Signals and Systems

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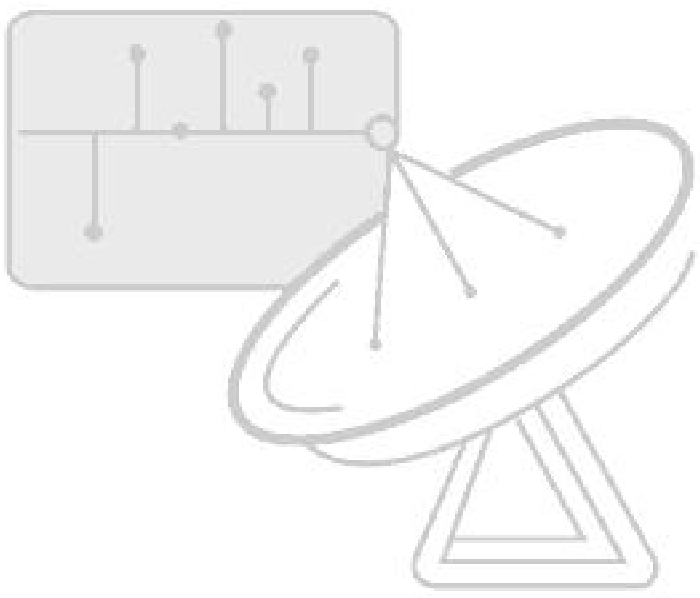
THIRD EDITION



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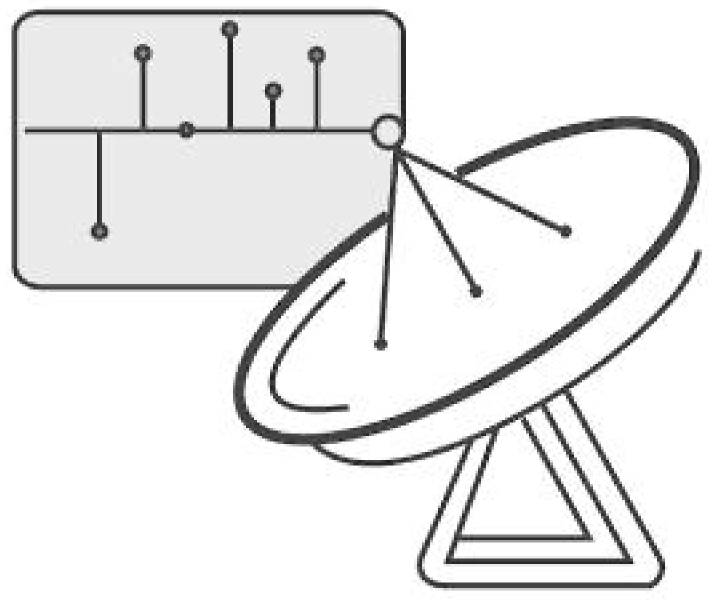

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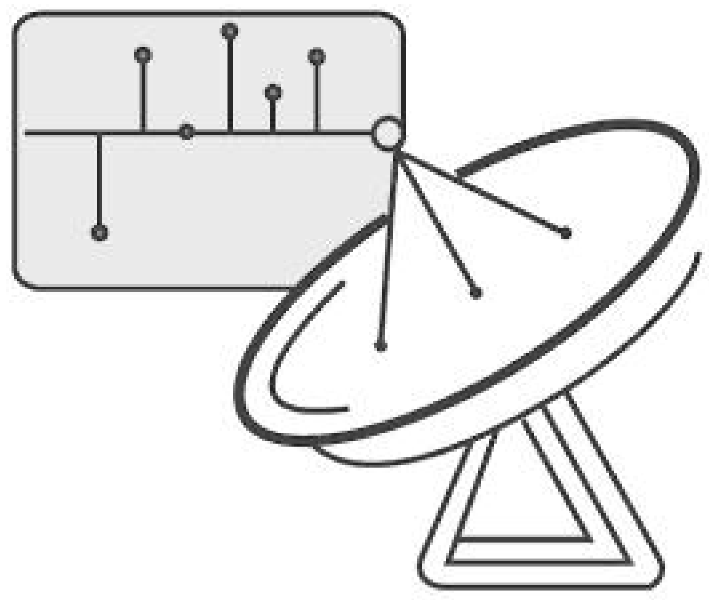

Preface

Reflecting over 37 years of experience in the class room, this comprehensive textbook on Signals and Systems is developed to provide a solid grounding in the foundations of this subject. Using a student-friendly writing style, the text introduces the reader to the concepts of signals and systems in a simple and lucid manner. The text is suitable for use as one semester course material by undergraduate students of Electronics and Communication Engineering, Telecommunication Engineering, Electronics and Instrumentation Engineering and Electrical and Electronics Engineering. It will also be useful to AMIE and grad IETE courses.

The third edition of this book is organised in 12 chapters. The outline of the book is as follows:

Signals constitute an important part of our daily life. Standard signals, basic operations on signals and classification of signals are discussed in Chapter 1. Also operations on signals and determination of the type of a given signal are illustrated with numerous examples.

A system is defined as an entity that acts on an input signal and transforms it into an output signal. Classification of the systems is covered in Chapter 2 along with the numerical examples on determination of the type of a given system.

A new problem can be understood very easily when an analogy can be found between this and a familiar phenomenon. Using concepts of vectors, signal analysis is performed in Chapter 3.

Periodic signals can be easily analysed using Fourier series. Fourier series representation of periodic signals in trigonometric form, cosine form and exponential form, and the conversion from one form to another are devoted to Chapter 4. The properties of Fourier series and Fourier spectrum are also discussed in this chapter.

Signal analysis becomes very easy in frequency domain. Fourier transform is a transformation technique to transform periodic and aperiodic signals from continuous time domain into frequency domain. Fourier transform of various signals, properties of Fourier transform, Fourier spectrum and system analysis using Fourier transform are discussed in Chapter 5. Hilbert transformation is also introduced.

Transmission of signals through linear systems is very important. The filter characteristics of linear time invariant systems, distortionless transmission through linear time invariant systems, signal bandwidth and system bandwidth, various types of filters and time domain and frequency domain criterion for physical realizability are described in Chapter 6.

Convolution and correlation of signals are very important in communication. Convolution is a mathematical way of combining two signals to form a third signal. Correlation, which is similar to convolution, compares two signals to determine the degree of similarity between them. Determination of convolution of two signals by algebraic method and graphical method, cross correlation and autocorrelation of signals, power spectral density and energy spectral density are also covered in Chapter 7.

Digital communication is more advantageous as compared to analog one. The process of converting a continuous-time signal into a discrete-time signal is called sampling. Sampling theorem, types of sampling, band pass sampling theorem, etc. are discussed in Chapter 8.

Laplace transform is a very powerful mathematical technique for analysis of continuous-time systems. Unilateral and bilateral Laplace transform, Inverse Laplace transform, ROC, its properties, properties and theorems of Laplace transform, solution of differential equations, waveform synthesis and circuit analysis using Laplace transform are discussed in Chapter 9.

Z-transform is a very powerful mathematical technique for analysis of discrete-time systems. Unilateral and bilateral Z-transform, Inverse Z-transform, ROC, its properties, properties and theorems of Z-transform, solution of difference equations using Z-transform are discussed in Chapter 10.

Systems may be continuous-time systems or discrete-time systems. Realization of a system means obtaining a network corresponding to the differential equation or difference equation or transfer function of the system. Various methods of realization of continuous-time systems are discussed in Chapter 11.

Discrete-time Fourier transform (DTFT) is a method of representing a discrete-time signal in frequency domain. It is popular for digital signal processing because using this the complicated convolution operation of two sequences in time domain can be converted into a much simpler operation of multiplication in frequency domain. The DTFT, its properties and its use in the analysis of signals are discussed in Chapter 12.

Matlab programs have been included at the end of each chapter to enable the students to practice and test and to get clear concepts.

A large number of typical examples have been worked out, so that the reader can understand the related concepts clearly. Extensive short questions with answers are given at the end of each chapter to enable the students to prepare for the examinations very thoroughly. Review questions, fill in the blank type questions, objective type multiple choice questions and numerical problems are included at the end of each chapter to enable the students to build a clear understanding of the subject matter discussed in the text and also to assess their learning. The answers to all these are also given at the end of the book. Almost all the solved and unsolved problems presented in this book have been class room tested.

I express my profound gratitude to all those without whose assistance and cooperation, this book would not have been successfully completed. I thank Smt. G. Bhavani of Sasi Institute of Technology and Engineering, Tadepalligudem for typing the manuscript and drawing all the figures.

