner

so that $\frac{t}{\sigma} = y$

or $dt = \sigma\, dy$

however, the integration limits will remain same.

Now, substituting these values in equation (*iv*), we get

$$E = \frac{1}{2\pi\sigma^2}\int_{-\infty}^{\infty} e^{-y^2}dy = \frac{1}{2\pi\sigma}\int_{-\infty}^{\infty} e^{-y^2}dy \quad ...(v)$$

Let us use the following standard result of integration:

$$\int_{-\infty}^{\infty} e^{-a^2x^2+bx}dx = \frac{\sqrt{\pi}}{a}e^{b^2/(4a^2)}, a > 0$$

Here, putting $b = 0$ and $a = 1$, we get

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi} \quad ...(vi)$$

Using the result of above equation, we can obtain equation (*v*) as,

$$E = \frac{1}{2\pi\sigma}\cdot\sqrt{\pi} = \frac{1}{2\sigma\sqrt{\pi}} \quad ...(viii)$$

Substituting this value of E in equation (*iii*), the maximum signal to noise ratio is given as,

$$\left(\frac{S}{N}\right)_{0\max} = \frac{2\cdot\frac{1}{2\sigma\sqrt{\pi}}}{N} = \frac{1}{N\sigma\sqrt{\pi}} \quad \textbf{Ans.}$$

Example 12.16. Two messages are transmitted by mark and space using a simple binary pulse shown in figure 12.45. Find error probability of the optimum receiver assuming that the probability of $x(t)$ being present is 0.5. Assume that the channel noise is a white noise of power spectral density $\frac{N}{2}$, where $N = 10^{-4}$.

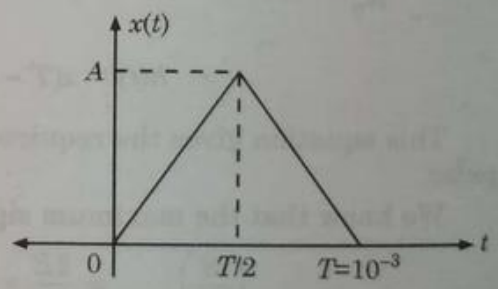


Fig. 12.45. *Transmitted signal x(t).*

Solution: The pulse of figure 12.46 can be represented mathematically as,

$$x(t) = \begin{cases} \frac{2A}{T}t & \text{for} \quad 0 < t \le T/2 \\ 2A - \frac{2A}{T}t & \text{for} \quad T/2 < t \le T \end{cases} \quad ...(i)$$

The energy of signal $x(t)$ can be calculated as under:

$$E = \int_{-\infty}^{\infty} x^2(t)\, dt$$

Substituting value of $x(t)$, we get

$$E = \int_{0}^{T/2}\left(\frac{2A}{T}t\right)^2 dt + \int_{T/2}^{T}\left(2A - \frac{2A}{T}t\right)^2 dt$$

Scanned by CamScanner

or $$E = \frac{4A^2}{T^2}\int_0^{T/2} t^2 dt + 4A^2 \int_{T/2}^{T}\left(1-\frac{t}{T}\right)^2 dt$$

or $$E = \frac{4A^2}{T^2}\left[\frac{t^3}{3}\right]_0^{T/2} + 4A^2\int_{T/2}^{T}\left(1-\frac{2t}{T}+\frac{t^2}{T^2}\right)dt$$

or $$E = \frac{4A^2}{T^2}\cdot\frac{T^3/8}{3} + 4A^2\left[t-\frac{2}{T}\cdot\frac{t^2}{2}+\frac{1}{T^2}\cdot\frac{t^3}{3}\right]_{T/2}^{T}$$

or $$E = \frac{4A^2}{T^2}\frac{T^3}{24} + 4A^2\left\{\left(T-\frac{T^2}{T}+\frac{1}{T^2}\cdot\frac{T^3}{3}\right)-\left(\frac{T}{2}-\frac{1}{T}\cdot\frac{T^2}{4}+\frac{1}{3T^2}\cdot\frac{T^3}{8}\right)\right\}$$

or $$E = \frac{A^2T}{6} + 4A^2\frac{T}{24} = \frac{A^2T}{3} \qquad ...(ii)$$

Since the channel noise is *white noise*, the optimum filter is same as matched filter. Hence, we can use the relations derived for matched filter. The probability of error of matched filter is given as,

$$P_e = \frac{1}{2}\,erfc\sqrt{\frac{E}{N_0}} \qquad ...(iii)$$

Here, it may be noted that this probability of error, P_e is obtained for the transmission and reception of two symbols $x_1(t)$ and $x_2(t)$. Even though not stated earlier, it is assumed that probability of occurrence of $x_1(t)$ and $x_2(t)$ is same. Since there are only two symbols, probability of occurrence of $x_1(t)$ and $x_2(t)$ both will be 0.5. Hence, this example probability of $x(t)$ [*i.e.*, one of the symbols] is given 0.5. Thus, the probability of other symbol which is represented by space will also be 0.5. Thus, the error probability given by equation (*iii*) above can be used to calculate P_e in this example.

Here $N_0 = N = 10^{-4}$ given

and $E = \frac{A^2T}{3} = \frac{A^2\times10^{-3}}{3}$ since $T = 10^3$ given.

Substituting these values in equation (*iii*) we get

$$P_e = \frac{1}{2}\,erfc\sqrt{\frac{A^2\times10^{-3}}{3}\cdot\frac{1}{10^{-4}}}$$

$$= \frac{1}{2}\,erfc\sqrt{\frac{10A^2}{3}}.$$

This is the required probability of error. **Ans.**

Example 12.17. Figure 12.46 shows signal $x(t)$

(*a*) Determine the impulse response of the filter matched to this signal and sketch it.

(*b*) Plot the output of the matched filter.

Solution: The impulse response of the matched filter is given as

$$h(t) = x(T-t) \text{ assuming } \frac{2k}{N_0} = 1$$

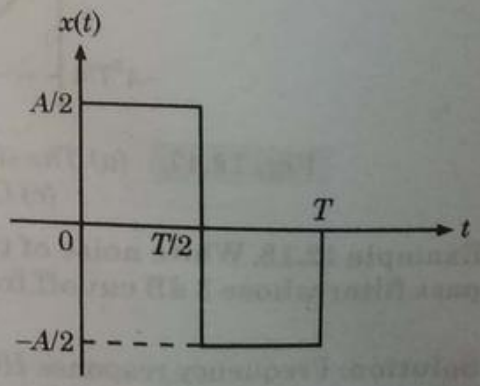


Fig. 12.46. *Input signal.*

The impulse response is sketched in figure 12.48(*b*), it can be described as,

Scanned by CamScanner

$$h(t) = \begin{cases} \dfrac{-A}{2} & \text{for} \quad 0 \le t \le \dfrac{T}{2} \\ \dfrac{A}{2} & \text{for} \quad \dfrac{T}{2} \le t \le T \end{cases}$$

The output of the matched filter is obtained by convolution of $x(t)$ and $h(t)$, *i.e.*,

$$y(t) = x(t) \otimes h(t)$$

Figure 12.47(*c*) shows the output signal. The waveform has maximum value at $t = T$, it is $\frac{A^2T}{4}$.

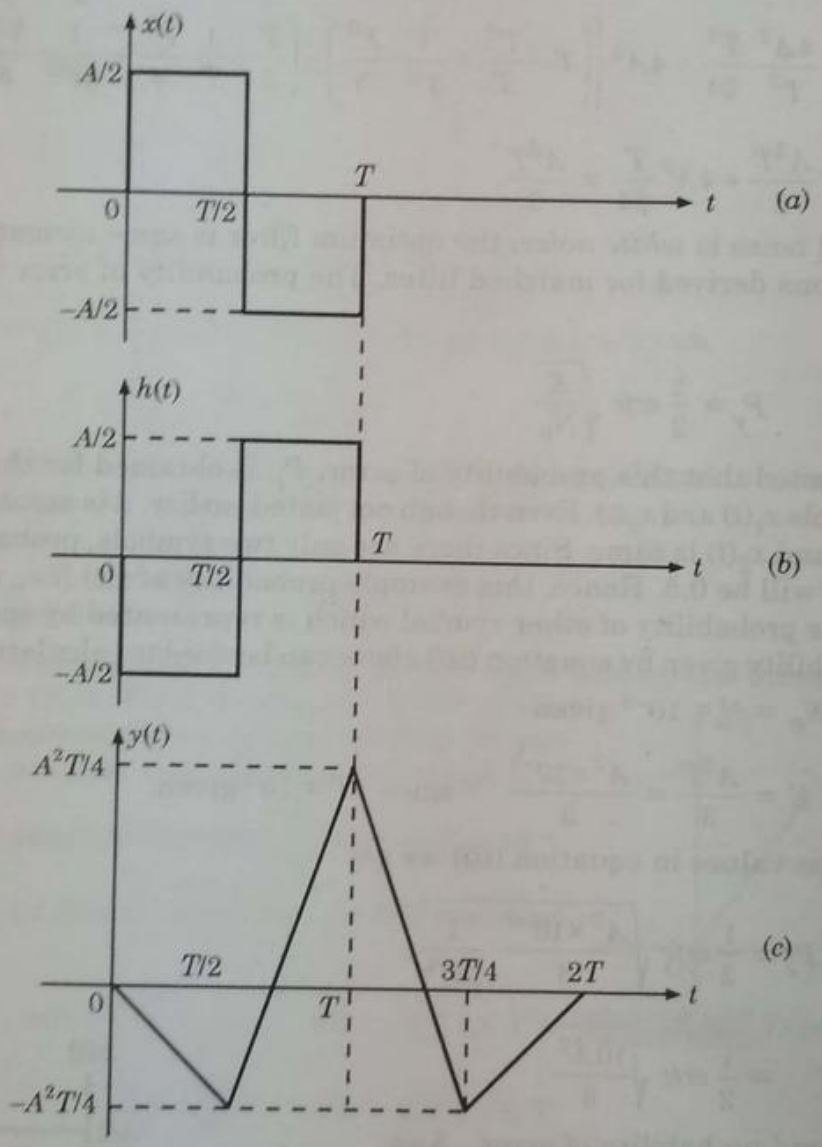


Fig. 12.47. *(a) The signal pulse x(t), (b) Impulse response of the signal of (a), (c) Output waveform of the matched filter*

Example 12.18. White noise of two-sided density 2×10^{-6} V^2/Hz is applied to a simple R-C low pass filter whose 3 dB cut off frequency is 4 kHz. Find the squared value of the noise output.

(GATE Examination, 1998)

Solution: Frequency response $H(\omega)$ of an RC low-pass filter with 3 dB cut of frequency of 4 kHz is given by

$$H(\omega) = \frac{1}{1 + j\omega / \omega_0}$$

Scanned by CamScanner

Now, psd at the output

$$S_{yy}(\omega) = |H(\omega)|^2 \cdot S_{xx}(\omega) = \frac{1}{1+(\omega/\omega_0)^2}\frac{\eta}{2}$$

Then, the mean square value of the noise output is,

$$N_0 = E[N_0^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{yy}(\omega)\cdot d\omega$$

or
$$N_0 = \frac{1}{2\pi}\cdot\int_{-\infty}^{\infty}\frac{\eta}{2}\cdot\frac{d\omega}{1+(\omega/\omega_0)^2} = \frac{\omega_0^2\,\eta}{4\pi}\int_{-\infty}^{\infty}\frac{d\omega}{\omega^2/\omega_0^2}$$

or
$$N_0 = \frac{\eta\omega_0^2}{4\pi}\frac{1}{\omega^0}\left[\tan^{-1}\frac{\omega}{\omega_0}\Big|_{-\infty}^{\infty}\right] = \frac{\eta\omega_o}{4\pi}.\pi$$

Given $\eta = 2\times10^{-6}\ V^2/Hz$

and $\omega_o = 2\pi\times4\times10^3$ Hz

then,
$$N_0 = \frac{2\times10^{-6}\times2p\times4\times10^3}{4} = 4\pi\times10^{-3}\ V^2$$

Example 12.19. A signal $s(t)$, $e^{-at}u(t)$ where $u(t)$ is the unit step function, is applied to the input of a low-pass filter having

$$|H(\omega)| = \frac{b}{\sqrt{\omega^2+b^2}}$$

Calculate the value of the ratio, $\frac{a}{b}$, for which 50% of the input energy is transferred to the output.

Solution: Input signal, $s(t) = e^{-at}.u(t)$.

Its power spectral density,

$$S_{xx}(\omega) = \frac{1}{\int\omega+a}$$

At the filter output, power spectral density.

$$S_{yy}(\omega) = S_{xx}(\omega)\times|H(\omega)|^2$$

$$S_{yy}(\omega) = \frac{b^2}{\omega^2+b^2}\cdot\frac{1}{S\omega+a}$$

Given, $$\frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}|S_{xx}(\omega)|^2\,d\omega\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}|S_{xx}(\omega)|^2\,d\omega$$

or
$$\frac{1}{2}\cdot\int_{-\infty}^{\infty}\frac{1}{\omega^2+a^2}\,d\omega = \frac{b^2}{b^2-a^2}\int_{-\infty}^{\infty}\left[\frac{1}{\omega^2+a^2}-\frac{1}{\omega^2+b^2}\right]d\omega$$

Scanned by CamScanner

or $$\frac{1}{2}\cdot\frac{1}{a}\left[\tan^{-1}\frac{\omega}{a}\int_{-\infty}^{\infty}\right] = \frac{b^2}{b^2-a^2}\left[\frac{1}{a}\tan^{-1}\frac{\omega}{a}\int_{-\infty}^{\infty}\right] - \frac{b^2}{(b^2-a^2)}\left[\frac{1}{b}\tan^{-1}\frac{\omega}{b}\int_{-\infty}^{\infty}\right]$$

$$\frac{1}{2a}\cdot\pi = \frac{b^2}{b^2-a^2}\pi\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$\frac{1}{2} = \frac{b}{a+b}$$

Hence $$\frac{a}{b} = 1$$

Example 12.20. The equivalent noise bandwidth of a system as

$$B_{eq} = \frac{1}{2\pi}\frac{\int_0^{\infty}|H(\omega)|^2\,d\omega}{|H(\omega)|^2_{max}}.Hz$$

where $H(\omega)|^2 \max = \max|H(\omega)|^2$

Determine the equivalent noise bandwidth of the ideal bandpass filter (BPF) and that of a low pass RC filter.

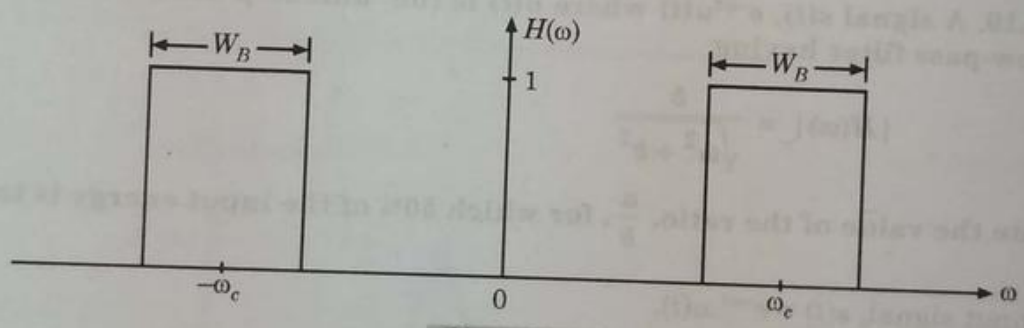


Fig. 12.48.

Solution: From figure 12.48, we have

$$\max|H(\omega)|^2 = 1$$

$$B_{eq} = \frac{1}{2\pi}\frac{\int_0^{\infty}|H(\omega)|^2\,d\omega}{|H(\omega)|^2_{max}}.Hz = \frac{1}{2\pi}\cdot\frac{W_B}{1} = \frac{W_B}{2\pi}$$

For the low-pass RC filter, frequency response is given as,

$$H(\omega) = \frac{1}{1+\int\omega RC}$$

then, $$\max|H(\omega)|^2 = |H(0)|^2 = 1$$

Thus, $$B_{eq} = \frac{1}{2\pi}\cdot\int_0^{\infty}|H(\omega)|^2\,d\omega = \frac{1}{2\pi}\cdot\int_0^{\infty}\frac{1}{1+(\omega RC)^2}\,d\omega$$

Scanned by CamScanner

$$B_{eq} = \frac{1}{2\pi} \cdot \frac{1}{RC}\left[\tan^{-1} \omega RC\Big|_0^\infty\right] = \frac{1}{4RC} \text{ Hz}$$

Example 12.21. A received binary NRZ signal assumes the voltages levels of 500 millivolts and 500 millivolts respectively for '1' and '0' transmission with a bit rate or r bits/second. The signal is corrupted by additive white Gaussian noise with a two-sided spectral density of 10^{-6} volts 2/Hz. The received signal is processed by an Integrate and Dump circuit in every bit interval and compared with a zero threshold to take a bit decision.

Assuming '1' and '0' transmission to be equally likely, the maximum value of r such that the bit error probability $\leq 10^{-5}$.

Given: $\frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-z^2/2} . dz = 10^{-5}$ at $x = 4.27$

$$erfc\ (x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-z^2/2} . dz = 2 \times 10^{-5} \text{ at } x = 3.02$$

Solution:

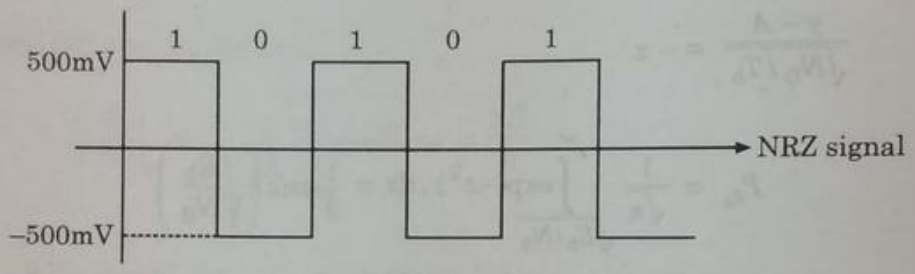


Fig. 12.49.

The transmitted signal is

$$S_i(t) = \begin{cases} s_1(t) = +A = +500 \text{ mV}, 0 \leq t \leq T \\ s_2(t) = -A = -500 \text{ mV}, 0 \leq t \leq T \end{cases}$$

then, energy of the difference signal at the filter input

$$E_d = \int_0^T [s_1(t) - s_2(t)]^2\, dt = \int_0^T (2A)^2\, dt = 4\, A^2 T.$$

Bit error probability,

$$P_s = Q\left(\sqrt{\frac{Ed}{2N_0}}\right) = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)$$

Given $\quad P_s \leq 10^{-5}$

then, $\quad Q\left(\sqrt{\frac{2A^2T}{N_0}}\right) \leq 10^{-5}$

or $\quad \frac{2A^2T}{N_0} = 4.27$

Now as $\quad \frac{N_o}{2} = 10^{-6}$ volts/Hz

Scanned by CamScanner

$$\therefore \qquad T = \frac{2 \times 10^{-6} \times (4.27)^2}{2 \times (500 \times 10^{-3})^2} = 7.29 \times 10{-}5$$

$$erfc(u) = \frac{2}{\sqrt{\pi}} \int_u^{\infty} \exp(-z^2)\, du$$

We have, $$P_{e_o} = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Now, the probability of error, on condition that sending symbol is 0,

$$P_{e_o} = \int_{-\infty}^{\infty} f_\gamma (y/1)\, dy = \frac{1}{\sqrt{\pi N_0 / T_b}} \int_{-\infty}^{\lambda} \exp\left(-\frac{(y-A)^2}{N_0 / T_b}\right) dy$$

Setting $\lambda = 0$, and putting

$$\frac{y - A}{\sqrt{N_0 / T_b}} = -z$$

we have, $$P_{e_o} = \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_b/N_0}}^{\infty} \exp(-z^2) \,.\, dz = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Now, the average probability of symbol error P_e in the receiver is given by,

$$P_e = P_0\, P_{e_0} + P_1, P_{e_1}$$

where P_e and P_1 are the prior probabilities of binary symbols 0 and 1, respectively, Since $P_{e_o} = P_{e_1}$ and assuring the symbols 0 and 1 occur with equal probability *i.e.*,

$$p_0 = p_1 = \frac{1}{2}$$

then, $$P_e = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Hence, average probability of symbol error in a binary coded PCM receiver depends solely on E_b/N_0, the ratio of transmitted signal energy per bit to noise spectral density.

Example 12.22. Prove that signal to noise ratio, $\left(\frac{S}{N}\right)_o \le \frac{2E}{\eta}$

where E is the energy of the input signal $s(t)$ and $\eta/2$ is the power spectral density of the input noise $\eta(t)$.

Solution: Suppose the frequency response of the linear filter is $H(\omega)$.

Then the matched filter output,

$$g_o(T) = k \int_{-\infty}^{\infty} H(\omega) \,.\, s(\omega) e^{j\omega T} d\omega$$

where $$s(\omega) = \int [s(t)]$$

Scanned by CamScanner

Output noise power,

$$N_0 = E[\eta_o^2(t)] = \frac{\eta}{2} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)^2 \, d\omega$$

$$\left(\frac{S}{N}\right)_o = \frac{\left[\frac{1}{(2\pi)}\right]^2 \left| \int_{-\infty}^{\infty} H(\omega) s(\omega) \, e^{j\omega T} \, d\omega \right|^2}{\left(\frac{\eta}{2}\right)\left[\frac{1}{2\pi}\right] \int_{-\infty}^{\infty} |H(\omega)^2 \, d\omega} \quad ...(i)$$

Now, using Schwarz inequality, we have

$$\left| \int_{-\infty}^{\infty} H(\omega) s(\omega) \, e^{j\omega T} \, d\omega \right|^2 \le \int_{-\infty}^{\infty} |H(\omega)^2 \, d\omega \int_{-\infty}^{\infty} |s(\omega)^2 \, d\omega \quad ...(ii)$$

Using equations (*ii*) and (*i*), we get

$$\left(\frac{S}{N}\right)_o \le \frac{2}{\eta} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} |s(\omega)^2 \, d\omega = \frac{2E}{\eta}$$

where $E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |s(\omega)^2 \, d\omega$, energy of input signal.

Example 12.23. Compute the matched filter output over (0, *T*) to the pulse waveform

$$s(t) = \begin{cases} e^{-t} & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

Solution: For the given $s(t)$, the impulse response of the matched filter is,

$$h(t) = s(T - t).$$

or $$h(t) = e^{-(T-t)}$$

Now, output $$z(t) = s(t) \otimes h(t)$$

or $$z(t) = \int_{-\infty}^{\infty} e^{-\tau} \cdot e^{-(T-1+\tau)} \cdot d\tau = e^{-T} \int_{-\infty}^{\infty} e^{-\tau} \cdot e^{-(T-t+\tau)} \cdot d\tau$$

or $$z(t) = -\frac{e^{-T}}{2} \cdot \left[e^{t-2\tau}\right]_0^t = e^{-T} \cdot \frac{e^t - e^{-t}}{2} = e^{-T} \sinh t$$

Example 12.24. Consider the binary sequence 0100101. Draw the waveforms for the following signalling formats:

(*i*) Unipolar NRZ signaling format

(*ii*) Bipolar RZ signaling format

(*iii*) AMI (alternate mark inversion) RZ signaling format.

(U.P. Tech-Semester Exam., 2002-2003)

Scanned by CamScanner

Solution: Figure 12.50 shows all the formats.

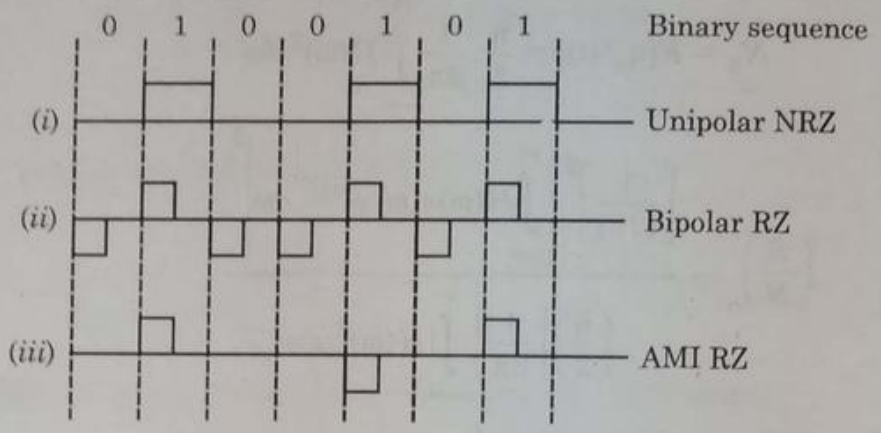


Fig. 12.50. *Signaling formats for example 12.24.*

Example 12.25. Discuss the advantages and disadvantages of the three signaling formats illustrated in figure 12.51 of example 12.24.

Solution: The unipolar NRZ signalling format, although conceptually simple, has certain disadvantages. There are no pulse transitions for long sequences of 0 s or 1 s, which are necessary if one wishes to extract timing or synchronizing information, and there is no way to detect when and if an error has occurred from the received pulse sequence.

The bipolar RZ signalling format guarantees the availability of timing information, but there is no error detection capability.

The AMI RZ signalling format has an error detection property, if two sequential pulses (ignoring intervening 0 s) are detected with the same polarity, it is evident, it is evident that an error has occurred. However, to guarantee the availability of timing information, it is necessary to restrict the allowable number of consecutive 0 s.

Example 12.26. Consider a binary sequence with a long sequence of 1 s followed by a single 0 and then a long sequence of 1 s. Draw the waveforms for this sequence, using the following signaling formats:

(*i*) Unipolar NRZ signaling
(*ii*) Bipolar NRZ signaling
(*iii*) AMI RZ signaling
(*iv*) Split-Phase (Manchester) signaling

Solution: Figure 12.51 shows all the required formats.

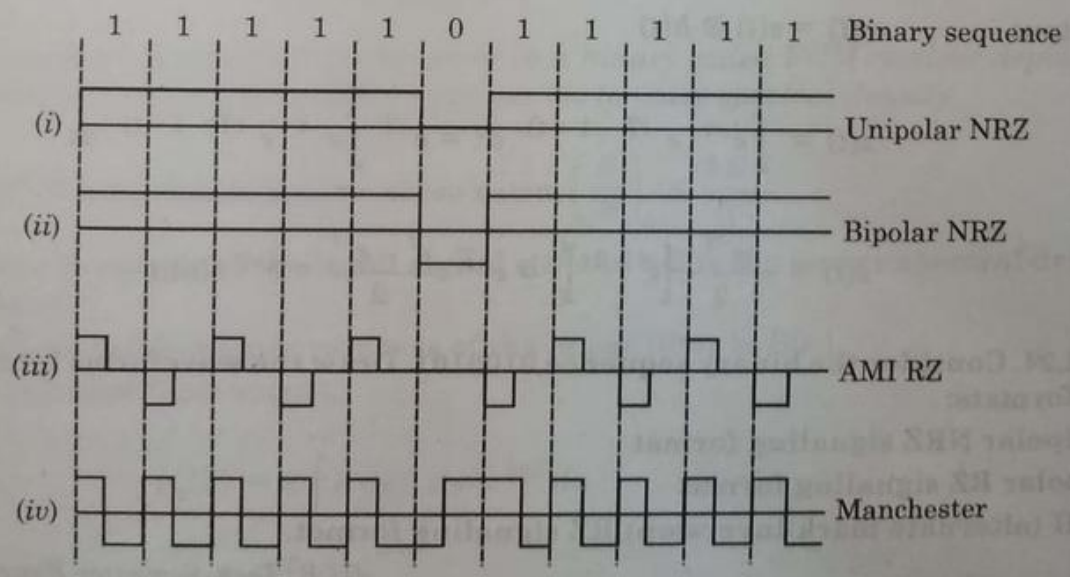


Fig. 12.51.

Scanned by CamScanner

Example 12.27. Show that (a pulse-shape, function) $h(t)$, with Fourier transform given by $H(\omega)$, that satisfies the criterion

$$\sum_{k=-\infty}^{\infty} H\left(\omega + \frac{2\pi k}{T}\right) = 1 \text{ for } |\omega| \le \frac{\pi}{T} \qquad ...(i)$$

has $h(nT)$ which is given by

$$h(nT) = \begin{cases} \frac{1}{T} & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases} \qquad ...(ii)$$

(GATE Examination-1999)

Solution: The criterion given in equation (*i*) is known as *Nyquist's pulse-shaping criterion.*

Taking the inverse Fourier transform of $H(\omega)$, we have

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{jwt}d\omega$$

The range of integration in the above equation can be divided into segments of length $2\pi/T$ as under:

$$h(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{(2k-1)\pi/T}^{(2k+1)\pi/T} H(\omega)e^{jwt}d\omega$$

and we can write $h(nT)$ as

$$h(nT) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{(2k-1)\pi/T}^{(2k+1)\pi/T} H(\omega)e^{jwnT}d\omega \qquad ...(iii)$$

Changing variable $u = \omega - 2\pi(k/T)$, equation (*iii*) becomes

$$h(nT) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\pi/T}^{\pi/T} H\left(u + \frac{2\pi k}{T}\right)e^{j(u+2\pi k/T)nT}du$$

Assuming that the integration and summation can be interchanged, we have

$$h(nT) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T}\sum_{k=-\infty}^{\infty} H\left(u + \frac{2\pi k}{T}\right)e^{junT}du$$

Finally, if equation (*i*) is satisfied, then we have

$$h(nT) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} e^{junT}du = \frac{1}{T}\frac{\sin n\pi}{n\pi} = \begin{cases} \frac{1}{T} & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

which verifies that h(t) with a Fourier transform H(ω) satisfying criterion in equation (i) produces zero ISI. **Ans.**

Example 12.28. A communication channel of bandwidth 75 kHz is required to transmit binary data at a rate of 0.1 Mb/s using raised-consine pulses. Determine the roll-off factor α.

Solution: We have

$$T_b = \frac{1}{0.1(10^6)} = 10^{-5} \text{ s}$$

Scanned by CamScanner

or $f_B = 75 \text{ kHz} = 75\ (10^3) \text{ Hz}$

We know that $1 + \alpha = 2 f_B T_b = 2(75)(10^3)(10^{-5}) = 1.5$

Hence, we obtain $\alpha = 0.5$ **Ans.**

Example 12.29. In a certain telemetry system, eight message signals having 2 kHz bandwidth each are time-division multiplexed using a binary PCM technique. The error in sampling amplitude cannot be greater than 1 per cent of the peak amplitude. Determine the minimum transmission bandwidth required if raised-consine pulses with roll-off factor $\alpha = 0.2$ are used. The sampling rate must be at least 25 per cent above the Nyquist rate.

Solution: We know that the maximum quantizing error must satisfy

$$\epsilon_{\max} = \frac{\Delta}{2} = \frac{m_p}{q} \le 0.01\, m_p$$

Hence, $q \ge 100$, and we choose $q = 128 = 2^7$. The number of bits per sample required is 7.

Since the Nyquist sampling rate is $2 f_m = 4000$ samples/s, the sampling rate for each signal is

$$f_s = 1.25(4000) = 5000 \text{ samples/s}$$

There are eight time-division multiplexed signals, requiring a total of

$$8(5000) = 40{,}000 \text{ samples/s}$$

Now, since each sample is encoded by 7 bits, therefore, the resultant bit rate will be

$$\frac{1}{T_b} = 7(40{,}000) = 280 \text{ kb/s}$$

Further, we know that the minimum transmission bandwidth required is

$$f_B = \frac{1 + 0.2}{2}(280) = 168 \text{ kHz} \quad \textbf{Ans.}$$

SUMMARY

1. The pulse code modulation (PCM), Delta modulation (DM), Adaptive delta modulation (ADM) etc. are used to convert an analog signal to digital data. This digital data is the sequence of binary symbols.
2. The output of a multiplexer is coded into electrical pulses or waveforms for the prupose of transmission over the channel. This process is known as line coding or transmission coding.
3. There are several possible ways of assigning waveforms (*i.e.* pulses) to the digital data. In the binary case (two symbols), for example, conceptually the simplest line code is on-off, where a 1 is transmitted by a pulse $p(t)$ and a 0 is transmitted by no pulse (*i.e.* zero signal).
4. Another line code that in the past appeared promising is the duobinary (and modified duobinary) proposed by Lender. Although this code is better than the bipolar in terms of bandwidth efficiency, it has lost its appeal due to some practical problems and will not be discussed here.
5. Regenerative repeaters are used at regularly spaced intervals along a digital transmission line to detect the incoming digital and regenerate new clean pulses for further transmission along the line. This process periodically eliminates, and thereby combats, the accumulation of noise and signal distortion along the transmission path.
6. The timing signal (*i.e.*, the resonant circuit output) is sensitive to the incoming bit pattern. In the on-off or bipolar case, a 0 is transmitted by "no pulse". Hence, if there are too many 0's in a sequence (no pulses), there is no signal at the input of the resonant circuit and the sinusoidal output of the resonant circuit starts decaying, thus causing error in the timing information.
7. The digital data may be transmitted by various transmission or line codes such as on-off, polar, bipolar and so on. This is called line-coding. Each type of line-code has its advantages and disadvantages.

Scanned by CamScanner

CHAPTER 14

Digital Modulation Techniques

Inside this Chapter

14.1. Introduction 697
14.2. Digital Modulation Formats 698
14.3. Types of Digital Modulation Techniques 699
14.4. Coherent Binary Modulation Techniques 699
14.5. Coherent Binary Amplitude Shift Keying or On-Off Keying 699
14.6. Binary Phase Shift Keying (BPSK) 702
14.7. Coherent Binary Frequency Shift Keying (BFSK) 710
14.8. Non-Coherent Binary Modulation Techniques 715
14.9. Non-Coherent Binary Amplitude Shift Keying (ASK) 715
14.10. Non-Coherent Detection of FSK 716
14.11. Differential Phase Shift Keying (DPSK) 716
14.12. Quadrature Phase Shift Keying (APSK) 721
14.13. Generation of QPSK 721
14.14. Minimum Shift Keying (MSK) 730
14.15. Comparison of Digital Modulation Techniques 739

14.1. Introduction

As discussed earlier, Modulation is defined as the process by which some characteristics of a carrier is varied in accordance with a modulating signal. In digital communications, the modulating signal consists of binary data or an M-ary encoded version of it. This data is used to modulate a carrier wave (usually sinusoidal) with fixed frequency. In fact, the input data may represent the digital computer outputs or PCM waves generated by digitizing voice or video signals. The channel may be a telephone channel, microwave radio link, satellite channel or an optical fiber. In digital communication, the modulation process involves switching or keying the amplitude, frequency or phase of the carrier in accordance with the input data.

Thus, there are three basic modulation techniques for the transmission of digital data. They are known as amplitude-shift keying (ASK), frequency shift keying (FSK) and phase-shift keying (PSK) which can be viewed as special cases of amplitude modulation frequency modulation and phase modulation respectively.

Scanned by CamScanner

The present chapter is devoted to detailed discussion of digital modulation techniques: their noise performance, spectral properties, their merits and limitations, applications and other related aspects.

14.2. Digital Modulation Formats

When we have to transmit a digital signal over a long distance, we need continuous-wave (CW) modulation. For this purpose, the transmission medium can be in form of radio, cable or other type of channel. Also, a carrier signal having some frequency f_c is used for modulation. Then the modulating digital signal modulates some prameter like frequency, phase or amplitude of the carrier. Due to this process, there is some deviation in carrier frequency f_c. This deviation is known as the bandwidth of the channel. This means that the channel has to transmit some range or band of frequencies. Such type of transmission is known as **bandpass transmission** and the communication channel is known as **bandpass channel**.

Here, the word bandpass is used since the range of frequencies does not start from zero Hz to f_m Hz. In fact, the range of frequencies from zero Hz to f_m Hz is known as **low-pass signal** and such channel is known as **low-pass channel.**

Now, when it is required to transmit digital signals on a bandpass channel, the amplitude, frequency or phase of the sinusoidal carrier is varied in accordance with the incoming digital data. Since the digital data is in discrete steps, the modulation of the bandpass sinusoidal carrier is also done in discrete steps. Due to this reason, this type of modulation (*i.e.*, Digital modulation) is also known as switching or signaling. Now, if an amplitude of the carrier is switched depending on the input digital signal, then it is called Amplitude shift keying (ASK).

This process is quite similar to analog amplitude modulation. If the frequency of the sinusoidal carrier is switched depending upon the input digital signal, then it is known as the frequency shift keying (FSK). This is very much similar to the analog frequency modulation. If the phase of the carrier is switched depending upon the input digital signal, then it is called phase shift keying (PSK). This is similar to phase modulation. Since the phase and frequency modulation has constant amplitude envelope, therefore FSK and PSK also has a constant amplitude envelope. Because of constant amplitude of FSK and PSK, the effect of non-linearities, noise interference is minimum on signal detection. However, these effects are more pronounced on ASK. Therefore, FSK and PSK are preferred over ASK.

Figure 14.1 shows the waveforms for amplitude-shift keying, phase-shift keying and frequency shift keying. In these waveforms, a single feature of the carrier (*i.e.*, amplitude, phase or frequency) undergoes modulation.

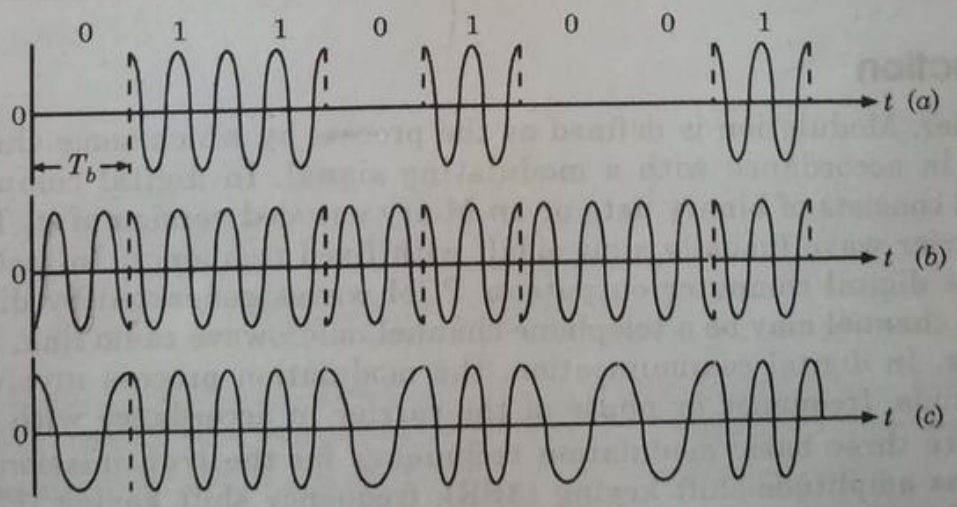


Fig. 14.1. *The three basic forms of signaling binary information, (a) Amplitude-shift keying, (b) Phase-shift keying, (c) Frequency shift keying with continuous phase*

Scanned by CamScanner

In digital modulations, instead of transmitting one bit at a time, we transmit two or more bits simultaneously. This is known as M-ary transmission. This type of transmission results in reduced channel bandwidth. However, sometimes, we use two quadrature carriers for modulation. This process is known as **Quadrature modulation.**

Thus, we see that there are a number of modulation schemes available to the designer of a digital communication system required for data transmission over a bandpass channel.

Every scheme offers system trade-offs of its own. However, the final choice made by the designer is determined by the way in which the available primary communication resources such as transmitted power and channel bandwidth are best exploited. In particular, the choice is made in favour of a scheme which possesses as many of the following design characteristics as possible:

(*i*) Maximum data rate,
(*ii*) Minimum probability of symbol error,
(*iii*) Minimum transmitted power,
(*iv*) Maximum channel bandwidth,
(*v*) Maximum resistance to interfering signals,
(*vi*) Minimum circuit complexity.

14.3. Types of Digital Modulation Techniques

(U.P. Tech., Sem. Examination, 2003-2004)

Basically, digital modulation techniques may be classified into coherent or non-coherent techniques, depending on whether the receiver is equipped with a phase-recovery circuit or not. The phase-recovery circuit ensures that the oscillator supplying the locally generated carrier wave receiver is synchronized* to the oscillator supplying the carrier wave used to originally modulate the incoming data stream in the transmitter.

(*i*) Coherent Digital Modulation Techniques

Coherent digital modulation techniques are those techniques which employ coherent detection. In coherent detection, the local carrier generated at the receiver is phase locked with the carrier at the transmitter. Thus, the detection is done by correlating received noisy signal and locally generated carrier. The coherent detection is a synchronous detection.

(*ii*) Non-coherent Digital Modulation Techniques

Non-coherent digital modulation techniques are those techniques in which the detection process does not need receiver carrier to be phase locked with transmitter carrier.

The advantage of such type of system is that the system becomes simple. But the drawback of such a system is that the error probability increases.

In fact, the different digital modulation techniques are used for various specific application areas.

14.4. Coherent Binary Modulation Techniques

As mentioned earlier, the binary (*i.e.*, digital) modulation has three basic forms amplitude-shift keying (ASK), phase-shift keying (PSK) and frequency-shift keying (FSK). In this section, let us discuss different coherent binary modulation techniques.

14.5. Coherent Binary Amplitude Shift Keying or On-Off Keying

Amplitude shift keying (ASK) or ON-OFF keying (OOK) is the simplest digital modulation technique. In this method, there is only one unit energy carrier and it is switched on or off depending upon the input binary sequence. The ASK waveform may be represented as,

$$s(t) = \sqrt{2P_s}\cos(2\pi f_c t) \quad \text{(To transmit '1')} \qquad ...(14.1)$$

* In both frequency and phase.

Scanned by CamScanner

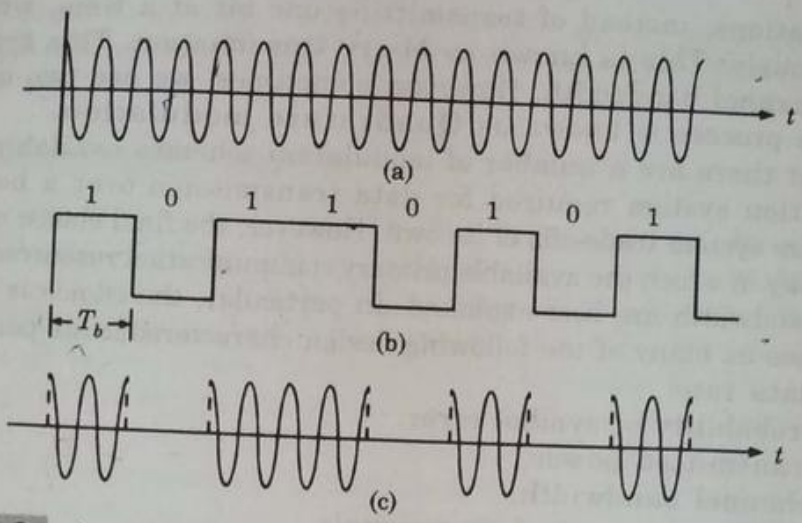


Fig. 14.2. *Amplitude-shift keying waveforms, (a) Unmodulated carrier, (b) Unipolar bit sequence, (c) ASK waveform.*

To transmit symbol '0', the signal $s(t) = 0$ *i.e.*, no signal is transmitted. Signal $s(t)$ contains some complete cycles of carrier frequency 'f_c'.

Hence, the ASK waveform looks like an ON-OFF of the signal. Therefore, it is also known as the **ON-OFF keying** (OOK). Figure 14.2 shows the ASK waveform.

14.5.1. Signal Space Diagram of ASK (Gromatic Representation)

The ASK waveform of equation (14.1) for symbol '1' can be represented as,

$$s(t) = \sqrt{P_s T_b} \cdot \sqrt{2/T_b} \cos(2\pi f_c t)$$

or

$$s(t) = \sqrt{P_s T_b}\, \phi_1(t) \qquad ...(14.2)$$

This means that there is only one carrier function $\phi_1(t)$. The signal space diagram will have two points on $\phi_1(t)$. One will be at zero and other will be at $\sqrt{P_s T_b}$. Figure 14.3 shows this aspect.

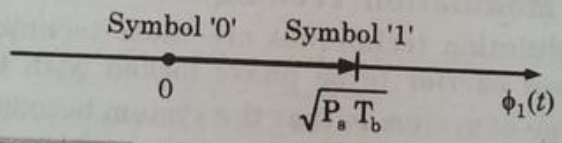


Fig. 14.3. *Signal space diagram of ASK.*

Thus, the distance between the two signal points is,

$$d = \sqrt{P_s T_b} = \sqrt{E_b} \qquad ...(14.3)$$

SNR ASK < SNR PSK

14.5.2. Generation of ASK Signal

ASK signal may be generated by simply applying the incoming binary data (represented in unipolar form) and the sinusoidal carrier to the two inputs of a product modulator (*i.e.*, balanced modulator). The resulting output will be the ASK waveform. This is shown in figure 14.4. Modulation causes a shift of the baseband signal spectrum. The ASK signal, which is basically the product of the binary sequence and the carrier signal, has a power spectral density (PSD) same as that of the baseband on-off signal but shifted in the frequency domain by $\pm f_c$. This is shown in figure 14.5. It may be noted that two impulses occur at $\pm f_c$. The spectrum of the ASK signal shows that it has an infinite bandwidth. However for practical purpose, the bandwidth is often defined as the bandwidth of an ideal bandpass filter centered at f_c whose output contains about 95% of the total average power content of the ASK signal. It may be proved that according to this criterion the bandwidth of the ASK signal

Scanned by CamScanner

is approximately $2/T_b$ Hz. The bandwidth of the ASK signal can however, be reduced by using smoothed versions of the pulse waveform instead of rectangular pulse waveforms.

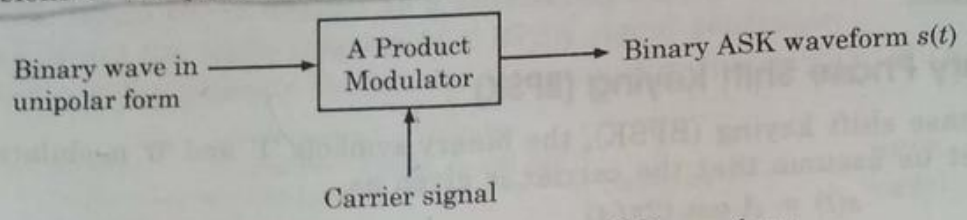


Fig. 14.4. *Generation of binary ASK waveform.*

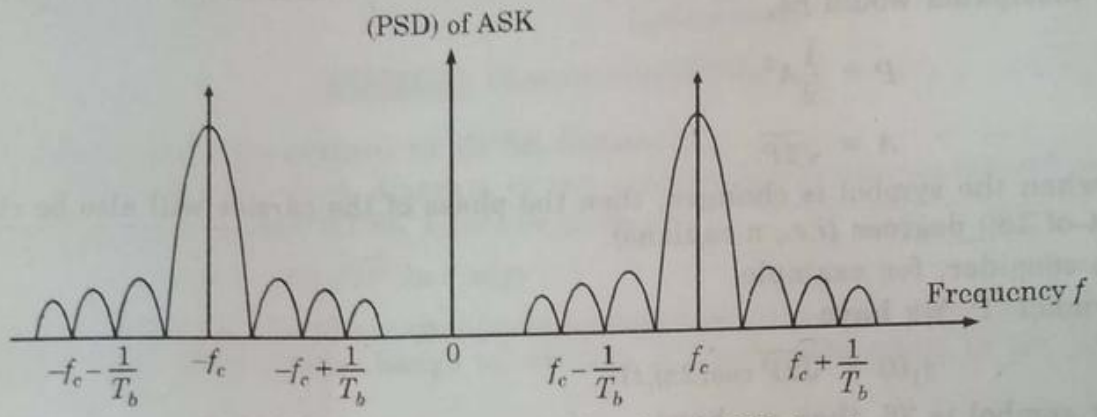


Fig. 14.5. *Power spectral density of ASK signal.*

14.5.3. Coherent Demodulation of Binary ASK

The demodulation of binary ASK waveform can be achieved with the help of *coherent detector* as shown in figure 14.6. It consists of a product modulator which is followed by an integrator and a decision-making device. The incoming ASK signal is applied to one input of the product modulator. The other input of the product modulator is supplied with a sinusoidal carrier which is generated with the help of a local oscillator. The output of the product modulator goes to input of the integrator. The integrator operates on the output of the multiplier for successive bit intervals and essentially performs a low-pass filtering action. The output of the integrator goes to the input of a decision-making device.

Now, the decision-making device compares the output of the integrator with a preset threshold. It makes a decision in favour of symbol 1 when the threshold is exceeded and in favour of symbol 0 otherwise. The *coherent detection* makes the use of linear operation. In this method we have assumed that the local carrier is in perfect synchronisation with the carriers used in the transmitter. This means that the frequency and phase of the locally generated carrier is same as those of the carriers used in the transmitter.

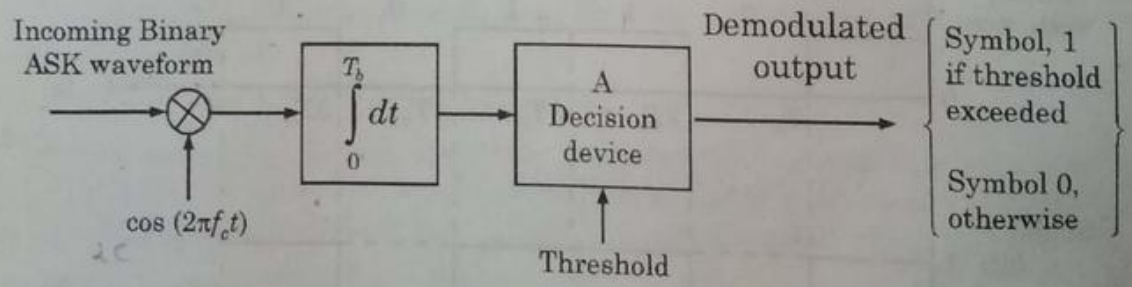


Fig. 14.6. *Coherent detection of binary ASK signals.*

The following two forms of synchronisation are required for the operation of coherent (or synchronous detector):

(i) *Phase synchronisation which ensures that carrier wave generated locally in the receiver is locked in phase with respect to one that is employed in the transmitter.*

Scanned by CamScanner

(ii) *Timing synchronisation which enable proper timing of the decision making operation in the receiver with respect to switching instants (switching between 1 and 0) in the original binary data.*

14.6. Binary Phase Shift Keying (BPSK)

In a binary phase shift keying (BPSK), the binary symbols '1' and '0' modulate the phase of the carrier. Let us assume that the carrier is given as,

$$s(t) = A \cos (2\pi f_c t) \quad ...(14.4)$$

Here 'A' represents peak value of sinusoidal carrier. For the standard 1 Ω load resistor, the power dissipated would be,

$$P = \frac{1}{2}A^2$$

or

$$A = \sqrt{2P} \quad ...(14.5)$$

Now, when the symbol is changed, then the phase of the carrier will also be changed by an amount of 180 degrees (*i.e.*, π radians).

Let us consider, for example,

For symbol '1', we have

$$s_1(t) = \sqrt{2P} \cos (2\pi f_c t) \quad ...(14.6)$$

If next symbol is '0', then we have

For symbol '0', we have

$$s_2(t) = \sqrt{2P} \cos (2\pi f_c t + \pi) \quad ...(14.7)$$

Now, because cos (θ + π) = – cos θ, therefore, the last equation can be written as,

$$s_2(t) = -\sqrt{2P} \cos (2\pi f_c t) \quad ...(14.8)$$

With the above equation, we can define BPSK signal combinely as,

$$s(t) = b(t) \sqrt{2P} \cos (2\pi f_c t) \quad ...(14.9)$$

where $b(t)$ = + 1 when binary '1' is to be transmitted.
– 1 when binary '0' is to be transmitted

Figure 14.7 illustrates binary signal and its equivalent signal $b(t)$.

Note: *It may be observed from figure 14.7(b) that the signal b(t) is a* ***NRZ bipolar signal.*** *In fact, this signal directly modulates the carrier signal cos (2π f_c t).*

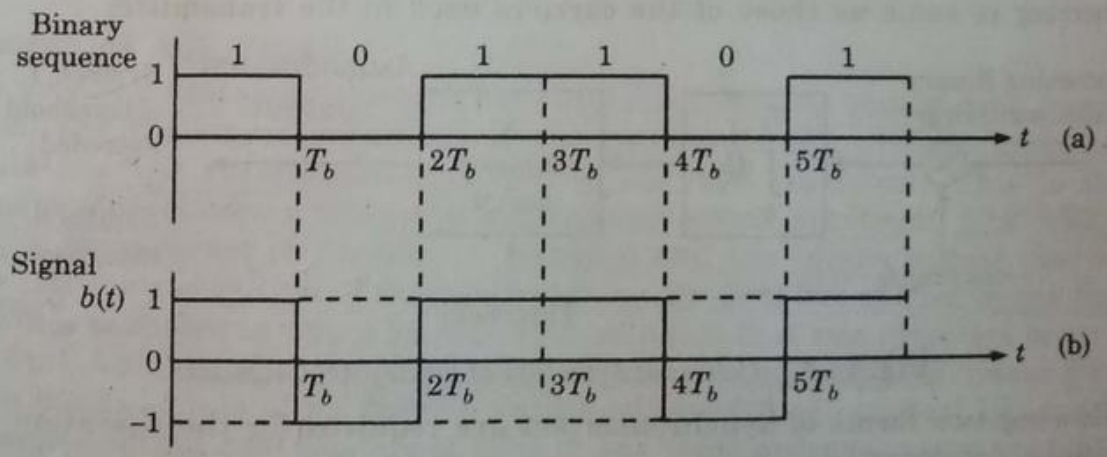


Fig. 14.7. *(a) Binary sequence, (b) The corresponding bipolar signal b(t).*

Scanned by CamScanner

14.6.1. Generation of BPSK Signal

The BPSK signal may be generated by applying carrier signal to a balanced modulator. Here, the bipolar signal $b(t)$ is applied as a modulating signal to the balanced modulator.

Figure 14.8 shows the block diagram of a BPSK signal generator.

A NRZ level encoder converts the binary data sequence into bipolar NRZ signal.

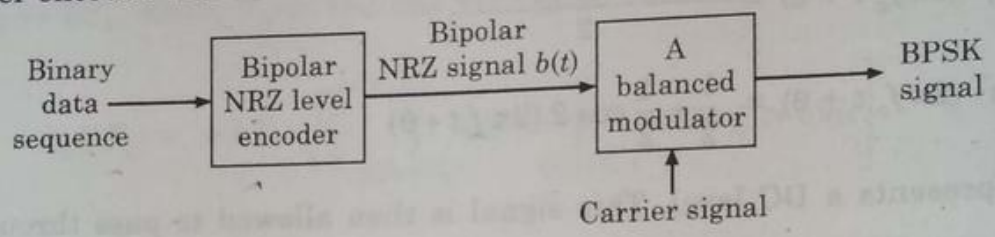


Fig. 14.8. *Generation of BPSK.*

14.6.2. Reception (*i.e.* Detection) of BPSK Signal

Figure 14.9 shows the block diagram of the scheme to recover baseband signal from BPSK signal. The transmitted BPSK signal is given as

$$s(t) = b(t)\sqrt{2P}\cos(2\pi f_c t)$$

This signal undergoes the phase change depending upon the time delay from transmitter end to receiver end. This phase change is, usually, a fixed phase shift in the transmitted signal.

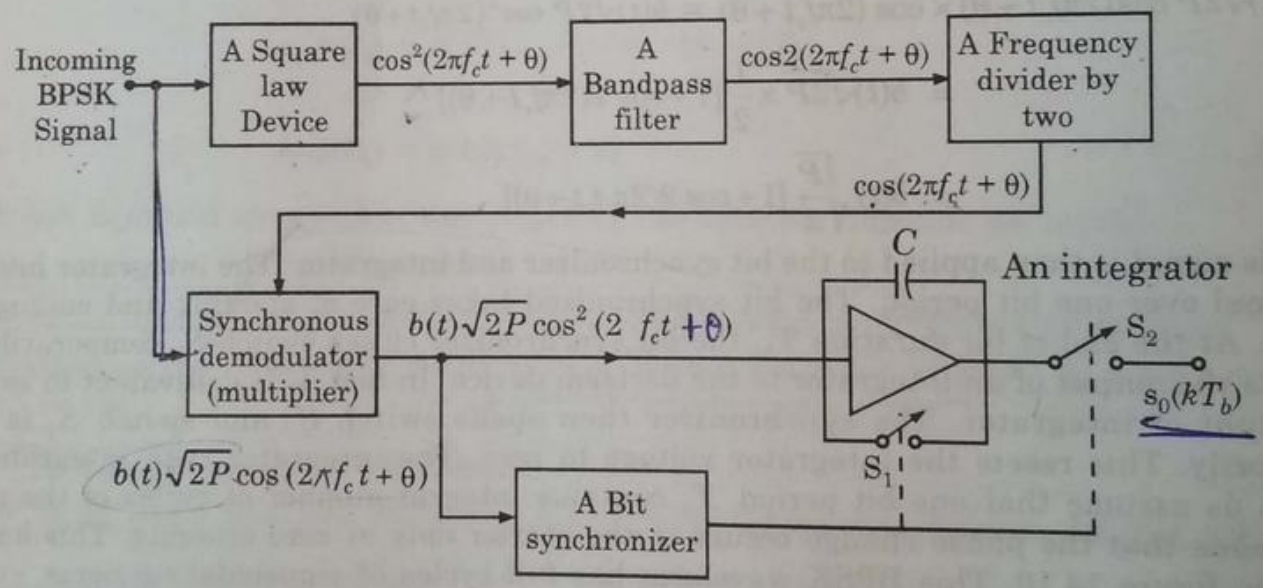


Fig. 14.9. *Reception of baseband signal in BPSK signal.*

Let us consider that this phase shift is θ. Because of this, the signal at the input of the receiver can be written as

$$s(t) = b(t)\sqrt{2P}\cos(2\pi f_c t + \theta) \quad \text{...(14.10)}$$

Now, from this received signal, a carrier is separated because this is coherent detection. As shown in the figure 14.9, the received signal is allowed to pass through a square law device. At the output of the square law device, we get a signal which is given as

$$\cos^2(2\pi f_c t + \theta)$$

Here, it may be noted that we have neglected the amplitude, since we are only interested in the carrier of the signal.