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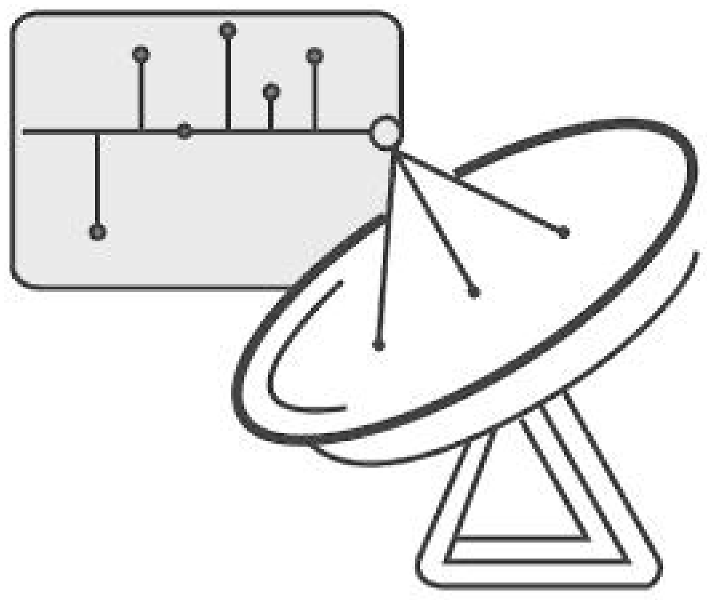
A. Anand Kumar

Symbols, Notations and Abbreviations



SYMBOLS AND NOTATIONS

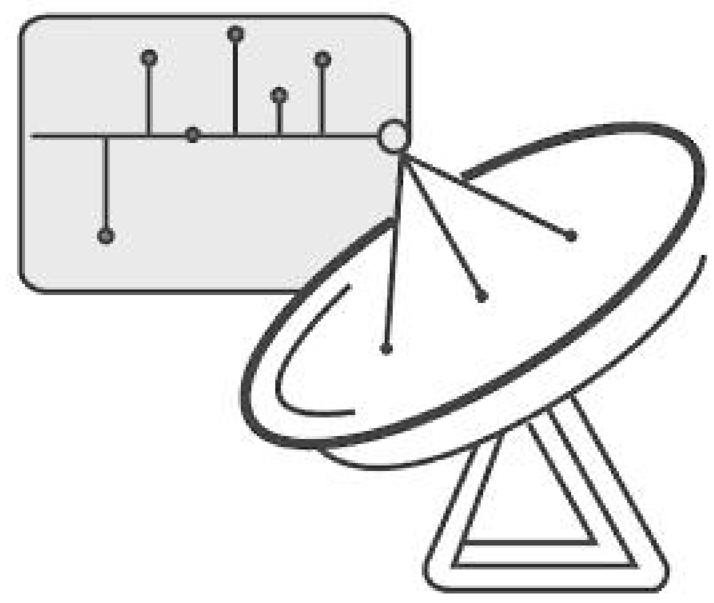
Ω	Ohm
H	Henry
F	Farad
$*$	Linear convolution
$\otimes$	Correlation
$\oint$	Integration over a closed contour
T	Transformation
$x^*(\mathrm{t})$	Complex conjugate of $x(t)$
L	Laplace transform
L^{-1}	Inverse Laplace transform
F	Fourier transform
F^{-1}	Inverse Fourier transform
Z	Z-transform
Z^{-1}	Inverse Z-transform
$\longleftrightarrow$	Used for indicating a transform pair
lc letter	$x(t)$ or $x(n)$ for time signal
uc letter	$X(\omega)$, $X(s)$ or $X(z)$ for the transformed signal
$\lvert a \rvert$	Magnitude of the complex quantity a. Absolute value of a, if a is real valued
arg (z) or $\angle z$	Angle of the complex quantity
$\delta(t)$	Unit-impulse function

$u(t)$	Unit-step function
$r(t)$	Unit-ramp function
$\delta(n)$	Unit-sample sequence
$u(n)$	Unit-step sequence
$r(n)$	Unit-ramp sequence
$\Pi(t/\tau)$ or rect (t/τ)	Unit rectangular pulse
$\Delta(t/\tau)$	Unit triangular pulse
sgn(t)	Signum function

ABBREVIATIONS

BIBO	Bounded-input Bounded-output
BPF	Band-pass filter
BRF	Band-reject filter
BW	Band width
CT	Continuous-time
CTFS	Continuous-time Fourier series
CTFT	Continuous-time Fourier transform
DFT	Discrete Fourier transform
DTFT	Discrete-time Fourier transform
FIR	Finite impulse response
FS	Fourier series
HPF	High-pass filter
Hz	Hertz
IDTFT	Inverse discrete-time Fourier transform
IIR	Infinite impulse response
Im(s)	Imaginary part of s
KVL	Kirchhoff's voltage law
LCM	Least common multiple
LPF	Low-pass filter
LSI	Linear shift invariant
LTI	Linear time invariant
LTV	Linear time variant
Re (s)	Real part of s
RMS	Root mean square
ROC	Region of convergence
S/H	Sample and hold
sps	Sample per second
ZOH	Zero-order hold

1



Signals

1.1 INTRODUCTION

Anything that carries information can be called a signal. Signals constitute an important part of our daily life. A signal is defined as a single-valued function of one or more independent variables which contain some information A signal may also be defined as any physical quantity that varies with time, space or any other independent variable. A signal may be represented in time domain or frequency domain. Human speech is a familiar example of a signal. Electric current and voltage are also examples of signals. A signal can be a function of one or more independent variables. A signal may be a function of time, temperature, position, pressure, distance, etc. If a signal depends on only one independent variable, it is called a *one-dimensional signal*, and if a signal depends on two independent variables, it is called a *two-dimensional signal*. In this book we discuss only about one-dimensional signals. In this chapter we discuss about various basic signals available, various operations on signals and classification of signals.

1.2 REPRESENTATION OF DISCRETE-TIME SIGNALS

In general signals may be continuous-time signals or discrete-time signals. Continuous-time signals are defined for all instants of time, whereas discrete-time signals are defined only at discrete instants of time. Continuous-time signals are represented by $x(t)$ and discrete-time signals are represented by $x(n)$ where t and n are independent variables in time domain. Continuous-time signals are represented by a function or a graph.

There are four ways of representing discrete-time signals. They are:

1. Graphical representation
2. Functional representation
3. Tabular representation
4. Sequence representation

1.2.1 Graphical Representation

Consider a signal $x(n)$ with values

$$x(-2) = -3,\quad x(-1) = 2,\quad x(0) = 0,\quad x(1) = 3,\quad x(2) = 1 \quad \text{and} \quad x(3) = 2$$

This discrete-time signal can be represented graphically as shown in Figure 1.1.

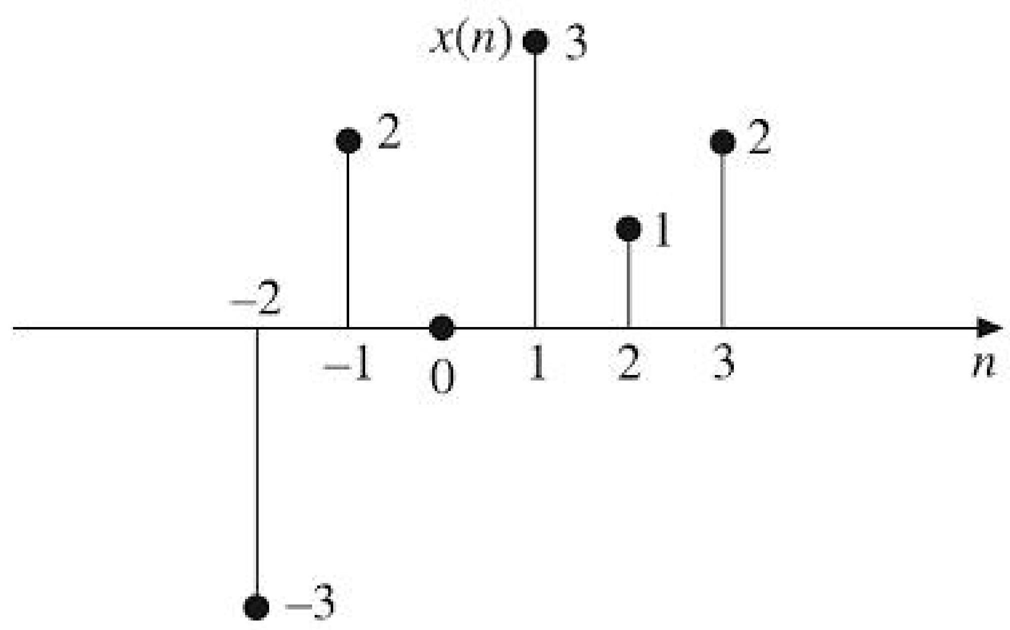


Figure 1.1 Graphical representation of discrete-time signal.

1.2.2 Functional Representation

In this, the amplitude of the signal is written against the values of n. The signal given in 1.2.1 can be represented using functional representation as given below.

$$x(n) = \begin{cases} -3 & \text{for } n = -2 \\ 2 & \text{for } n = -1 \\ 0 & \text{for } n = 0 \\ 3 & \text{for } n = 1 \\ 1 & \text{for } n = 2 \\ 2 & \text{for } n = 3 \end{cases}$$

Another example is

$$x(n) = \begin{cases} 2^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

1.2.3 Tabular Representation

In this, the sampling instant n and the magnitude of the signal at the sampling instant are represented in tabular form. The signal given in 1.2.1 can be represented in tabular form as shown below.

n	−2	−1	0	1	2	3
$x(n)$	−3	2	0	3	1	2

1.2.4 Sequence Representation

A finite duration sequence given in 1.2.1 can be represented as:

$$x(n) = \left\{ \begin{matrix} -3, 2, 0, 3, 1, 2 \\ \uparrow \end{matrix} \right\}$$

Another example is

$$x(n) = \left\{ \begin{matrix} \ldots 2, 3, 0, 1, -2, \ldots \\ \uparrow \end{matrix} \right\}$$

The arrow mark $\uparrow$ denotes the $n = 0$ term. When no arrow is indicated, the first term corresponds to $n = 0$

So a finite duration sequence, that satisfies the condition $x(n) = 0$ for $n < 0$ can be represented as $x(n) = \{3, 5, 2, 1, 4, 7\}$.

Sum and product of discrete-time sequences

The sum of two discrete-time sequences is obtained by adding the corresponding elements of sequences

$$\{C_n\} = \{a_n\} + \{b_n\} \quad \rightarrow \quad C_n = a_n + b_n$$

The product of two discrete-time sequences is obtained by multiplying the corresponding elements of the sequences.

$$\{C_n\} = \{a_n\}\{b_n\} \quad \rightarrow \quad C_n = a_n b_n$$

The multiplication of a sequence by a constant k is obtained by multiplying each element of the sequence by that constant.

$$\{C_n\} = k\{a_n\} \quad \rightarrow \quad C_n = k a_n$$

1.3 ELEMENTARY SIGNALS

There are several elementary signals which play vital role in the study of signals and systems. These elementary signals serve as basic building blocks for the construction of more complex signals. Infact, these elementary signals may be used to model a large number of physical signals which occur in nature. These elementary signals are also called standard signals.

The standard signals are:

1. Unit step function
2. Unit ramp function
3. Unit parabolic function
4. Unit impulse function
5. Sinusoidal function
6. Real exponential function
7. Complex exponential function, etc.

1.3.1 Unit Step Function

The step function is an important signal used for analysis of many systems. The step function is that type of elementary function which exists only for positive time and is zero for negative time. It is equivalent to applying a signal whose amplitude suddenly changes and remains constant forever after application.

If a step function has unity magnitude, then it is called unit step function. The usefulness of the unit-step function lies in the fact that if we want a signal to start at $t = 0$, so that it may have a value of zero for $t < 0$, we only need to multiply the given signal with unit step function $u(t)$. A unit step function is useful as a test signal because the response of the system for a unit step reveals a great deal about how quickly the system responds to a sudden change in the input signal.

The continuous-time unit step function $u(t)$ is defined as:

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

From the above equation for $u(t)$, we can observe that when the argument t in $u(t)$ is less than zero, then the unit step function is zero, and when the argument t in $u(t)$ is greater than or equal to zero, then the unit step function is unity.

The shifted unit step function $u(t - a)$ is defined as:

$$u(t-a) = \begin{cases} 1 & \text{for } t \geq a \\ 0 & \text{for } t < a \end{cases}$$

It is zero if the argument $(t - a) < 0$ and equal to 1 if the argument $(t - a) \geq 0$.
The graphical representations of $u(t)$ and $u(t - a)$ are shown in Figure 1.2[(a) and (b)].

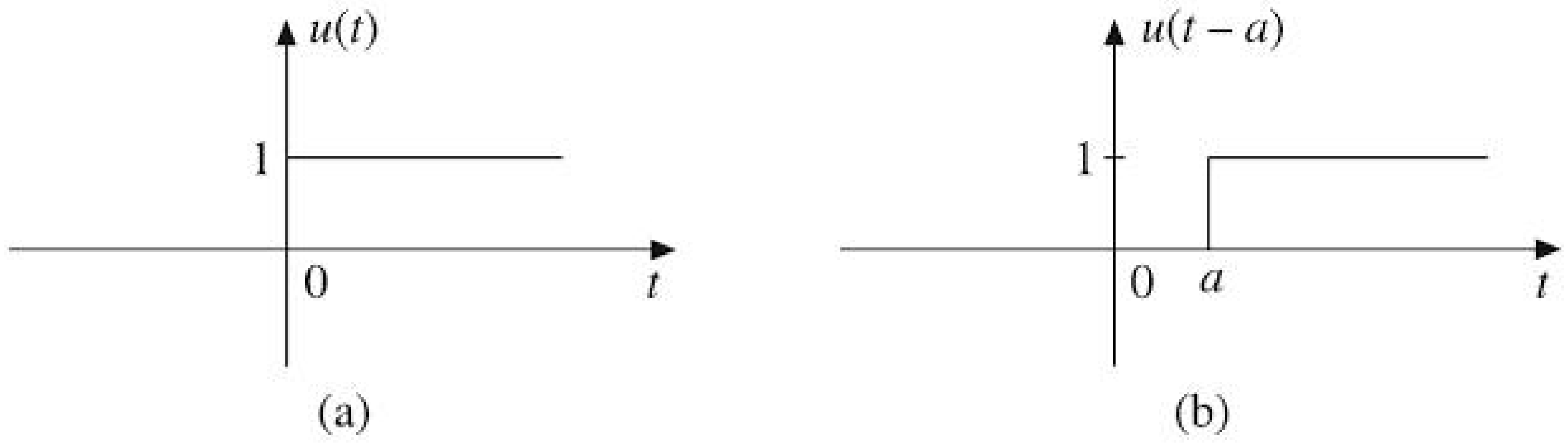


Figure 1.2 (a) Unit step function, (b) Delayed unit step function.

The discrete-time unit step sequence $u(n)$ is defined as:

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

The shifted version of the discrete-time unit step sequence $u(n - k)$ is defined as

$$u(n-k) = \begin{cases} 1 & \text{for } n \geq k \\ 0 & \text{for } n < k \end{cases}$$

The graphical representations of $u(n)$ and $u(n - k)$ are shown in Figure 1.3[(a) and (b)].

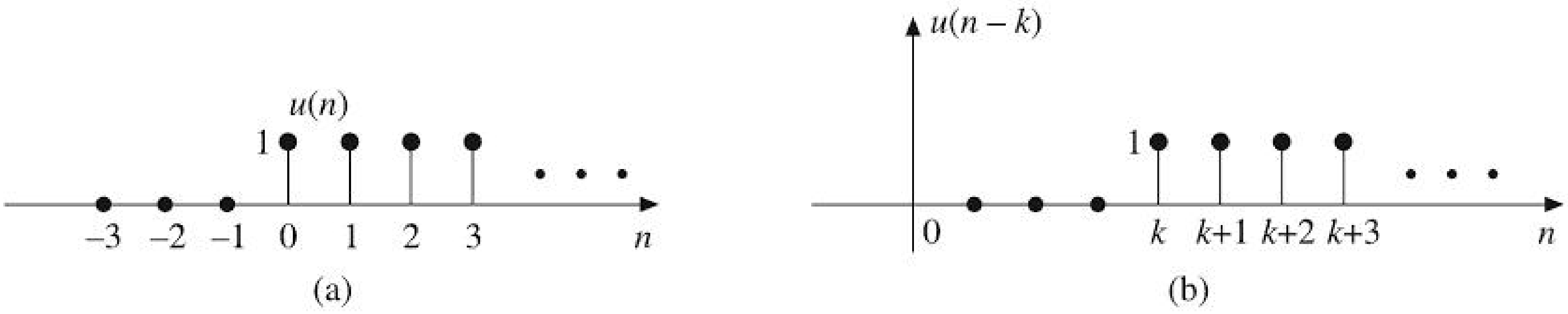


Figure 1.3 Discrete-time (a) Unit step function, (b) Shifted unit step function.

1.3.2 Unit Ramp Function

The continuous-time unit ramp function $r(t)$ is that function which starts at $t = 0$ and increases linearly with time and is defined as:

$$r(t) = \begin{cases} t & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

or

$$r(t) = t\,u(t)$$

The unit ramp function has unit slope. It is a signal whose amplitude varies linearly. It can be obtained by integrating the unit step function. That means, a unit step signal can be obtained by differentiating the unit ramp signal.

i.e.

$$r(t) = \int u(t)\, dt = \int dt = t \qquad \text{for } t \geq 0$$

$$u(t) = \frac{d}{dt} r(t)$$

The delayed unit ramp signal $r(t - a)$ is given by

$$r(t-a) = \begin{cases} t - a & \text{for } t \geq a \\ 0 & \text{for } t < a \end{cases}$$

or

$$r(t-a) = (t-a)\, u(t-a)$$

The graphical representations of $r(t)$ and $r(t - a)$ are shown in Figure 1.4[(a) and (b)].