Scanned by CamScanner

Again, we know that

$$\cos^2 \theta = \frac{1+\cos 2\theta}{2}$$

Therefore, we have

$$\cos^2 (2\pi f_c t + \theta) = \frac{1+\cos 2(2\pi f_c t+\theta)}{2}$$

$$\cos^2 (2\pi f_c t + \theta) = \frac{1}{2} + \frac{1}{2}\cos 2(2\pi f_c t+\theta)$$

Here, $\frac{1}{2}$ represents a DC level. This signal is then allowed to pass through a bandpass filter (BPF) whose passband is centred around $2f_c$. Bandpass filter removes the DC level of $\frac{1}{2}$ and at the output, we obtain,

$$\cos 2(2\pi f_c t + \theta)$$

This signal is having frequency equal to $2 f_c$. Hence, it is passed through a frequency divider by two. Thus, at the output of frequency divider, we get a carrier signal whose frequency is f_c *i.e.*, $\cos (2\pi f_c t + \theta)$.

The synchronous (*i.e.*, coherent) demodulator multiplies the input signal and the recovered carrier. Hence, at the output of multiplier, we get

$$b(t)\sqrt{2P}\cos(2\pi f_c t+\theta)\times\cos(2\pi f_c t+\theta) = b(t)\sqrt{2P}\cos^2(2\pi f_c t+\theta)$$

$$= b(t)\sqrt{2P}\times\frac{1}{2}[1+\cos 2(2\pi f_c t+\theta)]$$

$$= b(t)\sqrt{\frac{P}{2}}[1+\cos 2(2\pi f_c t+\theta)] \quad ...(14.11)$$

This signal is then applied to the bit synchronizer and integrator. The integrator integrates the signal over one bit period. The bit synchronized takes care of starting and ending times of a bit. At the end of bit duration T_b, the bit synchronizer closes switch S_2 temporarily. This connects the output of an integrator to the decision device. In fact, it is equivalent to sampling the output of integrator. The synchronizer then opens switch S_2 and switch S_1 is closed temperorily. This resets the integrator voltage to zero. The integrator then integrates next bit. Let us assume that one bit period 'T_b' contains integral number of cycles of the carrier. This means that the phase change occurs in the carrier only at zero crossing. This has been shown in figure 14.10. This BPSK waveform has full cycles of sinusoidal carrier.

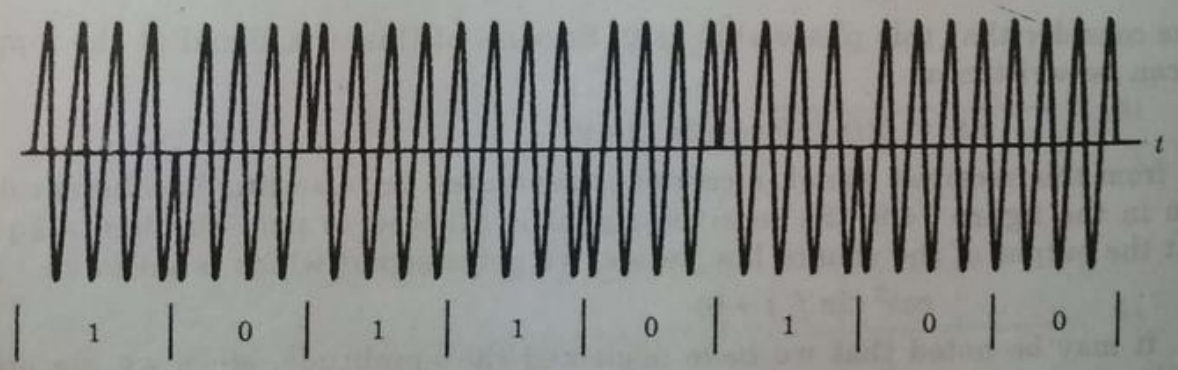


Fig. 14.10. *The BPSK waveform.*

Scanned by CamScanner

Also, in the kth bit interval, we can write output signal as under:

$$s_0(kT_b) = b(kT_b)\sqrt{\frac{P}{2}}\int_{(k-1)T_b}^{kT_b}[1+\cos(2\pi f_c t+\theta)]\,dt$$

This equation gives the output of an interval for k^{th} bit. Hence, integration is performed from $(k-1)T_b$ to kT_b. Here, T_b is the one bit period. We can write the above equation as under :

$$s_0(kT_b) = b(kT_b)\sqrt{\frac{P}{2}}\left[\int_{(k-1)T_b}^{kT_b} 1dt + \int_{(k-1)T_b}^{kT_b}\cos(2\pi f_c t+\theta)dt\right]$$

where $\int_{(k-1)T_b}^{kT_b}\cos 2(2\pi f_c t+\theta)\,dt = 0$, since average value of sinusoidal waveform is zero if integration is done over full cycles. Hence, we can write above equation as,

$$s_0(kT_b) = b(kT_b)\sqrt{\frac{P}{2}}\int_{(k-1)T_b}^{kT_b}1dt$$

or $$s_0(kT_b) = b(kT_b)\sqrt{\frac{P}{2}}\,[t]_{(k-1)T_b}^{kT_b}$$

or $$s_0(kT_b) = b(kT_b)\sqrt{\frac{P}{2}}\{kT_b-(k-1)T_b\}$$

or $$s_0(kT_b) = b(kT_b)\sqrt{\frac{P}{2}}\,T_b \qquad ...(14.12)$$

The last eqaution shows that the output of the receiver depends on input.

Thus, $s_0(kT_b)\,\alpha\, b(kT_b)$

Depending upon the value of $b(kT_b)$, the output $s_0(kT_b)$ is generated in receiver.

This signal is then applied to a decision device which decides whether transmitted symbol was zero or one.

14.6.3. The Spectrum of BPSK Signals

Type of we know that the waveform $b(t)$ is a NRZ binary waveform. In this waveform, there are rectangular pulses of amplitude $\pm V_b$. If we assume that each pulse is $\pm\frac{T_b}{2}$ around its centre, then it becomes easy to find Furrier transform of such pulse. The Fourier transform of this type of pulse is given as,

$$X(f) = V_b\,T_b\frac{\sin(\pi f T_b)}{(\pi f T_b)}\,* \qquad ...(14.13)$$

For a large number of such positive and negative pulses, the power spectral density $S(f)$ is expressed as

$$S(f) = \frac{\overline{|X(f)|^2}}{T_s} \qquad ...(14.14)$$

Here, $\overline{X(f)}$ denotes average value of $X(f)$ due to all the pulses in $b(t)$. And T_s is symbol duration. Substituting value of $X(f)$ from equation (13.13) in equation (14.14), we get

* Making use of standard relation.

Scanned by CamScanner

$$S(f) = \frac{V_b^2 T_b^2}{T_s}\left[\frac{\sin(\pi f T_b)}{\pi f T_b}\right]^2$$

For BPSK, because only one bit is transmitted at a time, therefore, symbol and bit durations are same *i.e.*, $T_b = T_s$. Then the last equation becomes,

$$S(f) = V_b^2\, T_b\left[\frac{\sin(\pi f T_b)}{\pi f T_b}\right]^2 \qquad ...(14.15)$$

This equation gives the power spectral density (psd) of baseband signal $b(t)$. The BPSK signal is generated by modulating a carrier by the baseband signal $b(t)$. Due to modulation of the carrier of frequency f_c, the spectral components are translated from f to $f_c + f$ and $f_c - f$. The magnitude of these components is divided by half.

Therefore, from equation (14.15) we can write the power spectral density of BPSK signal as under:

$$S_{BPSK}(f) = V_b^2\, T_b\left\{\frac{1}{2}\left[\frac{\sin\pi(f_c - f)\,T_b}{\pi(f_c - f)T_b}\right]^2 + \frac{1}{2}\left[\frac{\sin\pi(f_c + f)T_b}{\pi(f_c + f)\,T_b}\right]^2\right\}$$

It may be noted that this equation consists of two half magnitude spectral components of same frequency 'f' above and below f_c. Let us assume that the value of $\pm V_b = \pm\sqrt{P}$. This means that the NRZ signal is having amplitudes of $+\sqrt{P}$ and $-\sqrt{P}$. Then the last equation becomes,

$$S_{BPSK}(f) = \frac{PT_b}{2}\left\{\left[\frac{\sin\pi(f - f_c)T_b}{\pi(f - f_c)T_b}\right]^2 + \frac{1}{2}\left[\frac{\sin\pi(f_c + f)T_b}{\pi(f_c + f)T_b}\right]^2\right\} \qquad ...(14.16)$$

This equation gives power spectral density (psd) of BPSK signal for modulating signal $b(t)$ having amplitudes equal to $\pm\sqrt{P}$.

Further, we know that the modulated signal is given as

$$s(t) = \pm\sqrt{2P}\cos(2\pi f_c t) \qquad [\because A = \sqrt{2P}]$$

If $b(t) = \pm\sqrt{P}$, then the carrier becomes,

$$\phi(t) = \sqrt{2}\cos(2\pi f_c t) \qquad ...(14.17)$$

Equation (14.15) describes power spectral density (psd) of the NRZ waveform. For one rectangular pulse, the shape of $S(f)$ will be a sinc pulse as shown in figure 14.11.

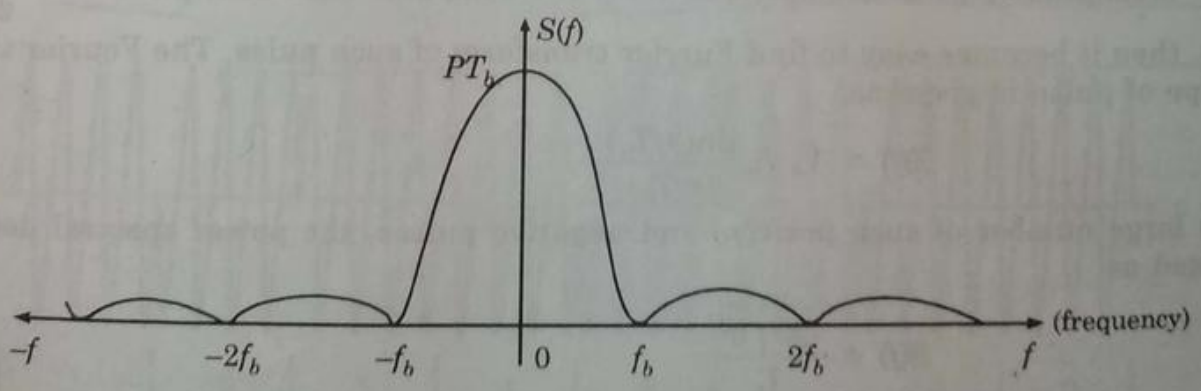


Fig. 14.11. *Plot of power spectral density (psd) of NRZ baseband signal.*

Scanned by CamScanner

It may be observed from this figure that the main lobe ranges from $-f_b$ to $+f_b$.* Because we have taken $\pm V_b = \pm\sqrt{P}$ in equation (14.15), therefore, the peak value of the main lobe is PT_b. Now let us consider the power spectral density (psd) of BPSK signal expressed by equation (14.16).

Figure 14.12 shows the plot of this equation. This figure, thus, clearly shows that there are two lobes, one at f_c and other at $-f_c$. The same spectrum of figure 14.11 has been placed at $+f_c$ and $-f_c$. However, the amplitudes of main lobes are $\frac{PT_b}{2}$ in figure 14.12.

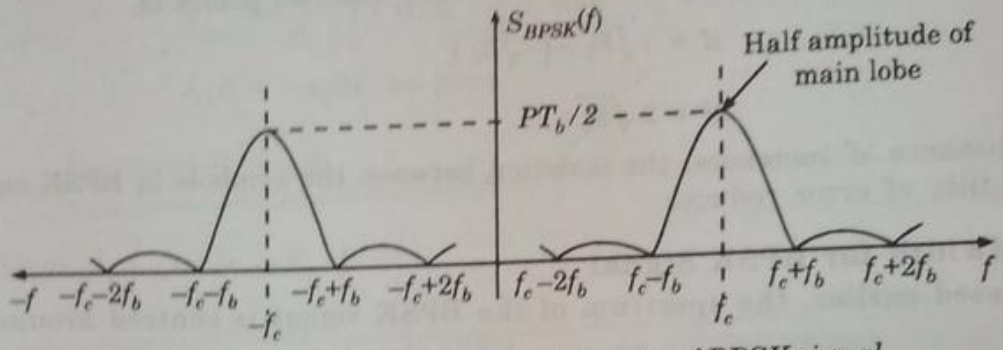


Fig. 14.12. *Plot of power spectral density of BPSK signal.*

Hence, they are reduced to half. The spectrum of $S(f)$ as well as $S_{BPSK}(f)$ extends overall the frequencies.

14.6.4. A Geometrical Representation for BPSK Signals

We know that BPSK signal carries the information about two symbols. These symbols are symbol '1' and symbol '0'. We can represent BPSK signal geometrically to show those two symbols. From equation (14.9), we know that BPSK signal is expressed as,

$$s(t) = b(t)\sqrt{2P}\cos(2\pi f_c t) \qquad ...(14.18)$$

Let us rearrange the last equation as,

$$s(t) = b(t)\sqrt{PT_b}\cdot\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t) \qquad ...(14.19)$$

Now, let $\phi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)$ represents an orthonormal carrier signal. Equation (14.17) also gives equation for carrier. It is slightly different than $\phi_1(t)$ defined here. Then, we may write equation (14.19) as,

$$s(t) = b(t)\sqrt{PT_b}\,\phi_1(t) \qquad ...(14.20)$$

The bit energy E_b is defined in terms of power 'P' and bit duration T_b as,

$$E_b = PT_b \qquad ...(14.21)$$

Therefore, equation (14.20) becomes,

$$s(t) = \pm\sqrt{E_b}\,\phi_1(t) \qquad ...(14.22)$$

Here, $b(t)$ is simply ± 1.

Thus, on the single axis of $\phi_1(t)$, there will be two points. One point will be located at $+\sqrt{E_b}$ and other point will be located at $-\sqrt{E_b}$. This has been shown in figure 14.13.

* Here, $f_b = \frac{1}{T_b}$.

Scanned by CamScanner

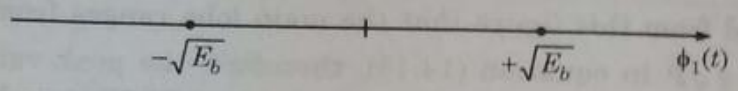


Fig. 14.13. *Geometrical representation of BPSK signal.*

At the receiver end, the point at $+\sqrt{E_b}$ on $\phi_1(t)$ represents symbol '1' and point at $-\sqrt{E_b}$ represents symbol '0'. The separation between these two points represents the isolation in symbols '1' and '0' in BPSK signal. This separation is generally called distance 'd'. From figure 14.13, it is obvious that the distance between the two points is,

$$d = +\sqrt{E_b} - \left(-\sqrt{E_b}\right)$$

or

$$d = 2\sqrt{E_b} \qquad ...(14.23)$$

As this distance 'd' increases, the isolation between the symbols in BPSK signal is more. Thus, probability of error reduces.

14.6.5. Bandwidth for BPSK Signal

As discussed earlier, the spectrum of the BPSK signal is centred around the carrier frequency f_c.

If $f_b = \frac{1}{T_b}$, then for BPSK, the maximum frequency in the baseband signal will be f_b as shown in figure 13.12. In this figure, the main lobe is centred around carrier frequency f_c and extends from $f_c - f_b$ to $f_c + f_b$.

Therefore Bandwidth of BPSK signal will be,

BW = Highest frequency – Lowest frequency in the main lobe

$$BW = f_c + f_b - (f_c - f_b)$$

or

$$BW = 2f_b \qquad ...(14.24)$$

Hence, the minimum bandwidth of BPSK signal is equal to twice of the highest frequency contained in baseband signal.

14.6.6. Error Probability of BPSK Signal Employing Coherent Reception or Matched Filter

The BPSK receiver shown in figure 14.9 is equivalent to the coherent receiver. It is nothing but coherent receiver. Let us say that the signal $x(t)$ represents $s_1(t)$ and $s_2(t)$. Then, the BPSK receiver of figure 14.9 can be approximated to coherent as shown in figure 14.14.

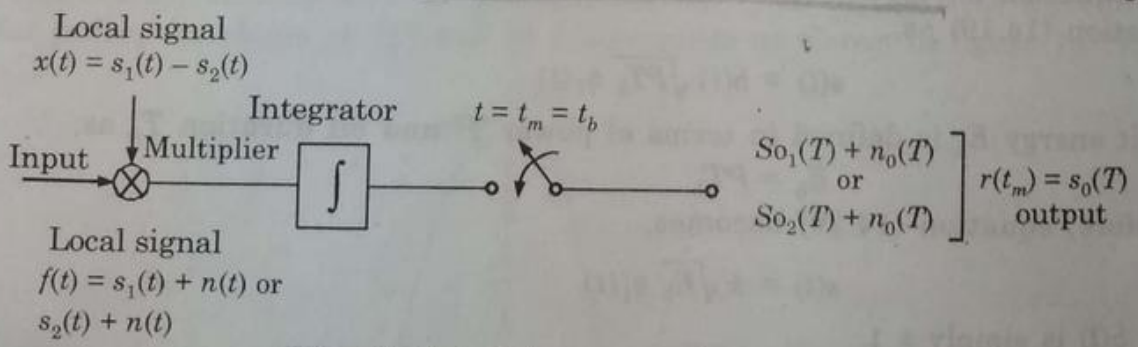


Fig. 14.14. *Coherent reception system*

Because, the correlator of figure 14.14 is equivalent to matched filter, we can apply the analysis of matched filter detection. We know that signal to noise ratio ρ_{max} at the output of receiver is given by

Scanned by CamScanner

$$\rho_{max} = \frac{2}{N_0}\int_{-\infty}^{\infty} x^2(t)\,dt \qquad ...(14.25)$$

Here, $x(t) = s_1(t) - s_2(t)$ *i.e.*, input difference signal, then we have

$$\rho_{max} = \frac{2}{N_0}\int_{-\infty}^{\infty}[s_1(t) - s_2(t)]^2 dt$$

or
$$\rho_{max} = \frac{2}{N_0}\left[\int_{-\infty}^{\infty} s_1^2(t)\,dt + \int_{-\infty}^{\infty} s_2^2(t)dt - 2\int_{-\infty}^{\infty} s_1(t)s_2(t)dt\right]$$

Since $s_1(t) = -s_2(t)$, we have

$$\int_{-\infty}^{\infty} s_1(t)\,s_2(t)\,dt = -E_b$$

and $\int_{-\infty}^{\infty} s_1^2(t)\,dt = \int_{-\infty}^{\infty} s_2^2(t)\,dt = E_b$

Hence,
$$\rho_{max} = \frac{2}{N_0}[E_b + E_b + 2E_b] \qquad \text{by putting values}$$

or
$$\rho_{max} = \frac{8E_b}{N_0}$$

For the correlator, we can write

$$\left[\frac{s_{01}(t) - s_{02}(t)}{\sigma}\right]^2 = \frac{8E_b}{N_0}$$

or
$$\frac{s_{01}(t) - s_{02}(t)}{\sigma} = 2\sqrt{2}\sqrt{\frac{E_b}{N_0}} \qquad ...(14.26)$$

The probability of error is given as,

$$P(e) = \frac{1}{2}\,erfc\left[\frac{s_{01}(T) - s_{02}(T)}{2\sqrt{2}\,\sigma}\right]$$

From equation (14.26), we can write above equation as under:

$$P(e) = \frac{1}{2}\,erfc\left[\frac{1}{2\sqrt{2}}\cdot 2\sqrt{2}\sqrt{\frac{E_b}{N_0}}\right]$$

or
$$\boxed{P(e) = \frac{1}{2}\,erfc\sqrt{\frac{E_b}{N_0}}} \qquad ...(14.27)$$

This is the required expression for error probability of BPSK reception using coherent/ matched filter detection.

14.6.7. Drawbacks of BPSK

Figure 14.9 shows the block diagram of BPSK receiver. To regenerate the carrier in the receiver, we start by squaring $b(t)\sqrt{2P}\cos(2\pi f_c t + \theta)$. If the received signal is $-b(t)\sqrt{2P}\cos(2\pi f_c t + \theta)$, then the squared signal remains same as before. Hence, the recovered carrier is unchanged even if the input signal has changed its sign. Therefore, it is not possible to determine whether the received signal is equal to $b(t)$ or $-b(t)$. Infact, this results in ambiguity in the output signal.

Scanned by CamScanner

This problem can be removed if we use differential phase shift keying (DPSK). However, differential phase shift keying (DPSK) also has some other problems. DPSK will be discussed in detail later on in this chapter. Other problems of BPSK are ISI and Interchannel interference. However, these problems can be reduced to some extent by making use of filters.

14.7. Coherent Binary Frequency Shift Keying (BFSK)

In binary frequency shift keying (BFSK), the frequency of the carrier is shifted according to the binary symbol. However, the phase of the carrier is unaffected. This means that we have two different frequency signals according to binary symbols. Let there be a frequency shift by Ω. Then we can write following equations.

If $\quad b(t) = \text{'1'}, \quad$ then $s_H(t) = \sqrt{2P_s}\cos(2\pi f_c + \Omega)t$ ...(14.28)

If $\quad b(t) = \text{'0'}, \quad$ then $s_L(t) = \sqrt{2P_s}\cos(2\pi f_c - \Omega)t$ ...(14.29)

Hence, there is increase or decrease in frequency by Ω. Let us use the following conversion table to combine above two FSK equations:

Table 14.1. Conversion table for BPSK representation

$b(t)$ Input	$d(t)$	$P_H(t)$	$P_L(t)$
1	+ 1V	+ 1V	0V
0	− 1V	0V	+ 1V

The equations (14.28) and (14.29) combinely may be written as

$$s(t) = \sqrt{2P_s}\cos\left[(2\pi f_c + d(t)\Omega)t\right] \quad ...(14.30)$$

Hence, if symbol '1' is to be transmitted, the carrier frequency will be $f_c + \left(\frac{\Omega}{2\pi}\right)$. If symbol '0' is to be transmitted, then the carrier frequency will be $f_c - \left(\frac{\Omega}{2\pi}\right)$.

Therefore, we have

Thus,

$$f_H = f_c + \frac{\Omega}{2\pi} \quad \text{for symbol '1'} \quad ...(14.31)$$

$$f_L = f_c - \frac{\Omega}{2\pi} \quad \text{for symbol '0'} \quad ...(14.32)$$

14.7.1. Generation of BFSK

It may be observed from Table 14.1 that $P_H(t)$ is same as $b(t)$ and also $P_L(t)$ is inverted version of $b(t)$. The block diagram for BFSK generation is shown in figure 14.15.

We know that input sequence $b(t)$ is same as $P_H(t)$. An inverter is added after $b(t)$ to get $P_L(t)$. The level shifter $P_H(t)$ and $P_L(t)$ are unipolar signals. The level shifter converts the '+1' level to $\sqrt{P_sT_b}$. Zero level is unaffected. Thus, the output of the level shifters will be either $\sqrt{P_sT_b}$ (if '+1') or zero (if input is zero). Further, there are product modulators after level shifter. The two carrier signals $\phi_1(t)$ and $\phi_2(t)$ are used. $\phi_1(t)$ and $\phi_2(t)$ are orthogonal to each other. In one bit period of input signal (*i.e.*, T_b), $\phi_1(t)$ or $\phi_2(t)$ have integral number of cycles.

Thus, the modulated signal is having continuous phase. Figure 14.16 shows such type of BFSK signal. The adder then adds the two signals.

Scanned by CamScanner

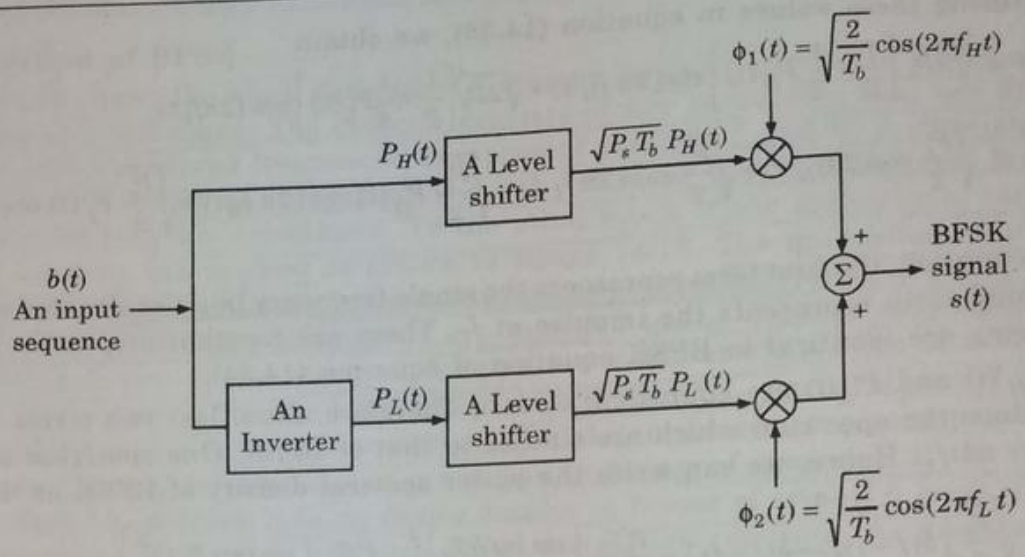


Fig. 14.15. *Block diagram for BFSK generation.*

Note: *Here it may be noted that outputs from both the multipliers are not possible at a time. This is because $P_H(t)$ and $P_L(t)$ are complementary to each other. Therefore, if $P_H(t) = 1$, then output will be only due to upper modulator and lower modulator output will be zero [since $P_L(t) = 0$)].*

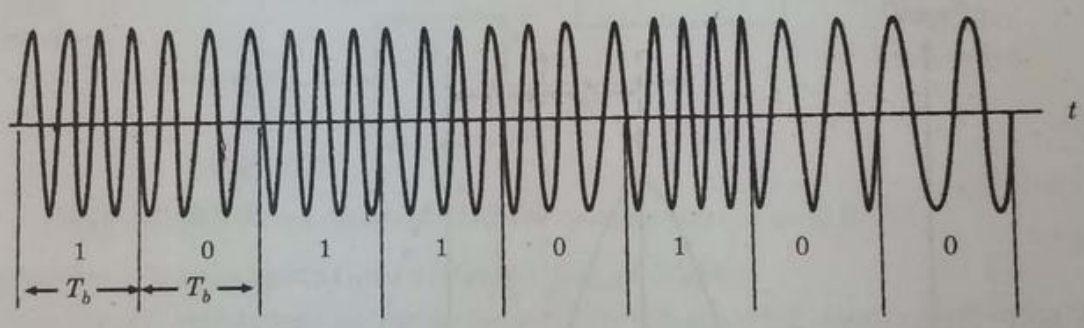


Fig. 14.16. *The BFSK signal.*

14.7.2. The Spectrum of BFSK Signal

In figure 14.15, the BFSK signal $s(t)$ may be written as,

$$s(t) = \sqrt{2P_s}\, P_H(t) \cos(2\pi f_H t) + \sqrt{2P_s}\, P_L(t) \cos(2\pi f_L t) \quad \text{...(14.33)}$$

This is the expression for BFSK signal. Let us compare this equation with BPSK equation which is written below:

$$S_{BPSK}(t) = b(t)\sqrt{2P} \cos(2\pi f_c t) \quad \text{...(14.34)}$$

It may be noted that this equation is identical to BFSK equation. In BPSK equation, $b(t)$ is a bipolar signal where as in BFSK, the similar coefficients $P_H(t)$ or $P_L(t)$ are unipolar. Hence, let us convert these coefficients in bipolar form as under:

$$P_H(t) = \frac{1}{2} + \frac{1}{2} P'_H(t) \quad \text{...(14.35)}$$

and

$$P_L(t) = \frac{1}{2} + \frac{1}{2} P'_L(t) \quad \text{...(14.36)}$$

where $P'_H(t)$ and $P'_L(t)$ will be bipolar (*i.e.*, + 1 or – 1).

Scanned by CamScanner

Substituting these values in equation (14.33), we obtain

$$s(t) = \sqrt{2P_s}\left[\frac{1}{2}+\frac{1}{2}P_H'(t)\right]\cos(2\pi f_H t)+\sqrt{2P_s}\left[\frac{1}{2}+\frac{1}{2}P_L'(t)\right]\cos(2\pi f_L t)$$

$$\text{or}\quad s(t) = \sqrt{\frac{P_s}{2}}\cos(2\pi f_H t)+\sqrt{\frac{P_s}{2}}\cos(2\pi f_L t)+\sqrt{\frac{P_s}{2}}P_H'(t)\cos(2\pi f_H t)+\sqrt{\frac{P_s}{2}}P_L'(t)\cos(2\pi f_L t) \quad ...(14.37)$$

In this equation, the first term represents the single frequency impulse situated at frequency f_H. The second term represents the impulse at f_L. These are constant amplitude pulses. The last two terms are identical to BPSK equation of equation (14.34).

Here $P_H'(t)$ and $P'_L(t)$ are equivalent to $b(t)$. Therefore, these last two terms in equation (14.37) produce the spectrum which are similar to that of BPSK. One spectrum is located at f_H and other at f_L. Hence, we can write the power spectral density of BFSK as under:

$$S(f) = \sqrt{\frac{P_s}{2}}\left[\delta(f-f_H)+\delta(f+f_L)+\frac{P_sT_b}{2}\left\{\frac{\sin(\pi f_H T_b)}{\pi f_H T_b}\right\}^2+\frac{P_sT_b}{2}\left\{\frac{\sin(\pi f_L T_b)}{\pi f_L T_b}\right\}^2\right] \quad ...(14.38)$$

Figure 14.17 illustrates the plot of power spectral density of BFSK signal expressed by equation (14.38).

Also, f_H and f_L are selected such that,

$$f_H - f_L = 2f_b \quad ...(14.39)$$

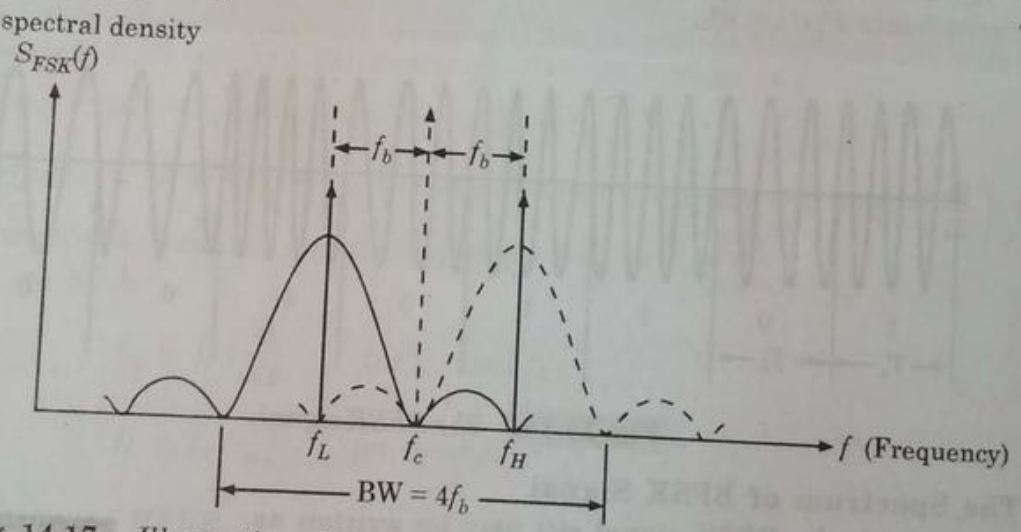


Fig. 14.17. *Illustration of Power spectral density (psd) of a BFSK signal.*

With such types of selection, it is obvious from the spectrums in the above figure that the two frequencies f_H and f_L may be identified properly. The interference between the spectrums is not ***much with the above assumption.***

14.7.3. Bandwidth of BFSK Signal

From figure 14.17, it is obvious that the width of one lobe is $2f_b$. The two main lobes due to f_H and f_L are placed such that the total width due to both main lobes is $4\,f_b$.

Therefore, we have

Bandwidth of BFSK = $2f_b + 2f_b$

or

$$BW = 4f_b \quad ...(14.40)$$

Now, if we compare this bandwidth with that of BPSK, we note that,

$$BW\,(BFSK) = 2 \times BW(BPSK) \quad ...(14.41)$$

Scanned by CamScanner

14.7.4. Detection of BFSK

Figure 14.18 shows the block diagram of a scheme for demodulation of BFSK wave using coherent detection technique. The detector consists of two correlators that are individually tuned to two different carrier frequencies to represent symbols '1' and '0'. A correlator consists of a multiplier followed by an integrator. Then, the received binary FSK signal is applied to the multipliers of both the correlators. To the other input of the multipliers, carriers with frequency f_{c1} and f_{c2} are applied as shown in figure 14.18. The multiplied output of each multiplier is subsequently passed through integrators generating output l_1 and l_2 in the two paths. The output of the two integrators are then fed to the decision making device. The decision making device is essentially a comparator which compares the output l_1 (in the upper path) and output l_2 (in the lower path). If the output l_1 produced in the upper path (associated with frequency f_{c1}) is greater than the output l_2 produced in the lower path (associated with frequency f_{c2}), the detector makes a decision in favour of symbol 1. If the output l_1 is less than l_2, then the decision making device decides in favour of symbol 0 (say). This type of digital communication receivers are also called *correlation receivers*. As discussed earlier, the detector based upon coherent detection requires phase and timing synchronisation.

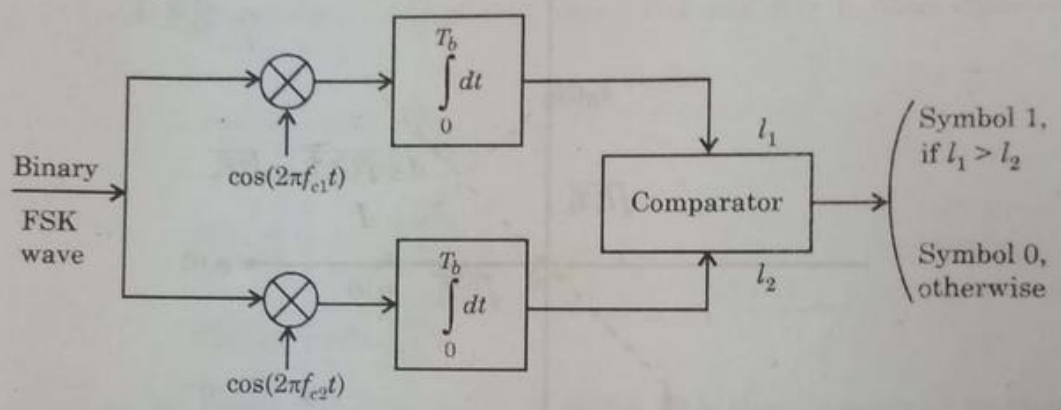


Fig. 14.18. *Block diagram of BFSK receiver (detection of BFSK).*

14.7.5. Geometrical Representation of Orthogonal BFSK

As a matter of fact, orthogonal carriers are used for M-ary PSK and QASK. The different signal points are represented geometrically in ϕ_1 ϕ_2-plane. For geometrical representation of BFSK signals, such orthogonal carriers are required. From figure 14.15, we know that two carriers $\phi_1(t)$ and $\phi_2(t)$ of two different frequencies f_H and f_L are used for modulation. To make $\phi_1(t)$ and $\phi_2(t)$ orthogonal, the frequencies f_H and f_L must be some integer multiple of band frequency 'f_b'.

Thus,
$$f_H = m\, f_b \qquad ...(14.42)$$
and
$$f_L = n\, f_b \qquad ...(14.43)$$

Here; $f_b = \dfrac{1}{T_b}$, then the carriers would be

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi m\, f_b\, t) \qquad ...(14.44)$$

and
$$\phi_2(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi\, m f_b\, t) \qquad ...(14.45)$$

The carriers $\phi_1(t)$ and $\phi_2(t)$ are orthogonal over the period T_b. We can write equation (14.28) and equation (14.29) as,

$$s_H(t) = \sqrt{P_s T_b} \sqrt{\frac{2}{T_b}} \cos(2\pi\, f_H t)$$

Scanned by CamScanner

and
$$s_L(t) = \sqrt{P_s T_b}\sqrt{\frac{2}{T_b}}\cos(2\pi f_L t)$$

Here
$$f_H = f_c + \frac{\Omega}{2\pi}$$

and
$$f_L = f_c - \frac{\Omega}{2\pi}$$

Using the relations in equations (14.42) to (14.45), we can write above equations as,

$$s_H(t) = \sqrt{P_s T_b}\cdot\phi_1(t) \quad ...(14.46)$$

and
$$s_L(t) = \sqrt{P_s T_b}\cdot\phi_2(t) \quad ...(14.47)$$

Thus, based on the above two equations, we can draw the signal space diagram as shown in figure 14.19.

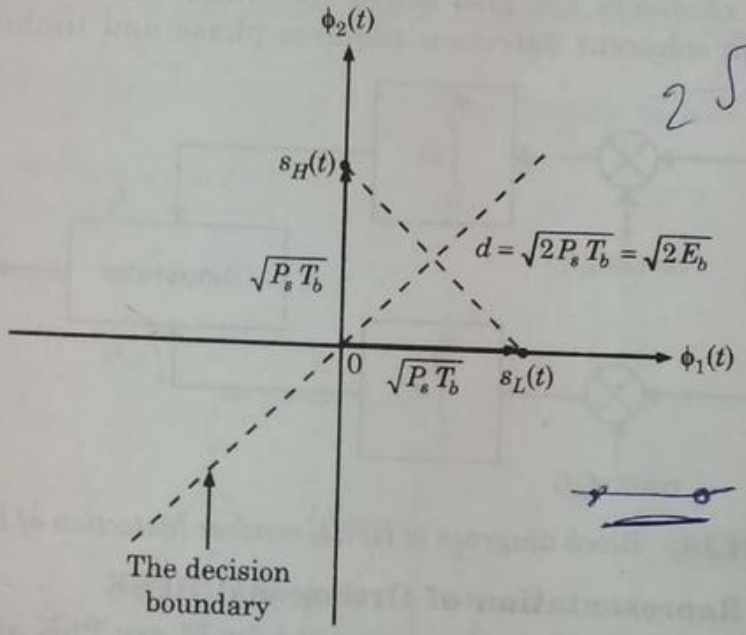


Fig. 14.19. *Illustration of signal space representation of orthogonal BFSK signal.*

14.7.5.1. Distance Between Signal Points

Note that there are two signal points in the signal space. The distance between these two points may be evaluated as under:

$$d^2 = \left(\sqrt{P_s T_b}\right)^2 + \left(\sqrt{P_s T_b}\right)^2$$

or
$$d^2 = 2P_s T_b \quad \text{or} \quad d = \sqrt{2P_s T_b} \quad ...(14.48)$$

Since $P_s T_b = E_b$, we can write above relation (*i.e.*, equation (14.48)) as under:

$$d = \sqrt{2E_b} \quad ...(14.49)$$

As compared to the distance of BPSK, we may observe that this distance is smaller than BPSK.

14.7.6. Geometrical Representation of Non-Orthogonal BFSK Signals

As a matter of fact, whenever the carriers $\phi_1(t)$ and $\phi_2(t)$ are non-orthogonal, then the signal point $S_H(t)$ or $S_L(t)$ would not lie exactly on the axes $\phi_1(t)$ and $\phi_2(t)$. Such a representation has been shown in figure 14.20.

Scanned by CamScanner

The distance 'd' for non-orthogonal signal shown in figure 14.20 may be given approximately as,

$$d^2 = 2E_b\left[1-\frac{\sin 2\pi(f_H-f_L)T_b}{2\pi(f_H-f_L)T_b}\right] \quad ...(14.50)$$

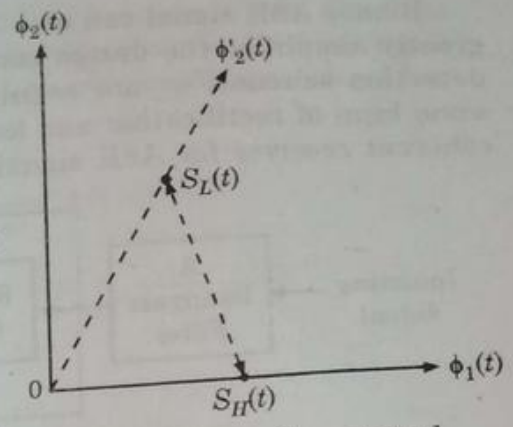


Fig. 14.20. *Geometrical representation of non-orthogonal BFSK signals.*

14.7.7. Probability of Error, P_e of BFSK Signals

The generated equation derived using union bound approximation gives probability of error as under:

$$P(e) = \frac{1}{2}\sum_{\substack{k=1\\k\neq i}}^{M} erfc\left(\frac{d_{ik}}{2\sqrt{N_0}}\right) \quad \text{for all } i \quad ...(14.50a)$$

Here, we have only two points in BFSK. Let $S_H(t)$ be $S_1(t)$ and $S_L(t)$ be $S_2(t)$. Then, distance between these two points can be written as,

$$d = d_{12} = \sqrt{2E_b}$$

Since there are only two points (*i.e.*, $M = 2$) take $i = 2$ and $k = 1$, then equation (14.50a) becomes,

$$P(e) = \frac{1}{2}erfc\left(\frac{d_{12}}{2\sqrt{N_0}}\right)$$

or

$$P(e) = \frac{1}{2}erfc\left(\frac{\sqrt{2E_b}}{2\sqrt{N_0}}\right)$$

or

$$P(e) = \frac{1}{2}erfc\sqrt{\frac{E_b}{2N_0}}$$

This equation shows that error probability of BFSK is higher compared to that of BPSK.

14.7.8. Advantages and Disadvantages of BFSK Signals

Even though the generation of BFSK is easier, it has many disadvantages compared to BPSK signal. Firstly, its bandwidth is greater than $4f_b$, which is almost double the bandwidth of BPSK. Also, the distance between the signal points is less in case of BFSK. Therefore, the error rate of BFSK is more compared to BPSK.

14.8. Non-Coherent Binary Modulation Techniques

As discussed earlier, coherent detection exploits knowledge of the carrier wave's phase reference, and thus providing the optimum error performance attainable with a digital modulation format of interest. However, when it is impractical to have knowledge of the carrier phase at the receiver, we make use of **non-coherent detection**. Thus, in this section, we shall study non-coherent binary modulation techniques *i.e.*, we shall study non-coherent detection of ASK and FSK. In the case of phase-shift keying (PSK), we cannot have "non-coherent PSK" since non-coherent means doing without phase information. However, there is a 'pseudo PSK' technique known as differential phase-shift keying (DPSK) which can be viewed as the non-coherent form of PSK.

14.9. Non-Coherent Binary Amplitude Shift Keying (ASK)

In the binary ASK case, the transmitted signal is defined as

$$s(t) = \sqrt{2P_s}\cos(2\pi f_c t)$$

Scanned by CamScanner

Binary ASK signal can also be demodulated non-coherently using envelope detector. This greatly simplifies the design consideration required in synchronous detection. Non-coherent detection schemes do not require a phase-coherent local oscillator. This method involves some form of rectification and low pass filtering at the receiver. The block diagram of a non-coherent receiver for ASK signal has been shown in figure 14.21.

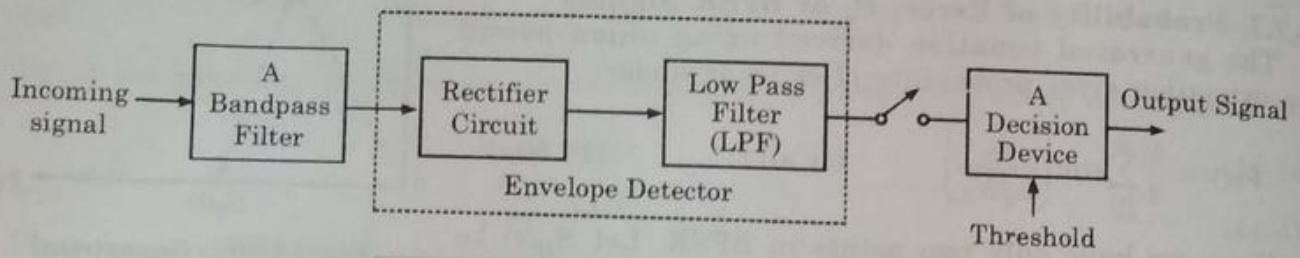


Fig. 14.21. *Non-coherent ASK detector.*

14.10. Non-Coherent Detection of FSK

Binary FSK waves may be demodulated non-coherently using envelope detector. The received FSK signal is applied to a bank of two bandpass filters, one tuned to frequency f_{c1} and the other tuned to f_{c2}. Each filter is followed by an envelope detector. The resulting outputs of the two envelope detectors are sampled and then compared with each other. The arrangement for non-coherent detection of FSK signal has been shown in figure 14.22.

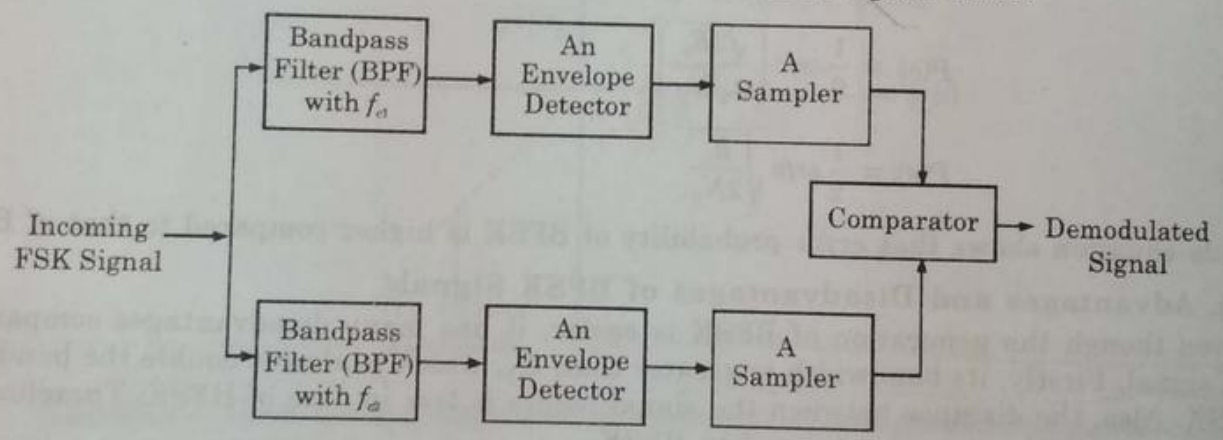


Fig. 14.22. *Non-coherent detection of FSK binary signals.*

A decision is made in favour of symbol '1' if the envelope detector output derived from the filter tuned to frequency f_{c1} is larger than that derived from the second filter. Otherwise, a decision is made in favour of the symbol 0.

14.11. Differential Phase Shift Keying (DPSK)

(U.P. Tech., Semester, Examination, 2003-2004)

We can view differential phase-shift keying as the non-coherent version of the PSK. Differential phase shift keying (DPSK) is differentially coherent modulation method. DPSK does not need a synchronous (coherent) carrier at the demodulator. The input sequence of binary bits is modified such that the next bit depends upon the previous bit. Therefore, in the receiver, the previous received bits are used to detect the present bit.

14.11.1. Generation of DPSK

Thus, in order to eliminate the need for phase synchronisation of coherent receiver with PSK, a differential encoding system can be used with PSK. The digital information content of the binary data is encoded in terms of signal transitions. As an example, the symbol 0 may

Scanned by CamScanner

be used to represent transition in a given binary sequence (with respect to the previous encoded bit) and symbol '1' to indicate no transition. This new signaling technique which combines differential encoding with phase-shift keying (PSK) is known as *differential phase-shift keying (DPSK).*

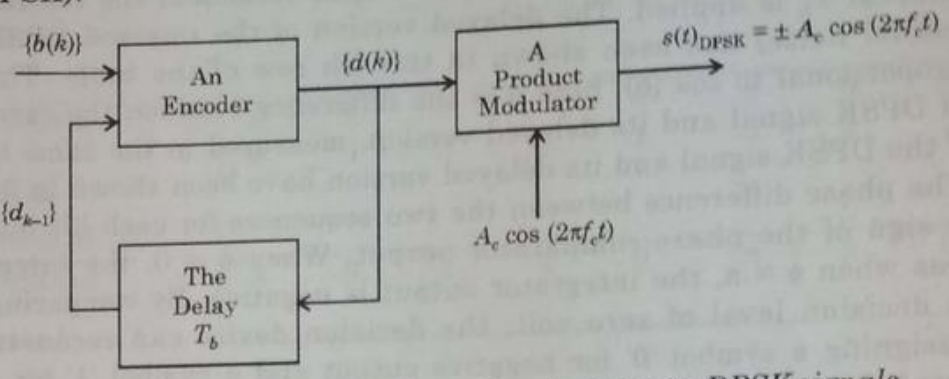


Fig. 14.23. *Illustration of the scheme to generate DPSK signals.*

A schematic arrangement for generating DPSK signal has been shown in figure 14.23. The data stream $b(t)$ is applied to the input of the encoder. The output of the encoder is applied to one input of the product modulator. To the other input of this product modulator, a sinusoidal carrier of fixed amplitude and frequency is applied. The relationship between the binary sequence and its differentially encoded version is illustrated in Table 14.2 for a assumed data sequence 0 0 1 0 0 1 0 0 1 1 1. In this illustration it has been assumed that the encoding has been done in such a way that *transition* in the given bianry sequence with respect to the previous encoded bit is represented by a symbol 0 and no *transition* by symbol '1'. It may be noted that an extra bit (symbol 1) has been arbitrarily added as an initial bit. This is essential to determine the encoded sequence. The phase of the generated DPSK signal has been shown in the third row of Table 14.2.

Table 14.2. Differentially encoded sequences with phase.

Binary data $\{b(k)\}$		0	0	1	0	0	1	0	0	1	1
Differentially encoded data $\{d(k)\}$	1*	0	1	1	0	1	1	0	1	1	1
Phase of DPSK	0	π	0	0	π	0	0	π	0	0	0
Shifted differentially encoded data $\{d_{k-1}\}$		1	0	1	1	0	1	1	0	1	1
Phase of shifted DPSK		0	π	0	0	π	0	0	π	0	0
Phase comparison output		−	−	+	−	−	+	−	−	+	+
Detected binary sequence		0	0	1	0	0	1	0	0	1	1

* Arbitrary starting reference bit.

Scanned by CamScanner

14.11.2. Detection of DPSK

For detection of the differentially encoded PSK (*i.e.*, DPSK), we can use the receiver arrangement as shown in figure 14.24. The received DPSK signal is applied to one input of the multiplier. To the other of the multiplier, a delayed version of the received DPSK signal by the time interval T_b is applied. The delayed version of the received DPSK signal (in the absence of channel noise) has been shown in the 4th row of the table. The output of the difference is proportional to cos (ϕ), here ϕ is the difference between the carrier phase angle of the received DPSK signal and its delayed version, measured in the same bit interval. The phase angle of the DPSK signal and its delayed version have been shown in 3rd and 5th rows respectively. The phase difference between the two sequences for each bit interval is used to determine the sign of the phase comparator output. When $\phi = 0$, the integrator output is positive whereas when $\phi = \pi$, the integrator output is negative. By comparing the integrator output with a decision level of zero volt, the decision device can reconstruct the binary sequence by assigning a symbol '0' for negative output and a symbol '1' for positive output. The reconstructed binary data is shown in the last row of the table. It is thus seen that in the absence of noise, the receiver can reconstruct the transmitted bianry data exactly. *DPSK may be viewed as a non-coherent version of PSK. It may also be noted that the reconstruction is invariant with the choice of the initial bit in the encoded data. This has been illustrated in the example 14.1 given below.*

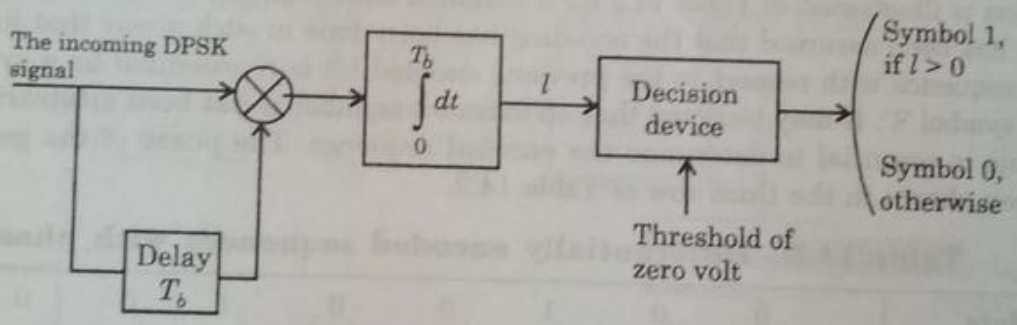


Fig. 14.24. *Receiver for the detection of DPSK signals.*

Example 14.1. A binary data strem 0 0 1 0 0 1 0 0 1 1 needs to be transmitted using DPSK technique. Prove that the reconstruction of the DPSK signal by the technique discussed in the previous article is independent of the choice of the extra bit.

Solution: In the last article, we have observed that DPSK signal can be detected accurately (in the absence of channel noise) without having a local oscillator for generation of synchronous carrier. The initial bit in the differentially encoded data was assumed to be '1'. In this example, we use the initial bit to be symbol '0' and verify that the reconstruction is invariant with the choice of the initial bit. The results obtained for this case are given in Table 14.3. It can be easily verified that the extra chosen bit 0 changes the phase of the DPSK sequence but the detected sequence remains invariant.

Scanned by CamScanner

Table 14.3. Differentially encoded sequences with phase.

Binary data $\{b(k)\}$		0	0	1	0	0	1	0	0	1	1
Differentially encoded data $\{d(k)\}$	0*	1	0	0	1	0	0	1	0	0	0
Phase of DPSK Shifted differentially encoded data $\{d_{k-1}\}$	π	0 0	π 1	π 0	0 0	π 1	π 0	0 0	π 1	π 0	π 0
Phase of shifted DPSK		π	0	π	π	0	π	π	0	π	π
Phase comparison output		–	–	+	–	–	+	–	–	+	+
Detected binary sequence		0	0	1	0	0	1	0	0	1	1

* Starting reference bit.

14.11.3. Evaluation of Bandwidth of DPSK Signal

As discussed earlier that one previous bit is used to decide the phase shift of next bit. Thus, change in $b(t)$ occurs only if input bit is at level '1'. No change happens if input bit is at level '0'. Because, one previous bit is always used to define the phase shift in next bit, therefore, the symbol can be said to have two bits. Hence, one symbol duration (T) is equivalent to two bits duration ($2T_b$) *i.e.*,

Symbol duration $T = 2T_b$ ...(14.51)

Bandwidth is expressed as

$$BW = \frac{2}{T} = \frac{1}{T_b} = f_b \quad ...(14.52)$$

Hence, the minimum bandwidth in DPSK is equal to f_b, *i.e.*, maximum baseband signal frequency.

14.11.4. Error Probability of DPSK

Figure 14.25(*a*) shows the phasor diagram of DPSK signal when no noise is present. This means that in the absence of noise and transmission delay, the phase shift of the DPSK signal is either '0' or 'π'. Therefore, a decision boundary is drawn at $\frac{\pi}{2}$ as shown in figure 14.25(*a*). Therefore, we consider that the transmitted symbol is '1', if the phase difference between two consecutive bits differs by less than $\frac{\pi}{2}$. If the pahse difference between two consecutive bits differs by more than $\frac{\pi}{2}$, then decision is taken in favour of zero.