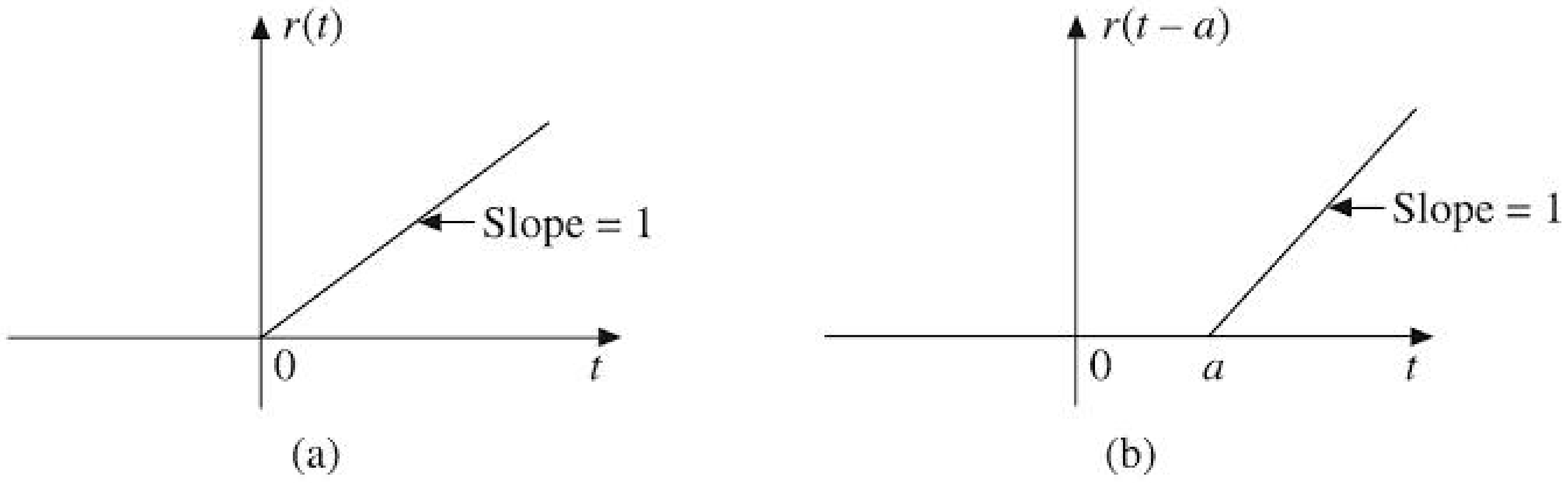


Figure 1.4 (a) Unit ramp signal, (b) Delayed unit ramp signal.

The discrete-time unit ramp sequence $r(n)$ is defined as

$$r(n) = \begin{cases} n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

or

$$r(n) = n\,u(n)$$

The shifted version of the discrete-time unit-ramp sequence $r(n - k)$ is defined as

$$r(n-k) = \begin{cases} n - k & \text{for } n \geq k \\ 0 & \text{for } n < k \end{cases}$$

or

$$r(n-k) = (n-k)\, u(n-k)$$

The graphical representations of $r(n)$ and $r(n-2)$ are shown in Figure 1.5[(a) and (b)].

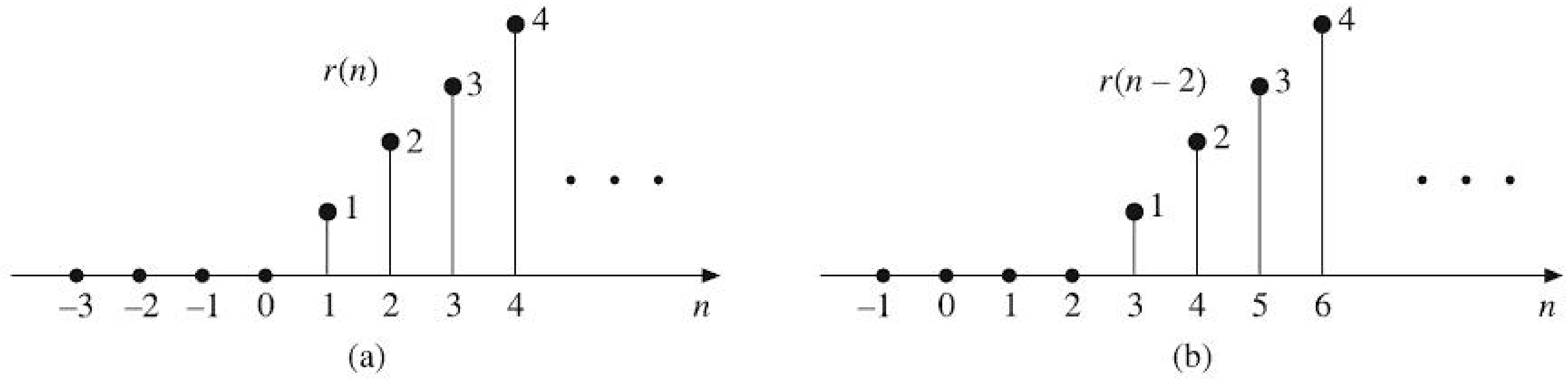


Figure 1.5 Discrete-time (a) Unit-ramp sequence, (b) Shifted-ramp sequence.

1.3.3 Unit Parabolic Function

The continuous-time unit parabolic function $p(t)$, also called unit acceleration signal starts at $t = 0$, and is defined as:

$$p(t) = \begin{cases} \dfrac{t^2}{2} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

or
$$p(t) = \frac{t^2}{2}u(t)$$

The shifted version of the unit parabolic sequence $p(t-a)$ is given by

$$p(t-a) = \begin{cases} \dfrac{(t-a)^2}{2} & \text{for } t \geq a \\ 0 & \text{for } t < a \end{cases}$$

or
$$p(t-a) = \frac{(t-a)^2}{2}u(t-a)$$

The graphical representations of $p(t)$ and $p(t-a)$ are shown in Figure 1.6[(a) and (b)].

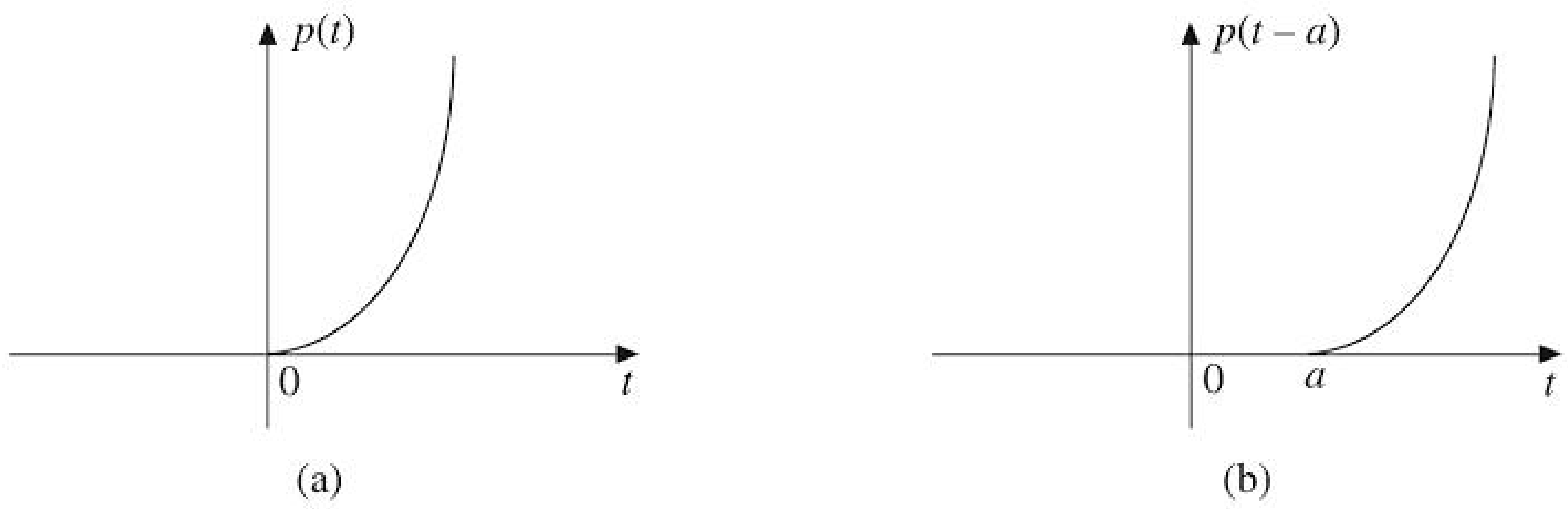


Figure 1.6 (a) Unit parabolic signal, (b) Delayed parabolic signal.

The unit parabolic function can be obtained by integrating the unit ramp function or double integrating the unit step function.

$$p(t) = \iint u(t)\,dt = \int r(t)\,dt = \int t\,dt = \frac{t^2}{2} \quad \text{for } t \geq 0$$

The ramp function is derivative of parabolic function and step function is double derivative of parabolic function

$$r(t) = \frac{d}{dt} p(t); \quad u(t) = \frac{d^2}{dt^2} p(t)$$

The discrete-time unit parabolic sequence $p(n)$ is defined as:

$$p(n) = \begin{cases} \frac{n^2}{2} & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

or

$$p(n) = \frac{n^2}{2} u(n)$$

The shifted version of the discrete-time unit parabolic sequence $p(n - k)$ is defined as:

$$p(n-k) = \begin{cases} \frac{(n-k)^2}{2} & \text{for } n \geq k \\ 0 & \text{for } n < k \end{cases}$$

or

$$p(n-k) = \frac{(n-k)^2}{2} u(n-k)$$

The graphical representations of $p(n)$ and $p(n - 3)$ are shown in Figure 1.7[(a) and (b)].

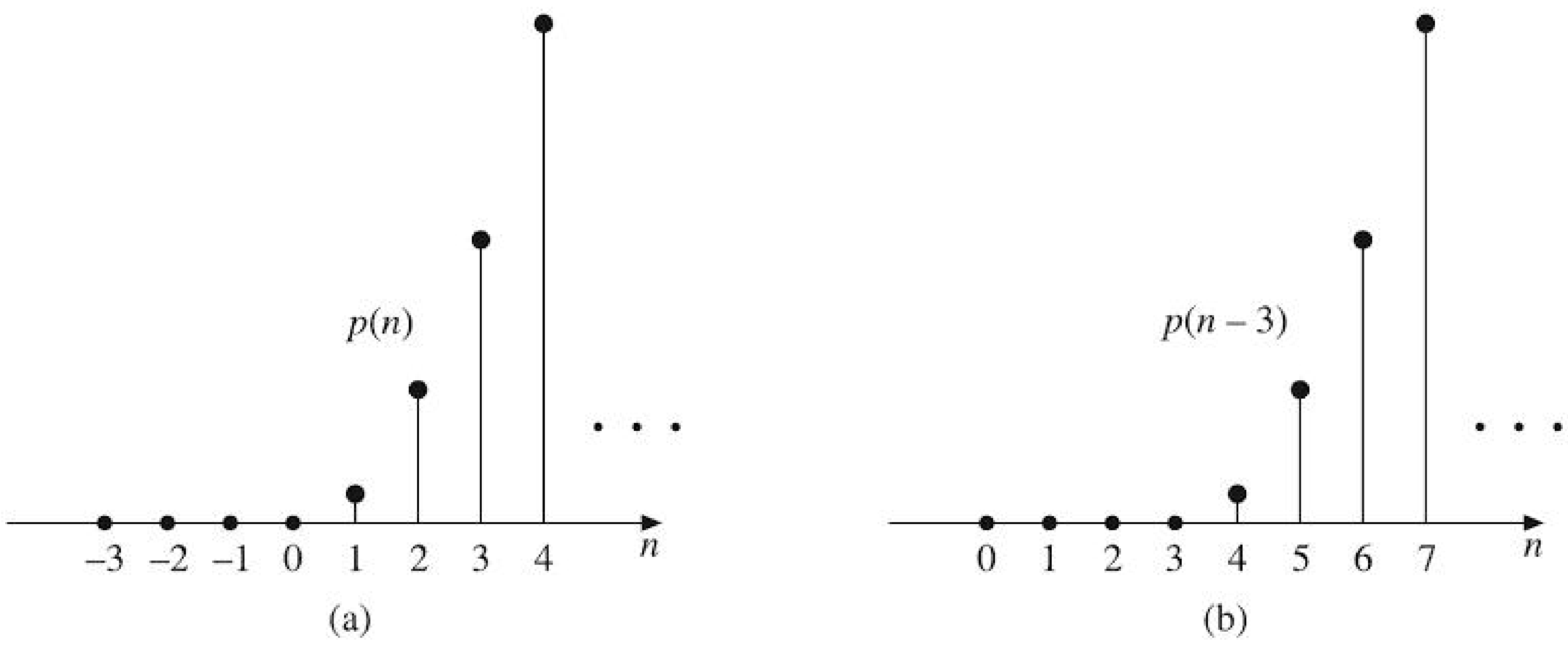


Figure 1.7 Discrete-time (a) Parabolic sequence, (b) Shifted parabolic sequence.

1.3.4 Unit Impulse Function

The unit impulse function is the most widely used elementary function used in the analysis of signals and systems. The continuous-time unit impulse function $\delta(t)$, also called Dirac delta function, plays an important role in signal analysis. It is defined as:

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

and $$\delta(t) = 0 \quad \text{for } t \neq 0$$

i.e. as $$\delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$

That is, the impulse function has zero amplitude everywhere except at $t = 0$. At $t = 0$, the amplitude is infinity so that the area under the curve is unity. $\delta(t)$ can be represented as a limiting case of a rectangular pulse function.

As shown in Figure 1.8(a),

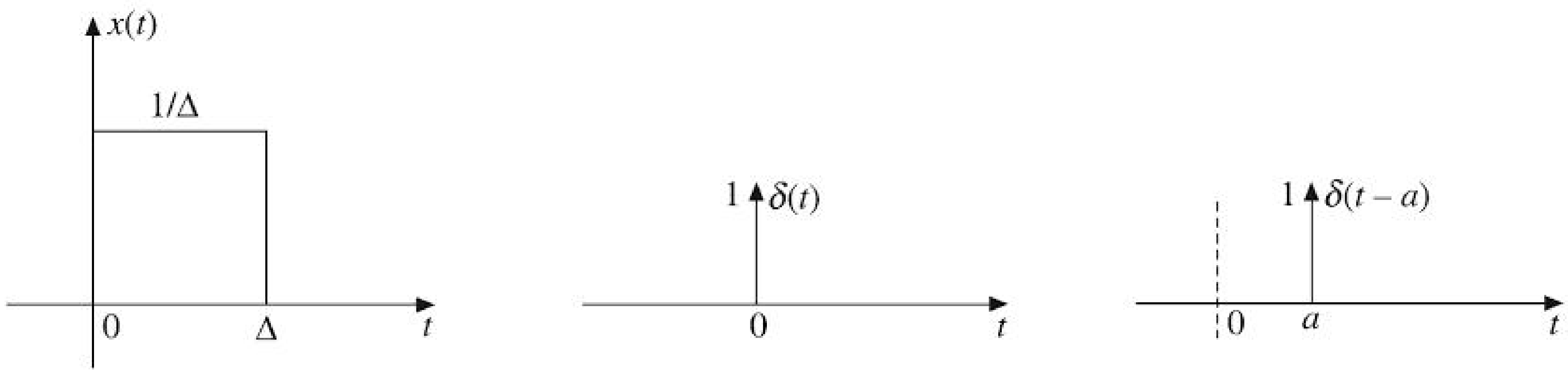
$$x(t) = \frac{1}{\Delta}[u(t) - u(t - \Delta)]$$

$$\delta(t) = \underset{\Delta \to 0}{\text{Lt}}\ x(t) = \underset{\Delta \to 0}{\text{Lt}}\ \frac{1}{\Delta}[u(t) - u(t - \Delta)]$$

A delayed unit impulse function $\delta(t - a)$ is defined as:

$$\delta(t - a) = \begin{cases} 1 & \text{for } t = a \\ 0 & \text{for } t \neq a \end{cases}$$

The graphical representations of $\delta(t)$ and $\delta(t - a)$ are shown in Figure 1.8[(b) and (c)].



(a) (b) (c)

Figure 1.8 (a) $\delta(t)$ as limiting case of a pulse, (b) Unit impulse, (c) Delayed unit impulse.

If unit impulse function is assumed in the form of a pulse, then the following points may be observed about a unit impulse function.

(i) The width of the pulse is zero. This means the pulse exists only at $t = 0$.
(ii) The height of the pulse goes to infinity.
(iii) The area under the pulse curve is always unity.
(iv) The height of arrow indicates the total area under the impulse.