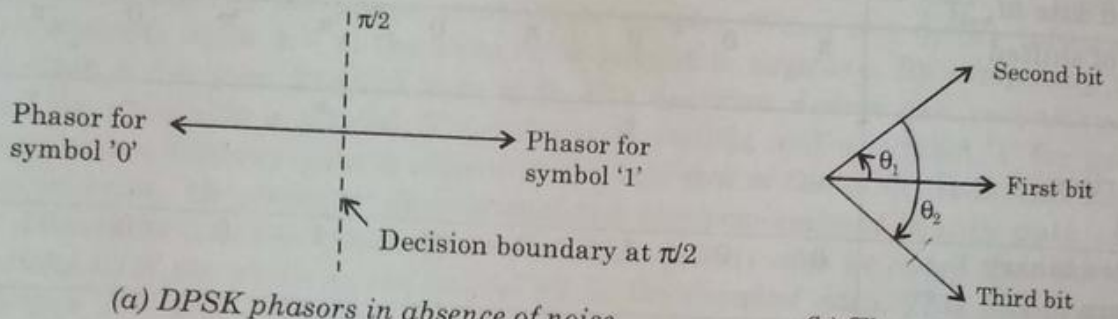
Figure 14.25(*b*) shows three consecutive bits. The first bit signal contains no noise, hence its phasor is along the horizontal line. Therefore, the symbol transmitted in first bit is assumed to be '1'. Because of noise, there is a phase difference of 'θ_1' between first and second bit. Since $\theta_1 < \frac{\pi}{2}$, second bit is also taken as symbol '1'. The phase difference between

Scanned by CamScanner

second and third bit is θ_2. From figure, it is clear that $\theta_2 > \frac{\pi}{2}$, hence third bit is taken as symbol zero. Since the phase differences are not exact between two successive bits, some error is introduced in the decision. Therefore, DPSK system is called 'sub-optimum' in nature. If the synchronization is used to make phase diferences exact, then it becomes PSK system.

Because of the sub-optimum nature of DPSK, the error probability is higher than that of BPSK. The average probability of error of non-coherent receiver is given as,

$$P(e) = \frac{1}{2} e^{-E/2N_0}$$



(a) DPSK phasors in absence of noise *(b) Three consecutive bits*

Fig. 14.25.

Here, E is the energy per symbol and $\frac{N_0}{2}$ is spectral density of white Gaussian noise. We know that symbol duration $T = 2T_b$. Hence, energy of the symbol will be

$$E = 2E_b$$

Hence, expression for $P(e)$ becomes,

$$P(e) = \frac{1}{2} e^{-E_b/N_0}$$

This is called average probability of error or bit error rate (BER) of DPSK system.

14.11.5. Advantages and Disadvantages of DPSK

We have observed in above discussion that DPSK has some advantages over BPSK, however, at the same time, it has some drawbacks.

(i) Advantages

(*i*) DPSK does not need carrier at the receiver end. This means that the complicated circuitry for generation of local carrier is not required.

(*ii*) The bandwidth requirement of DPSK is reduced as compared to that of BPSK.

(ii) Disadvantages

(*i*) The probability or error (*i.e.*, bit error rate) of DPSK is higher than that of BPSK.

(*ii*) Because DPSK uses two successive bits for its reception, error in the first bit creates error in the second bit. Therefore, error propagation in DPSK is more. On the other hand, in BPSK single bit can go in error since detection of each bit is independent.

(*iii*) Noise interference in DPSK is more.

Note: *In DPSK, previous bit is used to detect next bit. Hence, if error is present in previous bit, detection of next bit can also be wrong. Hence, error is created in next bit also. Therefore, there is tendency of appearing errors in pairs in DPSK.*

Scanned by CamScanner

14.12. Quadrature Phase Shift Keying (QPSK)

As a matter of fact, in communication systems, we have two main resources. These are the transmission power and the channel bandwidth. The channel bandwidth depends upon the bit rate or signaling rate f_b. In digital bandpass transmission, we use a carrier for transmission. This carrier is transmitted over a channel. If two or more bits are combined in some symbols, then the signaling rate will be reduced. Thus, the frequency of the carrier needed is also reduced. This reduces the transmission channel bandwidth. Hence, because of grouping of bits in symbols, the transmission channel bandwidth can be reduced. In quadrature phase shift keying (QPSK), two successive bits in the data sequence are grouped together. This reduces the bits rate or signaling rate (*i.e.*, f_b) and thus reduces the bandwidth of the channel.

In case of BPSK, we know that when symbol changes the level, the phase of the carrier is changed by 180°. Because, there were only two symbols in BPSK, the phase shift occurs in two levels only. However, in QPSK, two successive bits are combined. Infact, this combination of two bits forms four distinct symbols. When the symbol is changed to next symbol, then the phase of the carrier is changed by 45° ($\pi/4$ radians). Table 14.4 shows these symbols and their phase shifts.

Table 14.4. Symbol and corresponding phase shifts in QPSK

S. No.	*Input successive bits*		*Symbol*	*Phase shift in carrier*
$i = 1$	1(1 V)	0 (–1 V)	S_1	$\pi/4$
$i = 2$	0(–1 V)	0(–1 V)	S_2	$3\pi/4$
$i = 3$	0(–1 V)	1(1 V)	S_3	$5\pi/4$
$i = 4$	1(1 V)	1(1 V)	S_4	$7\pi/4$

Hence as shown in Table 14.4, there are four symbols and the phase is shifted by $\pi/4$ for each symbol.

14.13. Generation of QPSK

Figure 14.26 shows the block diagram of QPSK transmitter. Here, the input binary sequence is first converted to a bipolar NRZ type of signal. This signal is denoted by $b(t)$. It represents binary '1' by + 1 V and binary '0' by – 1 V. This signal has been shown in figure 14.27(*a*). The demultiplexer divides $b(t)$ into two separate bit streams of the odd numbered and even numbered bits. Here, $b_e(t)$ represents even numbered sequence and $b_0(t)$ represents odd numbered sequence. The symbol duration of both of these odd and even numbered sequences is $2T_b$. Hence, each symbol consists of two bits. Figure 14.27(*b*) and (*c*) illustrate the waveform of $b_e(t)$ and $b_0(t)$.

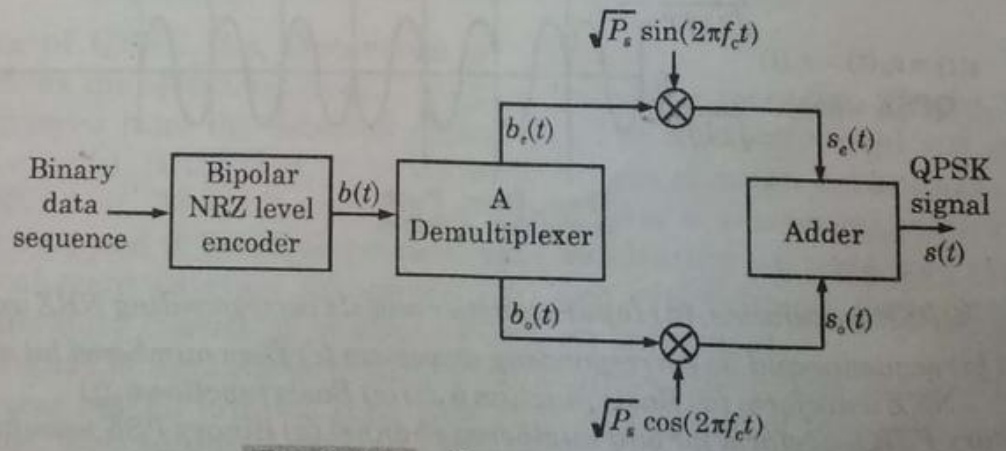


Fig. 14.26. *Generation of QPSK.*

Scanned by CamScanner

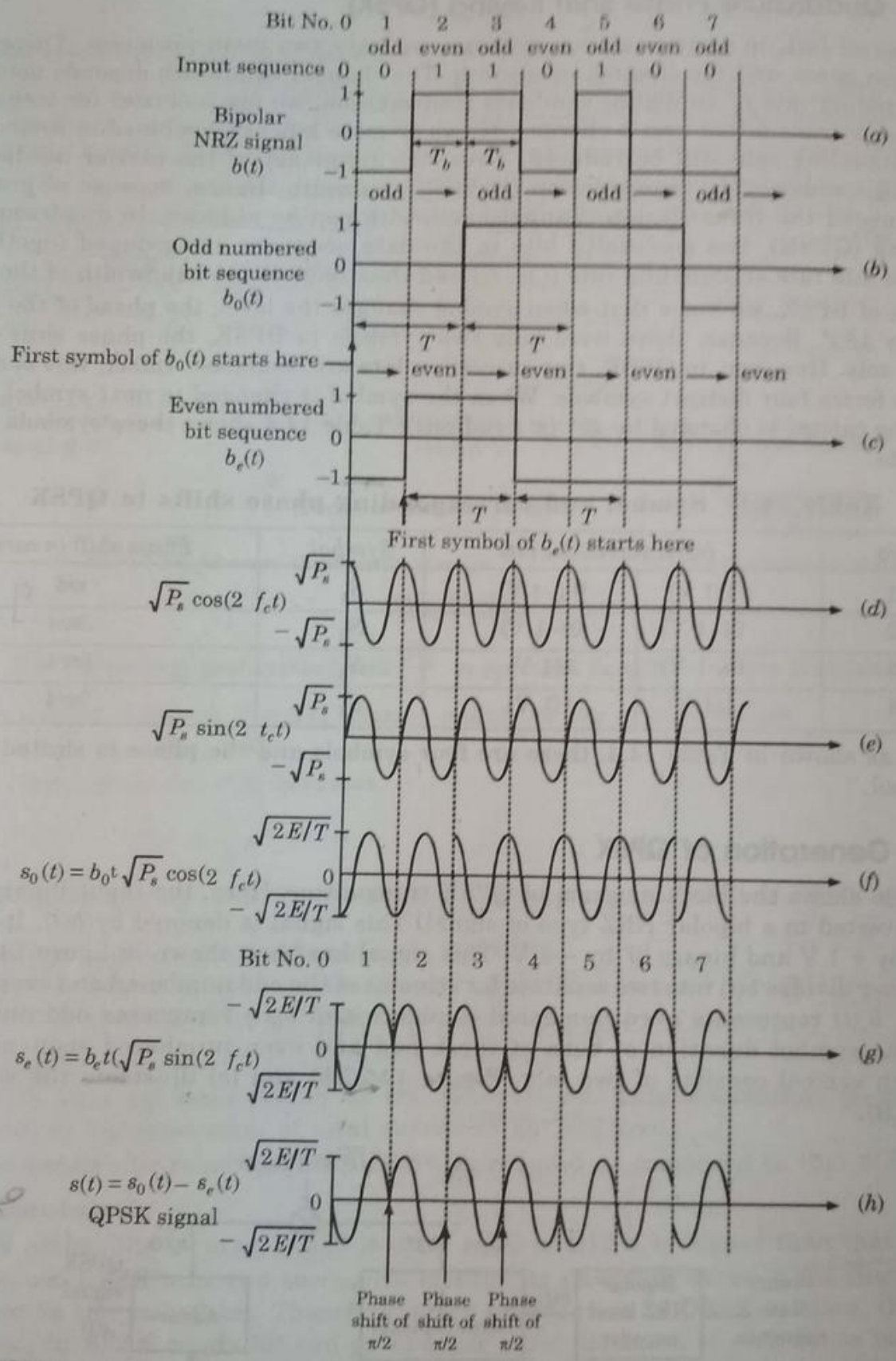


Fig. 14.27. *QPSK waveforms, (a) Input sequence and its corresponding NRZ waveform, (b) Odd numbered bit sequence and its corresponding waveform (c) Even numbered bit sequence and its NRZ waveform (d) Basis function $\phi_1(t)$ (e) Basis function $\phi_2(t)$ (f) Binary PSK waveform for odd numbered channel (g) Binary PSK waveform for even numbered channel (h) Final QPSK waveform.*

Scanned by CamScanner

It may be observed that the first even bit occurs after the first odd bit. Hence, even numbered bit sequence $b_e(t)$ starts with the delay of one bit period due to first odd bit. Thus, first symbol of $b_e(t)$ is delayed by one bit period 'T_b' with respect to first symbol of $b_0(t)$. This delay of T_b is known as offset. This shows that the change in levels of $b_e(t)$ and $b_0(t)$ cannot occur at the same time due to offset or staggering.

Also, the bit steam $b_e(t)$ modulates carrier $\sqrt{P_s}\cos(2\pi f_c t)$ and $b_0(t)$ modulates $\sqrt{P_s}\sin(2\pi f_c t)$. These modulators are the balanced modulators. The two carriers $\sqrt{P_s}\cos(2\pi f_c t)$ and $\sqrt{P_s}\sin(2\pi f_c t)$ have been shown in figure 14.27(*d*) and (*e*). There carriers are also known as **quadrature carriers.**

The two modulated sigals can be written as,

$$s_e(t) = b_e(t)\sqrt{P_s}\sin(2\pi f_c t) \quad ...(14.53)$$

and

$$s_0(t) = b_0(t)\sqrt{P_s}\cos(2\pi f_c t) \quad ...(14.54)$$

Hence, $s_e(t)$ and $s_0(t)$ are basically BPSK signals. The only difference is that $T = 2T_b$ here. The value of $b_e(t)$ and $b_0(t)$ would be + 1*V* or – 1*V*. Figure 14.27 (*f*) and (*g*) shows the waveforms of $s_e(t)$ and $s_0(t)$. The adder in figure 14.26 adds these two signals $b_e(t)$ and $b_0(t)$.

The output of the adder is QPSK signal and it is given by,

$$s(t) = s_0(t) + s_e(t)$$

or

$$s(t) = b_0(t)\sqrt{P_s}\cos(2\pi f_c t) + b_e(t)\sqrt{P_s}\sin(2\pi f_c t) \quad ...(14.55)$$

Figure 14.27(*h*) shows the QPSK signal represented by equation (14.55). In QPSK signal in figure 14.27(*h*), if there is any phase change, it occurs at minimum duration of T_b. This is because the two signals $s_e(t)$ and $s_0(t)$ have an offset of 'T_b'. Due to this offset, the phase shift in QPSK signal is $\frac{\pi}{2}$.

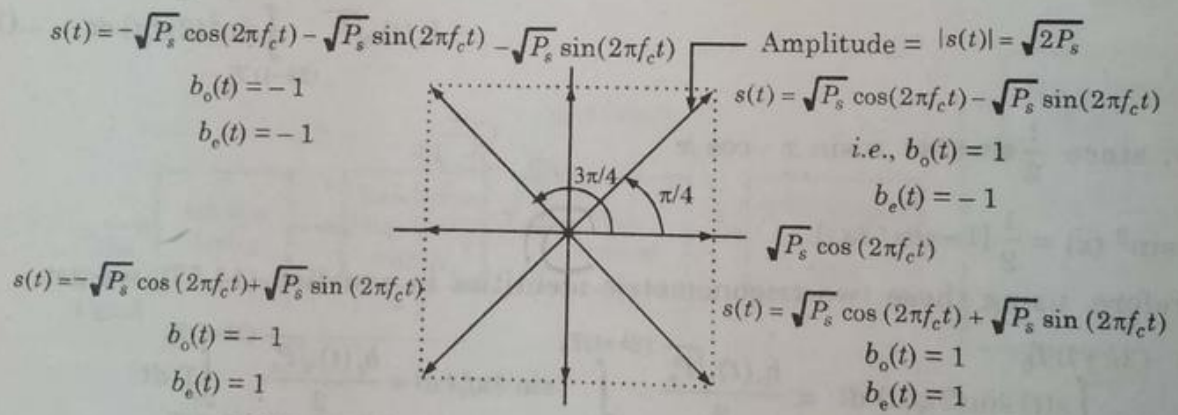


Fig. 14.28. *The Phasor diagram of QPSK signal.*

14.13.1. Reception of QPSK (*i.e.* Detection of QPSK)

Figure 14.29 shows the QPSK receiver. This is synchronous reception. Hence, the coherent carrier is to be recovered from the received signal $s(t)$. The received signal $s(t)$ is first raised to its 4th power, *i.e.*, $s^4(t)$. After that, it is allowed to pass through a bandpass filter (BPF) which is centred around $4f_c$. The output of the bandpass filter is a coherent carrier of frequency $4f_c$. This is divided by 4 and it provides two coherent quadrature carriers, *i.e.*, $\cos(2\pi f_c t)$ and $\sin(2\pi f_c t)$. These coherent carriers are applied to two synchronous demodulators. These synchronous demodulators consist of multiplier and an integrator.

The incoming signal is applied to both the multipliers. Here, the integrator integrates the product signal over two bit interval (*i.e.*, $T_s = 2T_b$). At the end of this period, the output of integrator is sampled. The outputs of the two integrators are sampled at the offset of one bit period, T_b. Hence, the output of the multiplexer is the signal $b(t)$. This means that the odd and even sequences are combined by multiplexer.

Scanned by CamScanner

Now, let us consider the product signal at the output of upper multiplier, *i.e.*,

$$s(t)\sin(2\pi f_c t) = b_0(t)\sqrt{P_s}\cos(2\pi f_c t)\sin(2\pi f_c t) + b_e(t)\sqrt{P_s}\sin^2(2\pi f_c t) \quad ...(14.56)$$

This signal is integrated by the upper integrator in figure 14.29.

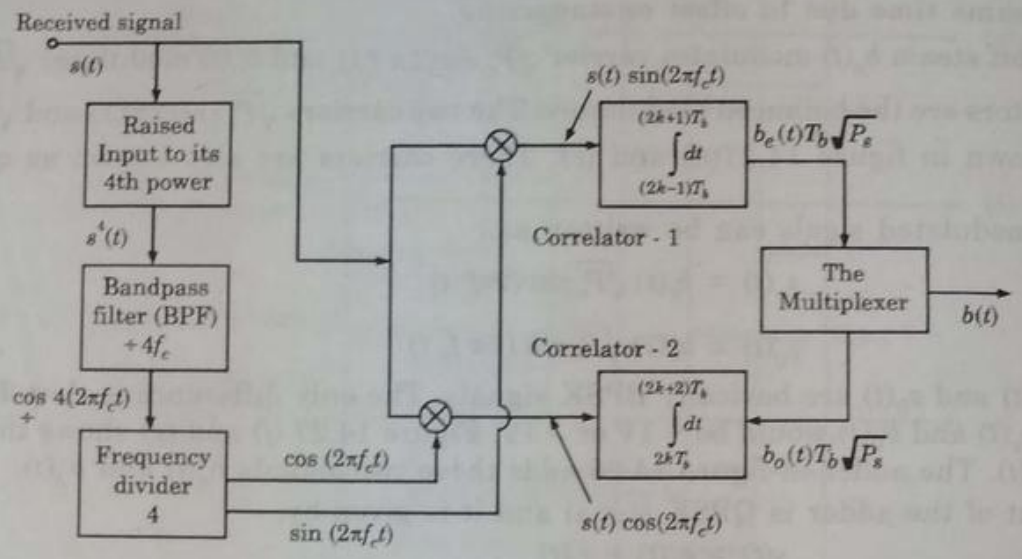


Fig. 14.29. *Reception of QPSK.*

Therefore, we have

$$\int_{(2k-1)T_b}^{(2k+1)T_b} s(t)\sin(2\pi f_c t)\,dt = b_0(t)\sqrt{P_s}\int_{(2k-1)T_b}^{(2k+1)T_b}\cos(2\pi f_c t)\sin(2\pi f_c t)\,dt$$

$$+ b_e(t)\sqrt{P_s}\int_{(2k-1)T_b}^{(2k+1)T_b}\sin^2(2\pi f_c t)\,dt \quad ...(14.57)$$

Now, since $\frac{1}{2}\sin(2x) = \sin x \cdot \cos x$

$2\sqrt{E_b}$

and $\sin^2(x) = \frac{1}{2}[1-\cos(2x)]$

Therefore, using these two trigonometric identities in equation (14.57), we get

$$\int_{(2k-1)T_b}^{(2k+1)T_b} s(t)\sin(2\pi f_c t)\,dt = \frac{b_o(t)\sqrt{P_s}}{2}\int_{(2k-1)T_b}^{(2k+1)T_b}\sin 4\pi f_c t\,dt + \frac{b_e(t)\sqrt{P_s}}{2}\int_{(2k-1)T_b}^{(2k+1)T_b}1\cdot dt$$

$$-\frac{b_e(t)\sqrt{P_s}}{2}\int_{(2k-1)T_b}^{(2k+1)T_b}\cos 4\pi f_c t\,dt$$

In this equation, the first and third integration terms involve integration of sinusoidal carriers over two bit period. They have full (integral number of) cycles over two bit periods and thus integration will be zero, *i.e.*,

$$\int_{(2k-1)T_b}^{(2k+1)T_b} s(t)\sin(2\pi f_c t)\,dt = \frac{b_e(t)\sqrt{P_s}}{2}[t]_{(2k-1)T_b}^{(2k+1)T_b}$$

$$= \frac{b_e(t)\sqrt{P_s}}{2}\times 2T_b = b_e(t)\sqrt{P_s}T_b \quad ...(14.58)$$

Scanned by CamScanner

Hence, the upper integrator responds to even sequence only. Similarly, we can obtain the output of lower integrator as $b_0(t)\sqrt{P_s}T_b$.

Note: *Even though bit synchronizer has not been shown in figure 14.29, it is assumed to be present with the integrator to locate starting and ending times of integration. The multiplexer is also operated by bit synchronizer. The amplitudes of voltage marked in figure 14.29 are arbitrary. They can change depending upon the gains of the integrator.*

14.13.2. Concept of Carrier Synchronization in QPSK

Both the carriers are to be synchronized properly in coherent detection in QPSK. Figure 14.30 shows the PLL system for carrier synchronization in QPSK.

The fourth power of the input signal consists of discrete frequency component at $4f_c$.

We know that,

$$\cos 4(2\pi f_c t) = \cos(8\pi f_c t + 2\pi N)$$

where 'N' is the number of cycles over the bit period. It is always an integer value. When the frequency division by four takes place, the RHS of this equation becomes $\cos\left(2\pi f_c t + \frac{N\pi}{2}\right)$.

This indicates that the output has a fixed phase error of $\frac{N\pi}{2}$. Differential encoding can be used to nullify the phase error events. The PLL remains locked with the phase of '$4f_c$' and then output of PLL is divided by 4. This provides a coherent carrier. A 90° phase shift is added to this carrier to produce a quadrature carrier.

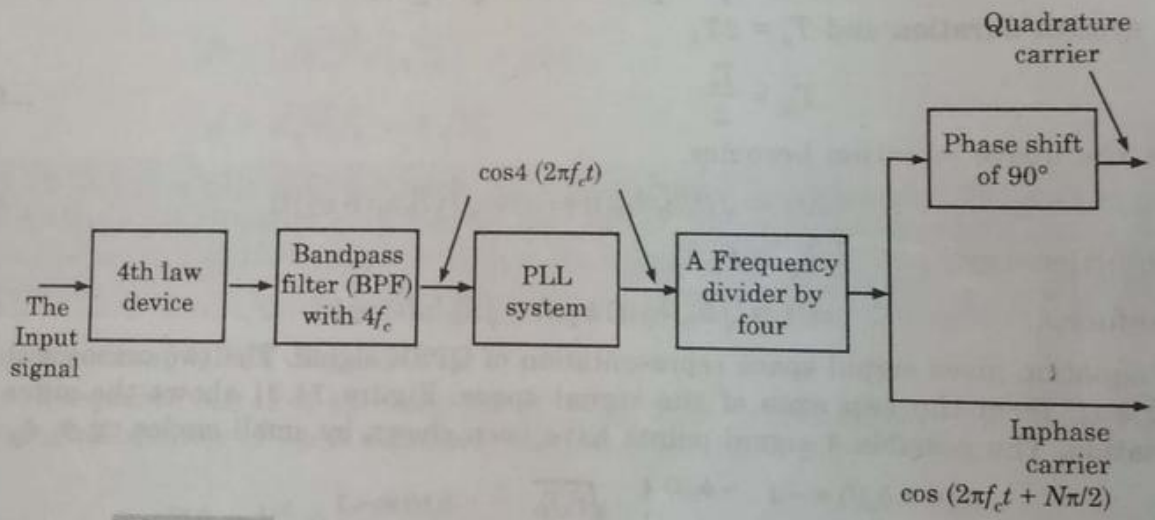


Fig. 14.30. *PLL system used for carrier synchronization in QPSK.*

14.13.3. Signal Space Representation in QPSK Signals

Figure 14.31 shows the phasor diagram of QPSK signal. Depending upon the combination of two successive bits, the phase shift occurs in carrier. This means that the QPSK signal in equation (14.55) can be written as,

$$s(t) = \sqrt{2P_s}\cos\left[2\pi f_c t + (2m+1)\frac{\pi}{4}\right] \quad . m = 0, 1, 2, 3 \qquad ...(14.60)$$

Here, this equation takes four values and they represent the phasors of figure 14.31. This equation can be expanded as under:

$$s(t) = \sqrt{2P_s}\cos(2\pi f_c t)\cos\left[(2m+1)\frac{\pi}{4}\right] - \sqrt{2P_s}\sin(2\pi f_c t)\sin\left[(2m+1)\frac{\pi}{4}\right]$$

Scanned by CamScanner

Let us rearrange the above equation as under:

$$s(t) = \left\{\sqrt{P_s T_s}\cos\left[(2m+1)\frac{\pi}{4}\right]\right\}\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t)$$

$$-\left\{\sqrt{P_s T_s}\sin\left[(2m+1)\frac{\pi}{4}\right]\right\}\sqrt{\frac{2}{T_s}}\sin(2\pi f_c t) \quad ...(14.61)$$

Again, let $\phi_1(t) = \sqrt{\frac{2}{T_s}}\cos(2\pi f_c t)$ ...(14.62)

and $\phi_2(t) = \sqrt{\frac{2}{T_s}}\sin(2\pi f_c t)$ ...(14.63)

These two signals are known as orthogonal signals and they are used as carriers in QPSK modulator.

Let $b_0(t) = \sqrt{2}\cos\left[(2m+1)\frac{\pi}{4}\right]$ ...(14.64)

and $b_e(t) = -\sqrt{2}\sin\left[(2m+1)\frac{\pi}{4}\right]$ ...(14.65)

With the use of equations (14.61) to (14.64) we can write equation (14.60) as under:

$$s(t) = \sqrt{P_s T_s}\cdot\frac{1}{\sqrt{2}}b_0(t)\,\phi_1(t) + \sqrt{P_s T_s}\cdot\frac{1}{\sqrt{2}}b_e(t)\,\phi_2(t)$$

or $$s(t) = \sqrt{P_s\cdot\frac{T_s}{2}}\,b_0(t)\,\phi_1(t) + \sqrt{P_s\frac{T_s}{2}}\,b_e(t)\,\phi_2(t)$$

T_s = symbol duration and $T_s = 2T_b$

or $$T_b = \frac{T_s}{2} \quad ...(14.66)$$

Then the above equation becomes,

$$s(t) = \sqrt{P_s T_b}\,b_0(t)\,\phi_1(t) + \sqrt{P_s T_b}\,b_e(t)\,\phi_2(t) \quad ...(14.67)$$

Since bit energy $E_b = P_s T_b$

Therefore, $$s(t) = \sqrt{E_b}\,b_0(t)\,\phi_1(t) + \sqrt{E_b}\,b_e(t)\,\phi_2(t) \quad ...(14.68)$$

This equation gives signal space representation of QPSK signal. The two orthogonal signals $\phi_1(t)$ and $\phi_2(t)$ form the two axes of the signal space. Figure 14.31 shows the signal space representation. The possible 4 signal points have been shown by small circles on ϕ_1 ϕ_2-plane.

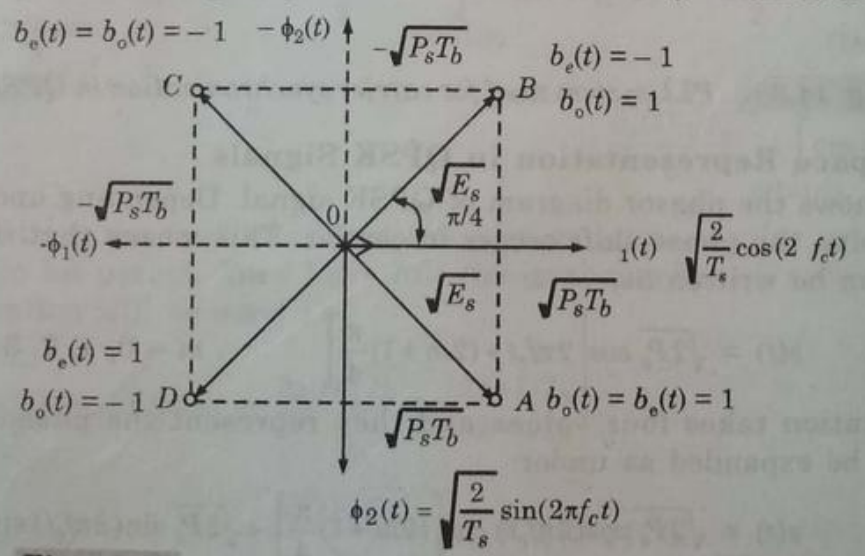


Fig. 14.31. *The signal space representation for QPSK signals.*

Scanned by CamScanner

From each signal point, we obtain two bits. For example, from point 'A', we obtain two bits as (1, 1) and from 'B' we obtain bits as (– 1, 1). The distance of any signal point from origin 'O', given as,

$$\text{Distance } 'OB' = \sqrt{P_sT_b + P_sT_b} = \sqrt{2P_sT_b} \quad ...(14.69)$$

$$= \sqrt{P_sT_s} \qquad [\because 2T_b = T_s] \quad ...(14.70)$$

or

$$'OB' = \sqrt{E_s} \qquad [\because P_s T_s = E_s]$$

Hence, the length of each signal point from origin is $\sqrt{E_s}$. We know that $b_e(t)$ and $b_0(t)$ represent two successive bits. There is an offset of 'T_b' between $b_e(t)$ and $b_o(t)$. Therefore, $b_e(t)$ and $b_o(t)$ both cannot change their levels simultaneously. Hence, either $b_e(t)$ or $b_o(t)$ can change at a time.

Let us say that $b_e(t) = b_o(t) = 1$ is representing signal point 'A' in figure 14.31. In the next bit interval, if $b_o(t) = -1$, then signal point will be '*D*'. Otherwise, if $b_e(t)$ changes its level [*i.e.*, $b_e(t) = -1$], then next signal point will be '*B*'. Hence, from signal point '*A*', then next signal points will be either '*D*' or '*B*'.

14.13.3.1. Distance Between Signal Points

As a matter of fact, the ability to determine a bit without error is measured by the distance between two nearest possible signal points in the signal space. Such points differed in signal bit. For example, signal points '*A*' and '*B*' are two nearest points since they differ by a signal bit $b_e(t)$. As '*A*' and '*B*' become closer to each other, the possibility of error increases. Therefore, this distance must be as large as possible. This distance is denoted by '*d*'. In figure 14.31, the distance between signal points '*A*' and '*B*' can be given by,

$$d^2 = \left(\sqrt{E_s}\right)^2 + \left(\sqrt{E_s}\right)^2 = \sqrt{2E_s} \quad ...(14.71)$$

or

$$d = 2\sqrt{P_sT_b} = 2\sqrt{E_b} \quad ...(14.72)$$

Note: *If we compare this distance with the distance of BPSK signals, then this shows that the distance for QPSK is the same as that for BPSK. Because, this distance represents noise immunity of the system, it shows that noise immunities of BPSK and QPSK are same.*

14.13.4. Spectrum of QPSK Signal

The input sequence $b(t)$ is of bit duration T_b. Also, it is a NRZ bipolar waveform. Recall, the power spectral density of such waveform can be given as,

$$S(f) = V_b^2\, T_b \left[\frac{\sin(\pi f\, T_b)}{\pi f\, T_b}\right]^2$$

Also, $V_b = \sqrt{P_s}$, then this equation becomes,

$$S(f) = P_sT_b \left[\frac{\sin(\pi f\, T_b)}{\pi f\, T_b}\right]^2 \quad ...(14.73)$$

This equation gives power spectral density (psd) of signal $b(t)$. This signal is divided into $b_e(t)$ and $b_0(t)$ each of bit period $2T_b$. If we consider that symbols 1 and 0 are equally likely, then we can write power spectral densities (psds) of $b_e(t)$ and $b_0(t)$ as,

$$S_e(f) = P_sT_s \left[\frac{\sin(\pi f T_s)}{\pi f T_s}\right]^2 \quad ...(14.74)$$

Scanned by CamScanner

and
$$S_0(f) = P_s T_s \left[\frac{\sin(\pi f T_s}{\pi f T_s}\right]^2 \quad ...(14.75)$$

In these two equations, we have just replaced T_b by T_s and T_s is the period of bit in $b_e(t)$ and $b_0(t)$. Because, inphase and quadrature components [$b_e(t)$ and $b_0(t)$] are statistically independent, the baseband power spectral density of QPSK signal equals the sum of the individual power spectral densities of $b_e(t)$ and $b_0(t)$ *i.e.*,

$$S_B(f) = S_e(f) + S_0(f)$$

or
$$S_B(f) = 2P_s T_s \left[\frac{\sin(\pi f T_s)}{\pi f T_s}\right]^2 \quad ...(14.76)$$

This equation gives baseband power spectral density of QPSK signal. Upon modulation of carrier of frequency f_c, the spectral density given by above equation is shifted at $\pm f_c$. Thus plots of power spectral density of QPSK will be similar to that BPSK.

14.13.5. Bandwidth of QPSK Signal

We have observed that the bandwidth of BPSK signal is equal of $2f_b$. Here, $T_b = \frac{1}{f_b}$ is the one bit period. In QPSK, the two waveforms $b_e(t)$ and $b_0(t)$ form the baseband signals. One bit period for both of these signals is equal to $2T_b$. Therefore, bandwidth of QPSK signal will be

$$BW = 2 \times \frac{1}{2T_b} = f_b \quad ...(14.77)$$

Hence, the bandwidth of QPSK signal is half of the bandwidth of BPSK signal. Earlier, we have observed that noise immunity of QPSK and BPSK is same. This shows that inspite of the reduction in bandwidth in QPSK, the noise immunity remains same as compared to BPSK. *BW* of QPSK can also be obtained by plotting equation (14.71) as shown in figure 14.32 below.

$$BW = \text{Highest frequency} - \text{Lowest frequency in main lobe}$$

$$BW = \frac{1}{T_s} - \left(-\frac{1}{T_s}\right) \text{ since carrier frequency } f_c \text{ cancels our}$$

$$BW = \frac{2}{T_s}$$

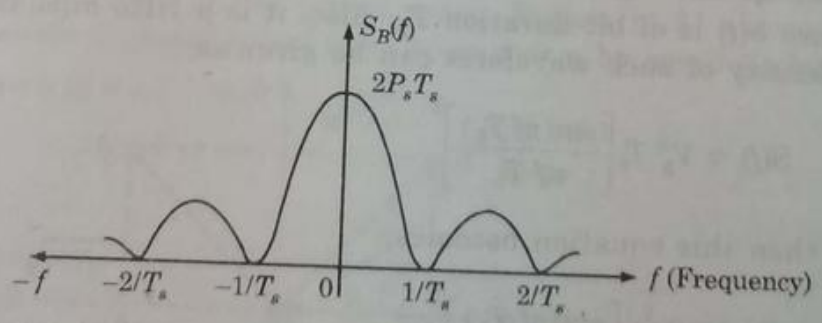


Fig. 14.32. *Plot of power spectral density (psd) of QPSK signal.*

We know that
$$T_s = 2T_b$$

or
$$BW = \frac{2}{2T_b} = \frac{1}{T_b}$$

or
$$BW = f_b \quad ...(14.78)$$

Scanned by CamScanner

14.13.6. Probability of Error of QPSK System

Observe figure 14.31 carefully. Between signal phasors 'OA' and 'OB', the axis $\phi_1(t)$ can be called decision boundary. Let us say signal vector 'OB' is present, but because of imperfect phase synchronization, it is detected as 'OA'. For this to happen, the phase shift must be at least $\frac{\pi}{4}$. The same thing can happen in case of other phasors also.

There are two correlators in the QPSK receiver. One correlator is used to detect even bits and other detects odd bits. Thus, any correlator can make a mistake if phase shift of $\frac{\pi}{4}$ occurs in the corresponding carrier. Therefore, the probability that correlator 1 or correlator 2 will make a mistake is given as,

$$P'_1(e) = P'_2(e) = \frac{1}{2} erfc \sqrt{\frac{E \cos^2 \theta}{N_0}}$$

This equation has been written because each correlator is independent PSK receiver. 'E_b' is replaced by 'E' in above equation, since energy of one bit in $b_e(t)$ or $b_0(t)$ is E (also written as E_s). Putting phase shift $\theta = \frac{\pi}{4}$ in above equation we obtain,

$$P'_1(e) = P'_2(e) = \frac{1}{2} erfc \sqrt{\frac{E \cos^2 \frac{\pi}{4}}{N_0}}$$

or
$$P'_1(e) = \frac{1}{2} erfc \sqrt{\frac{E}{2N_0}}$$

We can verify the above relation for BPSK as under:

We know that $E = E_s = P_s T_s$

and $T_s = 2 T_b$

Therefore, $E = P_s 2T_b = 2P_s T_b$

$E_b = P_s T_b$

$E = 2E_b$

By using the above relation, we obtain error probability of BPSK *i.e.*,

$$P'_1(e) = P'_2(e) = \frac{1}{2} erfc \sqrt{\frac{2E_b}{2N_0}}$$

or
$$P'_1(e) = \frac{1}{2} erfc \sqrt{\frac{E_b}{N_0}}$$

The above equation gives bit error probability of QPSK. Thus, bit error probability of QPSK and BPSK is same.

The probability $P(e)$ that the QPSK receiver will correctly detect the transmitted signal is equal to product of probabilities that both correlator 1 and correlator 2 will receive their bits correctly. The probability of correct reception of correlators 1 and 2 is,

$$P'_1(e) = 1 - P_1(e) \quad \text{and} \quad P_2(e) = 1 - P'_2(e)$$

$$\therefore \quad P(e) = P'_1(e) \times P'_2(e)$$
$$= [1 - P'_1(e)] \times [1 - P'_2(e)]$$
$$= 1 - P'_1(e) - P'_2(e) + P'_1(e) \times P'_2(e)$$

Scanned by CamScanner

$\therefore$ $P'_1(e) = P'_2(e)$ we can write above equation as,

$$P(e) = 1 - 2P'(e) + P'^2(e)$$

Here, $P'_1(e) = P'_2(e) = P'^2(e)$.

The term $P'^2(e)$ will be very very small and can be neglected. Hence,

$$P(e) = 1 - 2\,P'(e)$$

$\therefore$ Probability of error of QPSK system is,

$$P(e) = 1 - P'(e)$$

or $$P(e) = 1 - 1 + 2\,P'(e)$$

or $$P(e) = 2\,P'(e)$$

Putting value of $P'(e) = \frac{1}{2} erfc \sqrt{\frac{E}{2N_0}}$, we get

$$P(e) = erfc \sqrt{\frac{E}{2N_0}} = erfc \sqrt{\frac{E_b}{N_0}} \qquad \text{(since } E = 2E_b\text{)}$$

The individual probabilities $P'_1(e)$ and $P'_2(e)$ correlators are some times called as bit error probabilities or Bit Error Rate (BER). Thus bit error rate of QPSK is given as,

$$BER = \frac{1}{2} erfc \sqrt{\frac{E_b}{N_0}}$$

14.13.7. Advantages of QPSK

QPSK has some certain advantages as compared to BPSK and DPSK as under:

(*i*) For the same bit error rate, the bandwidth required by QPSK is reduced to half as compared to BPSK.

(*ii*) Because of reduced bandwidth, the information transmission rate of QPSK is higher.

14.14. Minimum Shift Keying (MSK)

(U.P. Tech., Sem., Examination 2003-2004)

We have discussed QPSK technique in last article. The bandwidth requirement of QPSK is high. Filters or other methods can overcome these problems, but they have other side effects. For example, filters alter the amplitude of the waveform.

MSK overcomes these problems. In MSK, the output waveform is continuous in phase hence there are no abrupt changes in amplitude. The sidelobes of MSK are very small hence bandpass filtering is not required to avoid interchannel interference. Figure 14.33 shows the waveform of MSK. The binary bit sequence is shown at the top. Figure 14.33(*a*) shows the corresponding NRZ waveform $b(t)$. From $b(t)$, two waveforms are generated for odd and even bits. $b_0(t)$ represents odd bits and $b_e(t)$ represents even bits. Figure 14.33(*b*) and (*c*) shows the waveform of $b_o(t)$ and $b_e(t)$. As shown in those waveforms b_1, b_3, b_5 etc. are represented by odd waveform *i.e.*, $b_o(t)$.

The duration of each bit in $b_o(t)$ or $b_e(t)$ is $2T_b$, whereas it is T_b in $b(t)$ *i.e.*,

$$T_s = 2T_b \qquad \text{...(14.79)}$$

The waveforms $b_o(t)$ and $b_e(t)$ have an offset of T_b. This offset is essential in MSK. Two waveforms $\sin 2\pi(t/4T_b)$ and $\cos 2\pi(t/4T_b)$ are generated as shown in figure 14.33(*d*). The waveform of $\sin 2\pi(t4T_b)$ passes through zero at the end of symbol time in $b_o(t)$. Hence, one symbol duration of $b_o(t)$ consists of complete half cycle of $\cos 2\pi(t/4T_b)$. This means that similarly, one symbol duration of $b_e(t)$ contains half cycle of $\sin 2\pi(t/T_b)$. Thus there is a phase shift of 'T_b' in sine and cosine waveforms. $b_e(t)$ is multiplied by $\sin 2\pi(t/4T_b)$ and $b_o(t)$ is multiplied by $\cos 2\pi(t/4T_b)$. These product waveforms are shown in figure 14.33 (*e*) and (*f*). The transmitted MSK signal is represented as under:

Scanned by CamScanner

$$s(t) = \sqrt{2P_s}\,[b_e(t)\sin(2\pi t/4T_b)]\cos(2\pi f_c t) + \sqrt{2P_s}\,[b_0(t)\cos(2\pi t/4T_b)]\sin(2\pi f_c t) \quad ...(14.80)$$

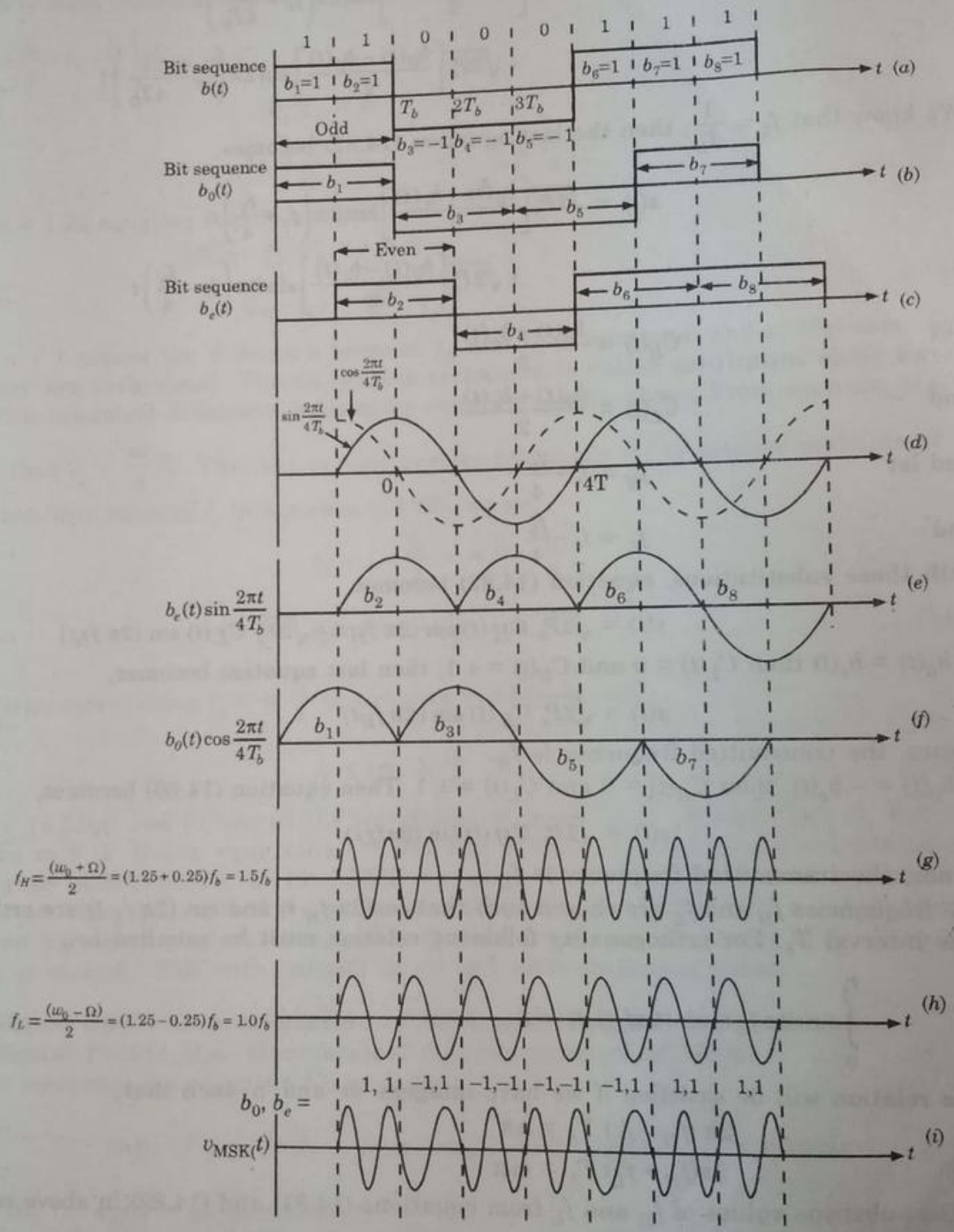


Fig. 14.33. *(a) Bipolar NRZ waveform representing bit sequence (b) Odd bit sequence waveforms $b_0(t)$ (c) Even bit sequence waveform $b_e(t)$ (d) Waveforms of frequency $f_b/4$ used for smoothing of $b_e(t)$ and $b_o(t)$ (e) Modulating waveform of even sequence (f) Modulating waveform of odd sequence (g) Waveform of frequency f_H (h) Waveform of frequency f_L (i) MSK waveform.*

Scanned by CamScanner

CHAPTER 15

Information Theory

Inside this Chapter

15.1. Introduction 744
15.2. What is information ? 745
15.3. Information Sources 745
15.4. Information content of a discrete memoryless source (DMS) 746
15.5. Information content of a symbol (*i.e*, Logarithmic Measure of Information) 746
15.6. Entropy (*i.e.* Average Information) 749
15.7. Information Rate 750
15.8. The Discrete Memoryless Channels (DMC) 757
15.9. Types of Channels 758
15.10. The Conditional and Joint Entropies 762
15.11. The Mutual Information 763
15.12. The Channel Capacity 768
15.13. Entropy Relations for a Continuous Channel 770
15.14. Capacity of an Additive White Gaussian Noise (AWGN) Channel: Shannon-Hartley Law 771
15.15. Channel Capacity 771
15.16. Transmission of Continuous Signals 773
15.17. Uncertainty in the Transmission Process 773
15.18. Exchange of Bandwidth for Signal-to-Noise Ratio 775
15.19. The Source Coding 780
15.20. Entropy Coding 785

15.1. Introduction

As discussed earlier in chapter 1, the purpose of a communication system is to carry information-bearing baseband signals from one place to another place over a communication channel. In the last few chapters, we have discussed a number of modulation schemes to accomplish this purpose. But what is the meaning of the word "Information". To answer this question we need to discuss information theory. Infact, information theory is a branch of probability theory which may be applied to the study of the communication systems. This broad mathematical discipline has made fundamental contributions, not only to communications, but also computer science, statistical physics and probability and statistics.

Scanned by CamScanner

Further, in the context of communications, information theory deals with mathematical modelling and analysis of a communication system rather than with physical sources and physical channels. As a matter of fact, Information theory was invented by communication scientists while they were studying the statistical structure of electronic communication equipments. When the communique is readily measurable, such as an electric current, the study of the communication system is relatively easy. But, when the communique is information, the study becomes rather difficult. How to define the measure for an amount of information? And also having described a suitable measure, how can it be applied to improve the communication of information? Information theory provides answers to all these questions. Thus, this chapter is devoted to a detailed discussion of information theory.

15.2. What is Information?

Before discussing the quantitative measure of information, let us review a basic concept about the amount of information in a message. Few messages produced by an information source contain more information than others. This may be best understood with the help of following example:

Consider you are planning a tour a city located in such an area where rain fall is very rare. To know about the weather forecast you will call the weather bureau and may receive one of the following information:

(*i*) It would be hot and sunny,

(*ii*) There would scattered rain,

(*iii*) There would be a cyclone with thunderstorm.

It may be observed that the amount of information received is clearly different for the three messages. The first message, just for instance, contains very little information because the weather in a desert city in summer is expected to be hot and sunny for maximum time. The second message forecasting a scattered rain contains some more information because it is not an event that occurs often. The forecast of a cyclonic storm contains even more information compared to the second message. This is because the third forecast is a rearest event in the city. Hence, on an conceptual basis the amount of information received from the knowledge of occurrence of a event may be related to the likelihood or probability of occurrence of that event. The message associated with the least likelihood event thus consists of maximum information. The above amount of information in a message depends only upon the uncertainty of the underlying event rather than its acutal content. Now, let us discuss few important concepts related to Information theory in the sections to follow.

15.3. Information Sources

An information source may be viewed as an object which produces an event, the outcome of which is selected at random according to a probability distribution. A practical source in a communication system is a device which produces messages, and it can be either analog or discrete. In this chapter, we deal mainly with the discrete sources since analog sources can be transformed to discrete sources through the use of sampling and quantization techniques, described in chapter 10. As a matter of fact, a discrete information source is a source which has only a finite set of symbols as possible outputs. The set of source symbols is called the **source alphabet**, and the elements of the set are called **symbols** or **letters.**

Information sources can be classified as having memory or being memoryless. A source with memory is one for which a current symbol depends on the previous symbols. A memoryless source is one for which each symbol produced is independent of the previous symbols.

A discrete memoryless source (DMS) can be characterized by the list of the symbols, the probability assignment to these symbols, and the specification of the rate of generating these symbols by the source.

Scanned by CamScanner

15.4. Information Content of a Discrete Memoryless Source (DMS)

The amount of information contained in an event is closely related to its uncertainty. Messages containing knowledge of high probability of occurrence convey relatively little information. We note that if an event is certain (that is, the event occurs with probability 1), it conveys zero information. Thus, a mathematical measure of information should be a function of the probability of the outcome and should satisfy the following axioms:

(*i*) Information should be proportional to the uncertainty of an outcome.

(*ii*) Information contained in independent outcomes should add.

15.5. Information Content of a Symbol (*i.e.*, Logarithmic Measure of Information)

Let us consider a discrete memoryless source (DMS) denoted by X and having alphabet $\{x_1, x_2, \ldots x_m\}$. The *information content* of a symbol x_i, denoted by $I(x_i)$ is defined by

$$I(x_i) = \log_b \frac{1}{P(x_i)} = -\log_b P(x_i) \quad \text{...(15.1)}$$

where $P(x_i)$ is the probability of occurrence of symbol x_i.

Note that $I(x_i)$ satisfies the following properties:

$$I(x_i) = 0 \text{ for } P(x_i) = 1 \quad \text{...(15.2)}$$

$$I(x_i) \geq 0 \quad \text{...(15.3)}$$

$$I(x_i) > I(x_j) \text{ if } P(x_i) < P(x_j) \quad \text{...(15.4)}$$

$$I(x_i, x_j) = I(x_i) + I(x_j) \text{ if } x_i \text{ and } x_j \text{ are indepedent} \quad \text{...(15.5)}$$

The unit of $I(x_i)$ is the bit (binary unit) if $b = 2$, Hartely or decit if $b = 10$, and nat (natural unit) if $b = e$. It is standard to use $b = 2$. Here the unit bit (abbreviated "b") is a measure of information content and is not to be confused with the term 'bit' meaning "binary digit". The conversion of these units to other units can be achieved by the following relationships.

$$\log_2 a = \frac{\ln a}{\ln 2} = \frac{\log a}{\log 2} \quad \text{...(15.6)}$$

According to our concept, the information content or amount of information of a symbol x_i, denoted by $I(x_i)$, must be inversely related to $P(x_i)$. Also from our intuitive concept $I(x_i)$ must staisfy the following requirements:

(*i*) $I(x_i)$ must approach 0 as $P(x_i)$ approaches infinity. For example, consider the message 'Sun will rise in the east'. This message does not contain any information since sun will rise in the east with probability 1.

(*ii*) The information content $I(x_i)$ must be a non-negative quantity since each message contains some information. In the worst case $I(x_i)$ can be equal to zero.

(*iii*) The information content of a message having higher probability of occurrence is less than the information content of a message having lower probability.

Now let us discuss few numerical examples to illustrate all the above concepts.

Example 15.1. A source produces one of four possible symbols during each interval having probabilities $P(x_1) = \frac{1}{2}$, $P(x_2) = \frac{1}{4}$, $P(x_3) = P(x_4) = \frac{1}{8}$. Obtain the information content of each of these symbols.

Solution: We know that the information content of each symbol is given as

$$I(x_i) = \log_2 \frac{1}{P(x_i)}$$

Scanned by CamScanner

Thus, we can write

$$I(x_1) = \log_2 \frac{1}{\frac{1}{2}} = \log_2 (2) = 1 \text{ bit}$$

$$I(x_2) = \log_2 \frac{1}{\frac{1}{4}} = \log_2 2^2 = 2 \text{ bits}$$

$$I(x_3) = \log_2 \frac{1}{\frac{1}{8}} = \log_2 2^3 = 3 \text{ bits}$$

$$I(x_4) = \log_2 \frac{1}{\frac{1}{8}} = 3 \text{ bits} \qquad \textbf{Ans.}$$

Example 15.2. Calculate the amount of information if it is given that $P(x_i) = \frac{1}{4}$.

Solution: We know that amount of information is given as,

$$I(x_i) = \log_2 \frac{1}{P(x_i)} = \frac{\log_{10} \frac{1}{P(x_i)}}{\log_{10} 2}$$

Substituting given value of $P(x_i)$ in above equation, we get

$$I(x_i) = \frac{\log_{10} 4}{\log_{10} 2} = 2 \text{ bits} \qquad \textbf{Ans.}$$

Example 15.3. Calculate the amount of information if binary digits (binits) occur with equal likelihood in a binary PCM system.

Solution: We know that in binary PCM, there are only two binary levels *i.e.*, 1 or 0. Since, they occur with equal likelihood, their probabilities of occurrence would be,

$$P(x_1) \text{ ('0' level)} = P(x_2) \text{ ('1' level)} = \frac{1}{2}$$

Therefore, the amount of information content will be given as

$$I(x_1) = \log_2 \frac{1}{P(x_1)}$$

and $$I(x_2) = \log_2 \frac{1}{P(x_2)}$$

or $$I(x_1) = \log_2 2$$

and $$I(x_2) = \log_2 2$$

or $$I(x_1) = I(x_2) = \frac{\log_{10} 2}{\log_{10} 2} = 1 \text{ bit} \qquad \textbf{Ans.}$$

Hence, the correct identification of binary digit (binit) in binary PCM carries 1 bit of information. **Ans.**

Example 15.4. In a binary PCM if '0' occur with probability $\frac{1}{4}$ and '1' occur with probability equal to $\frac{3}{4}$, then calculate the amount of information carried by each binit.

Solution: Here, given that the binit '0' has $P(x_1) = \frac{1}{4}$

Scanned by CamScanner

and binit '1' has $P(x_2) = \frac{3}{4}$

Then, amount of information is given as,

$$I(x_i) = \log_2 \frac{1}{P(x_i)}$$

with $P(x_1) = \frac{1}{4}$,

we have $I(x_1) = \log_2 4 = \frac{\log_{10} 4}{\log_{10} 2} = 2$ bits

and with $P(x_2) = \frac{3}{4}$,

we have $I(x_2) = \log_2\left(\frac{4}{3}\right) = \frac{\log_{10}(4/3)}{\log_{10} 2} = 0.415$ bits

Here, it may observed that binit '0' has probability $\frac{1}{4}$ and it carries 2 bits of information whereas binit '1' has probability $\frac{3}{4}$ and it carries 0.415 bits of information. Thus, this reveals the fact that if probability of occurrence is less, then the information carried is more, and vice-versa.

Example 15.5. If there are M equally likely and independent symbols, then prove that amount of information carried by each symbol will be,

$$I(x_i) = N \text{ bits.}$$

where $M = 2^N$ and N is an integer.

Solution: Since, it is given that all the M symbols are equally likely and independent, therefore, the probability of occurrence of each symbol must be $\frac{1}{M}$.