The integral of unit impulse function is a unit step function and the derivate of unit step function is a unit impulse function.

$$u(t) = \int_{-\infty}^{\infty} \delta(t)\, dt$$

and $$\delta(t) = \frac{d}{dt} u(t)$$

Properties of continuous-time unit impulse function

1. It is an even function of time t, i.e. $\delta(t) = \delta(-t)$
2. $\int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt = x(0)$; $\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$
3. $\delta(at) = \dfrac{1}{|a|}\delta(t)$
4. $x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0) = x(t_0)$; $x(t)\,\delta(t) = x(0)\,\delta(t) = x(0)$
5. $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$

The discrete-time unit impulse function $\delta(n)$, also called unit sample sequence, is defined as:

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

The shifted unit impulse function $\delta(n - k)$ is defined as:

$$\delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k \end{cases}$$

The graphical representations of $\delta(n)$ and $\delta(n - 3)$ are shown in Figure 1.9[(a) and (b)].

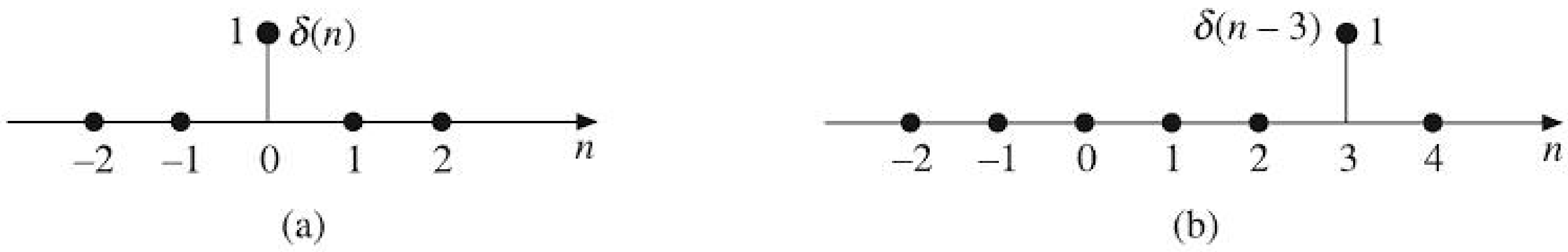


Figure 1.9 Discrete-time (a) Unit sample sequence, (b) Delayed unit sample sequence.

Properties of discrete-time unit sample sequence

1. $\delta(n) = u(n) - u(n-1)$
2. $\delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k \end{cases}$
3. $x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$
4. $\sum_{n=-\infty}^{\infty} x(n)\,\delta(n-n_0) = x(n_0)$

1.3.5 Sinusoidal Signal

A continuous-time sinusoidal signal in its most general form is given by

$$x(t) = A\sin(\omega t + \phi)$$

where

A = Amplitude

ω = Angular frequency in radians

ϕ = Phase angle in radians

Figure 1.10 shows the waveform of a sinusoidal signal. A sinusoidal signal is an example of a periodic signal. The time period of a continuous-time sinusoidal signal is given by

$$T = \frac{2\pi}{\omega}$$

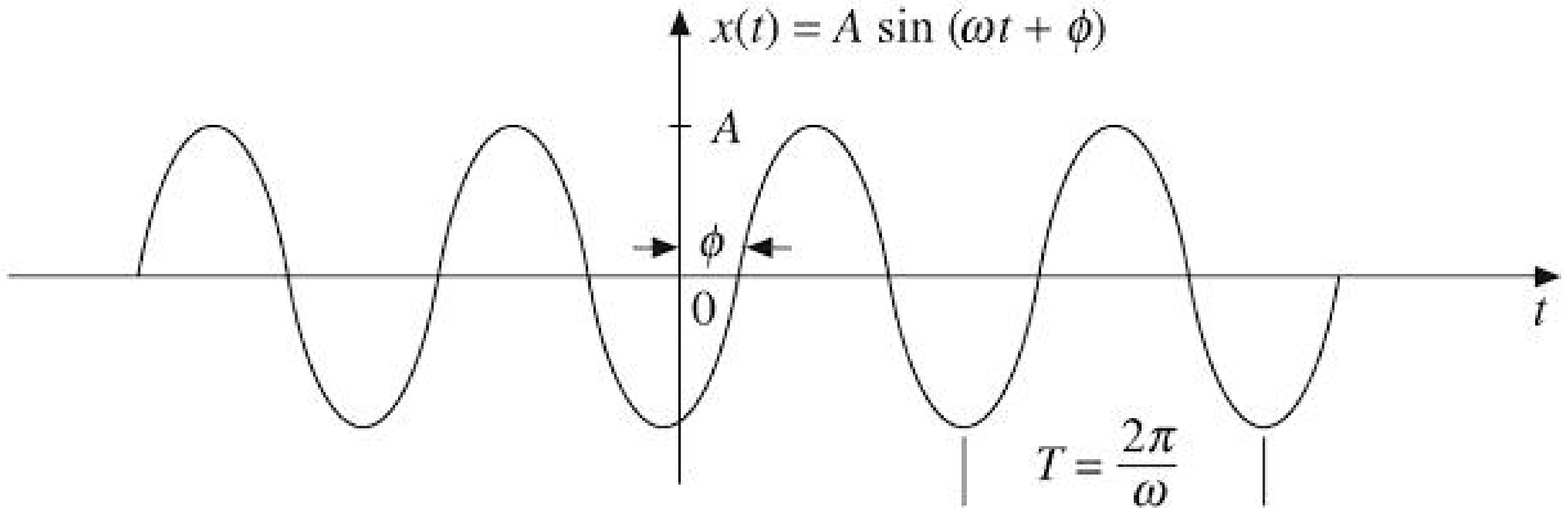


Figure 1.10 Sinusoidal waveform.

The discrete-time sinusoidal sequence is given by

$$x(n) = A \sin(\omega n + \phi)$$

where A is the amplitude, ω is angular frequency, ϕ is phase angle in radians and n is an integer.

The period of the discrete-time sinusoidal sequence is:

$$N = \frac{2\pi}{\omega} m$$

where N and m are integers.

All continuous-time sinusoidal signals are periodic but discrete-time sinusoidal sequences may or may not be periodic depending on the value of ω.

For a discrete-time signal to be periodic, the angular frequency ω must be a rational multiple of 2π.

The graphical representation of a discrete-time sinusoidal signal is shown in Figure 1.11.

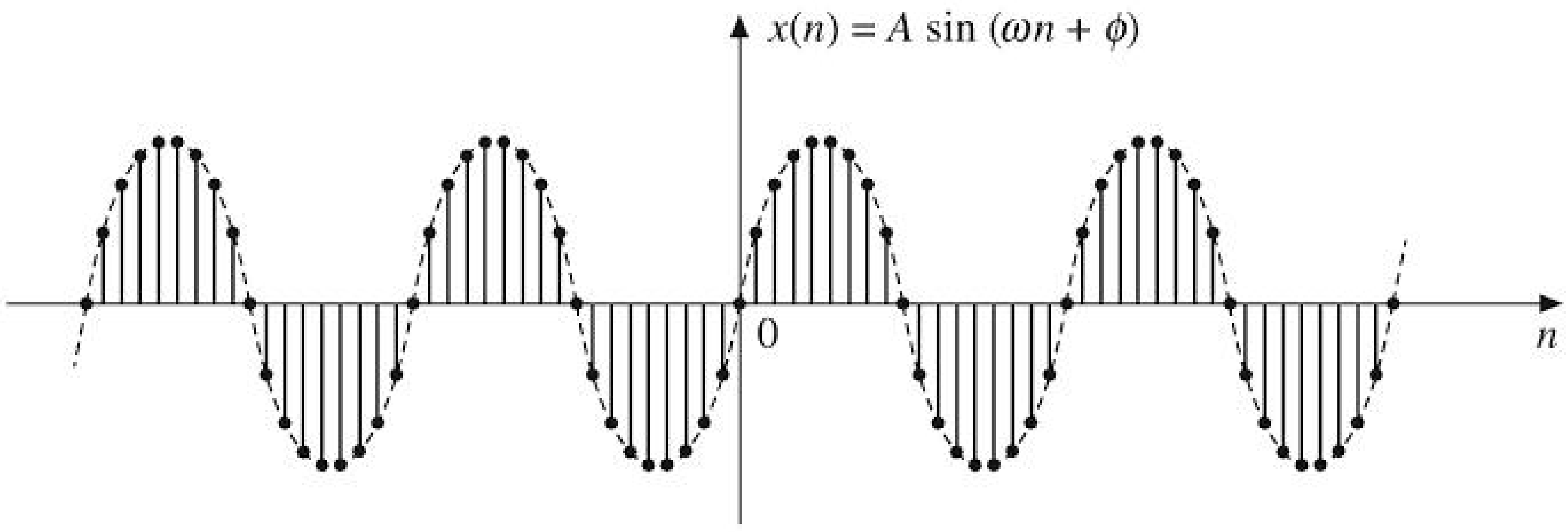


Figure 1.11 Discrete-time sinusoidal signal.

1.3.6 Real Exponential Signal

A continuous-time real exponential signal has the general form as:

$$x(t) = Ae^{\alpha t}$$

where both A and α are real.