We know that amount of information is given as,

$$I(x_i) = \log_2 \frac{1}{P(x_i)} \quad ...(i)$$

Here, probability of each message is, $P(x_i) = \frac{1}{M}$.

Hence, equation (*i*) will be,

$$I(x_i) = \log_2 M \quad ...(ii)$$

Further, we know that $M = 2^N$, hence equation (*ii*) will be,

$$I(x_i) = \log_2 2^N = N \log_2 2$$

$$= N \frac{\log_{10} 2}{\log_{10} 2} = N \text{ bits}$$

Hence, amount of information carried by each symbol will be 'N' bits. We know that $M = 2^N$. This means that there are 'N' binary digits (binits) in each symbol. This indicates that when the symbols are equally likely and coded with equal number of binary digits (binits), then the information carried by each symbol (measured in bits) is numerically same as the number of binits used for each symbol.

Example 15.6. Prove the statement stated as under:

"If a receiver knows the measage being transmitted, the amount of information carried will be zero."

Scanned by CamScanner

Solution: Here it is stated that receiver "knows" the message. This means that only one message is transmitted. Thus, probability of occurrence of this message will be $P(x_i) = 1$. This is because only one message and its occurrence is certain (probability of certain event is '1'). The amount of information carried by this type of message will be,

$$I(x_i) = \log_2 \frac{1}{P(x_i)} = \frac{\log_{10} 1}{\log_{10} 2} \quad \text{substituting } P(x_i) = 1$$

or $$I(x_i) = 0 \text{ bits}$$

This proves the statement that if the receiver knows message, the amount of information carried will be zero.

Also, as $P(x_i)$ is decreased from 1 to 0, $I(x_i)$ increases monotonically from 0 to infinity. This shows that amount of information conveyed is *greater* when receiver correctly identifies *less likely* messages.

Example 15.7. If $I(x_1)$ is the information carried by symbols x_1 and $I(x_2)$ is the information carried by message x_2, then prove that the amount of information carried compositely due to x_1 and x_2 is $I(x_1, x_2) = I(x_1) + I(x_2)$.

Solution: We know that the amount of information is expressed as

$$I(x_i) = \log_2 \frac{1}{P(x_i)}$$

The individual amounts carried by symbols x_1 and x_2 are,

$$I(x_1) = \log_2 \frac{1}{P(x_1)} \quad \text{...(i)}$$

and $$I(x_2) = \log_2 \frac{1}{P(x_2)} \quad \text{...(ii)}$$

Here $P(x_1)$ is probability of symbol x_1 and $P(x_2)$ is probability of symbol x_2. Since messages x_1 and x_2 are independent, the probability of composite message is $P(x_1)\,P(x_2)$. Therefore, information carried compositely due to symbols x_1 and x_2 will be,

$$I(x_1, x_2) = \log_2 \frac{1}{P(x_1)\,P(x_2)}$$

$$I(x_1, x_2) = \log_2 \left[\left(\frac{1}{P(x_1)}\right)\left(\frac{1}{P(x_2)}\right)\right]$$

or $$I(x_1, x_2) = \log_2 \left(\frac{1}{P(x_1)}\right) + \log_2 \left(\frac{1}{P(x_2)}\right) \quad \text{Since } \log [AB] = \log A + \log B$$

Using equations (*i*) and (*ii*), we can write RHS of above equation as,

$$I(x_1, x_2) = I(x_1) + I(x_2) \quad \textbf{Hence Proved.}$$

15.6. Entropy (*i.e.*, Average Information)

In a practical communication system, we usually transmit long sequences of symbols from an information source. Thus, we are more interested in the average information that a source produces than the information content of a single symbol.

In order to get the information content of the symbol, we take notice of the fact that the flow of information in a system can fluctuate widely because of randomness involved into the selection of the symbols. Thus, we require to talk about the average information content of the symbols in a long message.

Thus, for quantitative representation of average information per symbol we make the following assumptions:

Scanned by CamScanner

(*i*) The source is stationary so that the probabilities may remain constant with time.

(*ii*) The successive symbols are statistically independent and come form the source at a average rate of r symbols per second.

The mean value of $I(x_i)$ over the alphabet of source X with m different symbols is given by

$$H(X) = E[I(x_i)] = \sum_{i=1}^{m} P(x_i) I(x_i)$$

or

$$H(X) = -\sum_{i=1}^{m} P(x_i) \log_2 P(x_i) \text{ b/symbol} \quad ...(15.7)$$

The quantity $H(X)$ is called the **entropy** of source X. It is a measure of the *average information content per source symbol*. The source entropy $H(X)$ can be considered as the average amount of uncertainty within source X that is resolved by use of the alphabet.

It may be noted that for a binary source X which generates independent symbols 0 and 1 with equal probability, the source entropy $H(X)$ is

$$H(X) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2} = 1\text{b/symbol} \quad ...(15.8)$$

The source entropy $H(X)$ satisfies the following relation:

$$0 \le H(X) \le \log_2 m \quad ...(15.9)$$

where m is the size (number of symbols) of the alphabet of source X. The lower bound corresponds to no uncertainty, which occurs when one symbol has probability $P(x_i) = 1$ while $P(x_i) = 0$ for $j \neq i$, so X emits the same symbol x_i all the time. The upper bound corresponds to the maximum uncertainty which occurs when $P(x_i) = 1/m$ for all i, that is, when all symbols are equality likely to be emitted by X.

15.7. Information Rate

If the time rate at which source X emits symbols is r (symbols s), the *information rate* R of the source is given by

$$R = rH(X) \ b/s \quad ...(15.10)$$

Here R is information rate.

$H(X)$ is Entropy or average information

and r is rate at which symbols are generated.

Information rate R is represented in average number of bits of information per second. It is calculated as under:

$$R = \left[r \text{ in } \frac{\text{symbols}}{\text{second}}\right] \times \left[H(X) \text{ in } \frac{\text{Information bits}}{\text{symbol}}\right] \quad ...(15.11)$$

or

$$R = \text{Information bits/second} \quad ...(15.12)$$

Example 15.8. Verify the following expression:

$$I(x_i x_j) = I(x_i) + I(x_j) \text{ if } x_i \text{ and } x_j \text{ are independent}$$

Solution: If x_i and x_j indepedent, then we know that

$$P(x_i x_j) = P(x_i)P(x_j)$$

Also

$$I(x_i x_j) = \log\frac{1}{P(x_i x_j)}$$

or

$$I(x_i x_j) = \log\frac{1}{P(x_i)P(x_j)}$$

Scanned by CamScanner

$$I(x_i x_j) = \log \frac{1}{P(x_i)} + \log \frac{1}{P(x_j)}$$

or $$I(x_i x_j) = I(x_i) + I(x_j)$$ **Hence Proved.**

Example 15.9. A Discrete Memoryless Source (DMS) X has four symbols x_1, x_2, x_3, x_4 with probabilities $P(x_1) = 0.4$, $P(x_2) = 0.3$, $P(x_3) = 0.2$, $P(x_4) = 0.1$.

(i) Calculate $H(X)$.

(ii) Find the amount of information contained in the messages $x_1\, x_2\, x_1\, x_3$ and $x_4\, x_3\, x_3\, x_2$, and compare with the $H(X)$ obtained in part (i). *(U.P. Tech-Semester Exam. 2002-2003)*

Solution: (i) We know that entropy is given by

$$H(X) = -\sum_{i=1}^{4} P(x_i) \log_2 [P(x_i)]$$

Substituting values and simplifying, we get

$$H(X) = -0.4 \log_2 0.4 - 0.3 \log_2 0.3 - 0.2 \log_2 0.2 - 0.1 \log_2 0.1$$

Solving, we get

$$H(X) = 1.85 \text{ b/symbol}$$ **Ans.**

(ii) Now, we have

$$P(x_1\, x_2\, x_1\, x_3) = (0.4)(0.3)(0.4)(0.2) = 0.0096$$

Therefore,

$$I(x_1 x_2 x_1 x_3) = -\log_2 0.0096 = 6.70 \text{ b/symbol*}$$ **Ans.**

Thus, $$I(x_1 x_2 x_1 x_3) < 7.4\ [= 4H(X)] \text{ b/symbol}$$

Also, $$P(x_4 x_3 x_3 x_2) = (0.1)(0.2)^2(0.3) = 0.0012$$

Therefore

$$I(x_4 x_3 x_3 x_2) = -\log_2 0.0012 = 9.70 \text{ b/symbol}$$

We conclude that

$$I(x_4 x_3 x_3 x_2) > 7.4[= 4H(X)] \text{ b/symbol}$$ **Ans.**

Example 15.10. Consider a binary memoryless source X with two symbols x_1 and x_2. Prove that $H(X)$ is maximum when both x_1 and x_2 equiprobable.

Solution: Here, let us assume that

$$P(x_1) = \alpha \text{ so that } P(x_2) = 1 - \alpha.$$

We know that entropy is given by

$$H(X) = -\sum_{i=1}^{m} P(x_i) \log_2 P(x_i) \text{ bits/symbol}$$

Thus, $$H(X) = -\alpha \log_2 \alpha - (1-\alpha) \log_2 (1-\alpha) \quad \text{...(i)}$$

Differentiating above equation with respect to α, we get

or $$\frac{dH(X)}{d\alpha} = \frac{d}{d\alpha}[-\alpha \log_2 \alpha - (1-\alpha) \log_2 (1-\alpha)]$$

Using the relation

$$\frac{d}{dx} \log_b y = \frac{1}{y} \log_b e \frac{dy}{dx},$$

* Using the relation $I(x_i) = \log_2 \dfrac{1}{P(x_i)}$.

Scanned by CamScanner

We get
$$\frac{dH(X)}{d\alpha} = -\log_2 \alpha + \log_2(1-\alpha)$$
$$= \log_2 \frac{1-\alpha}{\alpha}$$

Note that the maximum value of $H(X)$ requires that
$$\frac{dH(X)}{d\alpha} = 0$$

This means that
$$\frac{1-\alpha}{\alpha} = 1 \quad \text{or} \quad \alpha = \frac{1}{2}$$

Note that $H(X) = 0$ when $\alpha = 0$ or 1.

When $P(x_1) = P(x_2) = \frac{1}{2}$, $H(X)$ is maximum and is given by
$$H(X) = \frac{1}{2}\log_2 2 + \frac{1}{2}\log_2 2 = 1 \text{ b/symbol} \quad ...(ii)$$

Hence Proved

Example 15.11. Verify the following expression:
$$0 \le H(X) \le \log_2 m$$
where m is the size of the alphabet of X.

Solution: Proof of the lower bound:

Since
$$0 \le P(x_i) \le 1,$$
$$\frac{1}{P(x_i)} \ge 1 \quad \text{and} \quad \log_2 \frac{1}{P(x_i)} \ge 0$$

Then, it follows that
$$P(x_i)\log_2 \frac{1}{P(x_i)} \ge 0$$

Thus,
$$H(X) = \sum_{i=1}^{m} P(x_i)\log_2 \frac{1}{P(x_i)} \ge 0 \quad ...(i)$$

Next, we note that
$$P(x_i)\log_2 \frac{1}{P(x_i)} = 0$$

If and only if $P(x_i) = 0$ or 1. Since
$$\sum_{i=1}^{m} P(x_i) = 1$$

when $P(x_i) = 1$ then $P(x_j) = 0$ for $j \ne i$. Thus, only in this case, $H(X) = 0$.

Proof of the upper bound

Let us consider two probability distributions $[P(x_i) = P_i]$ and $[Q(x_i) = Q_i]$ on the alphabet $\{x_i\}$, $i = 1, 2, \ldots m$, such that
$$\sum_{i=1}^{m} P_i = 1 \quad \text{and} \quad \sum_{i=1}^{m} Q_i = 1 \quad ...(ii)$$

We know that
$$\sum_{i=1}^{m} P_i \log_2 \frac{Q_i}{P_i} = \frac{1}{\ln 2}\sum_{i=1}^{m} P_i \ln \frac{Q_i}{P_i}$$

Scanned by CamScanner

Next, using the inequality

$$\ln \alpha \le \alpha - 1 \quad \alpha \ge 0 \qquad ...(iii)$$

and noting that the equality holds only if $\alpha = 1$, we obtain

$$\sum_{i=1}^{m} P_i \ln \frac{Q_i}{P_i} \le \sum_{i=1}^{m} P_i \left(\frac{Q_i}{P_i} - 1 \right) = \sum_{i=1}^{m} (Q_i - P_i)$$

$$= \sum_{i=1}^{m} Q_i - \sum_{i=1}^{m} P_i = 0 \quad \text{by using equations } (ii). \qquad ...(iv)$$

Hence, we have

$$\sum_{i=1}^{m} P_i \log_2 \frac{Q_i}{P_i} \le 0 \qquad ...(v)$$

where the equality holds only if $Q_i = P_i$ for all i.

Setting $$Q_i = \frac{1}{m}, \quad i = 1, 2, ..., m \qquad ...(vi)$$

We get $$\sum_{i=1}^{m} P_i \log_2 \frac{1}{P_i m} = -\sum_{i=1}^{m} P_i \log_2 P_i - \sum_{i=1}^{m} P_i \log_2 m = H(X) - \log_2 m \sum_{i=1}^{m} P_i \qquad ...(vii)$$

$$= H(X) - \log_2 m \le 0$$

Therefore, we get $H(X) \le \log_2 m$

and the quality holds only if the symbols in X are equiprobable.

Example 15.12. A high-resolution black-and-white TV picture consists of about 2×10^6 picture elements and 16 different brightness levels. Pictures are repeated at the rate of 32 per second. All picture elements are assumed to be independent, and all levels have equal likelihood of occurrence. Calculate the average rate of information conveyed by this TV picture source. *(U.P. Tech. (C.O.), 2003)*

Solution: We know that entropy is given by

$$H(X) = -\sum_{i=1}^{m} P(x_i) \log_2 P(x_i) \text{ bits/symbol} \qquad ...(i)$$

Here, given that $m = 16, \quad P(x_i) = \frac{1}{16}$

Substituting all the values in equation (i), we get

$$H(X) = -\sum_{i=1}^{16} \frac{1}{16} \log_2 \frac{1}{16} = 4 \text{ bits/element}$$

The rate, r, at which symbols are generated is given by

$$r = 2(10^6)(32) = 64(10^6) \text{ elements/second}$$

Hence, the average rate of information conveyed is given by

$$R = rH(X) = 64(10^6)(4)$$

or $$R = 256(10^6) \text{ b/s} = 256 \text{ Mb/s} \quad \textbf{Ans.}$$

Example 15.13. Given a telegraph source having two symbols, dot and dash. The dot duration is 0.2 s. The dash duration is 3 times the dot duration. The probability of the dot's occurring is twice that of the dash, and the time between symbols is 0.2 s. Calculate the information rate of the telegraph source.

Scanned by CamScanner

Solution: Given that

$$P(\text{dot}) = 2P(\text{dash})$$

$$P(\text{dot}) + P(\text{dash}) = 3P(\text{dash}) = 1$$

From above two equations, we have

$$P(\text{dash}) = \frac{1}{3} \quad \text{and} \quad P(\text{dot}) = \frac{2}{3}$$

Further, we know that the entropy is given by

$$H(X) = -\sum_{i=1}^{m} P(x_i) \log_2 P(x_i) \quad \text{bits/symbol}$$

Expanding accordingly, we get

$$H(X) = -P(\text{dot}) \log_2 P(\text{dot}) - P(\text{dash}) \log_2 P(\text{dash})$$

Substituting all the values, we get

$$H(X) = 0.667(0.585) + 0.333(1.585) = 0.92 \text{ b/symbol}$$

Also, given that

$$t_{\text{dot}} = 0.2\text{s}, \; t_{\text{dash}} = 0.6\text{s}, \; t_{\text{space}} = 0.2 \text{ s}$$

Thus, the average time per symbol will be

$$T_s = P(\text{dot})t_{\text{dot}} + P(\text{dash})t_{\text{dash}} + t_{\text{space}}$$
$$= 0.5333 \text{ s/symbol}$$

Further, the average symbol rate will be given by

$$r = \frac{1}{T_s} = 1.875 \text{ symbols/s}$$

Therefore, the average information rate of the telegraph source will be

$$R = rH(X) = 1.875(0.92) = 1.725 \text{ b/s} \quad \textbf{Ans.}$$

Example 15.14. An analog signal is bandlimited to f_m Hz and sampled at Nyquist rate. The samples are quantized into 4 levels. Each level represent one symbols. Thus there are 4 symbols. The probabilities of occurrence of these 4 levels (symbols) are $P(x_1) = P(x_4) = \frac{1}{8}$ and $P(x_2) = P(x_3) = \frac{3}{8}$. Obtain information rate of the source. *(Delhi University, 2000)*

Solution: We are given four symbols with probabilities $P(x_1) = P(x_4) = \frac{1}{8}$ and $P(x_2) = P(x_3) = \frac{3}{8}$.

Average information $H(X)$ (or entropy) is expressed as,

$$H(X) = P(x_1)\log_2 \frac{1}{P(x_1)} + P(x_2)\log_2 \frac{1}{P(x_2)} + P(x_3)\log_2 \frac{1}{P(x_3)} + P(x_4)\log_2 \frac{1}{P(x_4)}$$

Substituting all the given values, we get

or $$H(X) = \frac{1}{8}\log_2 8 + \frac{3}{8}\log_2\left(\frac{8}{3}\right) + \frac{3}{8}\log_2\left(\frac{8}{3}\right) + \frac{1}{8}\log_2(8)$$

or $$H(X) = 1.8 \text{ bits/ symbol} \quad \textbf{Ans.}$$

It is given that the signal is sampled at Nyquist rate. Nyquist rate for f_m Hz bandlimited signal is,

$$\text{Nyquist rate} = 2f_m \text{ samples/sec.}$$

Since every sample generates one source symbol,

Therefore, symbols per second, $r = 2f_m$ symbols/sec.

Scanned by CamScanner

Information rate is given by

$$R = rH(X)$$

Putting, values of r and $H(X)$ in this equation, we get

$$R = 2f_m \text{ symbols/sec} \times 1.8 \text{ bits/symbol} = 3.6\, f_m \text{ bits/sec} \quad \textbf{Ans.}$$

In this example, there are four levels. Those four levels may be coded using binary PCM as shown in Table 15.1.

Hence, two binary digits (binits) are required to send each symbol. We know that symbols are sent at the rate of $2f_m$ symbols/sec. Therefore, transmission rate of binary digits will be,

Binary digits (binits) rate = 2 binits/symbol × $2f_m$ symbols/sec. = $4f_m$ binits/sec.

Because one binit is capable of conveying 1 bit of information therefore, the above coding seheme is capable of conveying $4f_m$ *bits of information per second*. But in this example, we have obtained that we are transmitting 3.6 f_m bits of information per second. This means that the information carrying ability of binary PCM is not completely utilized by the transmission scheme.

Table 15.1.

S.No.	*Symbol or level*	*Probability*	*Binary digits*
1.	Q_1	$\frac{1}{8}$	0 0
2.	Q_2	$\frac{3}{8}$	0 1
3.	Q_3	$\frac{3}{8}$	1 0
4.	Q_4	$\frac{1}{8}$	1 1

Example 15.15. In the transmission scheme of last example, obtain the information rate if all symbols are equally likely.

Also, comment on the result.

Solution: We are given that there are four symbols. Since they are equally likely, therefore, their probabilities will be equal to $\frac{1}{4}$ *i.e.*,

$$P(x_1) = P(x_2) = P(x_3) = P(x_4) = \frac{1}{4}$$

We know that, average information per symbol (Entropy) is given as,

$$H(X) = P(x_1)\log_2\frac{1}{P(x_1)} + P(x_2)\log_2\frac{1}{P(x_2)} + P(x_3).\log_2\frac{1}{P(x_3)} + P(x_4)\log_2\frac{1}{P(x_4)}$$

since $\quad P(x_1) = P(x_2) = P(x_3) = P(x_4)$

Therefore, $\quad H(X) = 4P(x).\log_2\frac{1}{P(x)}$

Again, since $\quad P(x_1) = P(x_2) = P(x_3) = P(x_4) = P(x) = \frac{1}{4}$

Thus, $\quad H(X) = \log_2(4) = 2$ bits/symbol

We know that the information rate is expressed as,

$$R = rH(X)$$

Scanned by CamScanner

Here, $r = 2f_m$ symbols/sec. as obtained in last example. Substituting, these values in above equation, we get,

$$R = 2f_m \text{ symbols/sec.} \times 2 \text{ bits/symbol} = 4\, f_m \text{ bits/sec.} \quad \textbf{Ans.}$$

Comments: *Just before this example we have seen that a binary coded PCM with 2 binits per message is capable of conveying 4B bits of information per second. The transmission scheme discussed in above example transmits 4B bits of information per second. This has been made possible since all the messages are equally likely. Thus with binary PCM clding, the maximum information rate is achieved if all messages are equally likely.*

Example 15.16. A discrete source emits one of five symbols once every millisecond with probabilities 1/2, 1/4, 1/8, 1/16 and 1/16 respectively. Determine the source entropy and information rate.

Solution: We know that the source entropy is given as

$$H(X) = \sum_{i=1}^{m} P(x_i) \log_2 \frac{1}{P(x_i)} = \sum_{i=1}^{5} P(x_i) \log_2 \frac{1}{P(x_i)} \quad \text{bits/symbol}$$

or
$$H(X) = \frac{1}{2}\log_2(2) + \frac{1}{4}\log_2(4) + \frac{1}{8}\log_2(8) + \frac{1}{16}\log_2(16) + \frac{1}{16}\log_2(16)$$

or
$$H(X) = \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{1}{4} + \frac{1}{4} = \frac{15}{8}$$

or
$$H(X) = 1.875 \text{ bits/symbol} \quad \textbf{Ans.}$$

The symbol rate $r = \dfrac{1}{T_b} = \dfrac{1}{10^{-3}} = 1000$ symbols/sec.

Therefore, the information rate is expressed as

$$R = rH(X) = 1000 \times 1.875 = 1875 \text{ bits/sec.} \quad \textbf{Ans.}$$

Example 15.17. The probabilities of the five possible outcomes of an experiment are given as

$$P(x_1) = \frac{1}{2},\ P(x_2) = \frac{1}{4},\ P(x_3) = \frac{1}{8},\ P(x_4) = P(x_5) = \frac{1}{16}$$

Determine the entropy and information rate if there are 16 outcomes per second.

(Delhi University, 1999)

Solution: The entropy of the system is given as

$$H(X) = \sum_{i=1}^{5} P(x_i) \log_2 \frac{1}{P(x_i)} \text{ bits/symbol}$$

or
$$H(X) = \frac{1}{2}\log_2 2 + \frac{1}{4}\log_2 4 + \frac{1}{8}\log_2 8 + \frac{1}{16}\log_2 16 + \frac{1}{16}\log_2 16$$

or
$$H(X) = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{4}{16} = \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{1}{4} + \frac{1}{4}$$

or
$$H(X) = \frac{4+4+3+2+2}{8} = \frac{15}{8} = 1.875 \text{ bits/outcome} \quad \textbf{Ans.}$$

Now, rate of outcomes $r = 16$ outcomes/sec.

Therefore, the rate of information R will be

$$R = rH(X) = 16 \times \frac{15}{8} = 30 \text{ bits/sec} \quad \textbf{Ans.}$$

Scanned by CamScanner

Example 15.18. An analog signal bandlimited to 10 kHz is quantized in 8 levels of a PCM system with probabilities of 1/4, 1/5 1/5, 1/10, 1/10, 1/20, 1/20 and 1/20 respectively. Find the entropy and the rate of information.

Solution: We know that according to the sampling theorem, the signal must be sampled at a frequency given as

$$f_s = 10 \times 2 \text{ kHz} = 20 \text{ kHz}$$

Considering each of the eight quantized levels as a message, the entropy of the source may written as

$$H(X) = \frac{1}{4}\log_2 4 + \frac{1}{5}\log_2 5 + \frac{1}{5}\log_2 5 + \frac{1}{10}\log_2 10 + \frac{1}{10}\log_2 10 + \frac{1}{20}\log_2 20 + \frac{1}{20}\log_2 20 + \frac{1}{20}\log_2 20$$

or $$H(X) = \frac{1}{4}\log_2 4 + \frac{2}{5}\log_2 5 + \frac{7}{20}\log_2 20 = 2.84 \text{ bits/message}$$

As the sampling frequency is 20 kHz then the rate at which message is produced, will be given as

or $$r = 20000 \text{ messages/sec.}$$

Hence the information rate is

$$R = rH(X) = 20000 \times 2.84 = 56800 \text{ bits/sec.}$$ **Ans.**

15.8. The Discrete Memoryless Channels (DMC)

In this section, let us discuss different aspects related to discrete memoryless channels (DMC).

15.8.1. Channel Representation

A communication channel may be defined as the path or medium through which the symbols flow to the receiver end. A *discrete mamoryless channel* (DMC) is a statistical model with an input X and an output Y as shown in figure 15.1. During each unit of the time (signaling interval), the channel accepts an input symbol from X, and in response it generates an output symbol from Y. The channel is said to be "discrete" when the alphabets of X and Y are both finite. Also, it is said to be "memoryless" when the current output depends on only the current input and not on any of the previous inputs.

A diagram of a DMC with m inputs and n outputs has been illustrated in figure 15.1. The input X consists of input symbols $x_1, x_2, \ldots x_m$. The a priori probabilities of these source symbols $P(x_i)$ are assumed to be known. The outputs Y consists of output symbols $y_1, y_2, \ldots y_n$. Each possible input-to-output path is indicated along with a conditional probability $P(y_j | x_i)$, where $P(y_j | x_i)$ is the conditional probability of obtaining output y_j given that the input is x_i, and is called a **channel transition probability.**

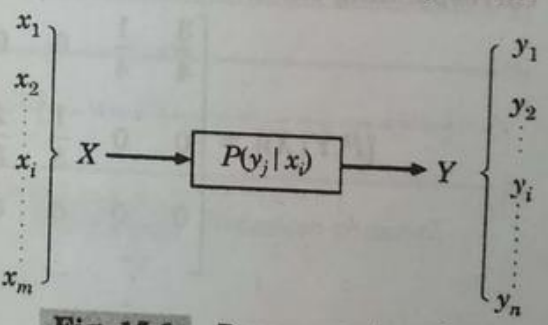


Fig. 15.1. *Representation of a discrete memoryless channel (DMC).*

15.8.2. The Channel Matrix

A channel is completely specified by the complete set of transition probabilities. Accordingly, the channel in figure 15.1 is often specified by the matrix of transition probabilities $[P(Y|X)]$. This matrix is given by

$$[P(Y|X)] = \begin{bmatrix} P(y_1|x_1) & P(y_2|x_1) & \cdots & P(y_n|x_1) \\ P(y_1|x_2) & P(y_2|x_2) & \cdots & P(y_n|x_2) \\ \cdots & \cdots & \cdots & \cdots \\ P(y_1|x_m) & P(y_2|x_m) & \cdots & P(y_n|x_m) \end{bmatrix} \quad \ldots(15.13)$$

This matrix $[P(Y|X)]$ is called the **channel matrix.**

Scanned by CamScanner

Since each input to the channel results in some output, each row of the channel matrix must sum to unity. This means that

$$\sum_{j=1}^{n} P(y_j \mid x_i) = 1 \text{ for all } i \qquad \text{...(15.14)}$$

Now, if the input probabilities $P(X)$ are represented by the row matrix, then we have

$$[P(X)] = [P(x_1)\ P(x_2) \ldots P(x_m)] \qquad \text{...(15.15)}$$

Also, the output probabilities $P(Y)$ are represented by the row matrix as under:

$$[P(Y)] = [P(y_1)\ P(y_2) \ldots P(y_n)] \qquad \text{...(15.16)}$$

then

$$[P(Y)] = [P(X)][P(Y \mid X)] \qquad \text{...(15.17)}$$

Now, if $P(X)$ is represented as a diagonal matrix, then we have

$$[P(X)]_d = \begin{bmatrix} P(x_1) & 0 & \cdots & 0 \\ 0 & P(x_2) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & P(x_m) \end{bmatrix} \qquad \text{...(15.18)}$$

then

$$[P(X,Y)] = [P(X)]_d\,[P(Y \mid X)] \qquad \text{...(15.19)}$$

where the (i, j) element of matrix $[P(X,Y)]$ has the form $P(x_i, y_j)$.

The matrix $[P(X, Y)]$ is known as the *joint probability matrix*, and the element $P(x_i, y_j)$ is the joint probability of transmitting x_i and receiving y_j.

15.9. Types of Channels

Other than continuous and discrete channels, there are some special type of channels with their own channel matrices. In this subsection, let us discuss various special channels.

15.9.1. Lossless Channel

A channel described by a channel matrix with only one non-zero element in each column is called a *lossless channel*. An example of a lossless channel has been shown in figure 15.2, and the corresponding channel matrix is given in equation (15.20) as under:

$$[P(Y \mid X)] = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{...(15.20)}$$

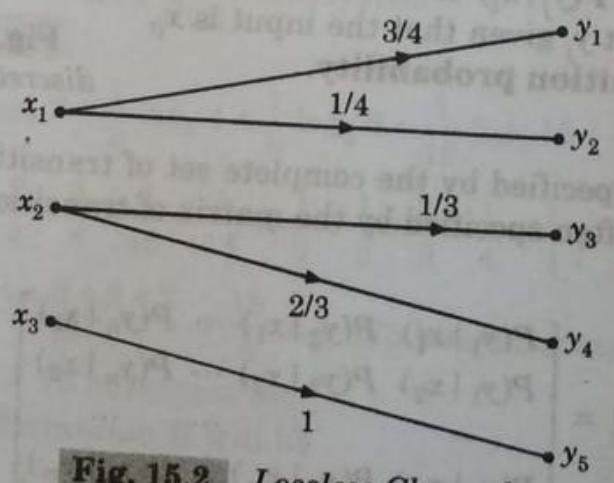


Fig. 15.2. *Lossless Channel.*

It can be shown that in the lossless channel, no source information is lost in transmission.

Scanned by CamScanner

15.9.2. Deterministic Channel

A channel described by a channel matrix with only one non-zero element in each row is called a *deterministic channel*. An example of a deterministic channel has been shown in figure 15.3, and the corresponding channel matrix is given by equation (15.21) as under:

$$[P(Y|X)] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ...(15.21)$$

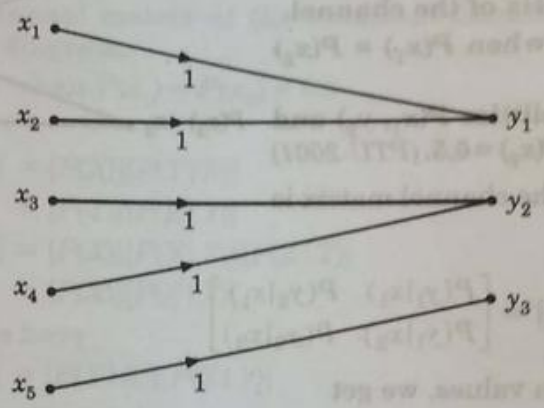


Fig. 15.3. *Deterministic Channel.*

Note: *It may be noted that since each row has only one non-zero element, therefore, this element must be unity by equation (15.14). Thus, when a given source symbol is sent in the deterministic channel, it is clear which output symbol will be received.*

15.9.3. Noiseless Channel

A channel is called *noiseless* if it is both lossless and deterministic. A noiseless channel has been shown in figure 15.4. The channel matrix has only one element in each row and in each column, and this element is unity. Note that the input and output alphabets are of the same size, that is, $m = n$ for the noiseless channel.

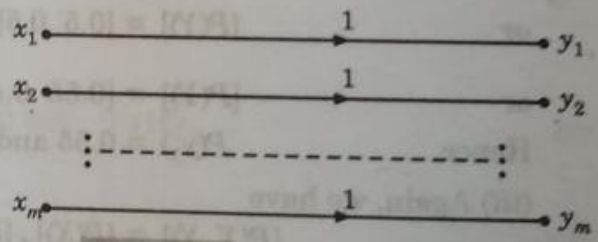


Fig. 15.4. *Noiseless channel.*

The matrix for a noiseless channel is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

15.9.4. Binary Symmetric Channel (BSC)

The binary symmetric channel (BSC) is defined by the channel diagram shown in figure 15.5, and its channel matrix is given by

$$[P(Y|X)] = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} \quad ...(15.22)$$

A BSC channel has two inputs ($x_1 = 0$, $x_2 = 1$) and two outputs ($y_1 = 0$, $y_2 = 1$). This channel is symmetric because the probability of receiving a 1 if a 0 is sent is the same as the probability of receiving a 0 if a 1 is sent. This common transition probability is denoted by p as shown in figure 7.5.

Scanned by CamScanner

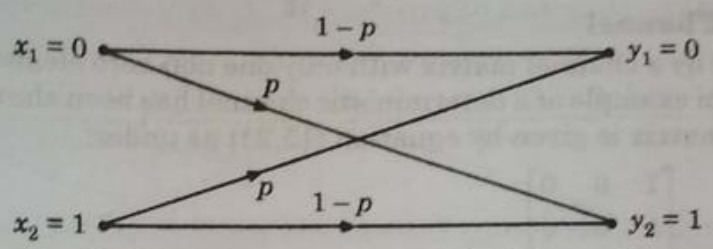


Fig. 15.5. *Binary symmetrical channel.*

Example 15.19. Given a binary channel shown in figure 15.6.

(*i*) Find the channel matrix of the channel.

(*ii*) Find $P(y_1)$ and $P(y_2)$ when $P(x_1) = P(x_2) = 0.5$.

(*iii*) Find the joint probabilities $P(x_1, y_2)$ and $P(x_2, y_1)$ when $P(x_1) = P(x_2) = 0.5$. *(PTU-2001)*

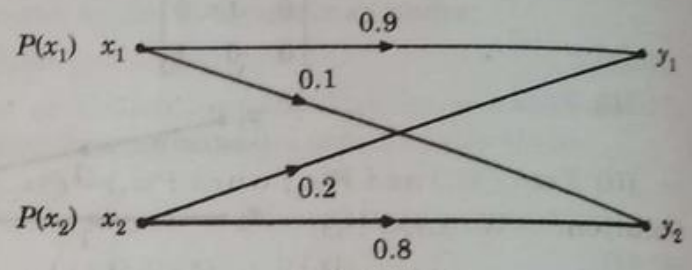


Fig. 15.6.

Solution: (*i*) We know that the channel matrix is given by

$$[P(Y|X)] = \begin{bmatrix} P(y_1|x_1) & P(y_2|x_1) \\ P(y_1|x_2) & P(y_2|x_2) \end{bmatrix}$$

Substituting all the given values, we get

$$[P(Y|X)] = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

(*ii*) We know that

$$[P(Y)] = [P(X)]\,[P(Y|X)]$$

Substituting all the values, we get

or $$[P(Y)] = [0.5 \;\; 0.5] \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

or $$[P(Y)] = [0.55 \;\; 0.45] = [P(y_1)\; P(y_2)]$$

Hence, $P(y_1) = 0.55$ and $P(y_2) = 0.45$ **Ans.**

(*iii*) Again, we have

$$[P(X, Y)] = [P(X)]_d\,[P(Y|X)]$$

Substituting all the values, we get

$$[P(X, Y)] = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

or $$[P(X, Y)] = \begin{bmatrix} 0.45 & 0.05 \\ 0.1 & 0.4 \end{bmatrix} \quad ...(i)$$

Now, let us write standard matrix form as under:

or $$[P(X, Y)] = \begin{bmatrix} P(x_1, y_1) & P(x_1, y_2) \\ P(x_2, y_1) & P(x_2, y_2) \end{bmatrix} \quad ...(ii)$$

Comparing (*i*) and (*ii*), we get

$P(x_1, y_2) = 0.05$ and $P(x_2, y_1) = 0.1$. **Ans.**

Scanned by CamScanner

Example 15.20. Two binary channels of problem 15.19 are connected in cascade as shown in figure 15.7

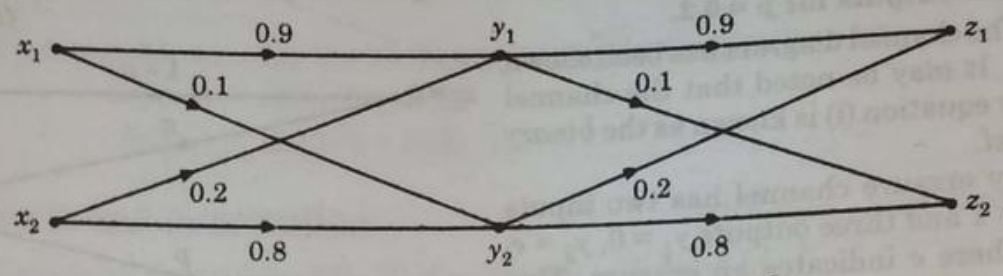


Fig. 15.7. *Two binary channels in cascade.*

(*i*) Find the overall channel matrix of the resultant channel, and draw the resultant equivalent channel diagram.

(*ii*) Find $P(z_1)$ and $P(z_2)$ when $P(x_1) = P(x_2) = 0.5$

Solution: (*i*) We know that

$$[P(Y)] = [P(X)][P(Y|X)]$$
$$= [P(Y)][P(Z|Y)]$$
$$[P(Z)] = [P(X)][P(Y|X)][P(Z|Y)]$$
$$= [P(X)][P(Z|X)]$$

Thus, from figure 15.7, we have

$$[P(Z|X)] = [P(Y|X)][P(Z|Y)]$$

Substituting all the values, we get

$$[P(Z|X)] = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

or
$$[P(Z|X)] = \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix}$$

The resultant equivalent channel diagram has been shown in figure 15.8.

(*ii*) Again, we have $[P(Z)] = [P(X)][P(Z|X)]$

Substituting all the values, we get

$$[P(Z)] = [0.5 \;\; 0.5] \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix} = [0.585 \;\; 0.415] \qquad ...(i)$$

Let us write standard expression as under:

$$[P(Z)] = [P(z_1)\; P(z_2)] \qquad ...(ii)$$

Comparing equation (*i*) and equation (*ii*), we obtain

$P(z_1) = 0.585$ and $P(z_2) = 0.415$. **Ans.**

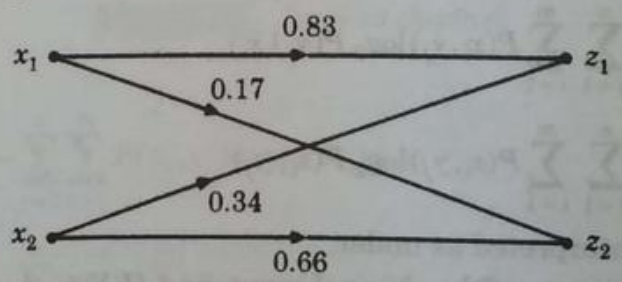


Fig. 15.8. *The resultant equivalent channel diagram.*

Example 15.21. A channel has the following channel matrix:

$$[P(Y|X)] = \begin{bmatrix} 1-p & p & 0 \\ 0 & p & 1-p \end{bmatrix} \qquad ...(i)$$

Scanned by CamScanner

(i) Draw the channel diagram.

(ii) If the source has equally likely outputs, compute the probabilities associated with the channel outputs for $p = 0.2$. *(Haryana-1998)*

Solution: (*i*) The channel diagram has been shown in figure 15.9. It may be noted that the channel represented by equation (*i*) is known as the *binary erasure channel.*

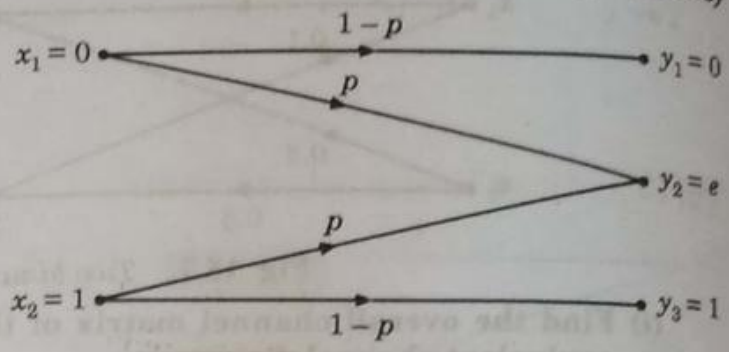


Fig. 15.9. *Binary erasure channel.*

The binary erasure channel has two inputs $x_1 = 0$ and $x_2 = 1$ and three outputs $y_1 = 0$, $y_2 = e$, and $y_3 = 1$, where e indicates an erasure. This means that the output is in doubt, and hence it should be erased.

(*ii*) We know that $[P(Y)]$ is expressed as

$$[P(Y)] = [P(X)][P(Y|X)]$$

Substituting all the values, we get

$$[P(Y)] = [0.5 \;\; 0.5]\begin{bmatrix} 0.8 & 0.2 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix}$$

Simplifying, we get

$$[P(Y)] = [0.4 \;\; 0.2 \;\; 0.4]$$

Thus, we get $P(y_1) = 0.4$,

$P(y_2) = 0.2$,

and $P(y_3) = 0.4$ **Ans.**

15.10. The Conditional and Joint Entropies

Using the input probabilites $P(x_i)$, output probabilities $P(y_i)$, transition probabilities $P(y_j|x_i)$, and joint probabilities $P(x_i, y_j)$, let us define the following various entropy functions for a channel with m inputs and n outputs:

$$H(X) = -\sum_{i=1}^{m} P(x_i)\log_2 P(x_i) \qquad ...(15.23)$$

$$H(Y) = -\sum_{j=1}^{n} P(y_j)\log_2 P(y_j) \qquad ...(15.24)$$

$$H(X|Y) = -\sum_{j=1}^{n}\sum_{i=1}^{m} P(x_i, y_j)\log_2 P(x_i|y_j) \qquad ...(15.25)$$

$$H(Y|X) = -\sum_{j=1}^{n}\sum_{i=1}^{m} P(x_i, y_j)\log_2 P(y_j|x_i) \qquad ...(15.26)$$

$$H(X, Y) = -\sum_{j=1}^{n}\sum_{i=1}^{m} P(x_i, y_j)\log_2 P(x_i, y_j) \qquad ...(15.27)$$

These entropies can be interpreted as under:

$H(X)$ is the average uncertainty of the channel input, and $H(Y)$ is the average uncertainty of the channel output.

The conditional entropy $H(X|Y)$ is a measure of the average uncertainty remaining about the channel input after the channel output has been observed.

Also, $H(X|Y)$ is sometimes called the *equivocation* of X with respect to Y.

Scanned by CamScanner

The conditional entropy $H(Y|X)$ is the average uncertainty of the channel output given that X was transmitted. The joint entropy $H(Y|Y)$ is the average uncertainty of the communication channel as a whole.

Two useful relationships among the above various entropies are as under:

$$H(X, Y) = H(X|Y) + H(Y) \quad ...(15.28)$$

$$H(X,Y) = H(Y|X) + H(X) \quad ...(15.29)$$

15.11. The Mutual Information

(U.P. Tech., Sem. Exam., 2003-2004)

The *mutual information* denoted by $I(X;Y)$ of a channel is defined by

$$I(X;Y) = H(X) - H(X|Y) \text{ b/symbol} \quad ...(15.30)$$

Since $H(X)$ represents the uncertainty about the channel input before the channel output is observed and $H(X|Y)$ represents the uncertainty about the channel input after the channel output is observed, the mutual information $I(X;Y)$ represents the uncertainty about the channel input that is resolved by observing the channel output.

Properties of Mutual Information I(X; Y)

(*i*) $I(X;Y) = I(Y;X)$ ...(15.31)

(*ii*) $I(X;Y) \geq 0$ ...(15.32)

(*iii*) $I(X;Y) = H(Y) - H(Y|X)$ ...(15.33)

(*iv*) $I(X;Y) = H(X) + H(Y) - H(X,Y)$ ...(15.34)

Example 15.22. For a lossless channel prove that

$$H(X|Y) = 0 \quad ...(i)$$

Solution: When we observe the output y_j in a lossless channel shown in figure 15.10, it is clear which x_i was transmitted. This means that

$$P(x_i|y_j) = 0 \text{ or } 1 \quad ...(ii)$$

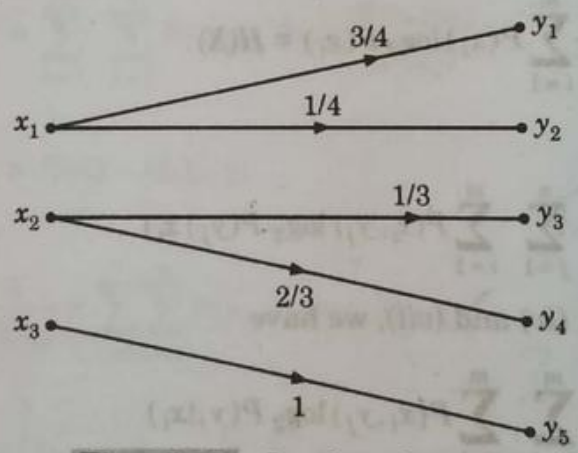


Fig. 15.10. *Lossless channel.*

Further, we know that

$$H(X|Y) = -\sum_{j=1}^{n}\sum_{i=1}^{m} P(x_i, y_j)\log_2 P(x_i|y_j)$$

or

$$H(X|Y) = -\sum_{j=1}^{n} P(y_j)\sum_{i=1}^{m} P(x_i|y_j)\log_2 P(x_i|y_j)$$

Note that all the terms in the inner summation are zero because they are in the form of $1 \times \log_2 1$ or $0 \times \log_2 0$. Hence, we conclude that for a lossless channel

$$H(X|Y) = 0 \qquad \textbf{Hence Proved.}$$

Scanned by CamScanner

Example 15.23. Given a noiseless channel with m input symbols and m output symbols as shown in figure 15.11. Prove that

$$H(X) = H(Y) \quad \text{...(i)}$$

and

$$H(Y|X) = 0 \quad \text{...(ii)}$$

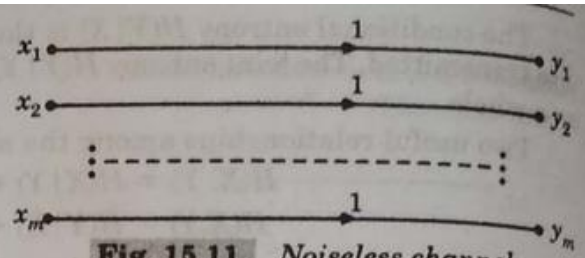


Fig. 15.11. *Noiseless channel.*

Solution: For a noiseless channel, the transition probabilities are given by

$$P(y_j|x_i) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \quad \text{...(iii)}$$

Hence,

$$P(x_i, y_j) = P(y_j|x_i)P(x_i) = \begin{cases} P(x_i) & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \quad \text{...(iv)}$$

and

$$P(y_j) = \sum_{i=1}^{m} P(x_i, y_j) = P(x_j) \quad \text{...(v)}$$

Also, we have

$$H(X) = -\sum_{i=1}^{m} P(x_i) \log_2 P(x_i) \text{ bits/symbol} \quad \text{...(vi)}$$

Thus, using equations (*v*) and (*vi*), we have

$$H(Y) = -\sum_{j=1}^{m} P(y_j) \log_2 P(y_j)$$

or

$$H(Y) = -\sum_{i=1}^{m} P(x_i) \log_2 P(x_i) = H(X)$$

Further, we know that

$$H(Y|X) = -\sum_{j=1}^{n} \sum_{i=1}^{m} P(x_i, y_j) \log_2 P(y_j|x_i) \quad \text{...(vii)}$$

Now, using equations (*iii*), (*iv*) and (*vii*), we have

$$H(Y|X) = -\sum_{j=1}^{m} \sum_{i=1}^{m} P(x_i, y_j) \log_2 P(y_j|x_i)$$

or

$$H(Y|X) = -\sum_{i=1}^{m} P(x_i) \sum_{j=1}^{m} \log_2 P(y_j|x_i)$$

or

$$H(Y|X) = -\sum_{i=1}^{m} P(x_i) \log_2 1 = 0 \quad \textbf{Hence Proved.}$$

Example 15.24. Verify the following expression:

$$H(X,Y) = H(X|Y) + H(Y)$$

Scanned by CamScanner

Solution: We know that

$$P(x_i, y_j) = P(x_i | y_j)\, P(y_j)$$

and $$\sum_{i=1}^{m} P(x_i, y_j) = P(y_j)$$

Also, we have

$$H(X,Y) = -\sum_{j=1}^{n}\sum_{i=1}^{m} P(x_i, y_j) \log P(x_i, y_j)$$

$$H(X,Y) = -\sum_{j=1}^{n}\sum_{i=1}^{m} P(x_i, y_j) \log[P(x_i \mid y_j) P(y_j)]$$

or $$H(X,Y) = -\sum_{j=1}^{n}\sum_{i=1}^{m} P(x_i, y_j) \log P(x_i \mid y_j) - \sum_{j=1}^{n}\left[\sum_{i=1}^{m} P(x_i, y_j)\right] \log P(y_j)$$

or $$H(X,Y) = H(X|Y) - \sum_{j=1}^{n} P(y_j) \log P(y_j)$$

or $$H(X,Y) = H(X|Y) + H(Y)$$ **Hence Proved**

Example 15.25. Show that the mutual information I(X;Y) of the channel described, by equation (15.11) with the input probabilities $P(x_i)$, $i = 1, 2, \ldots m$ and the output probabilities $P(y_j)$, $j = 1, 2, \ldots, n$, can be expressed as

$$I(X;Y) = \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i, y_j) \log_2 \frac{P(x_i|y_j)}{P(x_i)} \quad \text{...(i)}$$

Solution: We know that

$$I(X;Y) = H(X) - H(X|Y)$$

Also, we have

$$I(X;Y) = \sum_{i=1}^{m} P(x_i) \log_2 \frac{1}{P(x_i)} + \sum_{j=1}^{n}\sum_{i=1}^{m} P(x_i, y_j) \log_2 P(x_i|y_j)$$

$$= \sum_{i=1}^{m}\left[\sum_{j=1}^{n} P(x_i, y_j)\right] \log_2 \frac{1}{P(x_i)} + \sum_{j=1}^{n}\sum_{i=1}^{m} P(x_i, y_j) \log_2 P(x_i|y_j)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i, y_j)\left[\log_2 \frac{1}{P(x_i)} + \log_2 P(x_i|y_j)\right] = \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i, y_j) \log_2 \frac{P(x_i|y_j)}{P(x_i)}$$ **Ans.**

Example 15.26. Verify the following expression:

$$I(X;Y) = I(X;Y)$$

Solution: We can express $I(Y; X)$ as

$$I(Y; X) = \sum_{i=1}^{m}\sum_{j=1}^{n} P(y_j, x_i) \log_2 \frac{P(y_j|x_i)}{P(y_j)} \quad \text{...(i)}$$

Scanned by CamScanner

We know that $P(y_j, x_i) = P(x_i, y_j)$

and $$\frac{P(y_j|x_i)}{P(y_i)} = \frac{P(x_i|y_j)}{P(x_i)}$$

Thus, comparing equation (*i*) with equation (*i*), in problem 15.25, we conclude that

$$I(X;Y) = I(Y;X)$$ **Hence Proved**

Example 15.27. Verify the following expression:

$$I(X;Y) \geq 0$$

Solution: We have

$$-I(X;Y) = \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i, y_j) \log_2 \frac{P(x_i)}{P(x_i|y_j)} \quad ...(i)$$

Using Bayes' rule, we have

$$\frac{P(x_i)}{P(x_i|y_j)} = \frac{P(x_i)P(y_j)}{P(x_i, y_j)}$$

We can write equation (*i*) as under:

$$-I(X;Y) = \frac{1}{ln2}\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i, y_j) ln \frac{P(x_i)P(y_j)}{P(x_i, y_j)} \quad ...(ii)$$

Also, we know that

$$ln\,\alpha \leq \alpha - 1$$

Therefore, we have

$$-I(X;Y) \leq \frac{1}{ln2}\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i, y_j)\left[\frac{P(x_i)P(y_j)}{P(x_i, y_j)} - 1\right]$$

or

$$-I(X;Y) \leq \frac{1}{ln2}\left[\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i)\,P(y_j) - \sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i, y_j)\right] \quad ...(iii)$$

Since $$\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i)P(y_j) = \sum_{i=1}^{m} P(x_i) \sum_{j=1}^{n} P(y_j) = (1)(1) = 1$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} P(x_i, y_j) = \sum_{i=1}^{m}\left[\sum_{j=1}^{n} P(x_i, y_j)\right] = \sum_{i=1}^{m} P(x_i) = 1$$

Equation (*iii*) reduces to

$$-I(X;Y) \leq 0$$

$$I(X;Y) \geq 0.$$ **Hence Proved.**

Example 15.28. Given a binary symmetric channel (BSC) (figure 15.12) with $P(x_1) = \alpha$.

(*i*) Show that the mutual information $I(X;Y)$ is given by

$$I(X;Y) = H(Y) + p \log_2 p + (1-p) \log_2 (1-p) \quad ...(i)$$

(*ii*) Calculate $I(X;Y)$ for $\alpha = 0.5$ and $p = 0.1$.

(*iii*) Repeat part (*ii*) for a = 0.5 and $p = 0.5$, and comment on the result.

Figure 15.12 shows the diagram of the binary symmetric channel (BSC) with associated input probabilities.

Scanned by CamScanner

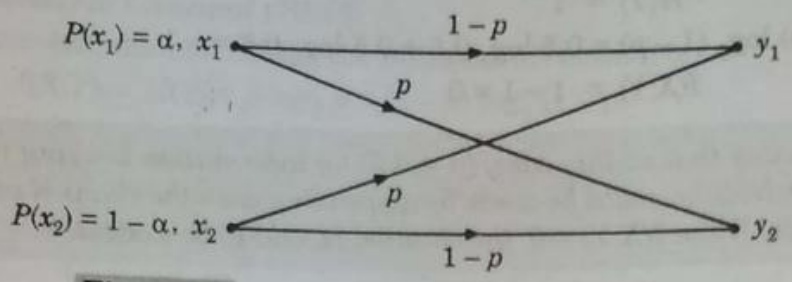


Fig. 15.12. *Binary symmetric channel (BSC).*

Solution: (*i*) Using equations (15.18), (15.19) and (15.22), we have

$$[P(X,Y)] = \begin{bmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{bmatrix} \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$$

or $$[P(X,Y)] = \begin{bmatrix} \alpha(1-p) & \alpha p \\ (1-\alpha)p & (1-\alpha)(1-p) \end{bmatrix}$$

or $$[P(X,Y)] = \begin{bmatrix} P(x_1,y_1) & P(x_1,y_2) \\ P(x_2,y_1) & P(x_2,y_2) \end{bmatrix}$$

We know that

$$H(Y|X) = -\sum_{j=1}^{n}\sum_{i=1}^{m} P(x_i, y_j) \log_2 P(y_j | x_i)$$

By above expression, we have

$$H(Y|X) = -P(x_1, y_1) \log_2 P(y_1|x_1) - P(x_1, y_2) \log_2 P(y_2|x_1)$$
$$- P(x_2, y_1) \log_2 P(y_1|x_2) - P(x_2, y_2) \log_2 P(y_2|x_2)$$

or $$H(Y|X) = -\alpha(1-p)\log_2(1-p) - \alpha p \log_2 p - (1-\alpha)p \log_2 p - (1-\alpha)(1-p)\log_2(1-p)$$

or $$H(Y|X) = -p\log_2 p - (1-p)\log_2(1-p) \quad \text{...(ii)}$$

We know that

$$I(X;Y) = H(Y) - H(Y|X) = H(Y) + p\log_2 p + (1-p)\log_2(1-p)$$

(*ii*) We know that

$$P(Y)] = [P(X)]\,[P(Y|X)]$$

when $\alpha = 0.5$ and $p = 0.1$, then, we have

$$[P(Y)] = [0.5 \quad 0.5]\begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix} = [0.5 \ \ 0.5]$$

Thus, $$P(y_1) = P(y_2) = 0.5$$

Now, using the expression

$$H(Y) = -\sum_{j=1}^{n} P(y_j)\log_2 P(y_j), \text{ we have}$$

$$H(Y) = -P(y_1)\log_2 P(y_1) - P(y_2)\log_2 P(y_2)$$

or $$H(Y) = -0.5\log_2 0.5 - 0.5\log_2 0.5 = 1$$

$$p\log_2 p + (1-p)\log_2(1-p) = 0.1\log_2 0.1 + 0.9\log_2 0.9 = -0.469$$

Thus, $$I(X;Y) = 1 - 0.469 = 0.531 \quad \textbf{Ans.}$$

(*iii*) When $\alpha = 0.5$ and $p = 0.5$, we have

$$[P(Y)] = [0.5 \quad 0.5]\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} = [0.5 \ \ 0.5]$$

Scanned by CamScanner

$$H(Y) = 1$$

$$p \log_2 p + (1-p) \log_2 (1-p) = 0.5 \log_2 0.5 + 0.5 \log_2 0.5 = -1$$

Thus,

$$I(X;Y) = 1 - 1 = 0$$

Note: *It may be noted that in this case, ($p = 0.5$) no information is being transmitted at all. An equally acceptable decision could be made by dispensing with the channel entirely and "flipping a coin" at the receiver. When $I(X;Y) = 0$, the channel is said to be useless.*

15.12. The Channel Capacity

(U.P. Tech, Semester Examination, 2003-04)

In this section, let us discuss various aspects regarding channel capacity.

15.12.1. Channel Capacity Per Symbol C_s

The *channel capacity per symbol* of a discrete memoryless channel (DMC) is defined as

$$C_s = \max_{\{P(x_i)\}} I(X;Y) \text{ b/symbol} \quad ...(15.35)$$

where the maximization is over all possible input probability distributions $\{P(x_i)\}$ on X. Note that the channel capacity C_s is a function of only the channel transition probabilities which define the channel.

15.12.2. Channel Capacity Per Second C

If r symbols are being transmitted per second, then the maximum rate of transmission of information per second is rC_s. This is the *channel capacity per second* and is denoted by $C(b/s)$, i.e.,

$$C = rC_s \text{ b/s} \quad ...(15.36)$$

15.12.3. Capacities of Special Channel

In this subsection, let us discuss capacities of various special channel.

15.12.3.1. Lossless Channel

For a lossless channel, $H(X|Y) = 0$, and

$$I(X;Y) = H(X) \quad ...(15.37)$$

Thus, the mutual information (information transfer) is equal to the input (source) entropy, and no source information is lost in transmission.

Consequently, the channel capacity per symbol will be

$$C_s = \max_{\{P(x_i)\}} H(X) = \log_2 m \quad ...(15.38)$$

where m is the number of symbols in X.

15.12.3.2. Deterministic Channel

For a deterministic channel, $H(Y|X) = 0$ for all input distributions $P(x_i)$, and

$$I(X;Y) = H(Y) \quad ...(15.39)$$

Thus, the information transfer is equal to the output entropy. The channel capacity per symbol will be

$$C_s = \max_{\{P(x_i)\}} H(Y) = \log_2 n \quad ...(15.40)$$

where n is the number of symbols in Y.

15.12.3.3. Noiseless Channel

Since a noiseless channel is both lossless and deterministic, we have

$$I(X;Y) = H(X) = H(Y) \quad ...(15.41)$$

and the channel capacity per symbol is

$$C_s = \log_2 m = \log_2 n \quad ....(15.42)$$

Scanned by CamScanner

15.12.3.4. Binary Symmetric Channel (BSC)

For the binary symmetric channel (BSC), the mutual information is

$$I(X;Y) = H(Y) + p \log_2 p + (1-p) \log_2 (1-p) \quad ...(15.43)$$

and the channel capacity per symbol will be

$$C_s = 1 + p \log_2 p + (1-p) \log_2 (1-p) \quad ...(15.44)$$

Example 15.29. Verify the following expression:

$$C_s = \log_2 m$$

where C_s is the channel capacity of a lossless channel and m is the number of symbols in X.

Solution: For a lossless channel, we have

$$H(X|Y) = 0$$

Then, by equation (15.30), we have

$$I(X;Y) = H(X) - H(X|Y) = H(X) \quad ...(i)$$

Hence, by equations (15.35) and (15.9), we have

$$C_s = \max_{\{P(x_i)\}} I(X;Y)$$

or

$$C_s = \max_{\{P(x_i)\}} H(X) = \log_2 m \quad \textbf{Hence proved.}$$

Example 15.30. Verify the following expression:

$$C_s = 1 + p \log_2 p + (1-p) \log_2 (1-p)$$

where C_s is the channel capacity of a BSC (figure 15.12)

Solution: We know that the mutual information $I(X: Y)$ of a BSC is given by

$$I(X;Y) = H(Y) + p \log_2 p + (1-p) \log_2 (1-p)$$

which is maximum when $H(Y)$ is maximum. Since, the channel output is binary, $H(Y)$ is maximum when each output has a probability of 0.5 and is achieved for equally likely inputs.

For this case $H(Y) = 1$, and the channel capacity is

$$C_s = \max_{\{p(X)\}} I(X;Y)$$

$$= 1 + p \log_2 p + (1-p) \log_2 (1-p) \quad \textbf{Hence Proved.}$$

Example 15.31. Find the channel capacity of the binary erasure channel of figure 15.13.

Solution: Let $P(x_1) = \alpha$. Then $P(x_2) = 1 - \alpha$.

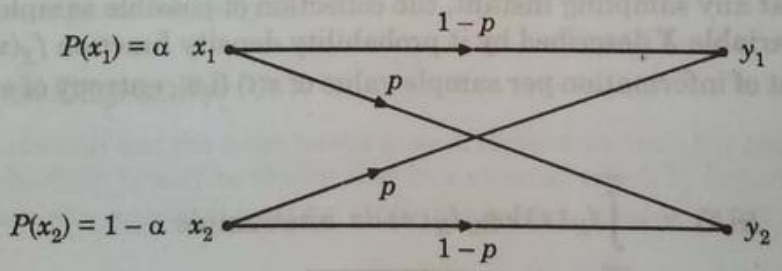


Fig. 15.13.

We have

$$[P(Y|X)] = \begin{bmatrix} 1-p & p & 0 \\ 0 & p & 1-p \end{bmatrix} = \begin{bmatrix} P(y_1|x_1) & P(y_2|x_1) & P(y_3|x_1) \\ P(y_1|x_2) & P(y_2|x_2) & P(y_3|x_2) \end{bmatrix}$$

Using equation (15.17), we have

$$[P(Y)] = \begin{bmatrix} \alpha & 1 & -\alpha \end{bmatrix} \begin{bmatrix} 1-p & p & 0 \\ 0 & p & 1-p \end{bmatrix}$$

$$= [\alpha(1-p)\ p\ (1-\alpha)(1-p)] = [P(y_1)\ P(y_2)\ P(y_3)]$$

Scanned by CamScanner

By using equation (15.19), we have

$$[P(X, Y)] = \begin{bmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{bmatrix} \begin{bmatrix} 1-p & p & 0 \\ 0 & p & 1-p \end{bmatrix}$$

or

$$[P(X, Y)] = \begin{bmatrix} \alpha(1-p) & \alpha p & 0 \\ 0 & (1-\alpha)p & (1-\alpha)(1-p) \end{bmatrix}$$

or

$$[P(X, Y)] = \begin{bmatrix} P(x_1, y_1) & P(x_1, y_2) & P(x_1, y_3) \\ P(x_2, y_1) & P(x_2, y_2) & P(x_2, y_3) \end{bmatrix}$$

In addition, from equations (15.24) and (15.26), we can calculate

$$H(Y) = -\sum_{j=1}^{3} P(y_j) \log_2 P(y_j)$$

$$= -\alpha(1-p) \log_2 \alpha(1-p) - p \log_2 p - (1-\alpha)(1-p) \log_2 [(1-\alpha)(1-p)]$$

$$= (1-p)[-\alpha \log_2 \alpha - (1-\alpha) \log_2 (1-\alpha)] - p \log_2 p - (1-p) \log_2 (1-p)$$

Also, we have $H(Y|X) = -\sum_{j=1}^{3}\sum_{i=1}^{2} P(x_i, y_j) \log_2 P(y_j|x_i)$

$$= -\alpha(1-p) \log_2 (1-p) - \alpha p \log_2 p - (1-\alpha) p \log_2 p - (1-\alpha)(1-p) \log_2 (1-p)$$

$$= -p \log_2 p - (1-p) \log_2 (1-p)$$

Thus, by equations (15.33) and (15.57), we have

$$I(X;Y) = H(Y) - H(Y|X)$$

$$= (1-p)[-\alpha \log_2 \alpha - (1-\alpha) \log_2 (1-\alpha)] = (1-p)H(X)$$

And by equations (15.35) and (15.58), we have

$$C_s = \max_{\{P(x_i)\}} I(X;Y) = \max_{\{P(x_i)\}} (1-p)H(X) = (1-p) \max_{\{P(x_i)\}} H(X) = 1-p \quad \textbf{Ans.}$$

15.13. Entropy Relations for a Continuous Channel

In a continuous channel, an information source produces a continuous signal $x(t)$. The set of possible signals is considered as an ensemble of waveforms generated by some ergodic random process. It is further assumed that $x(t)$ has a finite bandwidth so that $x(t)$ is completely characterized by its periodic sample values. Hence, at any sampling instant, the collection of possible sample values constitutes a continuous random variable X described by it probability density function $f_X(x)$.

The average amount of information per sample value of $x(t)$ (i.e., entropy of a continuous source) is measured by

$$H(X) = -\int_{-\infty}^{\infty} f_X(x) \log_2 f_X(x)\, dx \text{ bits/sample} \qquad \text{....(15.45)}$$

The entropy $H(X)$ defined by equation (15.45) is known as the *differential entropy* of X.

Also, the average mutual information in a continuous channel is defined (by analogy with the discrete case) as

$$I(X;Y) = H(X) - H(X|Y)$$

or

$$I(X;Y) = H(Y) - H(Y|X)$$

Scanned by CamScanner

where
$$H(Y) = -\int_{-\infty}^{\infty} f_Y(y)\log_2 f_Y(y)\,dy \qquad ...(15.46)$$

$$H(X|Y) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\log_2 f_X(x|y)\,dxdy \qquad ...(15.47)$$

$$H(Y|X) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\log_2 f_Y(y|x)\,dxdy \qquad ...(15.48)$$

15.14. Capacity of an Additive White Gaussian Noise (AWGN) Channel: Shannon-Hartley Law

In an additive white Gaussian noise (AWGN) channel, the channel output Y is given by

$$Y = X + n \qquad ...(15.48)$$

where X is the channel input and n is an additive bandlimited white Gaussian noise with zero mean and variance σ^2.

The capacity C_s of an AWGN channel is given by

$$C_s = \max_{\{f_X(x)\}} I(X;Y)$$

or
$$C_s = \frac{1}{2}\log_2\left(1+\frac{S}{N}\right) \text{ b/sample} \qquad ...(15.49)$$

where S/N is the signal-to-noise ratio at the channel output. If the channel bandwidth B Hz is fixed, then the output $y(t)$ is also a bandlimited signal completely characterized by its periodic sample values taken at the Nyquist rate $2B$ samples/s.

Then the capacity C(b/s) of the AWGN channel is given by

$$C = 2B \times C_s = B\log_2\left(1+\frac{S}{N}\right) \text{ b/s} \qquad ...(15.50)$$

Equation (15.50) is known as the **Shannon-Hartley law.**

The Shannon-Hartley law underscores the fundamental role of bandwidth and signal-to-noise ratio in communication. It also shows that we can exchange increased bandwidth for decreased signal power for a system with given capacity C.

15.15. Channel Capacity

We know that the bandwidth and the noise power place a restriction upon the rate of information that can be transmitted by a channel. It may be shown that in a channel which is disturbed by a white Gaussian noise, one can transmit information at a rate of C bits per second, where C is the the channel capacity and is expressed as

$$C = B\log_2\left(1+\frac{S}{N}\right) \qquad ...(15.51)$$

In this expression,

B = channel bandwidth in Hz
S = Signal power
N = Noise power

Scanned by CamScanner

It may be noted that the expression (equation 15.50) for channel capacity is valid for white Gaussian noise. However, for other types of noise, the expression is modified.

Proof: Let us present a proof of channel capacity formula based upon the assumption that if a signal is mixed with noise, the signal amplitude can be recognized only within the root main square noise voltage. In other words, we can say that the uncertainty in recognizing the exact signal amplitude is equal to the root mean square noise voltage.

Again, let us assume that the average signal power and the noise power are S watts and N watts respectively. This means that the root mean square value of the received signal is $\sqrt{S+N}$ volts and the root mean square value of the noise voltage is $\sqrt{N}$ volts.

Now, we have to distinguish the received signal of the amplitude $\sqrt{S+N}$ volts in the presence of the noise amplitude $\sqrt{N}$ volts.

As a matter of fact, the input signal variation of less than $\sqrt{N}$ volts will not be distinguished at the receiver end.

Therefore, the number of the distinct levels that can be distinguished without error can be expressed as

$$M = \frac{\sqrt{S+N}}{\sqrt{N}} = \sqrt{1+\frac{S}{N}} \qquad ...(15.51)$$

Thus, equation (15.51) expresses the maximum value of M.

Now, the maximum amount of information carried by each pulse having $\sqrt{1+\frac{S}{N}}$ distinct levels is given by

$$I = \log_2 \sqrt{1+\frac{S}{N}} = \frac{1}{2}\log_2\left(1+\frac{S}{N}\right) \text{ bits} \qquad ...(15.52)$$

Now, after establishing expression in equation (8.15), we can determine the channel capacity. In fact, the channel capacity is the maximum amount of information that can be transmitted per second by a channel.

If a channel can transmit a maximum of K pulses per second, then, the channel capacity C is given by

$$C = \frac{K}{2}\log_2\left(1+\frac{S}{N}\right) \text{ bits per second} \qquad ...(15.53)$$

Recall that for bandwidth requirements of PAM signals, it has been shown that a system of bandwidth nf_m Hz can transmit $2n\,f_m$ independent pulses per second. Further, under these conditions, the received signal will yield the correct values of the amplitudes of the pulses but will not reproduce the details of the pulse shapes.

Now, since, we are interested only in the pulse amplitudes and not their shapes, it is concluded that a system with bandwidth B Hz can transmit a maximum of $2B$ pulses per second. Further, since, each pulse can carry a maximum information of $\frac{1}{2}\log_2\left(1+\frac{S}{N}\right)$ bits, it follows that a system of bandwidth B can transmit the information at a following maximum rate:

$$C = B\log_2\left(1+\frac{S}{N}\right) \text{ bits per second} \qquad ...(15.54)$$

Scanned by CamScanner

Therefore, the channel capacity C is limited by the bandwidth of the channel (or system) and the noise signal.

For a noiseless channel, $N = 0$ and the channel capacity will be infinite. However, practically, N is always finite and therefore, the channel capacity is finite.*

The expression in equation (15.54) is also known as the **Hartley-Shannon law** and is treated as the central theorem of information theory.

From Hartley-Shannon law, it is obvious that the bandwidth and the signal power can be exchanged for one another. To transmit the information at a given rate, we may reduce the signal power transmitted provided that the bandwidth is increased correspondingly. In a similar manner, the bandwidth may be reduced if we have to increase the signal power.

As a matter of fact, the process of modulation is actually a means of effecting this exchange between the bandwidth and the signal-to-noise ratio.**

Note: *It may be noted that the channel capacity represents the maximum amount of information that can be transmitted by a channel per second. To achieve this rate of transmission, the information has to be processed properly or coded in the most efficient manner.*

15.16. Transmission of Continuous Signals

Now, let us further illustrate the Hartley-Shannon law for the exchange of bandwidth and signal-to-noise ratio by a continuous signal which is bandlimited to f_m Hz.

According to the sampling theorem, the information of a continuous-time signal, which is bandlimited to f_m Hz, is completely specified by $2f_m$ samples per second.

Hence, to transmit the information of such a signal, it is necessary to transmit only these discrete samples.

Theoretical, Aspect

Now, one important question arises. How much information does each sample contain.

It depends upon how many discrete levels or values the samples may assume. In fact, these samples can assume any value and hence to transmit such samples, we require pulses capable of assuming infinite levels. Clearly, the information carried by each sample is infinite bits.

Therefore, the information contained in a continuous bandlimited signal is infinite.

---

* This is true even if the bandwidth B is infinite. The noise signal is a white noise with a uniform power density spectrum over the entire frequency range. Therefore, as the bandwidth B is increased, N also increases and hence the channel capacity remains finite even if $B = \infty$.

If $\eta/2$ is the power density, then we have

$$N = \eta B, \text{ and}$$

$$C = B \log_2 \left(1 + \frac{S}{\eta B}\right)$$

and
$$\lim_{B \to \infty} C = \frac{S}{\eta} \frac{\eta B}{S} \log_2 \left(1 + \frac{S}{\eta B}\right)$$

The above limit may be found with the help of following standard expression:

$$\lim_{x \to 0} \frac{1}{x} \log_2 (1 + x) = \log_2 e = 1.44$$

Therefore, we have

$$\lim_{B \to \infty} C = \frac{S}{\eta} \log_2 e = 1.44 \frac{S}{\eta}$$

** The improvement in the signal-to-noise ratio in wideband FM and PCM can be properly understood.

Scanned by CamScanner

In the presence of noise*, the channel capacity is finite. Therefore, it is impossible to transmit complete information in a bandlimited signal by a physical channel in the presence of noise**.

In the absence of noise, $N = 0$, the channel capacity is infinite and hence any desired signal can be transmitted.

It is quite obvious that it is impossible to transmit the complete information contained in a continuous signal unless the transmitted signal power is made infinite.

Because of presence of noise, there is always a certain amount of uncertainty in the received signal. In fact, the transmission of complete information in a signal will mean a zero amount of uncertainty. Actually, the amount of uncertainty can be made arbitrarily small by increasing the channel capacity***. However, it can never be made zero.

15.17. Uncertainty in the Transmission Process

As a matter of fact, the uncertainty is introduced in the process of transmission. Therefore, although, it is possible to transmit the complete information in a continuous signal at the transmitter end, it is impossible to recover this infinite amount of information at the receiver end. The amount of information that can be recovered per second at the receiver is C bits per second where C is the channel capacity.

In place of transmitting all of the information at the transmitter, we can approximate the signal so that its information contents are reduced to C bits per second and transmit this approximated signal which has a finite information content.

Now, it will be possible to recover all of the information that has been transmitted. In fact, this happens in pulse code modulation (PCM) system.

Now, the question arises: How can we approximate a signal so that the approximated signal has a finite information content per second?

In fact, this can be done by a process known as **quantization**.

For illustration, let us consider the continuous signal bandlimited to f_m Hz as shown in figure 15.14.

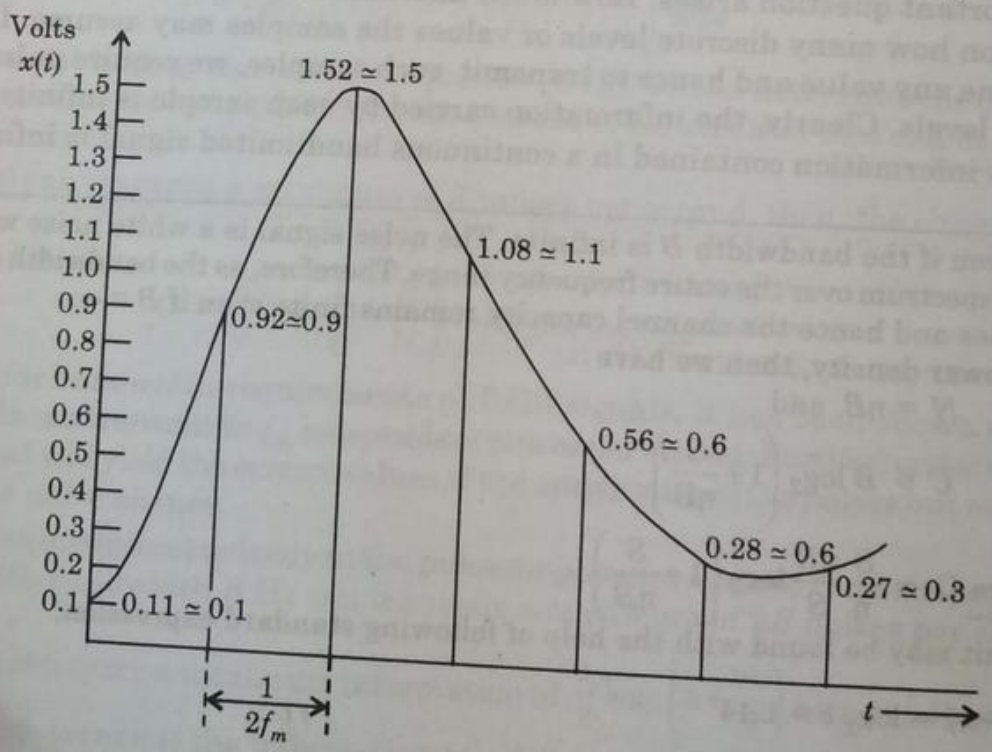


Fig. 15.14.

* That finite value of N.

** Existing over the same band.

*** By increasing the bandwidth and/or increasing the signal power.

Scanned by CamScanner

Now, to transmit the information in this signal, we require to transmit only $2f_m$ samples per second. Figure 15.14 also shows samples.

The samples can take any value, and to transmit them directly, we require pulses which can assume an infinite number of levels.

Therefore, instead of transmitting the exact values of these pulses, we round off the amplitudes to the nearest one of the finite number of permitted values.

In this example, all of the pulses are approximated to the nearest tenth of a volt.

It may be noted from the figure that each of the pulses transmitted assumes any one of the 16 levels, and thus, carries an amount of information of $\log_2 16 = 4$ bits. Also, since, there are $2f_m$ samples per second, the total information content of the approximated signal is $8\ f_m$ bits per second.

It the channel capacity is greater than or equal to $8\ f_m$ bits per second, all of the information that has been transmitted will be recovered completely without any uncertainty. This means that the received signal will be an exact replica of the approximated signal that was transmitted.

It can be shown that if the channel capacity is $8\ f_m$ bits per second, the process of transmission does not introduce an additional degree of uncertainty.

Now, let us consider that we are using a channel of bandwidth f_m Hz to transmit these samples, then, since the channel capacity required will be $8\ f_m$ bits per second, the required signal-to-noise power ratio will be given by

$$8\ f_m = f_m \log_2\left(\frac{S+N}{N}\right)$$

Therefore, $$\frac{S+N}{N} = 256$$

We have already discussed that the number of levels that can be distinguished at the receiver is $\sqrt{\frac{S+N}{N}}$.

It is obvious that in this case, the receiver can distinguish the 16 states without error. Thus, although the process of transmission introduces some noise in the desired signal, the levels are far enough apart to be distinguishable at the receiver.

In other words, we can say that a channel of the capacity of $8f_m$ bits per second can transmit information of $8\ f_m$ bits per second virtually without error.

15.18. Exchange of Bandwidth for Signal-to-Noise Ratio

As a matter of fact, a given signal can be transmitted with a given amount of uncertainty by a channel of finite capacity. We have already discussed that a given channel capacity may be obtained by any number of combinations of bandwidth and signal power. Actually, it is possible to exchange one for the other. Now, let us discuss how such an exchange can be affected.

Again, let us consider the transmission of signal $x(t)$ shown in figure 15.14, we have already observed that if an uncertainty of 0.1 volt is tolerated, the information content of the signal is given by $8\ f_m$ bits per second. Now, let us show that this information can be transmitted by various combinations of bandwidths and signal power.

One important and possible way of transmission is to send $2\ f_m$ samples per second directly. Each of the samples can assume any of the 16 states. In this case, we must have a signal-to-noise ratio which allows us to distinguish 16 states.

Clearly, $$\frac{\sqrt{S+N}}{\sqrt{N}} = 16.$$

Scanned by CamScanner

Moreover, in order to transmit $2 f_m$ pulses per second, we require a channel of bandwidth f_m Hz.

Therefore, the required channel capacity C is expressed as

$$C = f_m \log_2 \frac{S+N}{N}$$

or

$$C = f_m \log_2 (16)^2 = 8 f_m \text{ bits per second.}$$

Hence, the channel capacity C is exactly equal to the amount of information per second in the signal $x(t)$.

In other method of transmission, we may transmit the samples in figure 8.1 by quaternary pulse (pulses that can assume four states). Thus, it is obvious that we require a group of two quaternary pulses to transmit each sample that can assume 16 states.

In this case, the signal-to-noise ratio required at the receiver to distinguish pulses that assume four distinct states is $\frac{\sqrt{S+N}}{\sqrt{N}} = 4$.

Hence, in this mode of transmission, the required signal power is reduced. But, now, we have to transmit twice as many pulses per second *i.e.*, $4 f_m$ pulses per second.

Hence, the required bandwidth is $2 f_m$ Hz.

In this case, the channel capacity C will be

$$C = 2f_m \log_2 \frac{S+N}{N}$$

or

$$C = 2f_m \log_2 (4)^2 = 8f_m \text{ bits per second.}$$

Example 15.32. Find the differential entropy $H(X)$ of the uniformly distributed random variable X with the following probability density function (pdf):

$$f_X(x) = \begin{cases} \frac{1}{a} & \text{for} \quad 0 \le x \le a \\ 0 & \text{elsewhere} \end{cases}$$

for (*i*) $a = 1$, (*ii*) $a = 2$, and (*iii*) $a = \frac{1}{2}$.

Solution: We know that the differential entropy of X is given by

$$H(X) = -\int_{-\infty}^{\infty} f_X(x) \log_2 f_X(x)\, dx \quad \text{bits/sample}$$

Making use of given pdf, we have

$$H(X) = -\int_0^a \frac{1}{a} \log_2 \frac{1}{a} dx = \log_2 a \qquad \text{...(i)}$$

Now, we have

(*i*) $a = 1, H(X) = \log_2 1 = 0$

(*ii*) $a = 2, H(X) = \log_2 2 = 1$

(*iii*) $a = \frac{1}{2}, H(X) = \log_2 \frac{1}{2} = -\log_2 2 = -1$

It may be noted that the differential entropy $H(X)$ is not an absolute measure of information.

Scanned by CamScanner

Example 15.33. The differential entropy of a random variable X is defined by following equation:

$$H(X) = -\int_{-\infty}^{\infty} f_X(x) \log_2 f_X(x)\, dx$$

Find the probability density function $f_X(x)$ for which $H(X)$ is maximum.

Solution: We know that $f_X(x)$ must satisfy the following two conditions:

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1 \qquad ...(i)$$

$$\int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\, dx = \sigma^2 \qquad ...(ii)$$

where μ is the mean of X and σ^2 it its variance. Since the problem is the maximization of $H(X)$ under constraints of equations (*i*) and (*ii*), therefore, we use the method of *Lagrange multipliers* as under:

First, we form the function:

$$G[f_X(x), \lambda_1, \lambda_2] = H(X) + \lambda_1 \left[\int_{-\infty}^{\infty} f_X(x)dx - 1\right] + \lambda_2 \left[\int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)dx - \sigma^2\right]$$

$$= \int_{-\infty}^{\infty} [-f_X(x)\log_2 f_X(x) + \lambda_1 f_X(x) + \lambda_2 (x-\mu)^2 f_X(x)]\, dx - \lambda_1 - \lambda_2\sigma^2 \qquad ...(iii)$$

where the parameters λ_1 and λ_2 are the Lagrange multipliers. Then the maximization of $H(X)$ requires that

$$\frac{\partial G}{\partial f_X(x)} = -\log_2 f_X(x) - \log_2 e + \lambda_1 + \lambda_2 (x-\mu)^2 = 0 \qquad ...(iv)$$

Thus,

$$\log_2 f_X(x) - \log_2 e + \lambda_1 + \lambda_2 (x-\mu)^2$$

or

$$ln\, f_X(x) = -1 + \frac{\lambda_1}{\log_2 e} + \frac{\lambda_2}{\log_2 e}(x-\mu)^2$$

Hence, we obtain

$$f_X(x) = \exp\left[-1 + \frac{\lambda_1}{\log_2 e} + \frac{\lambda_2}{\log_2 e}(x-\mu)^2\right] \qquad ...(v)$$

In view of the constraints of equations (*i*) and (*ii*), it is required that $\lambda_2 < 0$. Let

$$\exp\left(-1 + \frac{\lambda_1}{\log_2 e}\right) = a \quad \text{and} \quad \left(\frac{\lambda_2}{\log_2 e}\right) = -b^2$$

Then, equation (*v*) can be rewritten as

$$f_X(x) = ae^{-b^2(x-\mu)^2} \qquad ...(vi)$$

Substituting equation (*vi*) into equations (*i*) and (*ii*), we get

$$a\int_{-\infty}^{\infty} e^{-b^2(x-\mu)^2} dx = a\frac{\sqrt{\pi}}{b} = 1 \qquad ...(vii)$$

$$a\int_{-\infty}^{\infty} (x-\mu)^2 e^{-b^2(x-\mu)^2} dx = a\frac{\sqrt{\pi}}{2b^3} = \sigma^2 \qquad ...(vii)$$

Scanned by CamScanner

Solving equations (*vii*) and (*viii*) for a and b^2, we get

$$a = \frac{1}{\sqrt{2\pi\sigma}} \quad \text{and} \quad b^2 = \frac{1}{2\sigma^2}$$

Substituting these values in equation (*vi*), we observe that the desired $f_X(x)$ is given by

$$f_X(x) = \frac{1}{\sqrt{2\,\pi\sigma}}\, e^{-(x-\mu)^2/(2\sigma^2)} \qquad \textbf{Hence proved.}$$

which is the probability density function of Gaussian random variable X of mean μ and variance σ^2.

Example 15.34. Show that the channel capacity of an ideal AWGN channel with infinite bandwidth is given by

$$C_\infty = \frac{1}{\ln 2}\cdot\frac{S}{\eta} \cong 1.44\frac{S}{\eta} \text{ b/s} \qquad \text{...(i)}$$

where S is the average signal power and $\eta/2$ is the power spectral density (psd) of white gaussian noise.

(U.P.S.C. I.E.S. Examination-1999)

Solution: We know that the noise power N is given by $N = \eta B$.

Also, according to Shannon-Hartley law, the channel capacity is given by

$$C = B\log_2\left(1 + \frac{S}{N}\right) b/s$$

In this expression, substituting $N = \eta B$, we get

$$C = B\log_2\left(1 + \frac{S}{\eta B}\right)$$

Let $S/(\eta B) = \lambda$. Then, we write

$$C = \frac{S}{\eta\lambda}\log_2(1+\lambda) = \frac{1}{\ln 2}\frac{S}{\eta}\frac{\ln(1+\lambda)}{\lambda} \qquad \text{...(ii)}$$

Now,

$$C_\infty = \lim_{B\to\infty} B\log_2\left(1 + \frac{S}{\eta B}\right)$$

$$C_\infty = \frac{1}{\ln 2}\cdot\frac{S}{\eta}\lim_{\lambda\to 0}\frac{\ln(1+\lambda)}{\lambda}$$

Since $\lim_{\lambda\to 0} [\ln(1+\lambda)]/\lambda = 1$, we get

$$C_\infty = \frac{1}{\ln 2}\cdot\frac{S}{\eta} \cong 1.44\frac{S}{\eta} \text{ b/s} \qquad \textbf{Hence proved.}$$

Note: *It may be noted that equation (i) can be used to estimate upper limits on the performance of any practical communication system whose transmission channel can be approximated by the AWGN channel.*

Example 15.35. Given an AWGN channel with 4 kHz bandwidth and the noise power spectral density $\eta/2 = 10^{-12}$ W/Hz. The signal power required at the receiver is 0.1 mW. Calculate the capacity of this channel.

Solution: Given that

$$B = 4000 \text{ Hz}$$
$$S = 0.1(10^{-3})W$$
$$N = \eta B = 2(10^{-12})(4000)$$
$$N = 8(10^{-9}) \text{ W}$$

Thus,

$$\frac{S}{N} = \frac{0.1(10^{-3})}{8(10^{-9})} = 1.25(10^4)$$

Scanned by CamScanner

And, by equation (14.50), we have

$$C = B\log_2\left(1+\frac{S}{N}\right) = 4000\log_2(1+1.25(10^4)] = 54.44(10^3)\text{ b/s}$$ **Ans.**

Example 15.36. An Analog signal having 4 kHz bandwidth is sampled at 1.25 times the Nyquist rate, and each sample is quantized into one of equally likely levels. Assume that the successive samples are statistically independent.

(*i*) What is the information rate of this source?

(*ii*) Can the output of this source be transmitted without error over an AWGN channel with a bandwidth of 10 kHz and an S/N ratio of 20 dB?

(*iii*) Find the S/N ratio required for error-free transmission for part (*i*).

(*iv*) Find the bandwidth required for an AWGN channel for error-free transmission of the output of this source if the S/N ratio is 20 dB? *(U.P.S.C. I.E.S. Examination, 1999)*

Solution: (*i*) Here, $f_m = 4(10^3)$Hz

We know that Nyquist rate f_s is given by

$$f_s = 2f_m$$

$$\text{Nyquist rate} = 2f_m = 8(10^3)\text{ samples/s}$$

Also, we have

$$r = 8(10^3)(1.25) = 10^4\text{ samples/s}$$

Further, we know that entropy is expressed as

$$H(X) = -\sum_{i=1}^{m} P(x_i)\log_2 P(x_i)\text{ bits/symbol}$$

Here $$P(x_i) = \frac{1}{256}$$

Hence, $$H(X) = \log_2 256 = 8\text{ bits/sample}$$

The information rate R of the source is given by

$$R = rH(X) = 10^4(8)\text{ b/s} = 80\text{ kb/s}$$ **Ans.**

(*ii*) Again, we know that channel capacity is given by

$$C = B\log_2\left(1+\frac{S}{N}\right)\text{ bits/s}$$

Hence, $$C = B\log_2\left(1+\frac{S}{N}\right) = 10^4\log_2(1+10^2) = 66.6(10^3)\text{ b/s}$$

Here, since $R > C$, *error*-free transmission is not possible.

(*iii*) The required S/N ratio can be found by

$$C = 10^4\log_2\left(1+\frac{S}{N}\right) \geq 8(10^4)$$

or $$\log_2\left(1+\frac{S}{N}\right) \geq 8$$

or $$1+\frac{S}{N} \geq 2^8 = 256$$

or $$\frac{S}{N} \geq 255\ (= 24.1\ dB)$$ **Ans.**

Thus, the required S/N ratio must be greater than or equal to 24.1 dB for error-free transmission.

Scanned by CamScanner

(*iv*) The required bandwidth B can be found by

$$C = B \log_2 (1 + 100) \geq 8(10^4)$$

or

$$B \geq \frac{8(10^4)}{\log_2(1+100)} \geq 1.2(10^4) \text{ Hz} \geq 12 \text{ kHz}$$

and the required bandwidth of the channel must be greater than or equal to 12 kHz. **Ans.**

15.19. The Source Coding

A conversion of the output of a DMS into a sequence of binary symbols (*i.e.*, binary code word) is called **source coding.** The device that performs this conversion is called the **source encoder** as shown in figure 15.15.

An objective of source coding is to minimize the average bit rate required for representation of the source by reducing the redundancy of the information source.

15.19.1. The Code Length and Code Efficiency

Let X be a DMS with finite entropy $H(X)$ and an alphabet $\{x_1, \ldots x_m\}$ with corresponding probabilities of occurrence $P(x_i)(i = 1,\ldots,m)$.

Let the binary codeword assigned to symbol x_i by the encoder have length n_i, measured in bits. The length of a codeword is the number of binary digits in the codeword. The average codeword length L, per source symbol is given by

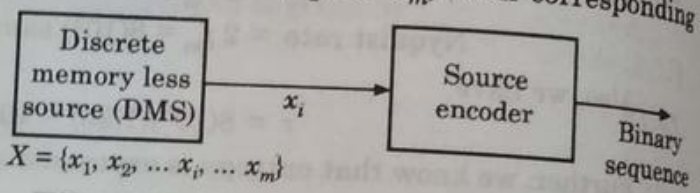


Fig. 15.15. *Block diagram for source Coding.*

$$L = \sum_{i=1}^{m} P(x_i) n_i \qquad \ldots(15.51)$$

The parameter L represents the average number of bits per source symbol used in the source coding process.

Also, the *code efficiency* η is defined as

$$\eta = \frac{L_{\min}}{L} \qquad \ldots(15.52)$$

where $L_{\min}$ is the minimum possible value of L. When η approaches unity, the code is said to be *efficient.*

The code *redundancy* γ is defined as

$$\gamma = 1 - \eta \qquad \ldots(15.53)$$

15.19.2. The Source Coding Theorem

The source coding theorem states that for a DMS X, with entropy $H(X)$, the average codeword length L per symbol is bounded as

$$L \geq H(X) \qquad \ldots(15.54)$$

and further, L can be made as close to $H(X)$ as desired for some suitably chosen code.

Thus, with $L_{\min} = H(X)$, the code efficiency can be rewritten as

$$\eta = \frac{H(X)}{L} \qquad \ldots(15.55)$$

15.19.3. Classification of Codes

Classification of codes is best illustrated by an example. Let us consider Table 15.2 where a source of size 4 has encoded in binary codes with symbol 0 and 1.

Scanned by CamScanner

Table 15.2. Binary Codes

x_i	Code 1	Code 2	Code 3	Code 4	Code 5	Code 6
x_1	00	00	0	0	0	1
x_2	01	01	1	10	01	01
x_3	00	10	00	110	011	001
x_4	11	11	11	111	0111	0001

15.19.3.1. Fixed-Length Codes

A *fixed-length* code is one whose codeword length is fixed. Code 1 and code 2 of Table 15.2 are fixed-length with length 2.

15.19.3.2. Variable-Length Codes

A *variable-length* code is one whose codeword length is not fixed. All codes of Table 15.2 except codes 1 and 2 are variable-length codes.

15.19.3.3. Distinct codes

A code is *distinct* if each codeword is distinguishable from other codewords. All codes of Table (15.2) except code 1 are distinct codes – notice the codes for x_1 and x_3.

15.19.3.4. Prefix-Free Codes

A code in which no codeword can be formed by adding code symbols to another codeword is called a *prefix-free code*. Thus, in a prefix-free code, no codeword is a *prefix* of another. Codes 2, 4, and 6 of Table 15.2 are prefix-free codes.

15.19.3.5. Uniquely Decodable Codes

A distinct code *uniquely decodable* if the original source sequence can be reconstructed perfectly from the encoded binary sequence. Note that code 3 of Table 15.2 is not a uniquely decodable code. For, example, the binary sequence 1001 may correspond to the source sequences $x_2x_3x_2$ or $x_2x_1x_1x_2$. A sufficient condition to ensure that a code is uniquely decodable is that no code word is a prefix of another. Thus, the prefix-free codes 2, 4, and 6 are uniquely decodable codes. Note that the prefix-free condition is not a necessary condition for unique decodability. For example, code 5 of Table 15.2 does not satisfy the prefix-free condition, and yet it is uniquely decodable since the bit 0 indicates the beginning of each codeword of the code.

15.19.3.6. Instantaneous Codes

A uniquely decodable code is called an *instantaneous code* if the end of any codeword is recognizable without examining subsequent code symbols. The instantaneous codes have the property previously mentioned that no codeword is a prefix of another codeword. For this reason, prefix-free codes are sometimes known as instantaneous codes.

15.19.3.7. Optimal Codes

A code is said to be *optimal* if it is instantaneous and has minimum average L for a given source with a given probability assuagement for the sourcesymbols.

15.19.4. The Kraft Inequality

Let X be a DMS with alphabet $\{x_i\}$ $(i = 1, 2, \ldots m)$. Assume that the length of the assigned binary codeword corresponding to x_i is n_i.

A necessary and sufficient condition for the existence of an instantaneous binary code is

$$K = \sum_{i=1}^{m} 2^{-n_i} \leq 1 \qquad \text{...(15.56)}$$

which is known as the **Kraft inequality.**

Scanned by CamScanner

Note: It may be noted that Kraft inequality assures us of the existence of an instantaneously decodable code with codeword lengths that satisfy the inequality. But it does not show us how to obtain these codewords, nor does it say any code satisfies the inequality is automatically uniquely decodable.

Example 15.37. Consider a DMS X with two symbols x_1 and x_2 and $P(x_1) = 0.9$, $P(x_2) = 0.1$. Symbols x_1 and x_2 are encoded as follows (Table 15.3)

Table 15.3.

x_i	$P(x_i)$	Code
x_1	0.9	0
x_2	0.1	1

Find the efficiency η and the redundancy γ of this code.

Solution: We know that the average code length L per symbol is

$$L = \sum_{i=1}^{2} P(x_i)n_i = (0.9)(1) + (0.1)(1) = 1 \text{ b}$$

Using equation (15.7), we have

$$H(X) = -\sum_{i=1}^{2} P(x_i)\log_2 P(x_i)$$

$$= -0.9 \log_2 0.9 - 0.1 \log_2 0.1 = 0.469 \text{ b/symbol}$$

Also, the code efficiency η is

$$\eta = \frac{H(X)}{L} = 0.469 = 46.9\%$$

And, the code redundancy γ is given by

$$\gamma = 1 - \eta = 0.531 = 53.1\% \text{ Ans.}$$

Example 15.38. The second-order extension of a DMS X, denoted by X^2, is formed by taking the source symbols two at a time. The coding of this extension has been shown in Table 15.4. Find the efficiency η and the redundancy γ of this extension code.

Solution:

Table 15.4.

a_i	$P(a_i)$	Code
$a_1 = x_1x_1$	0.81	0
$a_2 = x_1x_2$	0.09	10
$a_3 = x_2x_1$	0.09	110
$a_4 = x_2x_2$	0.01	111

We have

$$L = \sum_{i=1}^{4} P(a_i)n_i = 0.81(1) + 0.09(2) + 0.09(3) + 0.01(3) = 1.29 \text{ b/symbol}$$

The entropy of the second-order extension of X, $H(X^2)$, is given by

$$H(X^2) = -\sum_{i=1}^{4} P(a_i)\log_2 P(a_i)$$

$$= -0.81 \log_2 0.81 - 0.09 \log_2 0.09 - 0.09 \log_2 0.09 - 0.01 \log_2 0.01$$

or

$$H(X^2) = 0.938 \text{ b/symbol}$$

Scanned by CamScanner

Therefore, the code efficiency η is

$$\eta = \frac{H(X^2)}{L} = \frac{0.938}{1.29} = 0.727 = 72.7\%$$

Also, the code redundancy γ will be

$$\gamma = 1 - \eta = 0.273 = 27.3\% \quad \textbf{Ans.}$$

Note that $H(X^2) = 2H(X)$.

Example 15.39. Consider a DMS X with symbols x_i, $i = 1, 2, 3, 4$. Table 15.5 lists four possible binary codes.

Table 15.5

x_i	*Code A*	*Code B*	*Code C*	*Code D*
x_1	00	0	0	0
x_2	01	10	11	100
x_3	10	11	100	110
x_4	11	110	110	111

(*i*) Show that all the codes except code *B* satisfy the Kraft inequality.

(*ii*) Show that codes *A* and *D* are uniquely decodable but codes *B* and *C* are not uniquely decodable.

Solution: We have

(*i*) For code *A*, $n_1 = n_2 = n_3 = n_4 = 2$

therefore, $$K = \sum_{i=1}^{4} 2^{-n_i} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$$

For code *B*, $n_1 = 1, n_2 = n_3 = 2, n_4 = 3$

therefore, $$K = \sum_{i=1}^{4} 2^{-n_i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} = 1\frac{1}{8} > 1$$

For code *C*, $n_1 = 1, n_2 = 2, n_3 = n_4 = 3$

therefore, $$K = \sum_{i=1}^{4} 2^{-n_i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = 1$$

For code *D*, $n_1 = 1, n_2 = n_3 = n_4 = 3$

therefore, $$K = \sum_{i=1}^{4} 2^{-n_i} = \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{7}{8} < 1$$

Thus, all codes except code *B* satisfy the Kraft inequality.

(*ii*) Codes *A* and *D* are prefix-free codes. They are, therefore, uniquely decodable. Code *B* does not satisfy the Kraft inequality, and it is not uniquely decodable. Although code *C* does satisfy the Kraft inequality, but it is not uniquely decodable. This can be seen by the following example:

Given the binary sequence 0110110. This sequence may correspond to the source sequences $x_1x_2x_1x_4$ or $x_1x_4x_4$.

Example 15.40. Verify the following expression:

$$L \geq H(X)$$

where *L* is the average codeword length per symbol and *H*(*X*) is the source entropy.

Scanned by CamScanner

Solution: We know that

$$\sum_{i=1}^{m} P_i \log_2 \frac{Q_i}{P_i} \le 0$$

where the equality holds only if $Q_i = P_i$.

Let $$Q_i = \frac{2^{-n_i}}{K} \quad \text{...(i)}$$

where $$K = \sum_{i=1}^{m} 2^{-n_i} \quad \text{...(ii)}$$

Now, we have

$$\sum_{i=1}^{m} Q_i = \frac{1}{K}\sum_{i=1}^{m} 2^{-n_i} = 1 \quad \text{...(iii)}$$

and $$\sum_{i=1}^{m} P_i \log_2 \frac{2^{-n_i}}{KP_i} = \sum_{i=1}^{m} P_i\left(\log_2 \frac{1}{P_i} - n_i - \log_2 K\right)$$

$$= -\sum_{i=1}^{m} P_i \log_2 P_i - \sum_{i=1}^{m} P_i n_i - (\log_2 K)\sum_{i=1}^{m} P_i \quad \text{...(iv)}$$

$$= H(X) - L - \log_2 K \le 0$$

From the Kraft inequlity (14.56), we have

$$\log_2 K \le 0$$

Thus, $$H(X) - L \le \log_2 K \le 0 \quad \text{...(v)}$$

or $$L \ge H(X) \quad \text{...(vi)}$$

The equality holds when $K = 1$ and $P_i = Q_i$. **Hence proved.**

Example 15.41. Let X be a DMS with symbols x_i and corresponding probabilities $P(x_i) = P_i$, $i = 1,2, \ldots m$. Show that for the optimum source encoding, we require that

$$K = \sum_{i=1}^{m} 2^{-n_i} = 1 \quad \text{...(i)}$$

and $$n_i = \log_2 \frac{1}{P_i} = I_i \quad \text{...(ii)}$$

where n_i is the length of the codeword corresponding to x_i and I_i is the information content of x_i.

Solution: From the result of last problem, the optimum source encoding with $L = H(X)$ requires $K = 1$ and $P_i = Q_i$. Thus, equations using (*ii*) and (*i*) of last problem, we have

$$K = \sum_{i=1}^{m} 2^{-n_i} = 1 \quad \text{...(iii)}$$

and $$P_i = Q_i = 2^{-n_i} \quad \text{...(iv)}$$

Hence $$n_i = -\log_2 P_i = \log_2 \frac{1}{P_i} = I_i$$

Note: *Note that equation (ii) implies the following commonsense principle.*

Scanned by CamScanner

Symbols which occur with high probability should be assigned shorter codewords than symbols that occur with low probability. **Hence proved.**

Example 15.42. Consider a DMS X with symbols x_i and corresponding probabilities $P(x_i) = P_i$, $i = 1, 2, \ldots m$. Let n_i be the length of the codeword of x_i such that

$$\log_2 \frac{1}{P_i} \le n_i \le \log_2 \frac{1}{P_i} + 1 \qquad \ldots(i)$$

Show that this relationship satisfies the Kraft inequality (15.56), and find the bound on K in equation (15.56).

Solution: Equation (*i*) can be rewritten as

$$-\log_2 P_i \le n_i \le -\log_2 P_i + 1 \qquad \ldots(ii)$$

or

$$\log_2 P_i \ge -n_i \ge \log_2 P_i - 1$$

Then,

$$2\log_2 P_i \ge 2^{-n_i} \ge 2^{\log_2 P_i}\, 2^{-1}$$

or

$$P_i \ge 2^{-n_i} \ge \frac{1}{2} P_i \qquad \ldots(iii)$$

Thus,

$$\sum_{i=1}^{m} P_i \ge \sum_{i=1}^{m} 2^{-n_i} \ge \frac{1}{2} \sum_{i=1}^{m} P_i \qquad \ldots(iv)$$

or

$$1 \ge \sum_{i=1}^{m} 2^{-n_i} \ge \frac{1}{2} \qquad \ldots(v)$$

which indicates that the Kraft inequality (15.56) is satisfied, and the bound on K will be

$$\frac{1}{2} \le K \le 1 \qquad \textbf{Hence proved.} \qquad \ldots(vi)$$

Example 15.43. Consider a DMS X with symbols x_i and corresponding probabilities $P(x_i) = P_i$, i 1, 2, ..., m. Show that a code constructed in agreement with equation (*i*) in last problem will satisfy the following relation:

$$H(X) \le L \le H(X) + 1 \qquad \ldots(i)$$

where $H(X)$ is the source entropy and L is the average codeword length.

Solution: Multiplying equation (*ii*) in last problem by P_i and summing over i yields

$$-\sum_{i=1}^{m} P_i \log_2 P_i \le \sum_{i=1}^{m} n_i P_i \le \sum_{i=1}^{m} P_i(-\log_2 P_i + 1) \qquad \ldots(ii)$$

Now,

$$\sum_{i=1}^{m} P_i(-\log_2 P_i + 1) = -\sum_{i=1}^{m} P_i \log_2 P_i + \sum_{i=1}^{m} P_i = H(X) + 1$$

Thus, equation (*ii*) reduces to

$$H(X) \le L \le H(X) + 1 \qquad \textbf{Hence proved.}$$

15.20. Entropy Coding

The design of a variable-length code such that its average codeword length approaches the entropy of DMS is often referred to as **entropy coding.** In this section, we present two examples of entropy coding.

Scanned by CamScanner

15.20.1. Shannon-Fano Coding

An efficient code can be obtained by the following simple procedure, known as **Shannon-Fano algorithm:**

1. List the source symbols in order of decreasing probability.
2. Partition the set into two sets that are as close to equiprobables as possible, and assign 0 to the upper set 1 to the lower set.
3. Continue this process, each time partitioning the sets with as nearly equal probabilities as possible until further partitioning is not possible.

An example of Shannon-Fano encoding is shown in Table 15.6. Note in Shannon-Fano encoding the ambiguity may arise in the choice of approximately equiprobable sets.

Table 15.6. Shannon-Fano Encoding

x_i	$P(x_i)$	*Step 1*	*Step 2*	*Step 3*	*Step 4*	*Code*
x_1	0.30	0	0			00
x_2	0.25	0	1			01
x_3	0.20	1	0			10
x_4	0.12	1	1	0		110
x_4	0.08	1	1	1	0	1110
x_6	0.05	1	1	1	1	1111

$$H(X) = 2.36 \text{ b/symbol}$$
$$L = 2.38 \text{ b/symbol}$$
$$\eta = H(X)/L = 0.99$$

15.20.2. The Huffman Encoding

(U.P. Tech., Sem. Examination, 2003-04)

In general, Huffman encoding results in an optimum code. Thus, it is the code that has the highest efficiency. The Huffman encoding procedure is as follows:

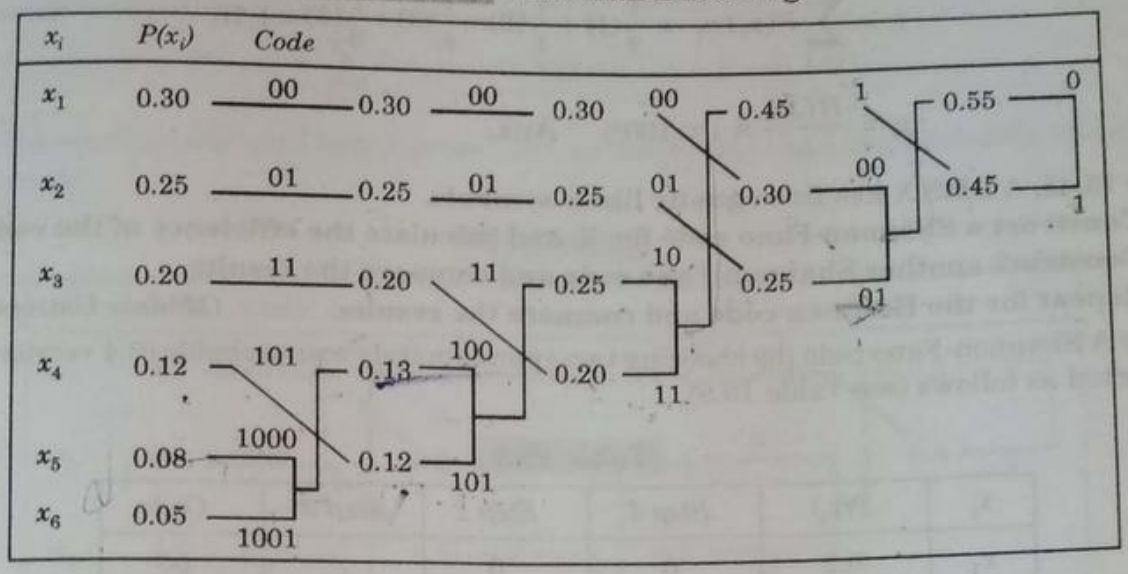
1. List the source symbols in order of decreasing probability.
2. Combine the probabilities of the two symbols having the lowest probabilities, and reorder the resultant probabilities, this step is called reduction 1. The same procedure is repeated until there are two ordered probabilities remaining.
3. Start encoding with the last reduction, which consist of exactly two ordered probabilities. Assign 0 as the first digit in the codewords for all the source symbols associated with the first probability; assign 1 to the second probability.
4. Now go back and assign 0 and 1 to the second digit for the two probabilities that were combined in the previous reduction step, retaining all assignments made in Step 3.
5. Keep regressing this way until the first column is reached.

An example of Huffman encoding is shown in Table 15.7

$$H(X) = 2.36 \text{ b/symbol}$$
$$L = 2.38 \text{ b/symbol}$$
$$\eta = 0.99$$

Scanned by CamScanner

Table 15.7. Huffman Encoding



Example 15.44. A DMS X has four symbols x_1, x_2, x_3, and x_4 with $P(x_1) = \frac{1}{2}$, $P(x_2) = \frac{1}{4}$, and $P(x_3) = P(x_4) = \frac{1}{8}$. Construct a Shannon-Fano code for X; show that this code has the optimum property that $n_i = I(x_i)$ and that the code efficiency is 100 percent.

Solution: The Shannon-Fano code is constructed as follows (see Table 15.8).

Table 15.8

x_i	$P(x_i)$	*Step 1*	*Step 2*	*Step 3*	*Code*
x_1	1/2	0			0
x_2	1/4	1	0		10
x_3	1/8	1	1	0	110
x_4	1/8	1	1	1	111

$$I(x_1) = -\log_2 \frac{1}{2} = 1 = n_1$$

$$I(x_2) = -\log_2 \frac{1}{4} = 2\ n_2$$

$$I(x_3) = -\log_2 \frac{1}{8} = 3 = n_3$$

$$I(x_4) = -\log_2 \frac{1}{8} = 3 = n_4$$

We know that,

$$H(X) = \sum_{i=1}^{4} P(x_i) I(x_i)$$

$$= \frac{1}{2}(1) + \frac{1}{4}(2) + \frac{1}{8}(3) + \frac{1}{8}(3) = 1.75$$

Scanned by CamScanner

$$L = \sum_{i=1}^{4} P(x_i)\, n_i = \frac{1}{2}(1) + \frac{1}{4}(2) + \frac{1}{8}(3) + \frac{1}{8}(3) = 1.75$$

Also $$\eta = \frac{H(X)}{L} = 1 = 100\% \quad \textbf{Ans.}$$

Example 15.45. A DMS X has five equally likely symbols.

(*i*) Construct a Shannon-Fano code for *X*, and calculate the efficiency of the code.

(*ii*) Construct another Shannon-Fano code and compare the results.

(*iii*) Repeat for the Huffman code and compare the results. *(Madras University, 1998)*

Solution: A Shannon-Fano code [by choosing two approximately equiprobable (0.4 versus 0.6) sets] is constructed as follows (see Table 15.9)

Table 15.9.

x_i	$P(x_i)$	*Step 1*	*Step 2*	*Step 3*	*Code*
x_1	0.2	0	0		00
x_2	0.2	0	1		01
x_3	0.2	1	0		10
x_4	0.2	1	1	0	110
x_5	0.2	1	1	1	111

$$H(X) = \sum_{i=1}^{5} P(x_i) \log_2 P(x_i) = 5(-0.2 \log_2 0.2) = 2.32$$

$$L = \sum_{i=1}^{5} P(x_i) n_i = 0.2(2 + 2 + 2 + 3 + 3)\ 2.4$$

The efficiency η is $$\eta = \frac{H(X)}{L} = \frac{2.32}{2.4} = 0.967 = 96.7\% \quad \textbf{Ans.}$$

(*ii*) Another Shannon-Fano code [by choosing another two approximately equiprobable (0.6 versus 0.4) sets] is constructed as follows (see Table 15.10)

Table 15.10.

x_i	$P(x_i)$	*Step 1*	*Step 2*	*Step 3*	*Code*
x_1	0.2	0	0		00
x_2	0.2	0	1	0	010
x_3	0.2	0	1	1	011
x_4	0.2	1	0		10
x_5	0.2	1	1		11

$$L = \sum_{i=1}^{5} P(x_i) n_i = 0.2(2 + 3 + 3 + 2 + 2) = 2.4$$

Since the average codeword length is the same as that for the code of part (*a*), the efficiency is the same. **Ans.**

Scanned by CamScanner

(*iii*) The Huffman code is constructed as follows (see Table 15.11)

$$L = \sum_{i=1}^{5} P(x_i)n_i = 0.2(2 + 3 + 3 + 2 + 2) = 2.4$$

Since the average code word length is the same as that for the Shannon-Fano code, the efficiency is also the same.