The parameter A is the amplitude of the exponential measured at $t = 0$. The parameter α can be either positive or negative. Depending on the value of α, we get different exponentials.

1. If $\alpha = 0$, the signal $x(t)$ is of constant amplitude for all times.
2. If α is positive, i.e. $\alpha > 0$, the signal $x(t)$ is a growing exponential signal.
3. If α is negative, i.e. $\alpha < 0$, the signal $x(t)$ is a decaying exponential signal.

These three waveforms are shown in Figure 1.12[(a), (b) and (c)].

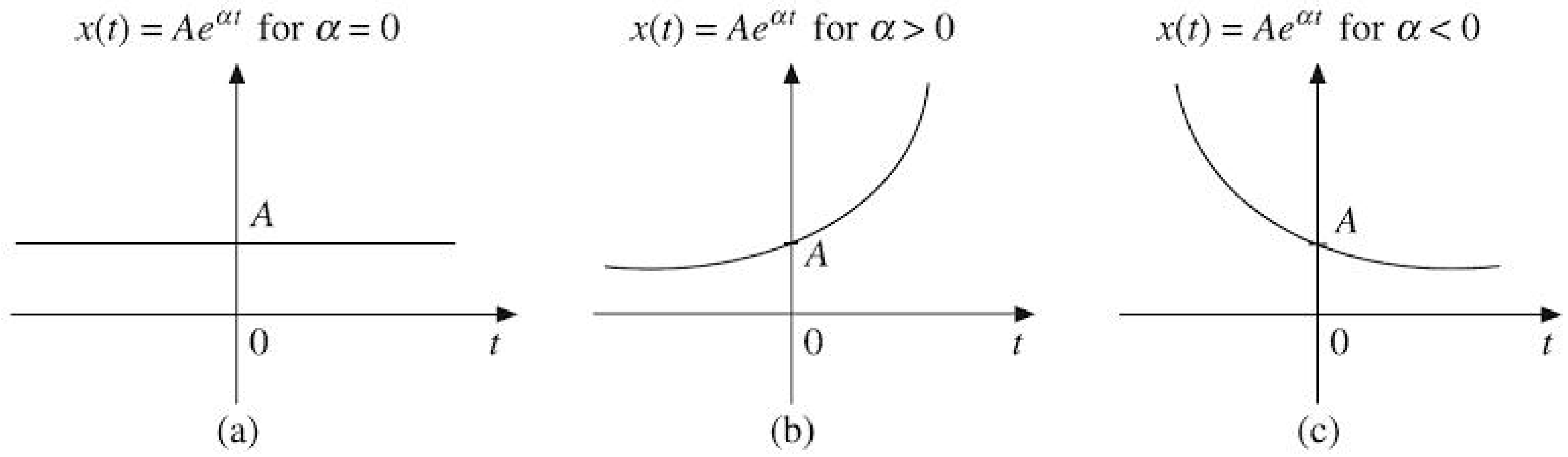


Figure 1.12 Continuous-time real exponential signals $x(t) = Ae^{\alpha t}$ for (a) $\alpha = 0$, (b) $\alpha > 0$, (c) $\alpha < 0$.

The discrete-time real exponential sequence a^n is defined as:

$$x(n) = a^n \quad \text{for all } n$$

Figure 1.13 illustrates different types of discrete-time exponential signals.

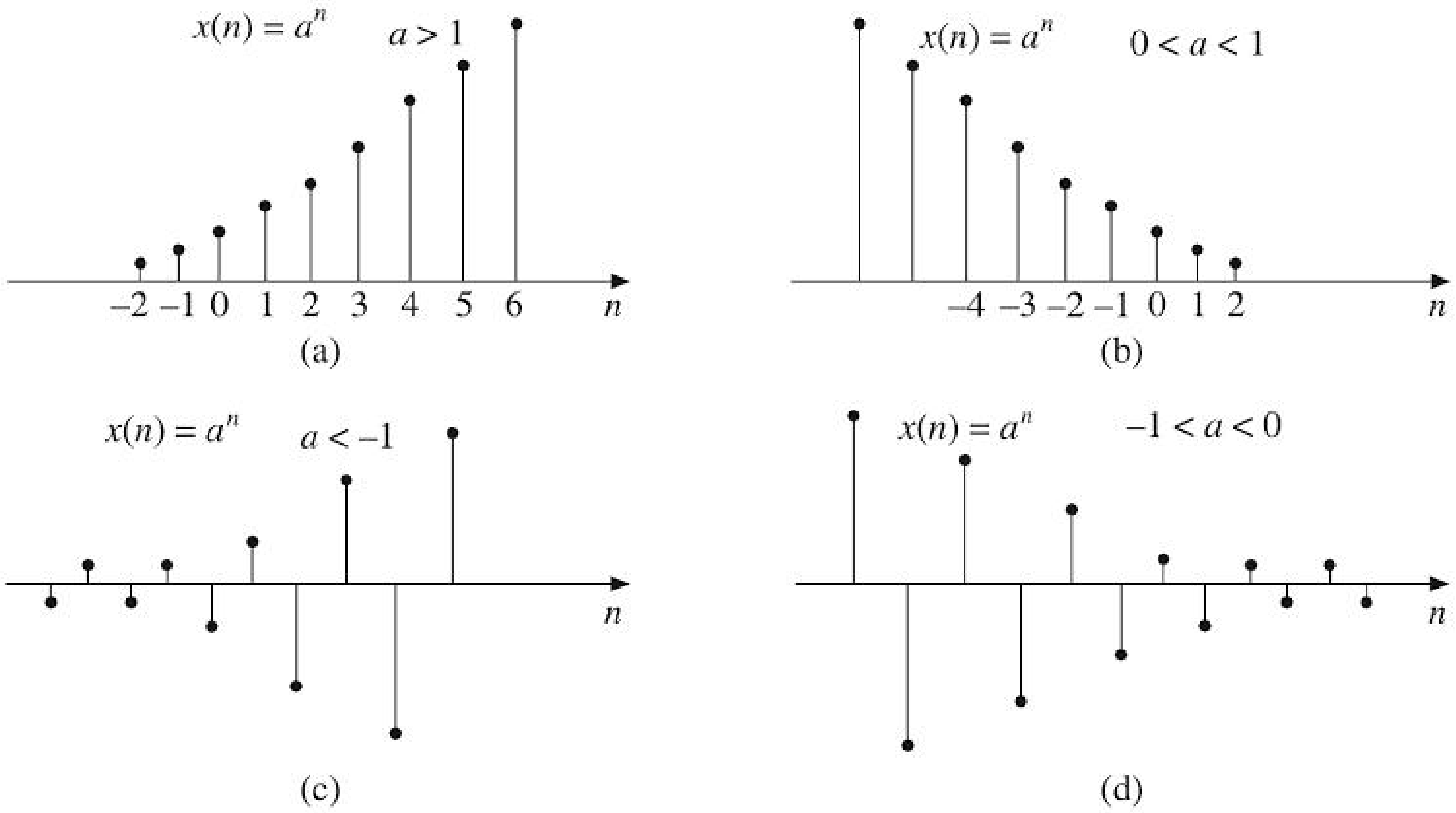


Figure 1.13 Discrete-time exponential signal a^n for (a) $a > 1$, (b) $0 < a < 1$, (c) $a < -1$, (d) $-1 < a < 0$.

When $a > 1$, the sequence grows exponentially as shown in Figure 1.13(a).

When $0 < a < 1$, the sequence decays exponentially as shown in Figure 1.13(b).

When $a < 0$, the sequence takes alternating signs as shown in Figure 1.13[(c) and (d)].

1.3.7 Complex Exponential Signal

The complex exponential signal has a general form as

$$x(t) = Ae^{st}$$

where A is the amplitude and s is a complex variable defined as

$$s = \sigma + j\omega$$

Therefore,
$$x(t) = Ae^{st} = Ae^{(\sigma + j\omega)t} = Ae^{\sigma t}e^{j\omega t}$$
$$= Ae^{\sigma t}[\cos \omega t + j \sin \omega t]$$

Depending on the values of σ and ω, we get different waveforms as shown in Figure 1.14.