Table 15.11.

x_i	$P(x_i)$	Code						
x_1	0.2	00	0.4	1	0.4	1	0.6	0
x_2	0.2	000	0.2	01	0.4	00	0.4	1
x_3	0.2	001	0.2	000	0.2	01		
x_4	0.2	10	0.2	001				
x_5	0.2	11						

Example 15.46 A DMS X has five symbols x_1, x_2, x_3, x_4, and x_5 with $P(x_1) = 0.4$, $P(x_2) = 0.19$, $P(x_3) = 0.16$, $P(x_4) = 0.15$, and $P(x_5) = 0.1$.

(*i*) Construct a Shannon-Fano code for X, and calculate the efficiency of the code.

(*ii*) Repeat for the Huffman code and compare the results.

Solution: The Shannon-Fano code is constructed as follows (see Table 15.12)

Table 15.12

x_i	$P(x_i)$	*Step 1*	*Step 2*	*Step 3*	*Code*
x_1	0.4	0	0		00
x_2	0.19	0	1		01
x_3	0.16	1	0		10
x_4	0.15	1	1	0	110
x_5	0.1	1	1	1	111

$$\cancel{H(X)}\ L = \sum_{i=1}^{5} P(x_i)n_i = 0.4(1) + 0.19(2) + 0.16(2) + 0.15(3) + 0.1(3) = 2.25$$

Also, $$\eta = \frac{H(X)}{L} = \frac{2.15}{2.25} = 0.956 = 9.5.6\% \qquad \textbf{Ans.}$$

(*ii*) The Huffman code is constructed as follows (see Table 15.13):

$$L = \sum_{i=1}^{5} P(x_i)n_i = 0.4(1) + (0.19 + 0.16 + 0.15 + 0.1)(3) = 2.2 \quad \textbf{Ans.}$$

Scanned by CamScanner

Table 15.13.

x_i	$P(x_i)$	Code											
x_1	0.4	0		0.4	1		0.4	1		0.6	0		
x_2	0.19	000		0.25	01		0.35	00		0.4	1		
x_3	0.16	001		0.19	000		0.25	01					
x_4	0.15	010		0.16	001								
x_5	0.1	011											

$$\eta = \frac{H(X)}{L} = \frac{2.15}{2.2} = 0.977 = 915.7\%$$

The average codeword length of the Huffman code is shorter than that of the Shannon-Fanco code, and thus the efficiency is higher than that of the Shannon-Fano code. **Ans.**

Example 15.47. Determine the Huffman code for the following messages with their probabilities given

x_1	x_2	x_3	x_4	x_5	x_6	x_7
0.05	**0.15**	**0.2**	**0.05**	**0.15**	**0.3**	**0.1**

Solution: Arranging and groping of messages is done as shown below:

x_i	$p(x_i)$	Code											
x_6	0.3	00	0.3	00	0.3	00	0.3	00	0.4	1	0.6	0	
x_3	0.2	1.0	0.2	10	0.2	10	0.3	01	0.3	0.4	1		
x_2	0.15	101	0.15	010	0.2	11	0.2	10	0.3	00			
x_5	0.15	011	0.15	011	0.15	010	0.2	11		01			
x_7	0.1	110	0.1	110	0.15	011							
x_1	0.05	1100	0.1	111									
x_4	0.05	1111											

Message	*Prob.*	*Code*	*No. of bits in code*
x_1	0.05	1110	4
x_2	0.15	010	3
x_3	0.2	10	2
x_4	0.05	1111	4
x_5	0.15	011	3
x_6	0.3	00	2
x_7	0.1	110	3

Scanned by CamScanner

∴ Average length L is given by

$$L = \sum_{i=1}^{7} n_i P(x_i)$$

or

$$L = 4(0.05 + 0.05) + 3(0.15 + 0.15 + 0.1) + 2(0.2 + 0.3)$$
$$L = 2.6 \text{ bits}$$

Entropy $H(X)$ is given by

$$H(X) = \sum_{i=1}^{7} P(x_i) \log_2 \frac{1}{P(x_i)}$$

$$= 0.3 \log_2\left(\frac{1}{0.3}\right) + 0.2 \log_2\left(\frac{1}{0.2}\right) + 0.3 \log\left(\frac{1}{0.15}\right) + 0.1 \log_2\left(\frac{1}{0.3}\right) + 0.1 \log_2\left(\frac{1}{0.05}\right)$$

$$= 2.57 \text{ bits}$$

$$\eta = \frac{H(X)}{L \log_2 M} = \frac{2.57}{2.6 \log_2 2} = \frac{2.57}{2.6} = 98.85\%$$

SUMMARY

1. The purpose of a communication system is to carry information-bearing baseband signals from one place to another place over a communication channel.
2. Information theory is a branch of probability theory which may be applied to the study of the communication systems.
3. In the context of communications, information theory deals with mathematical modeling and analysis of a communication system rather than with physical sources and physical channels.
4. Information theory was invented by communication scientists while they were studying the statistical structure of electronic communication equipments. When the communique is readily measurable, such as an electric current, the study of the communication system is relatively easy.
5. An information source may be an object produces an event, the outcome of which is selected at random according to a probability distribution. A practical source in a communication system is a device which produces messages, and it can be either analog or discrete.
6. A discrete information source is a source which has only a finite set of symbols as possible outputs. The set of source symbols is called the source alphabet, and the elements of the set are called symbols or letters.
7. Information sources can be classified as having memory or being memoryless. A source with memory is one for which a current symbol depends on the previous symbols. A memoryless source is one for which each symbol produced is independent of the previous symbols.
8. A discrete memoryless source (DMS) can be characterized by the list of the symbols, the probability assignment to these symbols, and the specification of the rate of generating these symbols by the source.
9. The amount of information contained in an event is closely related to its uncertainty. Messages containing knowledge of high probability of occurrence convey relatively little information.
10. In a practical communication system, we usually transmit long sequences of symbols from an information source. Thus, we are more interested in the average information that a source produces than the information content of a single symbol.
11. For quantitative representation of average information per symbol we make the following assumptions:
    (*i*) The source is stationary so that the probabilities may remain constant with time.
    (*ii*) The successive symbols are statistically independent and come form the source at a average rate of r symbols per second.

Scanned by CamScanner

CHAPTER 16

Error Control Coding

Inside this Chapter

16.1. Introduction 798
16.2. Few Terms Related to Error Control Coding 799
16.3. Parity Coding 801
16.4. Vertical Redundancy Check (VRC) 801
16.5. Linear Block Codes 804
16.6. Hamming Codes 808
16.7. Encoder of (7, 4) Hamming Code 812
16.8. Syndrome Decoding: Method of Correct Errors 812
16.9. Error Correction using Syndrome Vector 816
16.10. Syndrome Decoder for (n, k) Block Code 818
16.11. Other Linear Block Code 825
16.12. Cyclic Codes 829
16.13. Generator and Parity Check Matrices of Cyclic Codes 835
16.14. Encoders for Cyclic Codes 839
16.15. BCH Codes (Bose-Chaudhri-Hocquenghem Codes) 844
16.16. Convolutional Codes 844

16.1. Introduction

As discussed earlier that during the transmission process, the transmitted signal passes through some noisy channel. Due to noise interference, some errors are introduced in the received data. These errors can be detected and sometimes corrected using coding techniques. Generally, the error control methods are of two types:

(*i*) Error detection with retransmission, and

(*ii*) Forward acting error correction.

In case of first method, when an error is detected, the retransmission request (ARQ) is sent back to the transmitter. In the second method, the errors are detected and corrected by proper coding techniques at the receiver end. Whenever a single source transmits data to a number of receivers, the forward acting error correction is used. This is due to the fact that the retransmission is impossible in these cases.

In this chapter, we shall study some simple error detecting techniques such as parity checking. Some of the coding techniques which detect as well as correct errors will also be discussed in the subsequent sections. Coding techniques add some extra bits to the message bits and these bits

Scanned by CamScanner

are used to detect errors at the receiver end. Although these extra bits reduce the bit rate of the transmitter and its power but the advantage is that error probability is reduced.

Figure 16.1 shows the block diagram of the channel encoder and decoder.

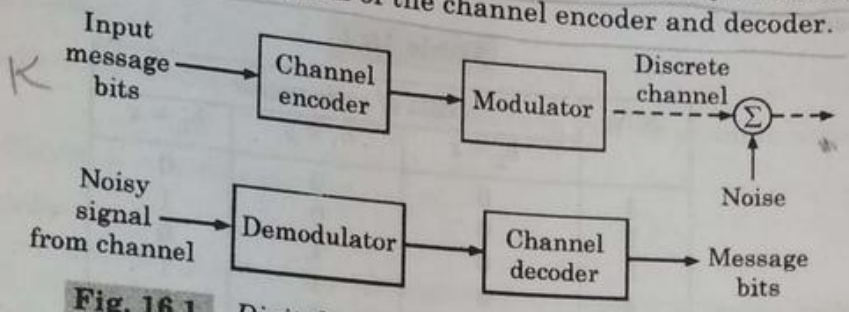


Fig. 16.1. *Digital communication system with channel encoding/decoding operations*

A channel encoder adds extra bits to the message signal. On the other hand, the channel decoder identifies these redundant bits and uses them to detect and correct the errors in the message bits if there is any. Let us consider that the 'k' message bits are combined in one block for encoding. Some bits are added by the encoder to this block and total encoded bits become 'n'. Then such a code is known as **(n, k) *block code***. Then we can write,

Redundant bits added to the block by encoder = (Bits at the output of encoder) – (Bits at the input of encoder)

$$= n - k \qquad ...(16.1)$$

These $(n - k)$ redundant bits (*i.e.*, code bits) are related to 'k' message bit depending on the coding technique used.

16.2. Few Terms Related to Error Control Coding

Some of the related terms are described here as under:

1. Codeword

The encoded block of 'n' bits is known as a **code word**. It consists of message bits and redundant bits.

2. Block Length

The number of bits 'n' after coding is known as the block length of the code.

3. Code Rate

The ratio of message bits (k) and the encoder output bits (n) is known as **code rate**. Usually, code rate is denoted by 'r' *i.e.*,

$$r = \frac{k}{n} \qquad ...(16.2)$$

we find that $0 < r < 1$.

4. Channel Data Rate

Channel data rate is the bit rate at the output of encoder. If the bit rate at the input of encoder is R_s, then channel data rate will be,

$$\text{Channel data rate } (R_0) = \frac{n}{k} R_s \qquad ...(16.3)$$

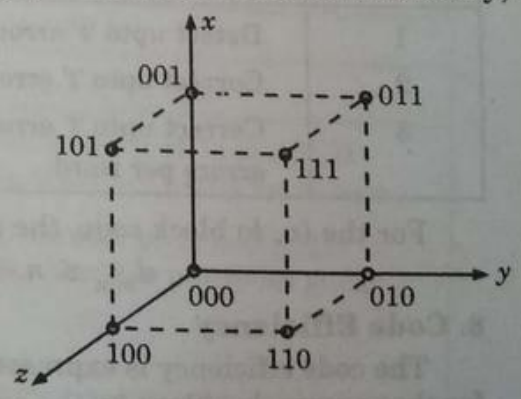


Fig. 16.2. *Code vectors representing 3-bit code words*

5. Code Vectors

An 'n' bit code word can be visualized in an n-dimensional space as a vector whose elements or co-ordinates are the bits in the code word. Let us visualize the 3-bit code words as it is quite simpler. Figure 16.2 shows the 3-bit code vectors. There will be

Scanned by CamScanner

distinct '8' code words (since number of code words = 2^k). If we let bits b_0 on x-axis, b_1 on y-axis and b_2 on z-axis, then the following Table 16.1 shows various points as code vectors in the 3-dimensional space.

Table 16.1

S.No	Bits of code vector		
	$b_2 = z$	$b_1 = y$	$b_0 = x$
1	0	0	0
2	0	0	1
3	0	1	0
4	0	1	1
5	1	0	0
6	1	0	1
7	1	1	0
8	1	1	1

6. Hamming Distance

The Hamming distance between the two code vectors is equal to the number of elements in which they differ.

As an example, let $X = 101$ and $Y = 110$. Since the two code vectors differ in second and third bits, therefore, Hamming distance between X and Y is 'two'. Hamming distance is denoted by $d(X, Y)$ or simply 'd'. *i.e*,

$$d(X, Y) = d = 2$$

Hence, it may be observed from figure 16.2 that the hamming distance between (100) and (011) is maximum, *i.e.* 3. This has been indicated by the vector diagram also.

7. Minimum Distance (d_{min})

Minimum distance (d_{min}) is the smallest Hamming distance between the valid code vectors.

Note: *It may be noted that the error detection is possible if the received vector is not equal to some other code vector. This shows that the transmission errors in the received code vector must be less than minimmum distance d_{min}. Table 16.2 shows some of the requirements of error capability of the code.*

Table 16.2.

S. No.	Name of errors detected /corrected	Distance requirement
1	Detect upto 's' errors per word	$d_{min} \geq s + 1$
2	Correct upto 't' errors per word	$d_{min} \geq 2t + 1$
3	Correct upto 't' errors and detect $s > t$ errors per word	$d_{min} \geq t + s + 1$

For the (n, k) block code, the minimum distance is expressed as

$$d_{min} \leq n - k + 1 \quad \text{...(16.4)}$$

8. Code Efficiency

The code efficiency is expressed as the ratio of message bits in a block to the transmitted bits for that particular block by the encoder *i.e.*,

$$\text{Code efficiency} = \frac{\text{Message bits in a block}}{\text{Transmitted bits for the block}}$$

Scanned by CamScanner

Further, since for an (n, k) block code, there are 'k' message bits and 'n' transmitted bits, therefore code efficiency will be

$$\text{Code efficiency} = \frac{k}{n} \quad \text{...(16.5)}$$

Now, if we compare the above expression with the code rate (r) given in equation (16.2), we note that,

$$\text{Code efficiency} = \text{code rate} = \frac{k}{n} \quad \text{...(16.6)}$$

9. Weight of the code

The number of non-zero elements in the transmitted code vector is known as **vector weight**. It is denoted by $\omega(X)$, where X is the code vector. As an example, if $X = 01110101$, then the weight of this code vector will be $\omega(X) = 5$.

16.3. Parity Coding

The simplest method of coding is a parity coding. In this method, parity bits are added to the message at the time of transmission. At the reception end, receiver checks these parity bits. Errors are detected if the expected pattern of parity bits is not received.

It may be noted that parity coding is used only to detect the errors.

Parity may be of two types:

(i) Even Parity

The parity of a binary word is known as 'even' if it contains even number of 1's. For example, 01101001 has even parity since there are four number of 1's in the word.

(ii) Odd Parity

The parity of a binary word is known as 'odd' if it contains odd number of 1's. For example, 10010001 has odd parity because there are three number of 1s in this word.

16.4. Vertical Redundancy Check (VRC)

In this method, the parity check is done on each character separately. This method is always used to ASCII code. In ASCII code, each character has seven binary digits. Also, one redundant digit is added to each character to make the parity odd or even. As an example, let us consider the coding shown in Table 16.3.

Table 16.3. Vertical redundancy check, even parity

Character	*ASCII code of the character*							*Additional parity bit*
	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
K	1	1	0	1	0	0	1	0 ← Even number of 1s in ASCII code, therefore parity bit is zero.
O	1	1	1	1	0	0	1	1 ← Odd number of 1s in ASCII code, therefore parity bit is one.
I	1	0	0	1	0	0	1	1
A	1	0	0	0	0	0	1	0

16.4.1. Few Points

From Table 16.3, let us note the following points:

(i) As depicted in the Table 16.3, the message consists of total 8 bits, *i.e.*, b_1 to b_16. Bits b_1 to b_7 are used for ASCII code of the character and the bit b_8 is used for parity bit.

Scanned by CamScanner

(*ii*) For example, the ASCII code of *K* is 1101001. This consists of even number of 1s. Therefore, the 8th parity bit has '0' value. Hence, the transmitted message for *K* will be 11010010, which has even parity.

(*iii*) The ASCII code for 0 is 1111001. This contains odd (*i.e*, five) number of 1's. Hence, the parity bit has value of 1. Hence, the transmitted message for 0 is 11110011, which has even parity.

(*iv*) During the transmission, if single error is created, it will make number of 1's odd. For example, let us consider that the received message for character *K* is as shown below:

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
Transmitted message for character K by Even Parity(see Table 16.3) →	1	1	0	1	0	0	1	0
Received message → It contains odd number of 1s. It means that there is an error	1	1	0	1	1	0	1	0

This digit is an error

Hence, as shown above, bit b_5 is in error and it makes number of 1s odd (*i.e.*, five). The receiver detects this error.

(*v*) However, it is not possible for the receiver to detect exactly which digit is in error. The receiver will only detect that *there are odd number of 1s and thus received message contains some error.*

(*vi*) Thus, the receiver has to send repeat request to the transmitter, to retransmit the message.

(*vii*) If two digits are in error, then number of 1's in the received message is even. In this case, receiver will detect no error. In fact, this is the major drawback of Vertical Redundancy Check (VRC).

16.4.2. Longitudinal Redundancy Check (LRC)

In the last subarticle, we have observed that VRC checks only in one direction. Hence, only single errors can be detected. But in LRC, large message block is divided into several characters. Now, parity check is applied in both row and columns. Thus, it is possible to detect upto tripplen errors. It is also possible to correct single errors with LRC. In LRC for checking the message block, a complete character known as '*Block Check Character*' (BCC) is added at the end of block of information. The block check character can be of even or odd parity. An even parity block check character (BCC) may be obtained by performing an exclusive OR operation on all the characters.

Figure 16.3 illustrates the structure of information block transmitted with block check character.

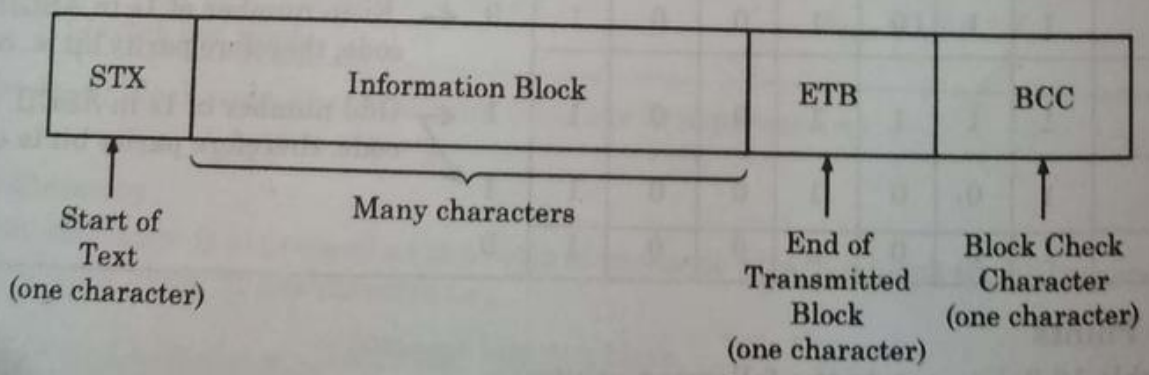


Fig. 16.3. *Structure of information block*

Scanned by CamScanner

As shown in the figure 16.3, start of text (STX) is one character in ASCII and implies that information block follows this character. After the information block, one character indicating End of Transmitted Block (ETB) is transmitted. It is also ASCII character. The Block Check Character (BCC) is transmited at the end. As an example, let us consider the transmitted and received message blocks as shown in Table 16.4 which uses LRC.

16.4.3. Few Points

(*i*) This method (*i.e.*, LRC) is used with ASCII codes only. Tables 16.4(*a*) shows the transmitted message block. The first row of this table shows characters which are transmitted.

(*ii*) It may be observed that the first character transmitted is STX. The ASCII value of STX is shown in its column. The ASCII value of STX is $(b_1b_2b_3b_4b_5b_6b_7) = (0100000)$.

(*iii*) The last row (for b_8) shows the even parity bits to be added for each character. Hence, $b_8 = 1$ for STX since ASCII code of STX contains only one (*i.e.* odd) binary 1. Hence, the transmitted character for STX is $(b_1b_2....b_8) = (01000001)$. This character is the first transmitted character according to figure 16.3.

(*iv*) As shown in Table 16.4 (*a*), the sequence of characters N, A, I, T,.... etc., is transmitted following STX. This is known as information block.

(*v*) In Table 16.4(*a*) it may be noted that the second last character transmitted is ETB (End of Transmitted Block). The ASCII value of ETB $(b_1,b_2,...b_7) = (1110100)$ is shown in its column. The parity bit b_8 is zero.

Table 16.4. LRC Using Block Check Character

Characters	STX	N	A	I	T	I	S	O	K	ETB	BCC ← Column Parity bits
Message Bits b_1	0	0	1	1	0	1	1	1	1	1	1
b_2	1	1	0	0	0	0	1	1	1	1	0
b_3	0	1	0	0	1	0	0	1	0	1	0
b_4	0	1	0	1	0	1	0	1	1	0	1
b_5	0	0	0	0	1	0	1	0	0	1	1
b_6	0	0	0	0	0	0	0	0	0	0	0
b_7	0	1	1	1	1	1	1	1	1	0	0
b_8 (Character parity Row Parity Bits)	1	0	0	1	1	1	0	1	0	0	1

(a) Transmitted Message Block

		N	A	I	~	I	S	O	K		
b_1	0	0	1	1	0	1	1	1	1	1	1 ← Column Parity Check
b_2	1	1	0	0	0	0	1	1	1	1	0
b_3	0	1	0	0	1	0	0	1	0	1	0
b_4	0	1	0	1	1	1	0	1	1	0	1 ×
b_5	0	0	0	0	1	0	1	0	0	1	1
b_6	0	0	0	0	0	0	0	0	0	0	0
b_7	0	1	1	1	1	1	1	1	1	0	0
b_8 (Row Parity)	1	0	0	1	1 ×	1	0	1	0	0	1

(b) Detected Message Block

Scanned by CamScanner

(*vi*) After ETB, BCC is transmitted. BCC is shown in last column of Table 16.4(*a*). The value of BCC in particular row is obtained by performing exclusive OR operation on all the digits in that row.

(*vii*) As an example, let us consider the row for b_1. This is,

	STX	N	A	I	T	I	S	O	K	ETB	BCC
$b_1 \equiv$	0	0	1	1	0	1	1	1	1	1	1

(*viii*) At this point, it may be observed that the row for b_1 contains odd (*i.e.*, seven) number of 1s. Thus, exclusive OR operation will provide a value of 1. Hence, in the row of b_1, BCC has value of 1. Note that the complete row of b_1 has even parity.

(*ix*) Similarly, in the row of b_2, BCC is zero. It has also got even parity. Block check character (BCC) is thus obtained by parity coding. Hence, the last row *i.e.*, row of b_8 gives parity check in vertical direction and BCC provides parity check in horizontal direction.

(*x*) Now, let us consider the received message block as shown in Table 16.4(*b*). This message block is received in response to that shown in Table 16.4(*a*) As shown in Table 16.4(*b*), the encircled digit is in error. This error is easily located in row and column paity checks.

(*xi*) Hence, single error can be detected and corrected due to LRC method. Double and tripple errors can also be detected.

16.4.4. Advantages of LRC Over VRC

Now let us summarize advantages of LRC method over VRC method as under:

(*i*) In LRC method, single errors can be detected and corrected.

(*ii*) In LRC method, double and tripple errors can be detected.

The drawback of LRC is that complexity in the method is increased.

Check Sum

The bits corresponding to row b_8 are character parity bits. They are also called row parity bits. These bits are appended to the data bits as an error check group. These bits are also called as ***checksum*** combinely.

16.5. Linear Block Codes

For the block of k message bits, $(n - k)$ parity bits or check bits are added. This means that the total bits at the output of channel encoder are 'n'. Such types of codes are known as **(n,k) block codes**. Figure 16.4 illustrates this concept.

1. Systematic Codes

In the systematic block code, the message bits appear at the beginning of the code word. Thus, as shown in figure 16.4, the message bits appear first and then check bits are transmitted in a block. This type of code is known as the **systematic code**.

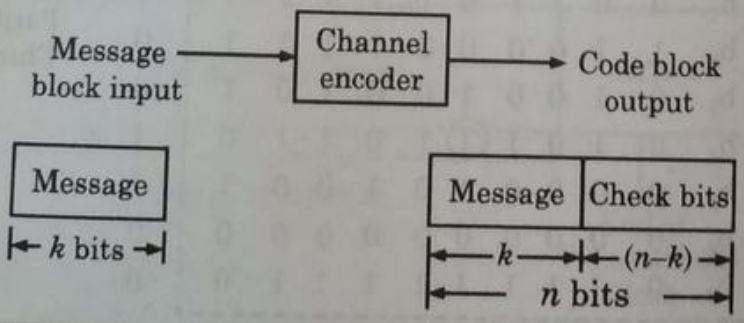


Fig 16.4. *Functional block diagram of a block coder*

In this section, we shall consider binary codes *i.e.*, in which all transmitted digits are binary.

Scanned by CamScanner

2. Linear Code

A code is known as 'linear' if the sum* of any two code vectors produces anothe code vector. This shows that any code vector may be expressed as a linear combination of other code vectors.

16.5.1. Steps for Determination of all the Codewords for a Linear Block Code

(*i*) Let us consider that the particular code vector consists of $m_1, m_2, m_3, \ldots m_k$ message bits and $c_1, c_2, c_3 \ldots\ldots c_q$ check bits. Then this code vector may be written as under:

$$X = (m_1, m_2, \ldots m_k\, c_1, c_2, \ldots c_q) \quad \ldots(16.7)$$

where $$q = n - k \quad \ldots(16.8)$$

(*ii*) This means that q are the number of redundant bits added by the encoder. The above code vector may also be written as,

$$X = (M \mid C) \quad \ldots(16.9)$$

where $M = k$-bit message vector and

and $C = q$-bit check vector

(*iii*) Here, the check bits play the role of error detection and correction. The function of the linear block code is to generate these 'check bits'. The code vector can be represented as,

$$X = MG \quad \ldots(16.10)$$

where X = Code vector of $1 \times n$ size or n bits

M = Message vector of $1 \times k$ size or k bits

and G = Generator matrix of $k \times n$ size.

(*iv*) Hence, equation (16.10) represents matrix form *i.e.*,

$$[X]_{1\times n} = [M]_{1\times k}\,[G]_{k\times n} \quad \ldots(16.11)$$

(*v*) The generator matrix depends upon the linear block code used. Normally, it is represented as,

$$G = \left[I_k \mid P_{k\times q}\right]_{k\times n} \quad \ldots(16.12)$$

Here $I_k = k \times k$ identity matrix, and

$P = k \times q$ submatrix

(*vi*) Now, the check vector may be obtained as,

$$C = MP \quad \ldots(16.13)$$

Hence, in the expanded form, the above expression can be written as

$$[c_1, c_2, \ldots\ldots\ldots c_q]_{1\times q} = [m_1, m_2, \ldots m_k]_{1\times k} \begin{bmatrix} P_{11} & P_{12} & \ldots & P_{1q} \\ P_{21} & P_{22} & \ldots & P_{2q} \\ \vdots & & & \\ \vdots & & & \\ P_{kl} & P_{k2} & \ldots & P_{kq} \end{bmatrix}_{k\times q} \quad \ldots(16.14)$$

(*vii*) On solving the above matrix equation, check vector may be obtained as under:

$$\left.\begin{aligned} c_1 &= m_1 P_{11} \oplus m_2 P_{21} \oplus m_3 P_{31} \oplus \ldots \oplus m_k P_{kl} \\ c_2 &= m_1 P_{12} \oplus m_2 P_{22} \oplus m_3 P_{32} \oplus \ldots \oplus m_k P_{k2} \\ c_3 &= m_1 P_{13} \oplus m_2 P_{23} \oplus m_3 P_{33} \oplus \ldots \oplus m_k P_{k3} \\ &\ldots\text{and so on} \end{aligned}\right\} \quad \ldots(16.14a)$$

Here, it may be noted that all the additions are mod-2 additions.

* Here 'sum' is performed according to mod-2 addition rules.
i.e., $1 \oplus 1 = 0$, $1 \oplus 0 = 1$; $0 \oplus 1 = 1$ and $0 \oplus 0 = 0$.

Scanned by CamScanner

Example 16.1. The generator matrix for a (6, 3) block code is shown below. Obtain all code of words of this code.

$$G = \begin{bmatrix} 1 & 0 & 0 & : & 0 & 1 & 1 \\ 0 & 1 & 0 & : & 1 & 0 & 1 \\ 0 & 0 & 1 & : & 1 & 1 & 0 \end{bmatrix}$$

Solution: We know that generation matrix is given by

$$G = \left[I_k : P_{k \times q}\right]_{k \times n} \quad ...(i)$$

Given generator matrix is

$$G = \begin{bmatrix} 1 & 0 & 0 & : & 0 & 1 & 1 \\ 0 & 1 & 0 & : & 1 & 0 & 1 \\ 0 & 0 & 1 & : & 1 & 1 & 0 \end{bmatrix} \quad ...(ii)$$

Comparing equation (*i*) with equation (*ii*), we get

$$I_k = I_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$P_{k \times q} = P_{3 \times 3} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Here $k = 3$, $q = 3$ and $n = 6$.

This means that the block size of the message vector is 3 bits. Thus, there will be total 8 possible message vectors as shown in the Table 16.5.

Table 16.5.

Sr. No.	*Bits of message vector in one block*		
	m_1	m_2	m_3
1	0	0	0
2	0	0	1
3	0	1	0
4	0	1	1
5	1	0	0
6	1	0	1
7	1	1	0
8	1	1	1

The P submatrix is given in the example which is reproduced here i.e.

$$P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

For the check bit vector, there would be three bits. These may be obtained according to equation (16.13) as under:

$$C = MP$$

Scanned by CamScanner

$$[c_1 c_2 c_3] = [m_1\ m_2\ m_3]\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Thus, from the above matrix multiplication, we get

$$c_1 = (0 \times m_1) \oplus (m_2) \oplus (m_3)$$
$$c_2 = (m_1) \oplus (0 \times m_2) \oplus (m_3)$$

or

$$c_3 = (m_1) \oplus (m_2) \oplus (0 \times m_3)$$

From the last three equations, we get

$$\left.\begin{aligned} c_1 &= m_2 \oplus m_3 \\ c_2 &= m_1 \oplus m_3 \\ c_3 &= m_1 \oplus m_2 \end{aligned}\right\} \quad \ldots(i)$$

There three equations give check bits for each block of m_1, m_2, m_3 message bits.

For example, for first block of $(m_1, m_2, m_3) = 000$, we have

$$c_1 = 0 \oplus 0 = 0$$
$$c_2 = 0 \oplus 0 = 0$$
$$c_3 = 0 \oplus 0 = 0$$

i.e., $(c_1, c_2, c_3) = 0\ 0\ 0$

For second block of $(m_1, m_2, m_3) = 001$, we have

$$c_1 = 0 \oplus 1 = 1$$
$$c_2 = 0 \oplus 1 = 1$$
$$c_3 = 0 \oplus 0 = 0$$

i.e., $(c_1, c_2, c_3) = 1\ 1\ 0$

The Table 16.6 shows all the message bits, their check bits and code vectors calculated as above.

Table 16.6. Code Vectors of (6, 3) Block Code.

Sr. No.	*Bits of message vector in one block*			*Check bits*			*Complete code vector or Complete codeword*					
	m_1	m_2	m_3	$c_1 = m_2 \oplus m_3$	$c_2 = m_1 \oplus m_3$	$c_3 = m_1 \oplus m_2$	m_1	m_2	m_3	C_1	C_2	C_3
1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	1	1	1	0	0	0	1	1	1	0
3	0	1	0	1	0	1	0	1	0	1	0	1
4	0	1	1	0	1	1	0	1	1	0	1	1
5	1	0	0	0	1	1	1	0	0	0	1	1
6	1	0	1	1	0	1	1	0	1	1	0	1
7	1	1	0	1	1	0	1	1	0	1	1	0
8	1	1	1	0	0	0	1	1	1	0	0	0

16.5.2. Concept of Parity Check Matrix (*H*) for Linear Block Code

As a matter of fact, for each block code, there is a $q \times n$ parity check matrix (*H*). This is described as under:

Scanned by CamScanner

$$H = \left[P^T : I_q\right]_{q \times n} \quad \text{...(16.15)}$$

Here, P^T is the transpose of P submatix. The P submatrix is defined as

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1q} \\ P_{21} & P_{22} & P_{23} & \cdots & P_{2q} \\ P_{31} & P_{32} & P_{33} & \cdots & P_{3q} \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ P_{k1} & P_{k2} & P_{k3} & \cdots & P_{kq} \end{bmatrix}_{k \times q} \quad \text{...(16.16)}$$

The transpose of this submatrix will become by changing rows to columns *i.e.*,

$$P^T = \begin{bmatrix} P_{11} & P_{21} & P_{31} & \cdots & P_{k1} \\ P_{12} & P_{22} & P_{32} & \cdots & P_{k2} \\ P_{13} & P_{23} & P_{33} & \cdots & P_{k3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{1q} & P_{2q} & P_{3q} & \cdots & P_{kq} \end{bmatrix}_{q \times K} \quad \text{...(16.17)}$$

With the above equation, we can write equation (16.15) as under:

$$H_{q \times n} = \begin{bmatrix} P_{11} & P_{21} & P_{31} & \cdots & P_{k1} & : & 1 & 0 & 0 & .. & 0 \\ P_{12} & P_{22} & P_{32} & \cdots & P_{k2} & : & 0 & 1 & 0 & .. & 0 \\ P_{13} & P_{23} & P_{33} & \cdots & P_{k3} & : & 0 & 0 & 1 & .. & 0 \\ \vdots & \vdots & \vdots & & \vdots & : & \vdots & \vdots & \vdots & .. & \vdots \\ \vdots & \vdots & \vdots & & \vdots & : & \vdots & \vdots & \vdots & .. & \vdots \\ P_{1q} & P_{2q} & P_{3q} & \cdots & P_{kq} & : & 0 & 0 & 0 & .. & 1 \end{bmatrix}_{q \times n} \quad \text{...(16.18)}$$

Note: *It may be noted that if generator matrix (G) is given, then parity check matrix (H) can be obtained and vice-versa.*

16.6. Hamming Codes

Hamming codes are defined as the (n, k) linear block codes.

These codes satisfy the following conditions:

(*i*) Number of check bits $q \geq 3$

(*ii*) Block length $n = 2^q - 1$

(*iii*) Number of message bits $k = n - q$

(*iv*) Minimum distance $d_{\min} = 3$

We know that the code rate is expressed as,

$$r = \frac{k}{n} = \frac{n-q}{n} \quad \text{(for Hamming code } k = n - q\text{)}$$

or

$$r = 1 - \frac{q}{n} \quad \text{...(16.19)}$$

Scanned by CamScanner

Now, substituting the value of $n = 2^q - 1$ in last equation, we get

$$r = 1 - \frac{q}{2^q - 1} \qquad ...(16.20)$$

From this equation, we may observe that $r \cong 1$ if $q >> 1$.

16.6.1. Error Detection and Correction Capabilities of Hamming Codes

Because the minimum distance (d_{min}) of Hamming code is 3, it can be used to detect double errors or correct single errors. This can also be obtained form the generalized Table 16.2.

For detecting double (2) errors $\Rightarrow d_{min} \geq 2 + 1$ *i.e.* $d_{min} \geq 3$

and for correcting upto one (1) errors $\Rightarrow d_{min} \geq 2(1) + 1$ *i.e.* $d_{min} > 3$

Example 16.2. The parity check matrix of a particular (7, 4) linear blok code is expressed as

$$[H] = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad ...(i)$$

(*i*) Obtain the generator matrix (*G*)

(*ii*) List all the code vectors

(*iii*) What will the minimum distance between code vector?

(*iv*) How many errors can be detected? How many errors can be corrected?

Solution: Here, given that $n = 7$ and $k = 4$

Number of check bits are $n - k = 7 - 4$ *i.e.* $q = 3$

Hence, $n = 2^q - 1 = 2^3 - 1 = 7$

This indicates that the given code is a Hamming code. The parity check matrix is of $q \times n$ size and is given by equation (16.18). It can be written as (with $q = 3$ and $n = 7$ and $k = 4$)

$$[H]_{3 \times 7} = \begin{bmatrix} P_{11} & P_{21} & P_{31} & P_{41} & \dots & 1 & 0 & 0 \\ P_{12} & P_{22} & P_{32} & P_{42} & \dots & 0 & 1 & 0 \\ P_{13} & P_{23} & P_{33} & P_{43} & \dots & 0 & 0 & 1 \end{bmatrix} \qquad ...(ii)$$

or

$$[H]_{3 \times 7} = \left[P^T : I_3\right] \qquad ...(iii)$$

On comparing parity check matrices of equations (*ii*) and (*iii*), we have

$$P^T = \begin{bmatrix} P_{11} & P_{21} & P_{31} & P_{41} \\ P_{12} & P_{22} & P_{32} & P_{42} \\ P_{13} & P_{23} & P_{33} & P_{43} \end{bmatrix} \qquad ...(iv)$$

Now, according to equations (*i*) and (*iv*), we have

$$P^T = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Hence, the *P* submatrix may be obtained as,

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \\ P_{41} & P_{42} & P_{43} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}_{4 \times 3} \qquad ...(v)$$

Scanned by CamScanner

Further, we know that generator matrix G is expressed as under:

$$G = \left[I_k : P_{k\times q}\right]_{k\times n}$$

with $k = 4$, $q = 3$ and $n = 7$, the above equation becomes,

$$G = \left[I_4 : P_{4\times 3}\right]_{4\times 7}$$

Substituting the identity matrix I_4 of size 4×4 and parity submatrix $P_{4\times 3}$ of size as obtained in equation (v), in above equation, we get

$$G = \left[\begin{array}{cccc:ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ \underbrace{0\ 0\ 0\ 1}_{I_{4\times 4}} & & & & \underbrace{0\ 1\ 1}_{P_{4\times 3}} & & \end{array}\right]_{4\times 7} \quad ...(vi)$$

This is the required generator matrix.

To find all the Codewords

The check bits can be obtained using equation (16.13), *i.e.*,

$$c = MP$$

In the more general form we can use equation (16.14) *i.e.*, (with $q = 3$, and $k = 4$), we have

$$[c_1\ c_2\ c_3]_{1\times 3} = [m_1\ m_2\ m_3\ m_4]_{1\times 4}\,[P]_{4\times 3}$$

$$[c_1\ c_2\ c_3] = [m_1\ m_2\ m_3\ m_4]\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}_{4\times 3}$$

Solving the above equation with mod-2 addition, we obtain,

$$c_1 = (1 \times m_1) \oplus (1 \times m_2) \oplus (1 \times m_3) \oplus (0 \times m_4)$$

$$c_2 = (1 \times m_1) \oplus (1 \times m_2) \oplus (0 \times m_3) \oplus (1 \times m_4)$$

and

$$c_3 = (1 \times m_1) \oplus (0 \times m_2) \oplus (1 \times m_3) \oplus (1 \oplus m_4)$$

Thus the above equation may be written as,

$$\left.\begin{aligned} c_1 &= m_1 \oplus m_2 \oplus m_3 \\ c_2 &= m_1 \oplus m_2 \oplus m_4 \\ \text{and}\quad c_3 &= m_1 \oplus m_3 \oplus m_4 \end{aligned}\right\} \quad ...(vii)$$

For example, if $(m_1\ m_2\ m_3\ m_4) = 1\ 0\ 1\ 1$, we obtain,

$$c_1 = 1 \oplus 0 \oplus 1 = 0$$

$$c_2 = 1 \oplus 0 \oplus 1 = 0$$

and

$$c_3 = 1 \oplus 1 \oplus 1 = 1$$

Hence, for message vector of (1 0 1 1) the check bits are $(c_1\ c_2\ c_3) = 001$. These, the systematic block code of the code vector (code word) may be written as,

$(m_1\ m_2\ m_3\ m_4\ c_1\ c_2\ c_3) = (1\ 0\ 1\ 1 : 0\ 0\ 1)$

Using the same procedure as given, we may obtain the other code sords or code vectors. Table 16.6 lists all the code vectors (code words). This table also lists the weight of each code word.

Scanned by CamScanner

Table 16.7.

Sr. No.	Message vector M				Check bits by eq. (v) c			Code vector orcode word X							Weight of code vector
	m_1	m_2	m_3	m_4	c_1	c_2	c_3	m_1	m_2	m_3	m_4	c_1	c_2	c_3	$w(X)$
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	1	0	1	1	0	0	0	1	0	1	1	3
3	0	0	1	0	1	0	1	0	0	1	0	1	0	1	3
4	0	0	1	1	1	1	0	0	0	1	1	1	1	0	4
5	0	1	0	0	1	1	0	0	1	0	0	1	1	0	3
6	0	1	0	1	1	0	1	0	1	0	1	1	0	1	4
7	0	1	1	0	0	1	1	0	1	1	0	0	1	1	4
8	0	1	1	1	0	0	0	0	1	1	1	0	0	0	3
9	1	0	0	0	1	1	1	1	0	0	0	1	1	1	4
10	1	0	0	1	1	0	0	1	0	0	1	1	0	0	3
11	1	0	1	0	0	1	0	1	0	1	0	0	1	0	3
12	1	0	1	1	0	0	1	1	0	1	1	0	0	1	4
13	1	1	0	0	0	0	1	1	1	0	0	0	0	1	3
14	1	1	0	1	0	1	0	1	1	0	1	0	1	0	4
15	1	1	1	0	1	0	0	1	1	1	0	1	0	0	4
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	7

The Table 16.7 lists $2^k = 2^4 = 16$ code vectors along with their weights. The smallest weight of any non-zero code vector is 3. We know that the minimum distance is $d_{min} = 3$.

Hence we can write

The minimum distance of a linear block code is equal to the minimum weigth of any non-zero code vector. This means that

$$d_{\min} = [w(X)]_{\min} \; ; X = (0\ 0 \ldots 0)$$

(*iii*) Minimum distance between code words is

$$d_{\min} = 3$$

(*iv*) Now, since

$$d_{\min} = 3.$$

We know that

$$d_{\min} \geq s + 1$$

or $$3 \geq s + 1$$

or $$s \leq 2$$

Hence, two errors will be detected.

Also we have

$$d_{\min} \geq 2t + 1$$

or $$3 \geq 2t + 1$$

or $$t \leq 1$$

Therefore one error will be corrected.

Scanned by CamScanner

Hence, with the help of Hamming code ($d_{min} = 3$), two errors can be detected and single error can be corrected by its property.

16.7. Encoder of (7, 4) Hamming Code

Figure 16.5 shows the encoder of (7, 4) Hamming code. This encoder is implemented for generator matrix of the example 16.2. The lower register contains check bits c_1, c_2, and c_3. These bits are obtained from the message bits by mod-2 additions. These additions are performed according to equation (*vii*). The mod-2 addition operation is nothing but exclusive-OR operation.

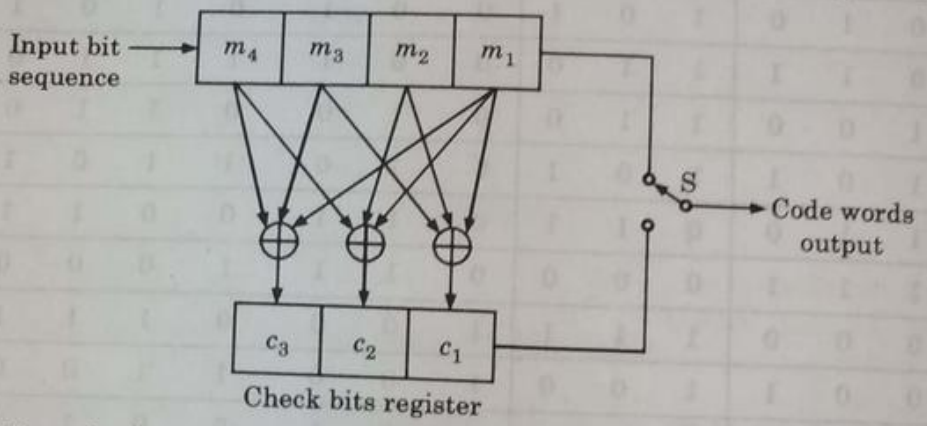


Fig. 16.5. *Encoder for (7, 4) Hamming code or (7, 4) linear block code for example 16.2.*

The switch '*S*' is connected to message register first and all message bits are transmitted. This switch is then connected to the check bit register and check bits are transmitted. This forms a block of '7' bits. After this input bits are taken for next block.

16.8. Syndrome Decoding: Method to Corect Errors

In this article, we shall discuss the method to correct errors in linear block coding. Let the transmitted code vector be '*X*' and corresponding received code vector be represented by '*Y*'.

Then, we can write,

$X = Y$ if there are no transmission errors

and $X \neq Y$ if there are errors produced during transmission

The decoder detects or corrects these errors in *Y* by using the stored bit pattern in the decoder about the code. For larger block lengths, more and more bits are needed to be stored in the decoder. This increases the memory requirement and adds to the complexity and cost of the system. Therefore, to avoid these problems, synchrome decoding is used in linear block codes. This method is shown in the paragraphs to follow:

(*i*) We know that with every (n, k) linear block code, there exists a parity check matrix (H) of size $q \times n$. It is defined as,

$$H = [P^T : I_q]_{q \times n}$$

The transpose of the above matrix may be obtained by interchanging the rows and the columns. Thus, we have

$$H^T = \begin{bmatrix} P \\ \cdots \\ I_q \end{bmatrix}_{n \times q}$$

Here P is the submatrix of size $k \times q$ and I_q is the identity matrix of size $q \times q$. We have defined P submatrix earlier.

Scanned by CamScanner

(*ii*) The transpose of parity check matrix (H^T) has very important property as under:

$$XH^T = (0\ 0\ 0\ \ldots 0) \qquad \ldots(16.21)$$

or

$$[H]_{1\times n}\,[H^T]_{n\times q} = (0\ 0\ 0\ \ldots 0)_{1\times q} \qquad \ldots(16.22)$$

This is true for all code vectors.

(*iii*) As an example, let us consider the parity check matrix and code vectors obtained in example 16.2. The parity check matrix is given by equation (*i*). The transpose of this matrix can be obtained as under:

$$H^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{7\times 3\ (n=7 \text{ and } q=3)} \qquad \ldots(16.23)$$

(*iv*) Table 16.7 lists all the code vectors for this parity check matrix. Let us consider the third code vector in this table *i.e.*,

$$X = (0\ 0\ 1\ 0\ 1\ 0\ 1) \qquad \ldots(16.24)$$

Now, let us apply the property of equation (16.21) *i.e.*,

$$XH^T = [0010101]_{1\times 7} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{7\times 3} \qquad \ldots(16.25)$$

(*v*) On solving the above two matrices with the rules of mod-2 addition (Exclusive - OR operation), we obtain,

$$XH^T = (0 \oplus 0 \oplus 1 \oplus 0 \oplus 1 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 1 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 1)$$
$$= (0\ \ 0\ \ 0)$$

This proves the property.

(*vi*) It can also be proved for other code vectors also. Hence, X belongs to the valid code vector at the transmitter end. At the receiver end the received code is Y.

Then, we can write,

$YH^T = (0\ 0\ \ldots 0)$, if $X = Y$ *i.e.*, no errors or Y is valid code vector

and YH^T = Non-zero, if $X \neq Y$ *i.e.*, some errors.

(*vii*) Also, when some errors are present in received vector Y, then it will not be from valid code vectors and it will not satisfy the property of equation (16.21). This shows that whenever YH^T is non-zero, some errors are present in Y. The non-zero output of the product YH^T is called *syndrome* and it is used to detect the errors in Y. Syndrome is represented by 'S' and may be written as,

$$S = YH^T \qquad \ldots(16.26)$$

or

$$[S]_{1\times q} = [Y]_{1\times n}\,[H^T]_{n\times q} \qquad \ldots(16.27)$$

(*viii*) The non-zero elements of 'S' represent error in the output. When all elements of 'S' are zero, then the two cases possible:

Scanned by CamScanner

(*a*) No error in the output and $Y = X$

(*b*) Y is some other valid code vector other than X. This means that the transmission errors are undetectable.

(*ix*) Now, let us consider an n-bit error vector E. Let, this vector represent the position of transmission errors in Y. As an example, consider,

$$X = (1\ 0\ \underset{\uparrow}{1}\ 1\ \underset{\uparrow}{0}) \quad \text{be a transmitted vector}$$

and

$$Y = (1\ 0\ \underset{\uparrow}{0}\ 1\ \underset{\uparrow}{1}) \quad \text{be a received vector}$$

Then

$$E = (0\ 0\ 1\ 0\ 1) \quad \text{represents the error vector}$$

The non-zero entries represent errors in Y.

(*x*) Using the mod-2 addition rules, we can write:

$$Y = X \oplus E$$

or

$$Y = (1 \oplus 0 \quad 0 \oplus 0 \quad 1 \oplus 1 \quad 1 \oplus 0 \quad 0 \oplus 1) \qquad \text{...(16.28)}$$

Bit by bit mod-2 addition

or

$$Y = (1\ 0\ 0\ 1\ 1)$$

or we can write

$$X = Y \oplus E$$

$$= (1 \oplus 0\ 0\ 0 \oplus 0\ 0 \oplus 1\ 1 \oplus 0\ 1 \oplus 1)$$

or

$$X = (1\ 0\ 1\ 1\ 0)$$

or we can write,

$$X = Y \oplus E$$

$$= (\oplus 0 \quad 0 \oplus 0 \quad 0 \oplus 1 \quad 1 \oplus 0 \quad 1 \oplus 1)$$

or X

$$= (1\ 0\ 1\ 1\ 0)$$

From equation (16.26), we know that syndrome vector is expressed as,

$$S = YH^T$$

Substituting the value of $Y = X \oplus E$ from equation (16.28) in this equation, we get

$$S = (X \oplus E)\, H^T$$

or

$$S = XH^T \oplus EH^T$$

From the property of equation (16.21) we know that $XH^T = 0$, then above equation will be

$$S = EH^T \qquad \text{...(16.29)}$$

(*xi*) This relation indicates that syndrome depends on the error pattern only. It does not depend upon a particular message. Syndrome vector 'S' is of size $1 \times q$. Hence, q bits of syndrome can only represent 2^q syndrome vectors. Each syndrome vector corresponds to a particular error pattern.

Example 16.3. The parity check matrix of a (7, 4) Hamming code is expressed as under:

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & : & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & : & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & : & 0 & 0 & 1 \end{bmatrix}_{3 \times 7}$$

Evaluate the syndrome vector for single bit errors.

Solution: This is a (7, 4) linear block code.

This means $n = 7$ and $k = 4$

$$q = n - k = 3$$

We know that syndorme vector is a q bit vector. For this example, syndrome will be a 3 bit vector. Hence, there will be $2^3 - 1 = 7$ non-zero syndromes. This shows that '7' single bit error patterns will be represented by these '7' non-zero syndromes. Error vector E is a n bit vector representing error pattern. For this example, E is '7' bit vector. Following Table 16.8 shows the single error patterns in a 7 bit error vector (Note that only single bit error patterns have been shown).

Scanned by CamScanner

Table 16.8. Single error pattern of 7 bit error vector

Sr.No.	Bit in error	Bits of Error vector (E), Non-zeros bit shows error						
1	1st	1	0	0	0	0	0	0
2	2nd	0	1	0	0	0	0	0
3	3rd	0	0	1	0	0	0	0
4	4th	0	0	0	1	0	0	0
5	5th	0	0	0	0	1	0	0
6	6th	0	0	0	0	0	1	0
7	7th	0	0	0	0	0	0	1

Calculation of Syndromes

From equation (16.29) the syndrome vector is given as,

$$S = EH^T$$

S is q bit, E is n bit and H^T is $n \times q$ bits size. For this example $n = 7$, $q = 3$, we can write above equation as,

$$[S]_{1\times3} = [E]_{1\times7}[H]_{7\times3} \quad ...(i)$$

Form the given parity check matrix H, we can obtain its transpose (H^T) by interchanging rows to column, *i.e.*,

$$H^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ...(ii)$$

Let us calculate syndrome for first bit error vector *i.e.*,

$$S = EH^T = [1\,0\,0\,0\,0\,0\,0]\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= (1 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \quad 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \quad 1 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0)$$

$$= [1 \quad 0 \quad 1]$$

This is the syndrome vector for first bit in error.

Let us evaluate syndrome for 2nd bit in error. In Table 16.7, the error vector for error in 2nd bit is given. We may write,

Scanned by CamScanner

$$S = EH^T = [0\,1\,0\,0\,0\,0\,0]\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or $$S = (0 \oplus 1 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \quad 0 \oplus 1 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \quad 0 \oplus 1 \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus 0)$$

or $$S = [1 \quad 1 \quad 1]$$

Hence, this is the syndrome vector for second bit in error. The Table 16.9 lists the error vector with single bit error and corresponding syndromes. Other syndromes can be calculated using the same procedure as above. The table also lists the syndrome for no error vector *i.e.*, E = (0 0 0 0 0 0 0). It may be observed that the corresponding syndrome is S = (0 0 0).

Table 16.9. Syndromes for (7, 4) Hamming code of single bit error

Sr. No.	*Error vector 'E' showing single bit error patterns*							*Syndrome Vector 'S'*			
1	0	0	0	0	0	0	0	0	0	0	
2	1	0	0	0	0	0	0	1	0	1	← 1st row of HT
3	0	1	0	0	0	0	0	1	1	1	← 2nd row of H^T
4	0	0	1	0	0	0	0	1	1	0	← 3rd row of H^T
5	0	0	0	1	0	0	0	0	1	1	← 4th row of H^T
6	0	0	0	0	1	0	0	1	0	0	← 5th row of H^T
7	0	0	0	0	0	1	0	0	1	0	← 6th row of H^T
8	0	0	0	0	0	0	1	0	0	1	← 7th row of H^T

This Table (16.8) shows that error in the first bit corresponds to a syndrome vector of S = (101). This syndrome vector is same as the first row of H^T. The syndrome vector (S = 111) for error in second bit is same as second row of H^T. This is same for remaining syndromes.

16.9. Error Correction Using Syndrome Vector

(*i*) Let us observe how single bit errors can be corrected using syndrome decoding. We will observe this for (7, 4) block code. Let the transmitted code vector be,

$$X = (1\,0\,0\,1\,1\;1\,0)$$

(*ii*) Let there be error created in the 3rd bit in the received code vector Y. Then Y will be

$$Y = (1\,0\,\textcircled{1}\,1\,1\,1\,0) \quad \text{encircled bit shows that it is in error.}$$

(*iii*) Let us use the parity check matrix and syndrome vectors of, example 16.2 for this illustration.

The receiver calculates $S = YH^T$ *i.e.*,

Scanned by CamScanner

$$S = YH^T = \underbrace{[1\,0\,1\,1\,1\,1\,0]}_{X} \underbrace{\begin{bmatrix} 1\,0\,1 \\ 1\,1\,1 \\ 1\,1\,0 \\ 0\,1\,1 \\ 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{bmatrix}}_{H^T} \quad \text{(from equation (16.26))}$$

or $S = (1 \oplus 0 \oplus 1 \oplus 0 \oplus 1 \oplus 0 \oplus 0 \quad 0 \oplus 0 \oplus 1 \oplus 1 \oplus 0 \oplus 1 \oplus 0 \quad 1 \oplus 0 \oplus 0 \oplus 1 \oplus 0 \oplus 0 \oplus 0)$

or $S = [1 \;\; 1 \;\; 0]$

(*iv*) Now, from equations (16.26) and (16.29), we can write

$$S = YH^T = EH^T$$

Here $S = YH^T = EH^T = (1\,1\,0)$.

(*v*) On comparing this syndrome with H^T, we observe that ($S = 1\,1\,0$) is the 3rd row of H^T. From Table 16.8, we can get the error pattern corresonding to this syndrome as under:

$$E = (0\,0\,1\,0\,0\,0\,0)$$

(*vi*) This shows that there is an error in the third bit of Y. We have also verifed that if syndrome vector is equal to 3rd row of H^T, then third bit of Y is in error. The correct vector can be obtained from equation (16.28) as under:

$$X = Y \oplus E$$

or $$X = [1\,0\,1\,1\,1\,1\,0) \oplus [0\,0\,1\,0\,0\,0\,0]$$

$$= (1\,0\,0\,1\,1\,1\,0)$$

which is same as transmitted code vector.

Thus, a single bit errors can be corrected using syndrome decoding.

16.9.1. What Happens if Double Error Occurs in Y

Let us observe the case of double error in Y. Let us consider the same message vector X, *i.e.*,

$$X = 1\,0\,0\,1\,1\,1\,0$$

Let us consider that the error is present in 3rd and 4th bits. Then, Y will be

$$Y = 1\,0\,1\,0\,1\,1\,0$$

encircled bits are in error.

Then $S = YH^T$ gives,

$$S = YH^T = [1\,0\,1\,0\,1\,1\,0] \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, we have $S = 1\,0\,1$

Scanned by CamScanner

From Table 16.8, we observe that the syndrome $S = 101$ corresponds to an error pattern of $E = 1000000$. This shows that there is an error in the first bit. Thus, the error detection and correction goes wrong. The probability of occurrence of multiple errors is less compared to single error. To correct multiple errors extended Hamming codes are used. In these codes, one more extra bit is provided to correct double errors.

We know that for (n, k) block code, there are $2^q - 1$ distinct non-zero syndromes. There are ${}^nC_1 = n$ single error patterns, ${}^nC_2$ double error patterns 8.7.2, ${}^nC_3$ tripple error patterns and so on. Therefore, to correct 't' errors per word, the following relation should be satisfied.

$$2^q - 1 \geq {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_t \quad \text{...(16.30)}$$

16.9.2. Hamming Bound

The last equation (*i.e.*, equation (16.30)) can be written as

$$2^q \geq 1 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_t$$

or

$$2^q \geq \sum_{i=0}^{t} {}^nC_i$$

We know that $q = n - k$. Therefore, the above equation becomes,

$$2^{n-k} \geq \sum_{i=0}^{t} {}^nC_i$$

By taking logarithm to base 2 on both sides, we get

$$n - k \geq \log_2 \sum_{i=0}^{t} {}^nC_i$$

Dividing both sides by n, we get

$$1 - \frac{k}{n} \geq \frac{1}{n} \log_2 \sum_{i=0}^{t} {}^nC_i$$

Since code rate $r = \frac{k}{n}$, the above equation would be,

$$1 - r \geq \frac{1}{n} \log_2 \sum_{i=0}^{t} {}^nC_i \quad \text{...(16.31)}$$

As indicated, this equation relates code rate 'r' with the error correction capability of 't' errors per code vector in a block of 'n' bits. We know that the error correcting capability of the code (i.e., 't' errors per code vector) is related to the minimum distance. This minimum distance is also called ***Hamming distance****. Equation (16.31) gives the relation between code rate, number of errors to be corrected and number of bits in a block. This equation is also known as* ***Hammng bound.***

16.10. Syndrome Decoder for (n, k) Block Code

Figure 16.6 illustrates the block diagram of a syndrome decoder for linear block code to correct errors. The received n-bit vector 'Y' is stored in an n-bit register. From this vector, a syndrome is calculated using,

$$S = YH^T \quad \text{...(16.32)}$$

Hence, H^T is stored in the syndrome calculator. The q-bit syndrome vector is then applied to a look-up table of error patterns. Depending upon the particular syndrome, an error pattern is selected. This error pattern's added (mod-2 addition) to the vector Y. Hence, the output will be

$$Y \oplus E = X \quad \text{...(16.33)}$$

Scanned by CamScanner

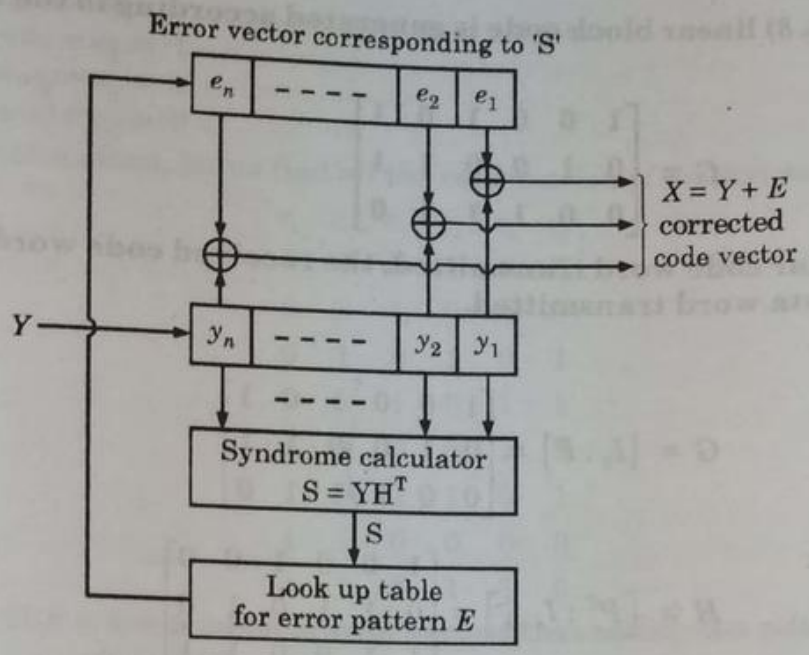


Fig. 16.6. *Syndrome decoder for linear block code*

The block diagram shown in figure 16.6 can correct only single errros in the above vectors.

Example 16.4. For a (6, 3) code, the parity check matrix is given by

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Determine whether a received code vector is erroneous. Received code vector is 100101.

Solution: We have

$$H^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The received code vector is $Y = 100101$.

Then,
$$S = YH^T = [100101]\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or
$$S = [0\,0\,0]$$

Because, syndrome vector is zero, there are no errors in the received code vector.

Scanned by CamScanner

Example 16.5. A(6, 3) linear block code is generated according to the following generator matrix:

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

For a particular code word transmitted, the received code word is 100011. Find the corresponding data word transmitted.

Solution: We have

Given
$$G = [I_k : P] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$H = \left[P^T : I_{n-k}\right] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Hence
$$H^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Here $n = 6, k = 3, n - k = 6 - 3 = 3$

$Y = [1\,0\,0\,0\,1\,1]$

We have

$$S = Y \cdot H^T = [1\,0\,0\,0\,1\,1] \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$S = [1\,1\,0]$

Also, $S = EH^T$

where $E = [e_1, e_2, e_3, e_4, e_5, e_6]$

i.e.,
$$S = [e_1, e_2, e_3, e_4, e_5, e_6] \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [1, 1, 0]$$

Scanned by CamScanner

On solving, we get

$$\therefore \quad e_1 + e_3 + e_4 = 1$$
$$e_2 + e_3 + e_5 = 1$$
$$e_1 + e_2 + e_6 = 0$$

By trial and error method, let us find all the combination of e's that satisfy these relations.

e_1	e_2	e_3	e_4	e_5	e_6
0	0	0	1	1	0
0	0	1	0	0	0
0	1	0	1	0	1
0	1	1	0	1	1
1	0	0	0	1	1
1	0	1	1	0	1
1	1	0	0	0	0
1	1	1	1	1	0

Thus there are $2^k (k = 3) = 8$ possible error vectors that satisfy the relation

$$S = EH^T$$

Among all the available error vectors, the vectors with minimum weight is selected (*i.e.*, the one having minimum number of 1's)

$$E = 0\,0\,1\,0\,0\,0$$
$$Y = 1\,0\,0\,0\,1\,1$$
$$X = Y + E = 1\ 0\ 0\ 0\ 1\ 1$$
$$0\ 0\ 1\ 0\ 0\ 0$$
$$\underbrace{1\ 0\ 1}_{M}\ \underbrace{0\ 1\ 1}_{C}$$

Data word transmitted is 101 or simply the row in H^T corresponding to $S = (110)$ is the 3rd row. Hence, the error is in the 3rd bit of the received vector.

Therefore, received codeword is 101011.

The first there bit are the transmitted data bits *i.e.*, 101.

Example 16.6. For a (7, 4) block code generated by [G] below, explain how the error syndrome helps in correcting a single error.

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Solution: We have

$$H = [P^T : I] = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ \underbrace{0\ 1\ 1\ 1}_{P_T} & & & & \underbrace{0\ 0\ 1}_{I} & & \end{bmatrix}$$

Let message be 1 1 0 1

$$X = MG = [\,1\,1\,0\,1\,0\,1\,0\,]$$

For this X, $S = XH^T = [0\ 0\ 0]$

Scanned by CamScanner

Let $Y = 1\,1\,1\,1\,0\,0\,1\,0$ (3rd bit in error)

$$Y = X \oplus E = [1\,1\,0\,1\,0\,1\,0] + [0\,0\,1\,0\,0\,0\,0]$$

$$S = YH^T$$

$$= [1111\ 010]\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [1\ 0\ 1] = EH^T$$

i.e., S for an error in the third bit = third row of H^T.

Also, for this code a single error in the ith bit of X leads to a syndrome vector *i.e.*, = i^{th} row of H^T.

Thus single error can be corrected at the receiver by comparing S with rows of H^T and correcting the ith received bit if $S = i^{th}$ row of H^T. When the syndrome vector = 0, it means no error has occured.

Example 16.7. *A* (7, 4) linear block code is generated according to the following parity check matrix *H*:

$$[H] = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The received codeword *Y* is 1000011 for a transmitted code word *X*. Find the corresponding data transmitted.

Solution: We have

$$H^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Y = [1\,0\,0\,0\,0\,1\,1]$$

$$S = YH^T = [1\,0\,0\,0\,0\,1\,1]\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [1\,0\,0]$$

fifth row

Scanned by CamScanner

$$S = EH^T = [e_1\ e_2\ e_3\ e_4\ e_5\ e_6\ e_7]\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = [100]$$

On solving, we get

$$e_1 + e_2 + e_3 + e_5 = 1$$
$$e_1 + e_2 + e_4 + e_6 = 0$$
$$e_1 + e_3 + e_4 + e_7 = 0$$

There will be $2^4 = 16$ possible error vectors that satisfy the above equations.

Among them all, the error vector with minimum of weight is 0 0 0 0 1 0 0 1 in the fifth column (or the fifth row in H^T indicates that this is the error vector.

Therefore, transmitted codeword is

$$X = Y \oplus E = \begin{array}{r} 1000011 \\ \underline{0000100} \\ = 1000111 \end{array}$$

Hence, data word will be $M = 1000$

Example 16.8. Given a (6, 3) linear block code with the following parity-check matrix *H*:

$$H = \begin{bmatrix}1 & 0 & 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0 & 1 & 0\\ 1 & 1 & 1 & 0 & 0 & 1\end{bmatrix}$$

(*i*) Find the generator matrix *G*.

(*ii*) Find the code word for the data bit 101.

Solution: Given Parity-check matrix is

$$H = \begin{bmatrix}1 & 0 & 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0 & 1 & 0\\ \underbrace{1\ \ 1\ \ 1}_{P} & & & \underbrace{0\ \ 0\ \ 1}_{I_k} & & \end{bmatrix}$$

(*i*) Since, H is a 6×3 matrix, $n = 6$, $k = 3$, then,

The transpose of the given parity matrix is given by

$$P^T = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$

Then, the generator matrix will be

$$G = \left[I_k\ P^T\right] = \begin{bmatrix}1 & 0 & 0 & 1 & 0 & 1\\ 0 & 1 & 0 & 0 & 1 & 1\\ 0 & 0 & 1 & 1 & 1 & 1\end{bmatrix}$$

Scanned by CamScanner

(*ii*) Codeword can be obtained as under:

$$X = MG$$

or
$$X = [1\ 0\ 1]\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

or
$$X = [1\ 0\ 1\ 0\ 1\ 0]$$ **Ans.**

Example 16.9. For the (7, 4) Hamming code, the parity-check matrix H is given by

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(*i*) Construct the Generator matrix.

(*ii*) The code word that begins with 1010.

(*iii*) If the received codeword Y is 0111100, then decode this received codeword.

Solution: (*i*) Since given parity check matrix $[H]$ is 7×3 matrix, therefore, we have $n = 7, k = 3,$ $q = 4$

then
$$P^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

We can obtain the generator matrix as under:

$$G = [I \quad P^T] = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(*ii*) Codeword can be found by the following expression:

$$X = MG$$

or
$$X = [1\ 0\ 1\ 0]\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

or
$$X = [1\ 0\ 1\ 0\ 0\ 1\ 1]$$

(*iii*) Given
$$Y = [0\ 1\ 1\ 1\ 1\ 0\ 0]$$

Then
$$S = YH^T = [0\ 1\ 1\ 1\ 1\ 0\ 0]\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [1\ 0\ 1]$$

Scanned by CamScanner

16.11. Other Linear Block Code

In this article, let us discuss other linear block codes

16.11.1. Single Parity Check Bit Codes

If $m_1, m_2, m_3, \ldots, m_k$ are the bits of the k-bit message word; then,

$$m_1 \oplus m_2 \oplus m_3 \oplus .. \oplus m_k \oplus c_1 = 0$$

In this equation, c_1 is the parity check bit added to the message bit. The above equation reavels the fact that if there are even number of 1's in the message word, then parity check bit $c_1 = 0$. If there are odd number of 1's in the message word, then parity check bit $c_1 = 1$. Thus, for this code, we have

$$n = k + 1 \qquad ...(16.34)$$

and

$$q = 1$$

At this point, it may be noted that this code only detects single error but does not correct it.

16.11.2. Repeated Codes

In this code, only a single message bit is transmitted and $q = 2t$ bit are parity bits. Hence $k = 1$.

Then the transmitted bits are,

$$n = 2t + 1 \qquad ...(16.35)$$

This code is known as **repeated code** became many redundant check bits are transmitted along with a single message bit. This code can correct 't' errors per block because this code uses many redundant check bits, therefore it requires a larger bandwidth.

16.11.3. Hadamard Code

The hadamard code is derived form a hadamard matrix. The hadamard matrix is the $n \times n$ square matrix. Rows of this hadamard matrix represent code vectors. Thus, a $n \times n$ hadamard matrix, represents 'n' code vectors of 'n' bits each.

If the block of message vector contains 'k' bits, then we have

$$n = 2^k \qquad ...(16.36)$$

This equation establishes the relationship between number of bits in the code vector and number of bits in the message vector. We know that number of check bits (q) in (n, k) block code are

$$q = n - k$$

Therefore

$$q = 2^k - k \qquad ...(16.37)$$

This equation shows that as number of bits in the message block (k) increases, the parity bits become very large. This reduces the code rate. The code rate is expressed as,

$$r = \frac{k}{n} = \frac{k}{2^k} \qquad ...(16.38)$$

This shows that with increase in 'k', the code rate will becomes very small.

16.11.3.1. Hadmard Matrix and Codewords

There are some important points as under:

(*i*) One code vector represented by hadmard matrix consists of all zero elements. This means that one row of hadamard matrix consists of all zero elements.

(*ii*) The other code vectors contain $\frac{n}{2}$ 1's and $\frac{n}{2}$ 0's. This means that other rows of hadamard matrix contains half number of 1's and half number of 0's.

(*iii*) Every code vector differs from other code vectors at $\frac{n}{2}$ places. This means that every row of hadamard matrix differs with other rows at $\frac{n}{2}$ places (*i.e.* half number of places).

Scanned by CamScanner

Let us consider the hadamard matrix with single message bit *i.e.*, $k = 1$.
Hence,

$$n = 2^k = 2^1 = 2$$

Hence, hadamard matrix for single message bit will be of $n \times n$ (*i.e.*, 2×2) size. The first row will be all zero elements.

This matrix is as shown below.

$$H_2 = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\substack{\text{These element satisfy} \\ \text{the points (ii) and (iii)} \\ \text{mentioned above}}} \underbrace{\leftarrow \text{All zero row}} \qquad \text{...(16.39)}$$

Here, H_2 is 2×2 size hadamard matrix. It may be observed that the second row contains half number of elements as zero and half as 1's. We know that the code words are the row of hadamard matrix. Here, the code words are 00 and 01. Let us consider the hadamard matrix for two message bits (*i.e.*, $k = 2$). Then, we have,

$$n = 2^k = 2^2 = 4$$

Hence, the hadamard matrix will be of size 4×4. This matrix has been shown below:

$$H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & \overline{H_2} \end{bmatrix} \qquad \text{...(16.40)}$$

Here, H_2 is the matrix given in equation (16.39)

$\overline{H_2}$ is the complement of matrix H_2 and will be given by

$$\overline{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{...(16.41)}$$

Thus, in this matrix, every element of H_2 is complemented. Then, the matrix H_4 given by equation (16.40) will be as under:

$$H_4 = \begin{array}{c} \begin{array}{cc} H_2 & H_2 \end{array} \\ \begin{bmatrix} \boxed{\begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}} & \boxed{\begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}} \\ \boxed{\begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}} & \boxed{\begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}} \end{bmatrix} \\ \begin{array}{cc} H_2 & \overline{H}_2 \end{array} \end{array} \qquad \text{...(16.42)}$$

The above matrix is of size 4×4 and it gives four code words. These code vectors are (0000), (0101), (0011) and (0110). It may be observed that the above code vectors and hadamard matrix satisfies all the three points discussed earlier. Since, every code word differs by $\frac{n}{2}$ places with the other code words, these code words are orthogonal to each other over complete 'n' bits. Equation (16.40) may be generalized further to higher number of bits as under:

$$H_{2n} = \begin{bmatrix} H_n & H_n \\ H_n & \overline{H}_n \end{bmatrix} \qquad \text{...(16.43)}$$

where $\overline{H}_n$ is the complement matrix of H_n.

Scanned by CamScanner

Because, each code word in hadamard matrix differs every other code word by $\frac{n}{2}$, therefore, the minimum distance between the code words will be,

$$d_{\min} = \frac{n}{2} \quad \text{...(16.44)}$$

or

$$d_{\min} = \frac{2^k}{2} = 2^{k-1} \quad \text{...(16.45)}$$

We know that to correct upto t errors per word, we have

$$d_{\min} \geq 2t + 1$$

Substituting the value of $d_{\min} = 2^{k-1}$ in above equation, we have

$$2^{k-1} \geq 2t + 1$$

or

$$t \leq \frac{2^{k-1} - 1}{2} \quad \text{...(16.46)}$$

Drawback

Because, Hadamard code uses many check bits, its code rate is very low. Hence, to transmit the signal at higher rates, higher bandwidths are required.

16.11.4. Extended Codes

We know that every (n,k) linear block code has a parity check matrix H. One column of zero elements (except last element) and one row of 1's is added to the parity check matrix as depicted below:

$$H_e = \begin{bmatrix} \text{Parity check} & 0 \\ & 0 \\ & \vdots \\ \text{matrix H} & \vdots \\ 1\ 1\ 1 \cdots & 1 \end{bmatrix}_{(q+1)\times(n+1)} \quad \text{...(16.47)}$$

The code turned by such parity check matrix is known as **extended code**. Hence, the code will be described as $(n + 1, k)$ linear block code.

The newly formed parity check matrix H_e will be of size $(q + 1)$ by $(n + 1)$. Let us consider the parity check matrix of (7, 4) Hamming code discussed earlier. It is reproduced here *i.e.*,

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}_{3\times 7}$$

The parity check matrix for extended code will be [using equation (16.47)]

$$H_e = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Newly added row and column

Scanned by CamScanner

With this extended parity check matrix, the minimum distance for extended code will be,

$$d_{e(min)} = d_{min} + 1 \quad \text{...(16.48)}$$

Hence, for extended Hamming code $d_{e(min)} = 4$.

Advantage

The advantage of this code is that it can detect more number of errors compared to normal (n, k) block code. However, it does not have any advantage of error correction.

Example 16.10. For a linear block code which corrects single error per code vector, show that

$$n \geq k = \log_2 (n + 1)$$

And hence design a linear block code with a minimum distance of three and a message block size of eight bits.

Solution: For linear block code, we know that

$$2^{q-1} \geq {}^nC_1 + {}^nC_2 + ... + {}^nC_t$$

This equation describes the condition for correction of 't' errors per word. For correction of single error per code vector (*i.e.*, $t = 1$), the above equation will have only first term on *RHS*; *i.e.*,

$$2^{q-1} \geq {}^nC_1$$

Since $q = n - k$ and ${}^nC_1 = n$

therefore $2^{(n-k)} - 1 \geq n$

or $(n - k) \geq \log_2 (n + 1)$

or $n \geq k + \log_2 (n + 1)$ ...(*i*)

which is proved.

Now, we have to design a linear block code with $d_{\min} = 3$ and $k = 16$. We know that the code with $d_{\min} = 3$ is a single error correcting code. Hence, we can use the relation given by equation (*i*) above.

Thus, for single error correction ($d_{\min} = 3$)

$$n \geq k + \log_2 (n + 1)$$

Substituting $k = 8$ as given, we get

$$n \geq 8 + \log_2 (n + 1)$$

On solving, we obtain $n = 12$. Hence, the code will be (12, 8).

Therefore $q = 12 - 8 = 4$. The parity check matrix will be of size $q \times n$, *i.e.*, parity check matrix size will be 4×12 *i.e.*,

$$[H]_{4 \times 12} = [P^T : I_4]_{4 \times 12}$$

The matrix P^T would be of size 4×8 and I_4 will be an identity matrix of size 4×4.

Select the P^T matrix in a manner such that

(*i*) Its size should be 4×16. This is P submatrix which will be of size 8×4.

(*ii*) No row must be zero, and

(*iii*) All rows must be distinct.

Such matrix is given below

$$P^T = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}_{4 \times 8}$$

Here, it may be noted that we have the freedom of which combination must form a particular column. Therefore, parity check matrix will be,

Scanned by CamScanner

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & : & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & : & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & : & 0 & 0 & 1 & 0 \\ \underbrace{0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1}_{p^T} & & & & & & & & : & \underbrace{0 \quad 0 \quad 0 \quad 1}_{I_4} & & & \end{bmatrix}_{4 \times 12}$$

From the above matrix, the generator matrix and code vectors may be obtained.

16.12. Cyclic Codes

Cyclic codes may be described as the subclass of linear block codes. They have the property that a cyclic shift of one code word produces another code word. As an example, let us consider an n-bit code vector as shown below:

$$X = (x_{n-1}, x_{n-2}, \ldots, x_1, x_0) \qquad \ldots(16.49)$$

Here, $x_{n-1}, x_{n-2}, \ldots, x_1, x_0$ etc. represent the individual bits of the code vector 'X'. If we shift the above code word cyclically, then we will get another code vector X' *i.e.*,

$$X' = (x_{n-2}, x_{n-3}, \ldots, x_1, x_0, x_{n-1}) \qquad \ldots(16.50)$$

Here, it may be observed that every bit is shifted to left by one bit position. Previously x_{n-1} was MSB but after cyclic shift it is at LSB position. One more cyclic shift provides another code vecdtor X'', *i.e.*,

$$X'' = (x_{n-3}, x_{n-4}, \ldots, x_1, x_0, x_{n-1}, x_{n-2}) \qquad \ldots(16.51)$$

Definition of Cyclic Code

A linear code is known as cyclic code if every cyclic shift of the code vector produces some other valid code vector.

The code vector may be defined with the help of polynomial of degree $< n - 1$.

Thus, we can define code vector of equation (16.45). 1 in terms of polynomial as under:

$$X(p) = x_{n-1}\, p^{n-1} + x_{n-2}\, p^{n-2} + \ldots + x_1 p + x_0 \qquad \ldots(16.52)$$

where p is an arbitrary variable and power of p represents the position of the codeword bits. i.e. $(n - 1)$ power of p represents MSB and p^0 represents LSB, p^1 represents bit at 1st position from LSB side and so on.

Now, multiplying the above equation (16.48) by p, we obtain

$$pX(p) = x_{n-1}\, p^n + x_{n-2}\, p^{n-1} + \ldots + x_1 p^2 + x_0 p \qquad \ldots(16.53)$$

Let us form the polynomial of code vector gven by equation (16.46) *i.e.*,

$$X'(p) = x_{n-2}\, p^{n-1} + x_{n-3}\, p^{n-2} + \ldots + x_1 p^2 + x_0 p + x_{n-1} \qquad \ldots(16.54)$$

Let us add equatons (16.49) and (16.50) by mod-2 addition rules to obtain

$$pX(p) + X'(p) = x_{n-1} p^n + (x_{n-2} \oplus x_{n-2}) p^{n-1} + \ldots + (x_1 \oplus x_1) p^2 + (x_0 \oplus x_0)\, p + x_{n-1} \qquad \ldots(16.55)$$

We know that in mod-2 addition if both bits are same, then result is zero.

This means that we have

$$x_{n-2} \oplus x_{n-2} = 0$$
$$x_1 \oplus x_1 = 0$$
$$x^0 \oplus x_0 = 0 \text{ and so on.}$$

Scanned by CamScanner

Then, the above equation (16.55) will become,

$$pX(p) \oplus X'(p) = x_{n-1}\, p^n + x_{n-1}$$
$$= x_{n-1}\,(p^n + 1) \quad ...(16.56)$$

We know that by mod-2 addition rules, there is no difference in addition and subtraction. This means that if $x \oplus y = z$ then $x = y \oplus z$ or $y = x \oplus z$. This is because mod-2 addition and subtraction is the same operation.

Applying this rule to above equation (16.56) and taking $p\,X(p)$ on RHS, we get

$$X'(p) = pX(p) \oplus x_{n-1}\,(p^n + 1) \quad ...(16.57)$$

Note that this is an iterative equaton. It gives another code vector $X'(p)$ from the code vector $X'(p)$. That is if we know one code vector polynomial $X'(p)$, then other code vector polynomial $X'(p)$ may be obtained by using above equation.

Now, let us represent $X(p)$ as,

$$X(p) = M(p)\,G(p) \quad ...(16.58)$$

Here, $M(p)$ is the message signal polynominal of degree $\le k$. This is due to the fact that there are k message bits and $G(p)$is the generating polynomial of degree q.

For an (n, k) cyclic code, the generating polynomial is expressed as,

$$G(p) = p^q + g_{q-1}\,p^{q-1} + ... + g_1\,p + 1 \quad ...(16.59)$$

Here $q = n - k$ *i.e.*, number of parity bits.

Also, the coefficients g_{q-1}, g_{q-2},..., g_2, g_1 are adjusted such that $G(p)$ is the factor of $p^n + 1$.

The message polynominal $M(p)$ is defined as,

$$M(p) = m_{k-1}\,p^{k-1} + m_{k-2}\,p^{k-2} + ... + m_1\,p + m_0 \quad ...(16.60)$$

This is the message polynomial of degree less than 'k' for 'k' message bits. Putting for $M(p)$ and $G(p)$ in equation (16.54), we get

$$X(p) = M(p)\,G(p)$$

or $$X(p) = (m_{k-1}\,p^{k-1} + ... + m_1 p + m_0)\,(p^q + g_{q-1}\,p^{q-1} + ... + g_1 p + 1)$$

Multiplying both sides by p, we get

or $$pX(p) = p(m_{k-1}\,p^{k-1} + ... + m_1 p + m_0)\,(p^q + g_{q-1} p^{q-1} + ... + g_1 p + 1)$$

or $$pX(p) = pm_{k-1}\,p^{k-1}.\,p^q + pm_{k-1}\,p^{k-1} g_{q-1}\,p^{q-1} + ... + m_0 p$$

or $$pX(p) = m_{k-1}\,p^{k+q} + (m_{k-2} + m_{k-1}\,g_{q-1})\,p^{k+q-1} + ... + m_0 p$$

We know that $k + q = n$, then above equation becomes,

$$pX(p) = m_{k-1} p^n + (m_{k-2} + m_{k-1} g_{q-1}) p^{n-1} + ... + m_0 p \quad ...(16.61)$$

Substituting this value of $p\,X(p)$ in equation (16.55), we obtain,

$$X'(p) = pX(p) \oplus x_{n-1}\,(p^n + 1)$$

or $$X'(p) = m_{k-1}\,p^n + (m_{k-2} + m_{k-1}\,g_{q-1}) p^{n-1} + ... + m_0 p \oplus x_{n-1}\,p^n \oplus x_{n-1}$$

Now, combining the coefficients of p^n together in above equation, we have

$$X'(p) = (m_{k-1} \oplus x_{n-1}) p^n + (m_{K-2} + m_{k-1}\,g_{q-1})\,p^{n-1} + ... + m_0 p + x_{n-1}$$

In the above equation, $G(p)$ is selected such that is the factor of p^{n+1}, and under such condition $m_{k-1} = x_{n-1}$.

Hence $m_{k-1} \oplus x_{n-1} = 0$

Then the above equation will be,

$$X'(p) = (m_{k-2} + m_{k-1}\,g_{q-1})\,p^{n-1} + ... + m_0 p + x_{n-1} \quad ...(16.62)$$

Scanned by CamScanner

When $G(p)$ is the factor of p^{n+1}, the above equation is equivalent to equation (16.50). Equation (16.50) gives the polynomial of the code word X' after one cyclic shift of X. Since both the equation are identical, we have proved that the code word polynomial obtained by product of equation (16.54) *i.e.*,

$$X(p) = M(p)\, G(p) \qquad ...(16.63)$$

Readly represents a valid cyclic code vector. But $G(p)$ should be a factor of $p^n + 1$. Also, both $X(p)$ and $X'(p)$ satisfy the iterative equation for multiple shifts given by equation (9.53) as,

$$X'(p) = p\, X(p) \oplus x_{n-1}\,(p^{n+1}) \qquad ...(16.64)$$

The other code vectors may be obtained from equation (16.58) as,

$$\begin{aligned} X_1(p) &= M_1(p)\, G(p) \\ X_2(p) &= M_2(p)\, G(p) \\ X_3(p) &= M_3(p)\, G(p) \end{aligned} \qquad ...(16.65)$$

and so on.

Here, all the code vectors x_2, x_3 ... and so on satisfy the cyclic shift condition and their polynomials also satisfy equation (16.61). It may be noted that generator polynomial remains same for all code vectors.

The code vectors obtained in equation (16.61) above are not in systematic form. The systematic cyclic code vectors can be obtained using the following procedure.

The n-bit code vector 'X' and its polynomial $X(p)$ are given by equation, (16.45) and (16.48) respectively as,

$$X = (x_{n-1}x_{n-2}x_{n-3} \cdots x_3x_2x_1x_0) \qquad ...(16.66)$$

and

$$X(p) = x_{n-1}p^{n-1} + x_{n-2}p^{n-2} + x_{n-3}p^{n-3} + \ldots + x_1\, p + x_0 \qquad ...(16.66a)$$

If we want to represent code vector 'X' to be in systematic form, then it will be,

$$X = (k \text{ message bits: } (n-k) \text{ check bits})$$

or

$$X = (m_{k-1}m_{k-2} \ldots m_1m_0 : c_{q-1}\, c_{q-2} \cdots c_1c_0) \qquad ...(16.67)$$

The above code vector may be written in polynomial form as under:

$$\begin{aligned} X(p) = {} & m_{k-1}p^{n-1} + m_{k-2}\, p^{n-2} + \ldots + m_1p^{n-k+1} + m_0p^{n-k} \\ & + c_{q-1}p^{n-k-1} + c_{q-2}\, p^{n-k-2} + \ldots + c_1p + c_0 \end{aligned}$$

We know that $n - k = q$ or $n = k + q$. Putting those values of 'n' and '$n - k$' in above equation, we obtain,

$$\begin{aligned} X(p) = {} & m_{k-1}\, p^{k+q-1} + m_{k-2}\, p^{k+q-2} + \ldots + m_1p^{q+1} + m_0p^q \\ & + c_{q-1}p^{q-1} + c_{q-2}\, p^{q-2} + \ldots + c_1p + c_0 \end{aligned}$$

Let us rearrange the above equation as under,

$$X(p) = p^q\left[m_{k-1}p^{k-1} + m_{k-2}p^{k-2} + \ldots + m_1p + m_0\right] + c_{q-1}p^{q-1} + c_{q-2}p^{q-2} + \ldots + c_1p + c_0 \qquad ...(16.68)$$

In the above equation, $m_{k-1}p^{k-1} + m_{k-2}p^{k-2} + \ldots + m_1p + m_0 = M(p)$

As we have defined in equation (16.56) earlier. Therefore above equation becomes,

$$X(p) = p^q\, M(p) + (c_{q-1}c_{q-2} \ldots c_1c_0) + p^{q-2} + \ldots + c_1p + c_0 \qquad ...(16.69)$$

* Here $x_{n-1}\,(p^n + 1) = x_{n-1}\, p^n + x_{n-1}$.

Scanned by CamScanner

Let us define the check bit polynomial of check bits

$$C = (c_{q-1} c_{q-2} \ldots c_1 c_0) \text{ as}$$

$$C(p) = c_{q-1}\, p^{q-1} + c_{q-2}\, p^{q-2} + \ldots + c_1 p + c_o \qquad \ldots(16.70)$$

The above equation is check bit polynomial of degree less than q. From equations (16.69) and (16.68), we get

$$X(p) = p^q\, M(p) + C(p) \qquad \ldots(16.71)$$

This equation gives a code word polynomial in systematic form. For this code vector to be cyclic, the above equation must be same as equation (16.59). Thus, for a cyclic code vector, we can equate equations (16.71) and (16.61) *i.e.*

$$p^q\, M(p) + C(p) = M(p)\, G(p)$$

$$\frac{p^q\, M(p)}{G(p)} \oplus \frac{C(p)}{G(p)} = M(p)$$

The above equation has the form of $z \oplus t = l$. We know that mod-2 addition and subtraction operation is same *i.e.*, $z \oplus t = l$, then we can write $z = t \oplus l$ or $t = z \oplus l$ or $z \oplus t \oplus l = 0$. Thus, there is no mod-2 subtraction as such. Mod-2 addition and subtraction yield same result. With these conclusions, we can write above equation as under:

$$\frac{p^q\, M(p)}{G(p)} = M(p) \oplus \frac{C(p)}{G(p)}$$

This equation has the form of

$$\frac{\text{Numerator}}{\text{Denominator}} = \text{Quotient} \oplus \frac{\text{Remainder}}{\text{Denominator}}$$

$$\left(\text{For example } \frac{14}{3} = 4 + \frac{2}{3}\right)$$

Hence, the check bit polynominal is obtained as remainder in dividing $p^q\, M(p)$ by $G(p)$ *i.e.*,

$$C(p) = rem\left[\frac{p^q\, M(p)}{G(p)}\right] \qquad \ldots(16.72)$$

Here, $C(p)$ check bit polynomial for systematic code.
$M(p)$ message bit polynomial,
and $G(p)$ generating polynomial, which is the factor of p^{n+1}.

Example 16.11. The generator polynomal of a (7, 4) cyclic code is $G(p) = P^3 + p + 1$. Obtain all the code vectors for the code in non-systematic and systematic form.

(U.P. Tech., Sem. Examination, 2003-2004)

Solution: Here $n = 7$ and $k = 4$ therefore $q = n - k = 3$.

There will be total $2^k = 2^4 = 16$ message vectors of 4 bits each. Let us consider any message vector as under:

$$M = (m_3\, m_2\, m_1\, m_0)$$
$$= (0\ 1\ 0\ 1)$$

Then, the message polynomial will be (by substituting $k = 4$ in equation (16.56),

$$M(p) = m_3 p^3 + m_2 p^2 + m_1 p + m_0$$

Scanned by CamScanner

$$\therefore \quad M(p) = p^2 + 1 \qquad ...(i)$$

Also, the given generator polynomial is,

$$G(p) = p^3 + p + 1 \qquad ...(ii)$$

To Obtain Non Systematic Code Vectors

The non-systematic cyclic code is given by (using equation (16.59))

$$X(p) = M(p)\, G(p)$$

or $$X(p) = \left(p^2 + 1\right)\left(p^3 + p + 1\right)$$

or $$X(p) = p^5 + p^3 + p^2 + p^3 + p + 1$$

or $$X(p) = p^5 + p^3 + p^3 + p^2 + p + 1$$

or $$X(p) = p^5 + (1 \oplus 1)\, p^3 + p^2 + p + 1$$

or $$X(p) = p^5 + p^2 + p + 1 \qquad \text{[Since } (1 \oplus 1)\, p^3 = op^3 = 0)$$

(Since $(1 \oplus |\ \ |^3 = 0P^3 = 0)$

$$= 0p^6 + p^5 + 0p^4 + 0p^3 + p^2 + p + 1 \qquad ...(iii)$$

Note that the degree of above polynomial is $n - 1 = 6$. The code vector corresponding to above polynomial is,

$$X = (x_6\ x_5\ x_4\ x_3\ x_2\ x_1\ x_0)$$

or $$X = (0\ 1\ 0\ 0\ 1\ 1\ 1)$$

This is the code vector for message vector 0101. This code vector is non-systematic cyclic code vector. Similarly, other code vectors may be obtained using the same procedure.

To Obtain Systematic Code Vectors

For systematic code vectors we will first find check bits. Those check bits are given by the check bit polynomial $C(p)$. From equation (16.68) $C(p)$ is expressed as,

$$C(p) = rem\left[\frac{p^q\, M(p)}{G(p)}\right] \qquad ...(v)$$

Since, $q = 3$, $p^q\, M(p)$ will be,

$$p^q\, M(p) = p^3\, M(p)$$

or $$p^q\, M(p) = p^3(p^2 + 1)$$

or $$p^q\, M(p) = p^5 + q^3 \qquad \text{(for message vector of 0101)}$$

or $$p^q\, M(p) = p^5 + 0p^4 + p^3 + 0p^2 + 0p + 0$$

and $$G(p) = p^3 + p + 1$$

or $$G(p) = p^3 + 0p^2 + p + 1$$

Now, we have $p^q\, M(p)$ and $G(p)$. Now let us perform the division to find remainder of this division as under:

$$\begin{array}{r|l} & p^2 + 0 + 0 \quad \leftarrow \text{Quotient} \\ \hline p^3 + 0p^2 + p + 1 & p^5 + 0p^4 + p^3 + 0p^2 + 0p + 0 \\ & p^5 + 0p^4 + p^3 + p^2 \\ \text{Mod-2 addition} \rightarrow & \oplus \quad \oplus \quad \oplus \quad \oplus \\ \hline & 0 \quad +0 \quad +0 \quad + \underbrace{p^2 \quad +0p + 0}_{\text{Remainder}} \end{array}$$

Scanned by CamScanner

Hence, the remainder polynomial is $p^2 + 0p + 0$ in the division of $p^3\,M(p)$ by $G(p)$. Therefore, equation (v) may be written as,

$$C(p) = rem\left[\frac{p^3\,M(p)}{G(p)}\right]$$

or $$C(p) = p^2 + 0p + 0$$

With $q = 3$, the polynomial $C(p)$ will be (using equation (16.65))

$$C(p) = c_2\,p^2 + c_1 p + c_0$$

Thus $$c_2\,p^2 + c_1 p + c_0 = p^2 + 0p + 0$$

Hence, the check bits are

$$C = (c_2\,c_1\,c_0) = (1\,0\,0)$$

The code vector is written in systematic form as under: (using equation (16.62(a))

$$X = (m_{k-1} m_{k-2} \ldots m_1\, m_0 : c_{q-1} c_{q-2} \ldots c_1 c_0)$$

In this example, $$X = (m_3\,m_2\,m_1\,m_0 : c_2\,c_1\,c_0)$$

or $$X = (0\,1\,0\,1 : 1\,0\,0)$$

This is the requred cyclic code vector in systematic form. The other code vectors can be obtained using the same procedure.

Table 16.10 lists all the systematic and non-systematic cyclic codes.

Table 16.10. Code vectors of a (7, 4) cyclic code for $G(p) = p^3 + p + 1$

Sr. No.	Message bits $M = m_3$	m_2	m_1	m_0	Nonsystematic code vectors $X = x_6$	x_5	x_4	x_3	x_2	x_1	x_0	Systematic code vectors $X = m_3$	m_2	m_1	m_0	c_2	c_1	c_0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	1	0	0	0	1	0	1	1	0	0	0	1	0	1	1
3	0	0	1	0	0	0	1	0	1	1	0	0	0	1	0	1	1	0
4	0	0	1	1	0	0	1	1	1	0	1	0	0	1	1	1	0	1
5	0	1	0	0	0	1	0	1	1	0	0	0	1	0	0	1	1	1
6	0	1	0	1	0	1	0	0	1	1	1	0	1	0	1	1	0	0
7	0	1	1	0	0	1	1	1	0	1	0	0	1	1	0	0	0	1
8	0	1	1	1	0	1	1	0	0	0	1	0	1	1	1	0	1	0
9	1	0	0	0	1	0	1	1	0	0	0	1	0	0	0	1	0	1
10	1	0	0	1	1	0	1	0	0	1	1	1	0	0	1	1	1	0
11	1	0	1	0	1	0	0	1	1	1	0	1	0	1	0	0	1	1
12	1	0	1	1	1	0	0	0	1	0	1	1	0	1	1	0	0	0
13	1	1	0	0	1	1	1	0	1	0	0	1	1	0	0	0	1	0
14	1	1	0	1	1	1	1	1	1	1	1	1	1	0	1	0	0	1
15	1	1	1	0	1	1	0	0	0	1	0	1	1	1	0	1	0	0
16	1	1	1	1	1	1	0	1	0	0	1	1	1	1	1	1	1	1

Scanned by CamScanner

16.13. Generator and Parity Check Matrices of Cyclic Codes

Because cyclic codes are subclass of linear block codes, generater and parity check matrices can also be defined for cyclic codes. The generator matrix has the size of $k \times n$. This means that there are 'k' rows and 'n' columns. Let the generator matrix $G(p)$ be given by equation (16.55) as,

$$G(p) = p^q + g_{q-1}p^{q-1} + \ldots + g_1\, p + 1 \qquad \ldots(16.73)$$

Multiplying both the sides of this polynomial by p^i, we get

$$p^i\, G(p) = p^{i+q} + g_{q-1}\, p^{i+q-1} + \ldots + g_1 p^{i+1} + p^i \qquad \ldots(16.74)$$

and $(k-1), (k-2), \ldots, 2, 1, 0$.

The above equation gives the polynomials for the rows of a generating polynomial. This procedure will be clear after the discussion of next exmaple.

Example 16.12. Find the generator matrix corresponding to $G(p) = p^3 + p^2 + 1$ for a (7, 4) cyclic code.

Solution: Here $n = 7$, $k = 4$ and $q = 7 - 4 = 3$

Let us evaluate $p^i\, G(p)$ as under:

$$p^i\, G(p) = p^{i+3} + p^{i+2} + p^i \quad \text{for givn } G(p)$$

Since $\quad k - 1 = 3$

$$i = 3, 2, 1, 0$$

Thus, we will obtain four polynomials corresponding to 4 values of i for four rows of generator matrix as under:

$$\left.\begin{array}{ll} \text{For row 1: } i = 3 \Rightarrow & p^3\, G(p) = p^6 + p^5 + p^3 \\ \text{For row 2: } i = 2 \Rightarrow & p^2\, G(p) = p^5 + p^4 + p^2 \\ \text{For row 3: } i = 1 \Rightarrow & pG(p) = p^4 + p^3 + p \\ \text{For row 4: } i = 0 \Rightarrow & G(p) = p^3 + p^2 + 1 \end{array}\right\} \ldots(i)$$

The generator matrix for (n, k) code is of size $k \times n$. For this (7, 4) cyclic code, the size will be 4×7. Corresponding to four rows, we have obtained four polynomials given by equation (i). Let us write each polynomial in the following manner:

$$\left.\begin{array}{ll} \text{Row 1} & p^3\, G(p) = p^6 + p^5 + 0p^4 + p^3 + 0p^2 + 0\,p + 0 \\ \text{Row 2} & p^2\, G(p) = 0p^6 + p^5 + p^4 + 0p^3 + p^2 + 0p + 0 \\ \text{Row 3} & pG(p) = 0p^6 + 0p^5 + p^4 + p^3 + 0p^2 + p + 0 \\ \text{Row 4} & G(p) = 0p^6 + 0p^5 + 0p^4 + p^3 + p^2 + 0p + 1 \end{array}\right\} \ldots(ii)$$

Let us transform the above set of polynomials into a matrix of 4×7

$$G_{4\times 7} = \begin{array}{c} \begin{array}{ccccccc} p^6 & p^5 & p^4 & p^3 & p^2 & p^1 & p^0 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}_{4\times 7} \end{array} \qquad \ldots(iii)$$

This is the generator matrix for given generator polynomials. **Ans.**

Example 16.13. Find out the possible generator polynomials (7, 4) cyclic code. Obtain the code vectors corresponding to these generator polynomials.

Solution: For this (7, 4) cyclic code, we have

$$n = 7, \quad k = 4 \quad \text{and} \quad q = n - k = 7 - 4 = 3$$

We know that the generator polynomial is the factor of $p^n + 1$. For this example, generator polynomial is the factor of $p^7 + 1$. The factors of $p^7 + 1$ are

Scanned by CamScanner

$$p^7 + 1 = (p \oplus 1)(p^3 \oplus p^2 \oplus 1)(p^3 \oplus p \oplus 1)$$

The valid generating polynomial is given by equation (16.55) *i.e.*, ...(i)

$$G(p) = p^q + g_{q-1}\, p^{q-1} + \ldots + g_1\, p + 1$$

Hence, the degree of the generating polynomial should be q. For this example $q = 3$. Therefore the valid generator polynomials for $p^7 + 1$ will be $p^3 + p^2 + 1$ and $p^3 + p + 1$. Also, $p + 1$ will not be a generator polynomial. Since its degree is not q (*i.e.*, 3). Thus generator polynomials for (7, 4) cyclic code are,

$$G_1(p) = p^3 + p^2 + 1$$

and $$G_2(p) = p^3 + p + 1 \quad \textbf{Ans.}$$

Example 16.14. Obtain the generator matrix corresponding to $G(p) = p^3 + p + 1$ and find out the code vectors for (7, 4) cyclic code.

Solution: The row of a generator matrix are given by $p^i G(p)$.

Here, $$p^i\, G(p) = p^{i+3} + p^{i+1} + p^i$$

and $$i = 3, 2, 1, 0 \quad \text{since} \quad k - 1 = 3$$

$$\left.\begin{aligned} &\text{For row 1: } i = 3 \Rightarrow & p^3\, G(p) &= p^6 + p^4 + p^3 \\ &\text{For row 2: } i = 2 \Rightarrow & p^2\, G(p) &= p^5 + p^3 + p^2 \\ &\text{For row 3: } i = 1 \Rightarrow & pG(p) &= p^4 + p^2 + p^1 \\ &\text{For row 4: } i = 0 \Rightarrow & G(p) &= p^3 + p + 1 \end{aligned}\right\} \quad \text{...(ii)}$$

The above set of polynomials is transformed into a generator matrix of size 4×7 (*i.e.*, $k \times n$) as shown below.

$$[G] = \begin{array}{c} \\ \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \\ \text{Row 4} \end{array} \begin{array}{c} \begin{array}{ccccccc} p^6 & p^5 & p^4 & p^3 & p^2 & p^1 & p^0 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}_{4\times 7} \end{array}$$

Since, cyclic is a subclass of linear block code, its code vectors can be obtained by

$$X = MG \quad \text{...(iii)}$$

Here, M is the $1 \times k$ message vector and G is generator matrix.

Here, $k = 4$. Let us consider any 4-bits message vector as under:

$$M = (m_3\, m_2\, m_1\, m_0) = (1\ 0\ 0\ 1)$$

The code vector corresponding to this message vector will be,

$$X = MG = [1001]\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} = (1\ 0\ 1\ 0\ 0\ 1\ 1)$$

Note: *Here, we perform matrix multiplication and additions by mod-2 rules. i.e.,*

$$1 \oplus 1 = 0,$$
$$1 \oplus 0 = 1$$
$$0 \oplus 1 = 1$$

and $$0 \oplus 0 = 0.$$

This code vector we have already obtained in example 16.5, and is listed in Table 16.9. This code vector is in non-systematic form. Also note that generator matrix is also in non-systematic form. In a same way other code vectors for cyclic code may be obtained.

Scanned by CamScanner

Systematic form of Generator Matrix

The systematic form of generator matrix is expressed as,

$$G = \left[I_k : P_{k\times q}\right]_{k\times n} \quad ...(iii)$$

The tth row of this matrix will be represented in the polynomial form as,

$$t^{th} \text{ row of } G = p^{n-t} + R_t(p)$$

where $t = 1, 2, 3... k$ ...(iv)

Let us divide p^{n-t} by a generator matrix $G(p)$. Then, we can express the result of this division in terms of quotient and remainder us under:

$$\frac{P^{n-t}}{G(-p)} = \text{Quotient} + \frac{\text{Remainder}}{G(p)} \quad ...(v)$$

Here, remainder will be a polynomial of degree less than 'q', since degree of $G(p)$ is 'q'. The degree of quotient will depend upon value of t.

Let us represent Remainder $= R_t(p)$

and Quotient $= Q_t(p)$

then equation (v) will be,

$$\frac{p^{n-t}}{G(p)} = Q_1(p) + \frac{R_t(p)}{G(p)} \quad ...(vi)$$

i.e., $p^{n-t} = Q_t(p)G(p) \oplus R_t(p)$

and $t = 1, 2,...k$ ...(vii)

Again, we know that if $z = y \oplus t$, then $z \oplus y = t$ or $z \oplus t = y$. This means that the mod-2 addition and subtraction yield same results. Then we can write equation (vii) as under:

$$p^{n-t} \oplus R_t(p) = Q_t(p)\, G(p) \quad ...(viii)$$

Thus, the above equation represents t^{th} row of systematic generator matrix. **Ans.**

Example 16.15. Obtain the generator matrix for a systematic (7, 4) cyclic code if $G(p) = p^3 + p + 1$. Also, obtain the partiy check matrix.

Solution: The tth row of generator matrix is given as,

$$p^{n-t} + R_t(p) = Q_t(p)\, G(p) \quad \text{and} \quad t = 1, 2,...k$$

We are given that $n = 7$, $k = 4$ and $q = n - k = 3$.

The above equation will be,

$$p^{7-t} + R_t(p) = Q_t(p)\,(p^3 + p + 1)$$

and $t = 1, 2, 3, 4$ ...(i

With $t = 1$, the above equation becomes,

$$p^6 + R_1(p) = Q_1(p)\,(p^3 + p + 1) \quad ...(ii$$

The RHS or LHS of this equation represents 1st row of systematic generator matrix. We hav to find $Q_1(p)$. From last example, we know that $Q_1(p)$ is obtained by dividing p^{n-t} by $G(p)$. Here t obtain $Q_1(p)$, we have to divide p^6 by $G(p) = p^3 + p + 1$.

Scanned by CamScanner

$$
\begin{array}{r|l}
 & p^3+p+1 \quad \longleftarrow \text{Quotient} \\
\hline
p^3+p+1 & p^6+0+0 \\
 & p^6+p^4+p^3 \\
\text{denotes} \longrightarrow & \oplus \quad \oplus \quad \oplus \\
\text{mod-2 addition} & 0+p^4+p^3+0+0 \\
 & p^4+0+p^2+p \\
 & \oplus \quad \oplus \quad \oplus \quad \oplus \\
 & p^3+p^2+p+0 \\
 & p^3+0+p+1 \\
 & \oplus \quad \oplus \quad \oplus \quad \oplus \\
 & p^2+1 \longleftarrow \text{Remainder}
\end{array}
$$

Here $\quad Q_1(p) = p^3 + p + 1$

and $\quad R_1(p) = p^2 + 1$

Putting these values in equation (*ii*), we get

$$p^6 + p^2 + 1 = (p^3 + p + 1)(p^3 + p + 1)$$

The RHS or LHS (actually both are same) of the above equation represents 1st row of generator matrix *i.e.*,

1st row polynomaial $= p^6 + p^2 + 1$ ...(*iii*)

Using the same procedure as discussed above the row polynomials are obtained and they are given below:

$$\left.\begin{array}{l}
2^{nd} \text{ row polynomial} = p^5 + p^2 + p + 1 \\
3^{rd} \text{ row polynomial} = p^4 + p^2 + p \\
4^{th} \text{ row polynomial} = p^3 + p + 1
\end{array}\right\} \quad \text{...(iv)}$$

The above equation can be transformed into generator matrix as shown below

$$G = \begin{array}{c} \\ \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \\ \text{Row 4} \end{array}
\begin{array}{c}
\begin{array}{cccccccc} p^6 & p^5 & p^4 & p^3 & & p^2 & p^1 & p^0 \end{array} \\
\left[\begin{array}{cccc|c|ccc}
1 & 0 & 0 & 0 & : & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & : & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & : & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & : & 0 & 1 & 1
\end{array}\right]_{4\times 7} \\
\underbrace{\qquad\qquad}_{I_3} \qquad\qquad \underbrace{\qquad\qquad}_{P_{4\times 3}}
\end{array} \quad \text{...(v)}$$

This is the required generator matrix in systematic form. The code vector can be obtained from

$$X = MG$$

Let us take any 4 bits message vector and find corresponding code vector. Let us take

$$M = (m_3\, m_2\, m_1\, m_0)$$
$$= (1\ 1\ 0\ 0)$$

Therefore, $\quad X = MG = [1\ 1\ 0\ 0] \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$

or $\quad X = (1\ 1\ 0\ 0\ 0\ 1\ 0)$

This code vector is obtained by performing matrix multiplication and mod-2 additions. Observe that the same systematic code vector is listed in Table 16.9.

Using the same procedure other code vectors may also be obtained.

Scanned by CamScanner

We know that $G\left[I_k : P_{k\times q}\right]_{k\times n}$

The P submatrix may be obtained from equation (v) as under:

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}_{4\times 3} \quad ...(vi)$$

The parity check matrix is given as,

$$H = [P^T : I_q]_{q\times n}$$

Here P^T is the transpose of P submatrix and

I_q is the $q \times q$ identity matrix.

By taking transpose of P submatrix of equation (vi), the parity check matrix will be given by

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & : & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & : & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & : & 0 & 0 & 1 \end{bmatrix}_{3\times 7}$$

$$\underbrace{\qquad\qquad}_{P^T} \qquad \underbrace{\qquad}_{I_3}$$

This is the required parity check matrix for (7, 4) cyclic code in systematic form. **Ans.**

16.14. Encoders for Cyclic Codes

In this article, we shall discuss the encoders for systematic cyclic codes. Figure 16.7 shows the block diagram of a generalized (n, k) cyclic code. Following symbols and blocks are used in the Fig. 16.7.

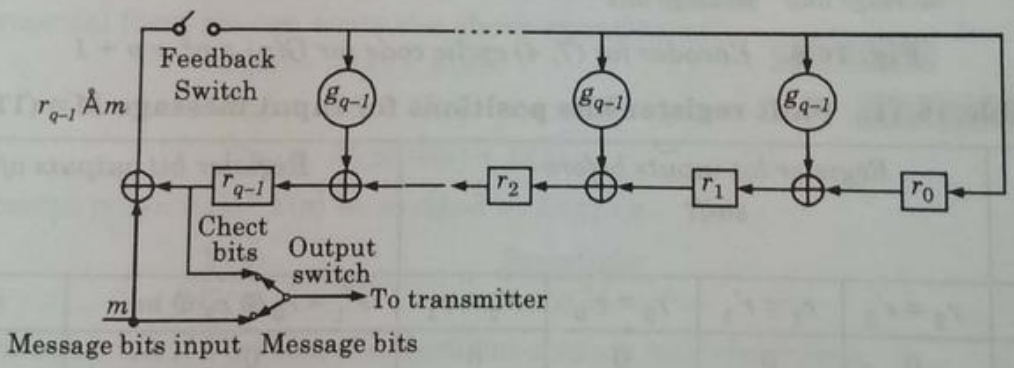


Fig. 16.7. *Encoder for systematic (n, k) cyclic code*

They represent the flip-flops. They are connected in sequential order to make a shift register.

⊕ They represent module-2 adders.

They represent closed path if $g = 1$ and open path if $g = 0$.

The contents of the register are shifted from input to output when clock pulse is applied.

Operation:

The feedback switch is first closed. The output switch is connected to message input. All the shift registers are initialized to all zero state. The k message bits are shifted to the transmitter as well as shifted into the registers.

After the shift of 'k' message bits the registers contian 'q' check bits. The feedback switch is now opened and output switch is connected to check bits position. With the every shift, the check bits are then shifted to the transmitter.

Scanned by CamScanner

Here it may be we observed that the block diagram performs the division operation and generates the remainder (*i.e.*, check bits). This remainder is stored in the shift after all message bits are shifted out.

Example 16.16. Design the encoder for the (7, 4) cyclic code generated by $G(p) = p^3 + p + 1$ and also verify its operation for any message vector.

Solution: The generator polynomial is,

$$G(p) = p^3 + 0\,p^2 + p + 1$$

and

$$G(p) = p^3 + g_2\,p^2 + g_1\,p + 1$$

On comparison of the two equation, we get

$$g_1 = 1 \quad \text{and} \quad g_2 = 0$$

and

$$q = n - k = 7 - 4 = 3$$

With these values the block diagram of figure 16.7 will be as shown in figure 16.8. Since $q = 3$, there are '3' flip-flops in shift register to hold check bits $c_1\,c_2$ and c_0 because $g_2 = 0$, its link is not connected $g_1 = 1$, hence its link is connected.

Now let's verify the operation of this encoder for message vector $M = (m_3\,m_2\,m_1\,m_0) = (1100)$. Table 16.11 shows the contents of shift registers before and after shifts.

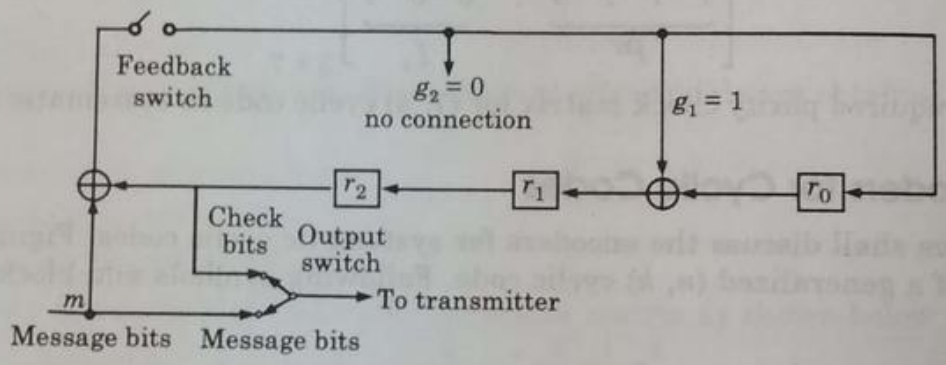


Fig. 16.8. *Encoder for (7, 4) cyclic code for $G(p) = p^3 + p + 1$*

Table 16.11. Shift register bits positions for input message $M = (1100)$

Input message bit m	*Register bit inputs before shift*				*Register bit outputs after shift*	
	$r_2 = r'_2$	$r_1 = r'_1$	$r_0 = r'_0$	$r'_2 = r_1$	$r'_1 = r_0 \oplus r_2 \oplus m$	$r'_0 = r_2 \oplus m$
	0	0	0	0	0	0
1	0	0	0	0	$0 \oplus 0 \oplus 1 = 1$	$0 \oplus 1 = 1$
1	0	1	1	1	$1 \oplus 0 \oplus 1 = 0$	$0 \oplus 1 = 1$
0	1	0	1	0	$1 \oplus 1 \oplus 0 = 0$	$1 \oplus 0 = 1$
0	0	0	1	0	$1 \oplus 0 \oplus 0 = 1$	$0 \oplus 0 = 0$

This table shows that at the end of last message bit the register bit outputs are $r_2 = 0$, $r'_1 = 1$ and $r'_0 = 0$. The feedback switch is opened and output switch is closed to check bits position. The check bits are then shifted to the transmitter. The check bits are shifted as $c_2 = r_2$, $c_1 = r'_1$ and $c_0 = r'_0$. The following table shows the shift operation of message and check bits. We know that the code vector is, t

$$X = (m_3\,m_2\,m_1\,m_0\,c_2\,c_1\,c_0)$$
$$= (1\,1\,0\,0\,0\,1\,0)$$

Scanned by CamScanner

Table 16.12. Operation of (7, 4) cylic code encoder

Shift clock	*Message bit m*	*Shift register outputs*			*Feedback switch on/off*	*Output switch position*	*Transmitted bits*
		r_2'	r_1'	r_0'			
1	1	0	1	1	on	message	1
2	1	1	0	1	on	message	1
3	0	0	0	1	on	message	0
4	0	0	1	0	on	message	0
5	-	0	1	0	off	check bits	0
6	-	1	0	0	off	check bits	1
		$r_2'=r_1'$	$r_1'=r_0$	$r_0=0$			
7	-	0	0	0	off	check bits	0

This table illustrated shows the bits are transmitted when input message is (1100).

16.14.1. Syndrome Decoding for Cyclic Codes

In cyclic codes also during transmission some errors may occur. Syndrome decoding can be used to correct those errors. Let us represent the received code vector by Y. If 'E' denotes an error vector then the correct code vector may be obtained as,

$$X = Y \oplus E \qquad ...(16.75)$$

or we can write the above equation as,

$$Y = X \oplus E \qquad ...(16.76)$$

We can write the above equation since it is mod-2 addition.

In the polynomial form we can write the above equation as,

$$Y(p) = Y(p) + E(p) \qquad ...(16.77)$$

Since $X(p) = M(p)\,G(p)$ the above equation will be,

$$Y(p) = M(p)\,G(p) + E(p) \qquad ...(16.78)$$

Let the received polynomial $Y(p)$ be divided by $G(p)$ *i.e.*,

$$\frac{Y(p)}{G(p)} = \text{Quotient} + \frac{\text{Remainder}}{G(p)} \qquad ...(16.79)$$

In the above equation if $Y(p)$ *i.e.*, if it does not contain any error then,

$$\frac{X(p)}{G(p)} = \text{Quotient} + \frac{\text{Remainder}}{G(p)} \qquad ...(16.80)$$

Since $X(p) = M(p)\,G(p)$, Quotient will be equal to $M(p)$ and remainder will be zero. This shows that if there is no error then remainder will be zero. Since $G(p)$ is factor of code vector polynomial. Let's represent Quotient by $Q(p)$ and Remainder by $R(p)$ then equation (16.80) becomes,

$$\frac{Y(p)}{G(p)} = Q(p) + \frac{R(p)}{G(p)} \qquad ...(16.81)$$

Clearly $R(p)$ will be the polynomial of degree less than or equal to $q - 1$. Multiply both sides of above equation by $G(p)$ *i.e.*,

$$Y(p) = Q(p)\,G(p) + R(p)$$

Compare equations (16.82) and (16.85) we obtain,

Scanned by CamScanner

$$M(p)\,G(p) \oplus E(p) = E(p)G(p) \oplus R(p)$$

or

$$E(p) = M(p)G(p) \oplus Q(p)\,G(p) \oplus R(p)$$

This equation has all mod-2 additions. Therefore subtraction and addition is same, *i.e.*

$$E(p) = [M(p) + Q(p)]\,G(p) + R(p) \quad \text{...(16.82)}$$

This equation shows that for a fixed message vector and generator polynomial, an error pattern or error vector 'E' depends R. For every remainder 'R' there will be specific error vector. Therefore we can call the remainder vector 'R' as syndrome vector 'S', or $R(p) = S(p)$. Therefore equation (16.85) will be,

$$\frac{Y(p)}{G(p)} = Q(p) + \frac{S(p)}{G(p)} \quad \text{...(16.83)}$$

Thus the syndrome vector is obtained by dividing received vector $Y(p)$ by $G(p)$, *i.e.*,

$$S(p) = rem\left[\frac{Y(p)}{G(p)}\right] \quad \text{...(16.84)}$$

Block Diagram of Syndrome Calculator

Figure 16.19 shows the generalized block diagram of a syndrome calculator.

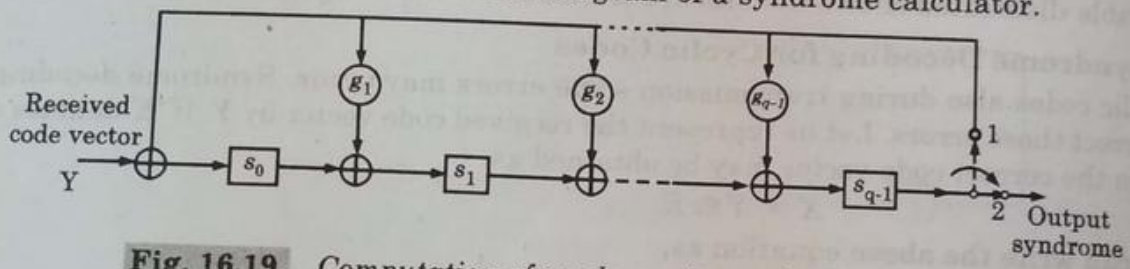


Fig. 16.19. *Computation of syndrome for an (n, k) cyclic code*

Observe in figure that there are 'q' stage shift register to generate 'q' bit syndrome vector. The operations as follows.

Initially all the shift register contents are zero and the switch is closed in position 1. The received vector Y is shifted bit by bit into the shift register. The contents of flip = flops keep on changing according to input bits of Y and values of g_1, g_2 etc. After all the bits of Y are shifted, the 'q' flip-flops of shift register contains the q-bit syndrome vector. The switch is then closed to position 2 and clocks are applied to the shift register. The output is a syndrome vector $S = (S_{q-1}, s_{q-2}, \ldots s_1 s_0)$.

Exmaple 16.17. Design a syndrome calculator for a (7, 4) cyclic Hamming code generated by the polynomial $G(p) = p^3 + p + 1$. Evaluate the syndrome for $Y = (1\ 0\ 0\ 1\ 1\ 0\ 1)$.

Solution: For the given code $n = 7$, $k = 4$, $q = n - k = 7 - 4 = 3$

The given generator polynomial is,

$$G(p) = p^3 + 0\,p^2 + p + 1$$

and

$$G(p) = p^3 + g_2\,p^2 + g_1\,p + 1 \quad \text{(generalized equation)}$$

On comparison of the above two equations, we get

$$g_1 = 1 \text{ and } g_2 = 0$$

With these values the block diagram of a syndrome calculator for (7, 4) cyclic code will be as shown in figure 16.11 below

$$G(p) = p^3 + p + 1$$

The switch is kept in position 1 until all the '7' bits of received vector Y are shifted into the shift register. The flip-flops of the shift register contain syndrome vector when all bits of 'Y' are shifted. The switch is then closed to position 2 and clock pulses are applied to shift register. This

Scanned by CamScanner

gives syndrome vector at the output. The following table show the operation of this syndrome calculator for received vector $Y = (1\ 0\ 0\ 1\ 1\ 0\ 1)$. The Table 16.13 shows the contents of flip-flops with every shift.

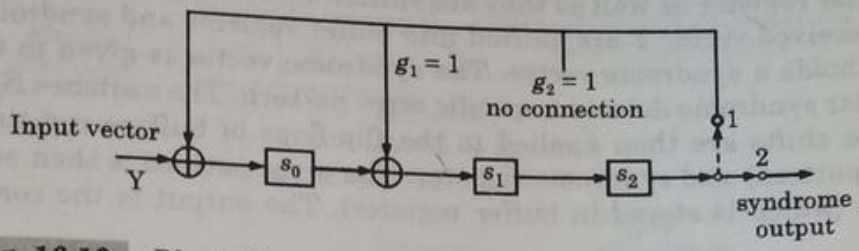


Fig. 16.10. *Block diagram of a syndrome calculator for (7, 4) cyclic code with $G(P) = p^3 + p + 1$*

Table 16.13.

Shift	*Received vector i.e., bits of Y*	*Contents of flip flops in shift register*		
		$s_0 = y \oplus s_2$	$s_1 = s_0 \oplus$	$s_2\ s_2 = s_1$
-	-	0	0	0
1	1	1	0	0
2	0	0	1	0
3	0	0	0	1
4	1	0	1	0
5	1	1	0	1
6	0	1	0	0
7	1	1	1	0

This table shows that at end of last shift the register contents are $(s_0\ s_1\ s_2) = (110)$.

Hence the calculated syndrome will be,

$$S = (s_2\ s_1\ s_0) = (0\ 1\ 1) \quad \textbf{Ans.}$$

16.14.2. Decoder for Cyclic Codes

Once the syndrome is calculated, then an error pattern is detected for that particular syndrome. When this error vector is added to the received vector Y, then it gives corrected code vector at the output. This decoding operation can be performed by the scheme shown in figure 16.11.

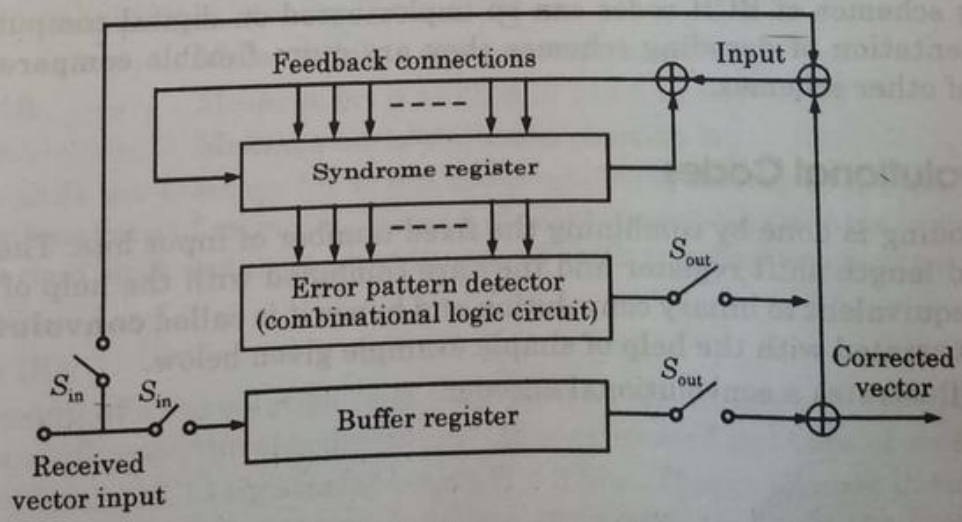


Fig. 16.11. *Generalized block diagram of decoder for cyclic codes.*

Scanned by CamScanner

The operation of the above decoder is as under:

The switches named S_{out} are opened and S_{in} are closed. The bits of the received vector Y are shifted into the buffer register as well as they are shifted into the syndrome calculator. When all the 'n' bits of the received vector Y are shifted into buffer register and syndrome calculator the syndrome register holds a syndrome vector. The syndrome vector is given to the error pattern detector. A particular syndrome detects a specific error pattern. The switches S_{in} are opened and S_{out} are closed. The shifts are then applied to the flip-flops of buffers register, error register (which holds error pattern) and syndrome register. The error pattern is then added bit by bit to the received vector (which is stored in buffer register). The output is the corrected error free vector.

16.14.3. Advantages and Disadvantages of Cyclic Codes

As we have seen that cyclic codes are the subclass of linear block codes, they have some advantages over noncyclic block codes as given under:

Advantages

1. The error correcting and decoding methods of cyclic codes are simpler and easy to implement. These methods eliminate the storage needed for lookup table decoding. Therefore the codes becomes powerful and efficient.
2. The encoders and decoders for cyclic codes are simpler compared to noncyclic codes.
3. Cyclic codes also detect error burst that span many successive bits.

Inspite of these advantages cyclic codes also have some advantages.

Disadvantages

1. The error detection in cyclic codes is simpler but error correction is little complicated since the combinational logic circuits in error detector are complex.

To abiod such complex circuits some special cyclic codes are used which are discussed next.

16.15. BCH Codes (Bose - Chaudhri - Hocquenghem Codes)

BCH codes are most extensive and powerful error correcting cyclic codes. The decoding of BCH codes is comparatively simpler. For any positive integer m and t (where $t < 2^{m-1}$) there exists a BCH code with following parameters:

Block length: $n = 2^{m-1}$

Number of parity check bits: $n - k \leq mt$

Minimum distance: $d_{min} \geq 2t + 1$

The decoding schemes of BCH codes can be implemented on digital computer. Because of software implementation of decoding schemes they are quite flexible compared to hardware implementation of other schemes.

16.16. Convolutional Codes

A convolutional coding is done by combining the fixed number of input bits. The input bits are stored in the fixed length shift register and they are combined with the help of mod-2 adders. This operation is equivalent to binary convolution and hence it is called **convolutional coding**. This concept is illustrated with the help of simple example given below.

Figure 16.12 illustrates a convolutional encoder.

Scanned by CamScanner

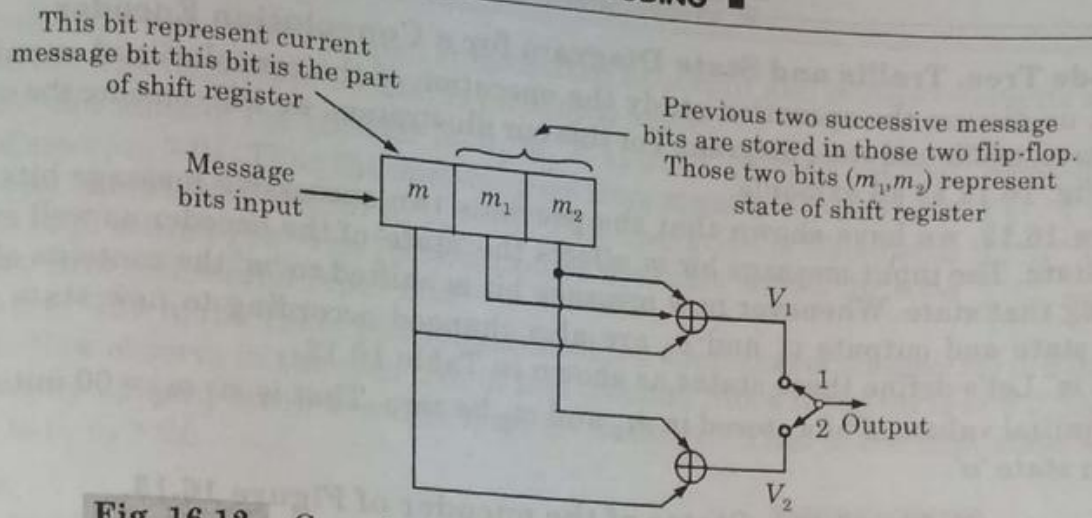


Fig. 16.12. *Convolutional encoder with K = 3, k = 1 and n = 2*

This convolutional encoder operates as under:

Operation

Whenever the message bit is shifted to position 'm', the new values of v_1 and v_2 are generated depending upon m, m_1 and m_2. m_1 and m_2 store the previous two message bits. The current bit is present in m. Hence, we can write,

$$v_1 = m \oplus m_1 \oplus m_2 \quad ...(16.85)$$

and

$$v_2 = m \oplus m_2 \quad ...(16.86)$$

The output switch first samples v_1 and then v_2. The shift register then shifts contents of m_1 to m_2 and contents of m to m_1. Next input bit is then taken and stored in m. Again v_1 and v_2 are generated according to this new combination of m, m_1 and m_2. Hence, output switch then samples v_1 then v_2. Hence, the output bit strem for successive input bits will be,

$$X = v_1\, v_2\, v_1\, v_2\, v_1\, v_2 \text{ ... and so on} \quad ...(16.87)$$

Here it may be noted that for every input message bit two encoded output bits v_1 and v_2 are transmitted. In other words, for a single message bit, the encoded code word is two bits *i.e.*, for this convolutional encoder,

Number of message bits, $k = 1$

Number of encoded output bits for on message bit, $n = 2$

Therefore the code rate of this encoder is,

$$r = \frac{k}{n} = \frac{1}{2} \quad ...(16.88)$$

In the encoder of figure 16.12, observe that whenever a particular message bit enters a shift register, it remains in the shift register for three shifts *i.e.*,

First shift → Message bit is entered in position 'm'.

Second shift → Message bit is shifted in position m_1.

Third shift → Message bit is shifted in position m_2.

And at the forth shift the message bit is discarded or simply lost by overwriting. We know that v_1 and v_2 are combinations of m, m_1, m_2. Since a single message bit remains in m during first shift, in m_1 during second shift and in m_2 during second shift; it influences output v_1 and v_2 for 'there' successive shifts.

Constraint Length (K):

The constraint length of a convolution code is defined as the number of shifts over which a single message bit can influence the encoder output. It is expressed in terms of message bits.

For the encoder of figure 16.13 constraint length $K = 3$ bits. This is because in this encoder, a single message bit influences encoder output for three successive shifts. At the fourth shift, the message bit is lost and it has no effect on the output.

Scanned by CamScanner

16.16.1. Code Tree, Trellis and State Diagram for a Convolution Encoder

Now let us discuss the operation study the operation of the convolutional encoder with the help of code tree, trellis and state diagram. For this our illustration, we will consider the convolutional encoder of Fig. 16.12 as an exmaple.

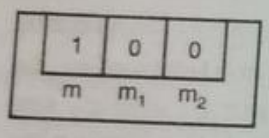
In figure 16.12, we have shown that the previous two successive message bits m_1 and m_2 represents state. The input message bit m affects the 'state' of the encoder as well as outputs v_1 and v_2 during that state. Whenever new message bit is shifted to 'm' the contents of m_1 and m_2 define new state and outputs v_1 and v_2 are also changed according to new state m_1, m_2 and message bit m. Let's define these states as shown in Table 16.12.

Let the initial values of bits stored in m_1 and m_2 be zero. That is $m_1\ m_2 = 00$ initially and the encoder is in state 'a'.

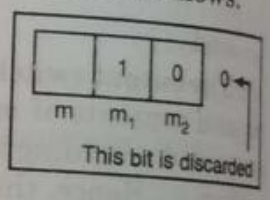
Table 16.14. States of the encoder of Figure 16.13.

m_2	m_1	*State of encoder*
0	0	a
0	1	b
1	0	c
1	1	d

Let the first message input be $m = 1$. With this input v_1 and v_2 will be calculated follows.



$$V_1 = 1 \oplus 0 \oplus 0 = 1$$
$$v_2 = 1 \oplus 0 = 1$$



The values of $v_1\ v_2 = 11$ are transmitted to the output and register contents are shifted to right by one bit position as shown.

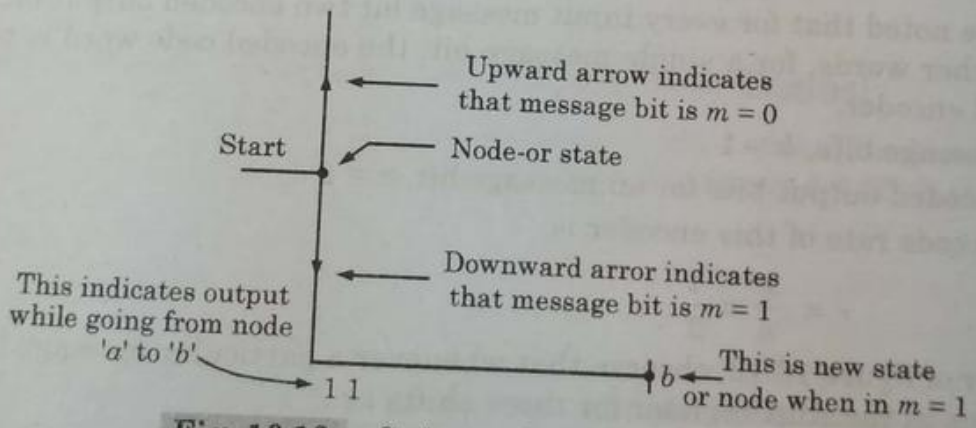


Fig. 16.13. *Code tree from node 'a' to 'b'.*

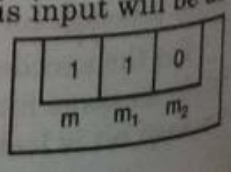
Hence, the new state of encoder is $m_2\ m_1 = 01$ or 'b' and output transmitted are $v_1\ v_2 = 11$. This shows that if encoder is in state 'a' and if input is $m = 1$ then the next state is 'b' and outputs are $v_1\ v_2 = 11$. The first row of Table 16.14 illustrates this operation.

The last column of this table shows the code free diagram. The code free diagram starts at node or state 'a'. This diagram is reproduced as shown in figure 16.13.

It may be observed that if $m = 1$, we go downward from node 'a'. Otherwise if $m = 0$. We go upward from node 'a'. It can be verifed that if $m = 0$ then next node (state) is 'a' only. Since $m = 1$ here we go downwards toward node b and output is 11 in this node (or state).

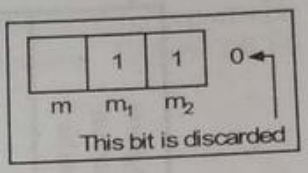
Now let the second message bit be 1. The contents of shift register with this input will be as shown below.

$$v_1 = 1 \oplus 1 \oplus 0 = 0$$
$$v_2 = 1 \oplus 0 = 1$$



Scanned by CamScanner

These values of $v_1\ v_2 = 01$ are then transmitted to output and register contents are shifted to right by one bit. The next state formed is as shown below.



Thus the new state of the encoder is $m_1\ m_2 = 11$ or 'd' and the outputs transmitted are $v_1\ v_2 = 01$. Thus the encoder goes from state 'b' to state 'd' if input is '1' and transmitted output $v_1\ v_2 = 01$. This operation is illustrated by Table 16.14 in second row. The last column of the table shows the code tree for those first and second input bits.

Similarly 3rd row of the Table 16.15 illustrated the operation of encoder for 3rd input message bit as $m = 0$. Now observe in the code tree of last column. Since input bit is $m = 0$, the path of the tree is shown by upward arrow towards node (or state) 'C'. That is the next state is 'C' (*i.e.*, 10) and output is $v_1\ v_2 = 0L$

Code Tree:

Figure 16.15 shows the code tree for this encoder. The code tree starts at node 'a'. If input message bit is '1' then path of the tree goes down towards node 'b' and output is 11. Otherwise if the input is $m = m\ 0$ at node 'a', then path of the tree goes upward towards node 'a' and output is 00. Similarly depending upon the input message bit, the path of the tree goes upward or downward. The nodes are marked with their states a, b, c or d. On the path between two nodes the outputs are shown. We have verified the part of this code tree for first there message bits as 110.

If one carefully observed the code tree of Fig. 16.14, one will find that the branch pattern begins to repeat after third bit. This is shown in figure. The repetition starts after 3rd bit, since any message bit is stored in the shift registers of the encoder for three shifts. If the length of the shift register is increased by one bit, then the pattern of code tree will repeat after fourth message bit.

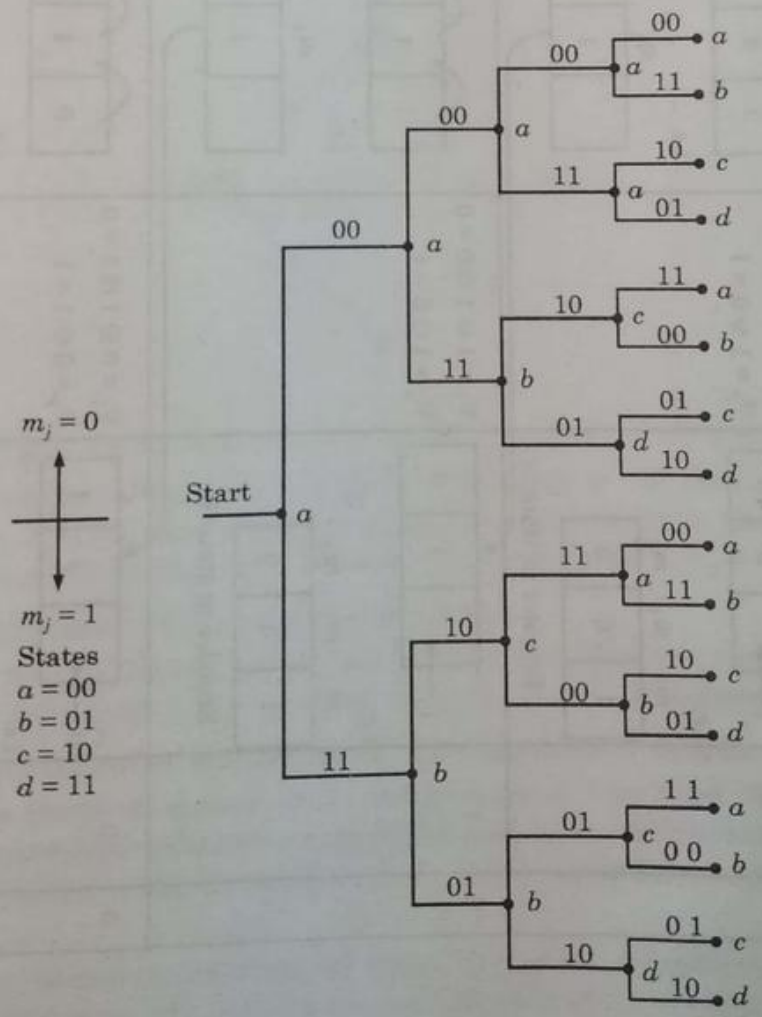
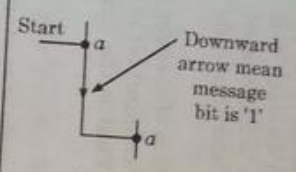
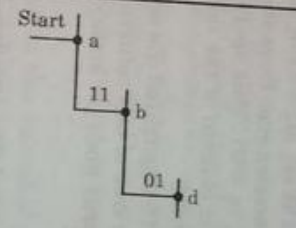
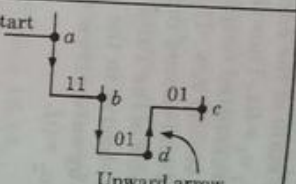


Fig. 16.14. *Code tree for convolutional encoder of figure 16.12.*

Scanned by CamScanner

Table 16.15

Sr. No.	Input message bit M	Status of shift register after entry of m	Calculation of outputs v_1 and v_2	Status of shift register after transmission of o/p and shift right by one bit	New state of recoder m_1 m_2	Trans-mitted output v_1 v_2	Code Tre diagram downward arrow indicated inputis 1
1	1	1 → [ ] [0] [0] i.e. m m_1 m_2 [1] [0] [0] Encoder in state 'a'	$V_1 = 1 \oplus 0 \oplus 0 = 1$ $V_2 = 1 \oplus 0 = 1$	[1] [0] [0] i.e. m_1 m_2 [ ] [1] [0] 0	01 or b	11	 Code tree for $m = 1$
2	1	1 → [ ] [1] [0] i.e. m m_1 m_2 [1] [1] [0] Encoder in state 'b'	$V_1 = 1 \oplus 1 \oplus 0 = 0$ $V_2 = 1 \oplus 0 = 1$	[1] [1] [0] i.e. m_1 m_2 [ ] [1] [1] 0	11 or d	01	 Code tree for $m = 11$
3	0	0 → [ ] [1] [1] i.e. m m_1 m_2 [0] [1] [1] Encoder in state 'c'	$V_1 = 0 \oplus 1 \oplus 1 = 0$ $V_2 = 0 \oplus 1 = 1$	[0] [1] [1] i.e. m_1 m_2 [ ] [0] [1] 1	10 or c	01	 Upward arrow means message bits is '0' Code tree for $m = 110$

Scanned by CamScanner

Code Trellis

Code trellis is the more compact representation of the code tree. We know that in the code tree there are four states (or nodes). Every state goes to some other state depending upon the input code. Trellis represents the signal, an unique diagram for such transitions. Figure 16.15 shows code trellis diagram.

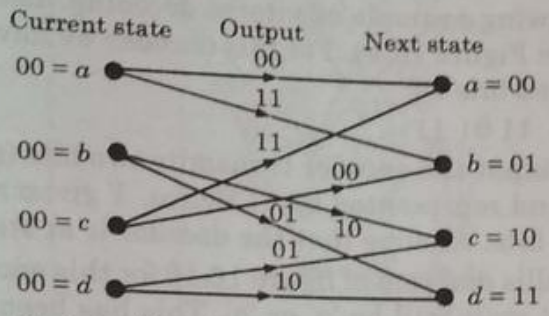


Fig. 16.15. *Code trellis of convolutional encoder of figure 16.13.*

The nodes on the left denote four possible current states and those on the right represent next state. The solid transition line represents input $m = 0$ and broken line represents input $m = 1$. Along with each transition line the output $v_1\ v_2$ is represented during that transition. For example let the encoder be in current state of 'a'. If input $m = 0$, then next state will be 'a' with outputs $v_1\ v_2 = 11$. Thus code trellis is the compact representation of code tree.

State Diagram

If we combine the current and next states, then we obtain state diagram.

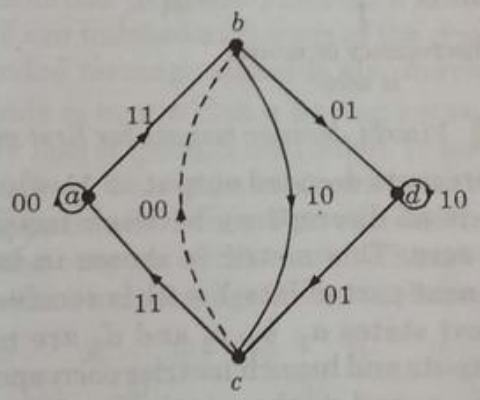


Fig. 16.16. *State diagram for convolutions encoder of figure 16.13.*

For example consider that the encoder in state 'a'. If input $m = 0$, then next state is same *i.e.*, a (*i.e.*, 00) with outputs $v_1\ v_2 = 00$. This is shown by self loop at node 'a' in the state diagram. If input $m = 1$, then state diagram shows that next state is 'b' with outputs $v_1\ v_2 = 11$.

16.16.2. Decoding Methods of Convolutional Codes

These methods are used for decoding of convolutional codes. They are viterbi algorithm, sequential decoding and feedback decoding. Let's consider them in details in subsequent sections.

16.16.2.1. Viterbi Algorithm for Decoding of Convolutional Codes

Let's represent the received signal by Y. Convolutional encoding operates continuously on input data. Hence there are non–code vectors and blocks as such. Let's assume that the transmission error probability of symbols 1's and 0's is same. Let's define an integren variable metric as follows.

Metric

It is the discrepancy between the received signal y and the decoded signal at particular node. This metric may be added over few nodes for a particular path.

Scanned by CamScanner

Surviving Path

This is the path of the decoded signal with minimum metric.

In viterbi decoding, a metric is assigned to each surviving path. (*i.e.* Metric of a particular path is obtained by adding individual metric on the nodes along that path). Y is decoded as the surviving path with smallest metric.

Let us consider the following example of viterbi decoding. Let the signal being received is encoded by the encoder shown Figure 16.14. For this encoder we have obtained code trellis in Fig. 16.16. Let the first six received bits be

$$Y = 11\ 01\ 11$$

Noted that for single bit input the encoder transmits two bits ($v_1\ v_2$) outputs. These outputs are received at the decoder and represented by Y. Hence, Y given above represents the outputs for three successive message bits. Assume that the decoder is at state a_0.

Let us look at the code trellis diagram of figure 16.16 for this encoder. It illustrates that if the current state is 'a', then next state will be 'a' or 'b'. This has been shown in figure 16.17. Two branches are shown from a_0. One branch is at next node a_1 representing decoded. Signal as 00 and other branch is at b_1 representing decoded signal as 11.

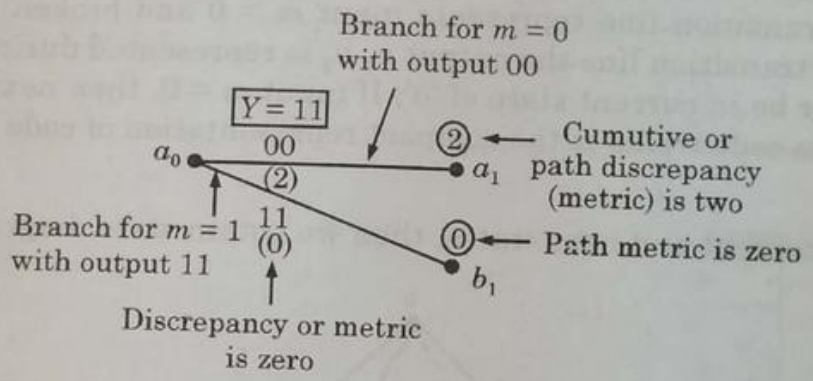


Fig. 16.17. *Viterbi decoder resuots for first message bit.*

The branch from a_0 to b_1 represents decoded output as 11 which is same as received signal at that node *i.e.*, 11. Hence, there is no discrepancy between received signal and decoded signal. Thus 'Metric' of that branch is zero. This metric is shown in brackets along that path metric reaching to the node. When the next part of bits $Y = 01$ is received at nodes a_1 and b_1 then from nodes a_1 and b_1 four possible next states a_2, b_2, c_2 and d_2 are possible. Figure 16.18 shows all these branches, their decoded outputs and branch metrics corresponding to those decoded outputs. The encircled number near a_2, b_2, c_2 and d_2 show path metric emerging from a_0. As an example the path metric of path a_0, a_1, a_2 is 'there'. The path metric of path $a_0\ b_1\ d_2$ is zero. Figure 16.18 shows the trellis diagram for all the six bits for Y.

Figure 16.18 illustrates the possible paths for decoder.

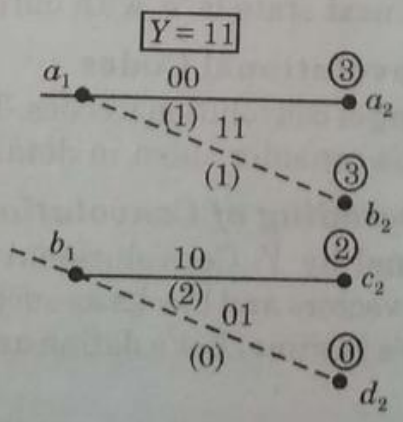


Fig. 16.18. *Viterbi decoder results for second message bit.*

Scanned by CamScanner

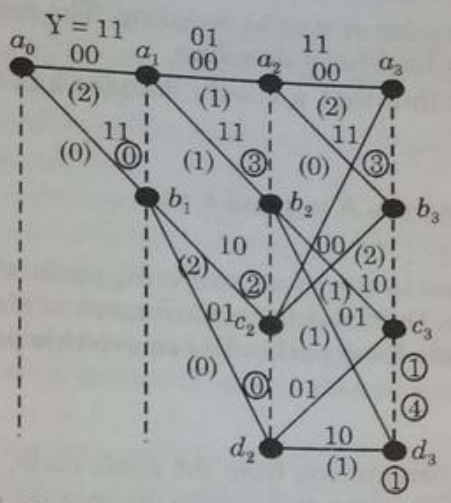


Fig. 16.19. *Paths and their metrics for viterbi decoding.*

Figure 16.19 illustrates the nodes with their path metrics on the right hand side at the end of sixth of Y. Thus two paths are common to node 'a'. One path is $a_0\ a_1\ a_2\ a_3$ with metric 5. The other path is $a_0\ a_1\ a_2\ a_3$ with metric 2. In the same way, there are two paths at other nodes also. According to viterbi decoding, only one path with lower metric should be retained at particular node. As shown in figure 16.19, the paths marked with × (cross) are cancelled since they have higher metrics than other path coming to that particular node. These four paths with lower metrics are stored in the decoder and the decoding continues to next received bits.

Figure 16.21 shows the continuation of figure 16.19 for a message 12 bits. It may be observe that in this figure, received bits Y are marked at the top of the decoded value of output *i.e.*, $Y + E$ is marked at the bottom and decoded message signal is also marked.

Only one path of particular node is kept which is having lower metric. In case if there are two paths have same metric, then any one of them is continued. It may be observed that a node 'a_{12}' only one path arrives with metric two. This path is shown by a thick line. Because this path is lowest metric, it is the surviving path and hence Y is decoded from this path. All the decoded values of output are taken from the outputs of this path. Whenever this path is broken it shows message bit $m = 1$ and if it is continuous, message bit $m = 0$ between two nodes.

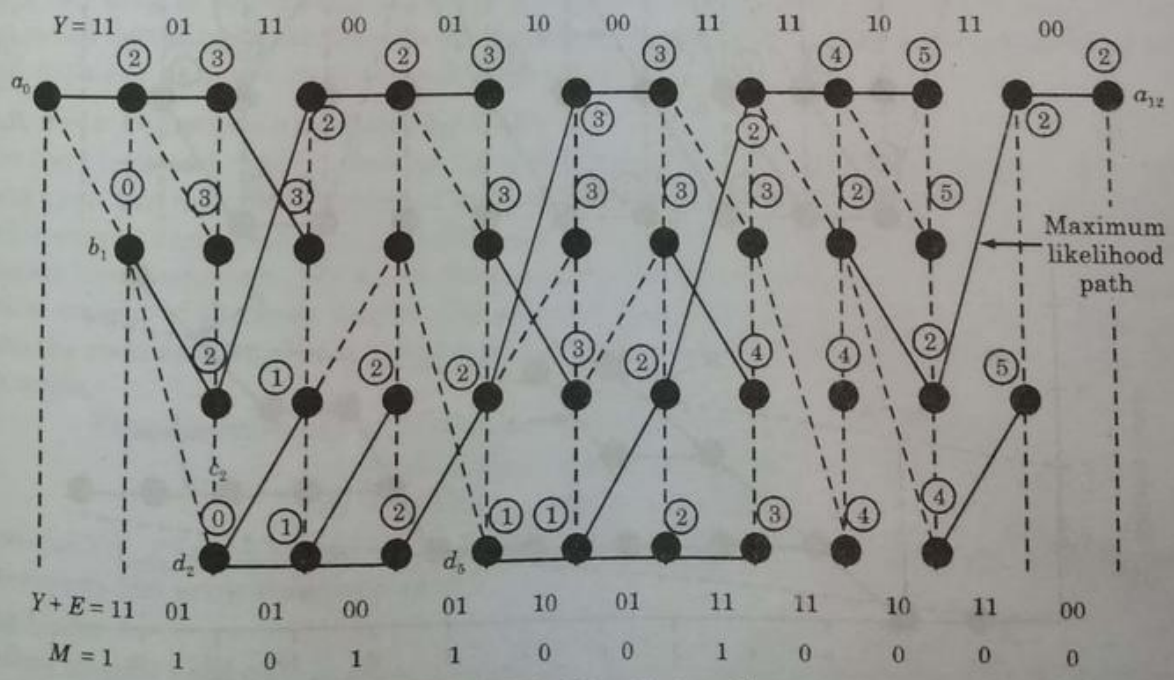


Fig. 16.20. *Viterbi decoding*

Scanned by CamScanner

This is the complete explanation of viterbi decoding. The method of decoding used in viterbi decoding is known as maximum likelihood decoding.

During decoding you would find that a viterbi decoder has to store four survivity paths for four nodes.

Surviving paths = $2^{(K-1)k}$

For a particular encoder, we have $K = 3$ and $k = 1$ ...(16.89)

Thus, Surviving paths = $2^{(3-1)\times 1} = 4$

Hence, the viterbi decoder has to store four surviving paths always. If the number of message bits to be decoded are very large, then storage requirement is also large since the decoder has to store multiple (in present example four) paths. To remove this problem metric diversion effect is used.

Metric Diversion Effect

For the two surviving paths originating from the same node, the running metric of less likely path tends to increase more rapidly than the metric divergence effect. For example consider the two paths coming from node b_1 in figure 16.21. One path comes at a_5 and other path comes at d_5. The path at a_5 is less likely and hence its metric is more compared to the path at d_5. Hence at node d_5 only the surviver path is selected and the message bits are decoded. The fresh paths are started from d_5. Due to this, the memory storage is reduced since complete path need not be stored.

16.16.2.2. Sequential Decoding for Convolutional Codes

(U.P. Tech., Sem. Examination, 2003-2004)

Sequential decoding uses metric divergence effect figure 16.21(*a*) shows the code trellis for the convolutional encoder. The same code trellis we have discussed in the last article following are the important points about sequential decoding.

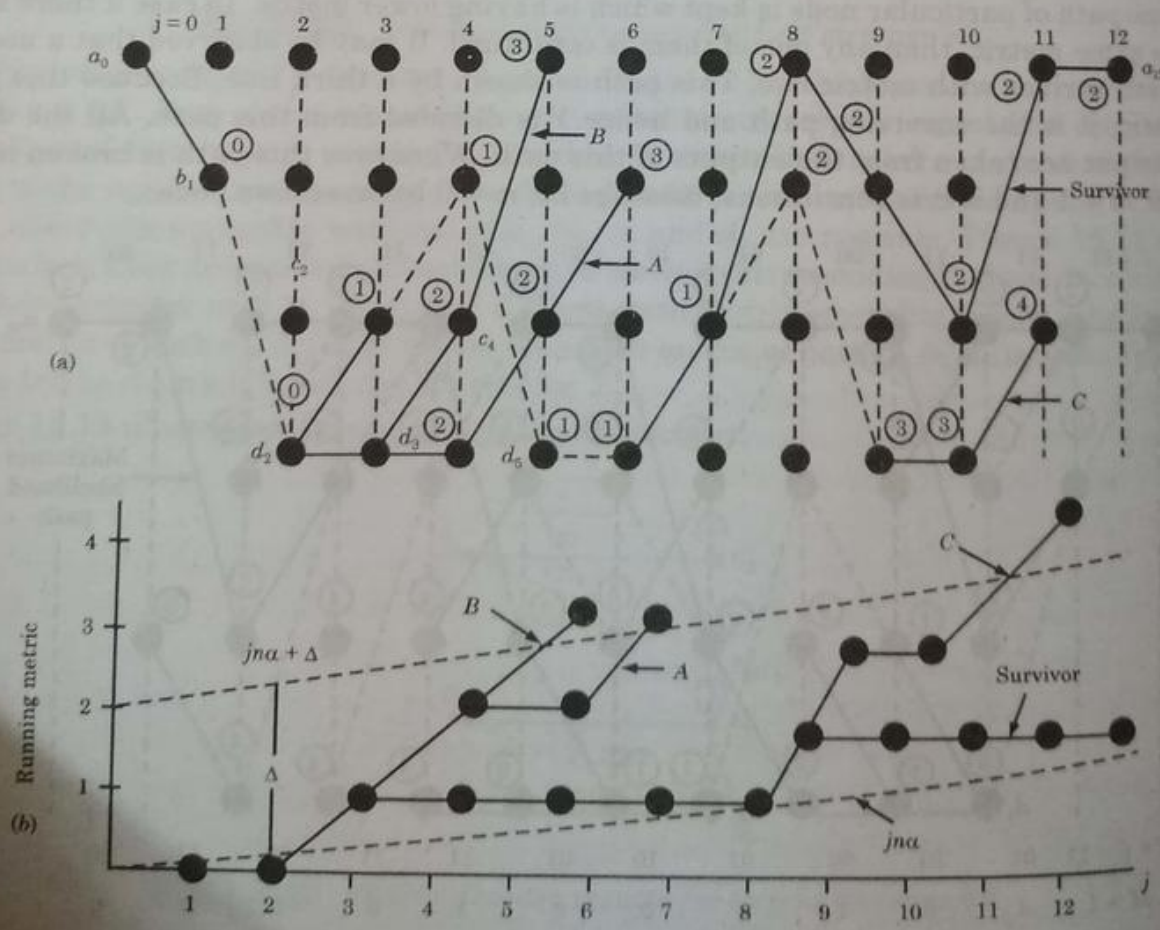


Fig. 16.21. *Sequential Decoding*

Scanned by CamScanner

(*i*) The decoding starts at a_0. It follows the single path by taking the branch with smallest metric. As an example as shown in figure 16.22(*a*), the path for first three nodes is $a_0\ b_1\ b_2$ since its metric is the lowest.

(*ii*) If there are two or more branches from the same node with same metric, then decoder selects any one branch and continues decoding.

(*iii*) From point (*ii*) above we know that if there are two branches from one nodes with equal matrices, then any one is selected at random. If the selected path is found to be unlikely with rapidly increasing merit, then decoder cancels that path and goes back to that node. It then selects other path emerging from that node. As an example observe in the figure 16.22(*a*) that two branches with same metric emerge from node d_2. One path is $d_2\ d_3\ c_4\ a_5$ (or path marked '*B*') with metric '3' at a_5. Hence, decoder drops this path and follows other path.

(*iv*) The decision about dropping a path is based on the expected value of running metric at a given node. Running metric at a particular j^{th} node is given as,

Running metric $= jn\alpha$ ...(16.90)

where j is the node at which metric is to be calculated.

n is the number of encoded output bits for one message bit.

and α is the transmission error probability per bit.

The sequential decoder abandons a path whenever its running metric exceeds $(j\ n\alpha + \Delta)$, where Δ is should be above ina at j^{th} node. Figure 16.22(*b*) shows the running metric at a particular node with respect to number of that node. The two dotted lines shows the range of threshold 'Δ' above $jn\alpha$ at a particular node. It may be observed that since metric of path 'β' exceeds the threshold at 5th node, it is abandoned and decoder starts from node '*z*' again. In the same way path '*A*' is also abandoned.

(*v*) If the running metric of every path goes out of threshold limits then the value of threshold 'Δ' is increased and decoder tries back again. In figure 16.22(*b*) the value of $\alpha = 1/16$, for encoder of figure 16.13, we know that $n = 2$. Lets calculate $jn\ \alpha$ at 8th node.

At 8th node $jn\alpha = 8 \times 2 \times 1/16 = 1$. The value of $\Delta = 2$. Hence, threshold will be,

Threshold $= jn\alpha + \Delta = 1 + 2 = 3$ at 8th node. Similarly the threshold at other nodes can be calculated. The computations involved in sequential decoding are less than viterbi decoding. However, the back tracking in sequential decoding is complex. The output error probability is more in case of sequential decoding. Both the viterbi decoding the sequential decoding methods may be implemented with the help of computer software efficiency.

16.16.3. Free Distance and Coding Gain

For the block and cyclic codes we have discussed that the error correction or detection power depends upon the minimum distance between the code vectors. A convolutional encoder does not divide the output encoded signal into different code vector. Let us consider that because X represent the transmitted sequence. We know that minimum distance between the code vector is equal to minimum weight of the code vector. Therefore a free distance is equal to the minimum distance between the code vectors. Since minimum distance is equal to minimum weight of the code vector we can write,

$$\text{Free distance } (d_f) = \text{Minimum distance between code vectors}$$
$$= \text{Minimum weight of the code vectors}$$

or

$$d_f = [w(X)]_{min} \text{ and } X \text{ is non-zero} \quad ...(16.91)$$

Here $[w(X)]_{min}$ is the minimum weight of code vector. For convolutional coding free distance (d_f) represents the error control power.

Coding Gain

Coding gain is used as a basis of comparison for different coding methods. To get the same bit error rate the coding gain is described as,

Scanned by CamScanner

$$A = \frac{\left(\frac{E_b}{N_0}\right) Encoded}{\left(\frac{E_b}{N_0}\right) coded} \quad ...(16.92)$$

For convolutional coding the coding gain is expressed as,

$$A = \frac{rd_f}{2} \quad ...(16.93)$$

here 'r' is the code rate
and d_f is the free distance.

Example 16.18. The figure 16.22 depicts a rate $\frac{1}{2}$, constraint length $N = 2$ convolutional code encoder. Sketch the code tree for the same.

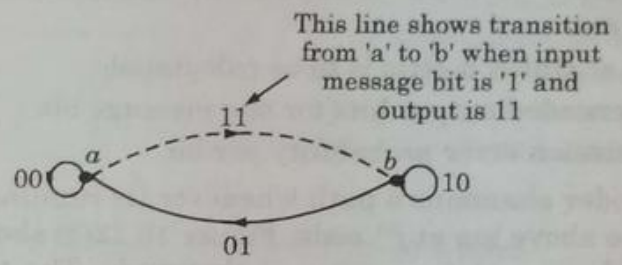


Fig. 16.22. *Convolutional encoder*

Solution: The constraint length of the given convolutional encoder is $K = 2$. Its rate is $\frac{1}{2}$ means for single message bit input, two bits v_1 and v_2 are encoded at the output 'S_1' represents the input message bit and S_2 stores the previous message bit. Because, only one previous message bit is stored, this encoder can have states depending on this stored message bit. Let us represent.

$s_2 = 0$ state '0'

and $s_2 = 1$ state 'b'

Let us assume that the contents of s_1 of s_2, are zero initially. From encoder of, it is clear that outputs v_1 and v_2 are expressed as,

$$v_1 = s_1$$
$$v_2 = s_1 \oplus s_2 \quad ...(i)$$

Before drawing code tree, we would first prepare the state diagram for this encoder. State diagram represents the compact version of code tree. Table 16.15 shows the present and next states corresponding to different inputs of message signal. First row of the table shows that if present state of encoder (which is defined by s_2) is 'a' and if input s_1 (*i.e.*, m) = 0, then outputs v_1 v_2 = 00 and next state of encoder is 'a' and input is 1 then contents of s_1 s_2 = 10. Then the output v_1 v_2 = 11 and next state of encoder is 'b' similarly other two rows represent how encoder operates if its present state is 'b'.

We can draw the state diagram as shown in figure 16.23.

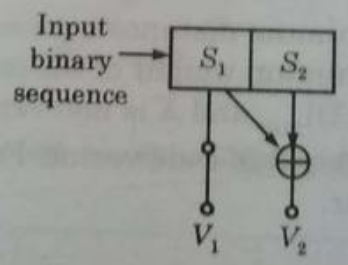


Fig. 16.23. *State diagram for the encoder of figure 16.22. Continuous line represent input as '0' and dotted line represents input as '1'*

Scanned by CamScanner

As shown in state diagram, encoder remains in state 'a' if input is zero. In this diagram continuous lines are used when input message bit is '0' and dotted lines are used when message bit is '1'. The arrow on the line shows the transition towards next state. The numbers marked on line represent outputs v_1 v_2 during that transition.

The code tree is derived from the state diagram is shown in figure 16.23. Observe that the branch pattern of the code tree repeats after two successive message bits. This is due to the fact that any message bit remains stored in the encoder register for two successive shifts.

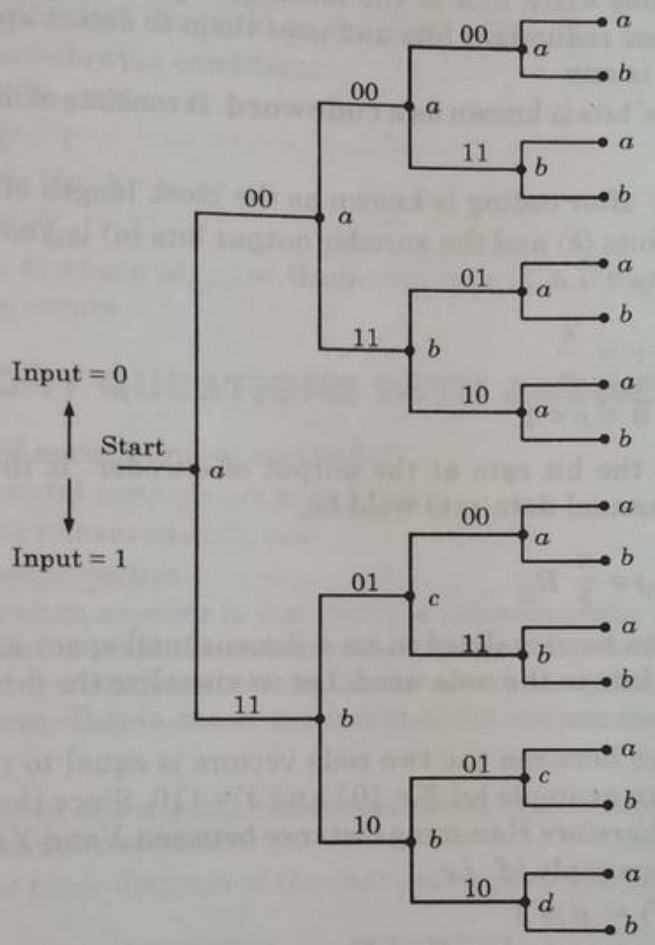


Fig. 16.24. *Code tree for encoder of figure 16.22.*

16.16.4. Advantages and Disdvantages of Convolutional Codes

Convolutional codes may be designed to detect or correct the errors. Some convolutional codes are available which are used to correct random errors and bursts. Convolutional codes have some advantages over block codes.

Advantages

(i) The decoding delay is small in convolutional codes since they operate on smaller blocks of data.

(ii) The storage hardware required by convolutional decoder is less since the block sizes are smaller.

(iii) Synchronization problem does not affect the performance of convolutional codes.

Disadvantages

(i) Convolutional codes are difficult to analyse since their analysis is complex.

(ii) Convolutional codes are not developed much as compared to block codes.

Scanned by CamScanner

SUMMARY

1. The transmitted signal passes through some noisy channel. Due to noise interference, some errors are introduced in the received data. These errors can be detected and sometimes corrected using coding techniques. Generally the error control methods are of two types:
   (*i*) Error detection with retransmission, and
   (*ii*) Forward acting error correction.
2. A channel encoder adds extra bits to the message signal. On the other hand, the channel decoder identifies these redundant bits and uses them to detect and correct the errors in the message bits if there is any.
3. The encoded block of 'n' bits is known as a **codeword**. It consists of message bits and redundant bits.
4. The number of bits 'n' after coding is known as the block length of the code.
5. The ratio of message bits (k) and the encoder output bits (n) is known as **code rate**. Usually code rate is denoted by 'r' *i.e.*,

$$r = \frac{k}{n}$$

   We find that $0 < r < 1$.
6. Channel data rate is the bit rate at the output of encoder. If the bit rate at the input of encoder is R_S, then channel data rate wold be,

$$\text{Channel data rate } (R_0) = \frac{n}{k} R_S$$

7. An 'n' bit code word can be visualized in an n-dimensional space as a vector whose elements or coordinates are the bits in the code word. Let us visualize the 3-bit code words as it is quite simpler.
8. The Hamming distance between the two code vectors is equal to the number of elements in which they differ. As an example let $X = 101$ and $Y = 110$. Since the two code vectors differ in second and third bits therefore Hamming distance between X and Y is 'two'. Hamming distance is denoted by $d(X, Y)$ or simply 'd'. *i.e*,

$$d(X, Y) = d = 2.$$

9. Minimum distance (d_{min}) is the smallest Hamming distance between the valid code vectors.
10. It may be noted that the error detection is possible if the received vector is not equal to some other code vector. This shows that the transmission errors in the received code vector must be less than minimum distance d_{min}. The Table 16.2 shows some of the requirements of error control capability of the code.
11. The code efficiency is expressed as the ratio of message bits in a block to the transmitted bits for that particular block by the encode *i.e.*,

$$\text{Code efficiency} = \frac{\text{Message bits in a block}}{\text{Transmitted bits for the block}}.$$

12. The number of non-zero elements in the transmitted code vector is known as **vector weight**.
13. Parity may be of two types:
    (*i*) Even Parity: The parity of a binary word is known as 'even' if it contains even number of 1's. For example 01101001 has even parity since there are four number of 1's in the word.
    (*ii*) Even Parity: The parity of a binary word is known as 'odd' if it contains odd number of 1's. For example 10010001 has odd parity because there are three number of 1s in this word.

Scanned by CamScanner