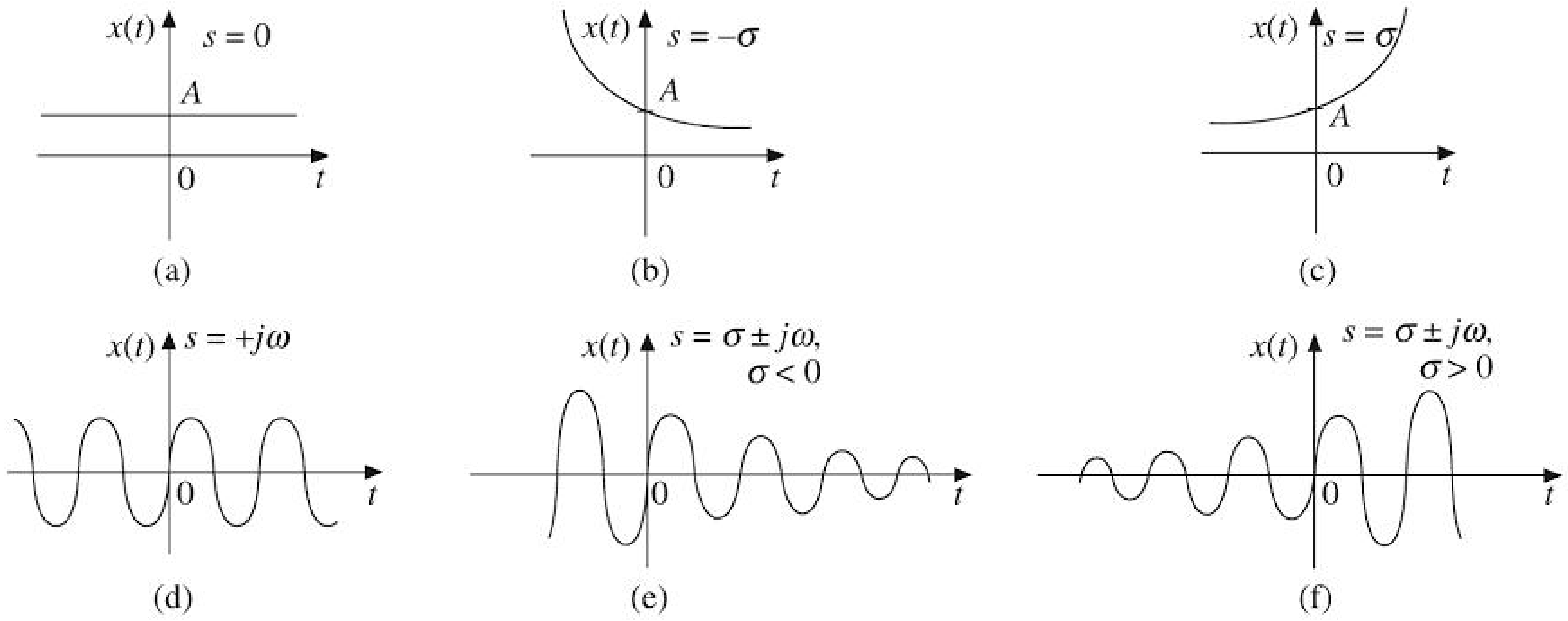


Figure 1.14 Complex exponential signals.

The discrete-time complex exponential sequence is defined as

$$x(n) = a^n e^{j(\omega_0 n + \phi)}$$
$$= a^n \cos(\omega_0 n + \phi) + ja^n \sin(\omega_0 n + \phi)$$

For $|a| = 1$, the real and imaginary parts of complex exponential sequence are sinusoidal.

For $|a| > 1$, the amplitude of the sinusoidal sequence exponentially grows as shown in Figure 1.15(a).

For $|a| < 1$, the amplitude of the sinusoidal sequence exponentially decays as shown in Figure 1.15(b)

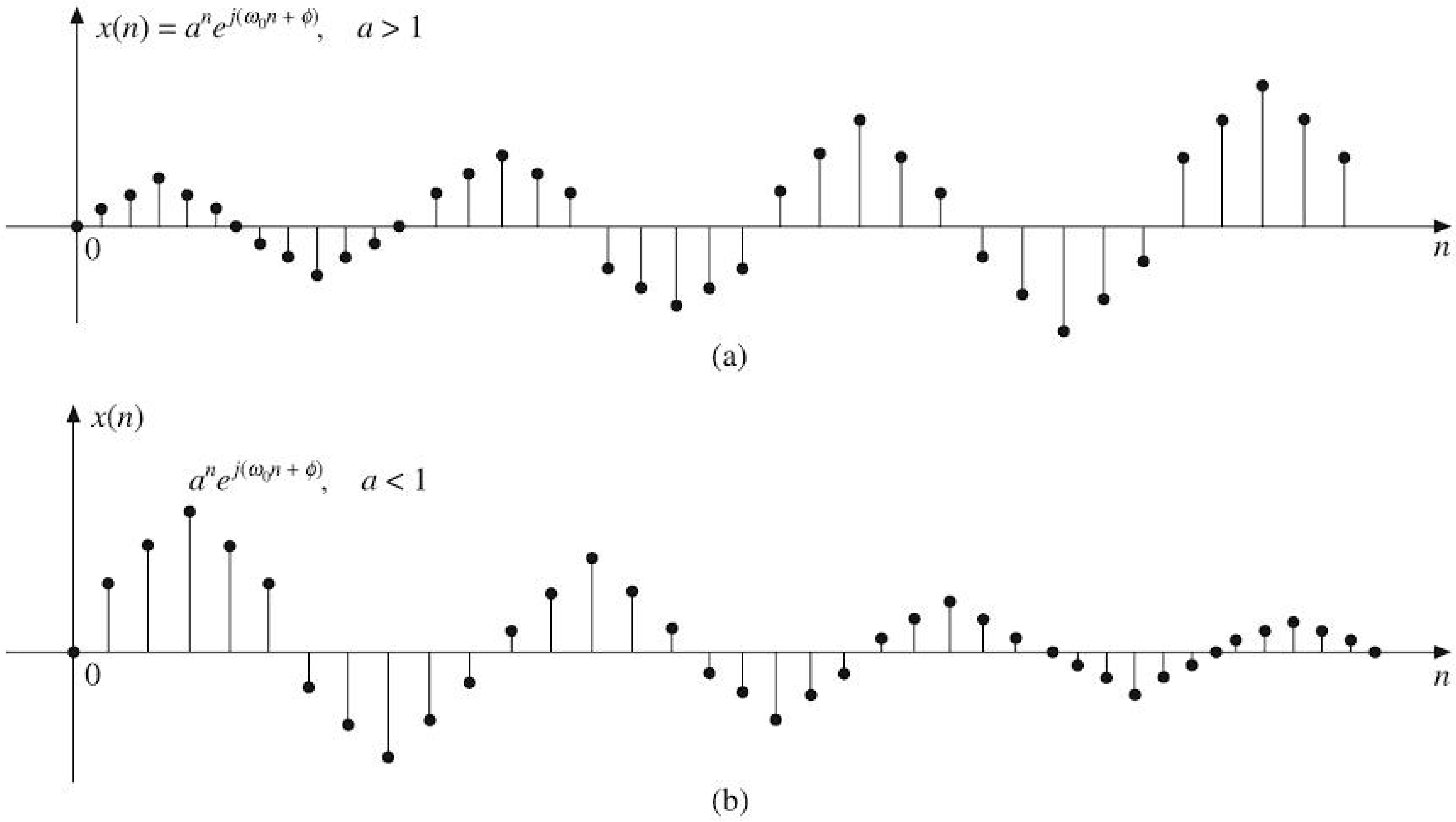


Figure 1.15 Complex exponential sequence $x(n) = a^n e^{j(\omega_0 n + \phi)}$ for (a) $a > 1$, (b) $a < 1$.

1.3.8 Rectangular Pulse Function

The unit rectangular pulse function $\Pi(t/\tau)$ shown in Figure 1.16 is defined as

$$\Pi\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \text{for } |t| \le \dfrac{\tau}{2} \\ 0 & \text{otherwise} \end{cases}$$

It is an even function of t.

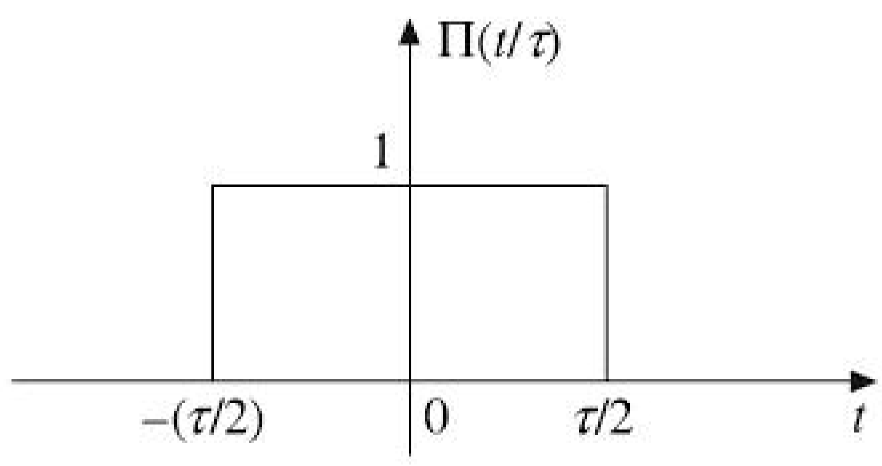


Figure 1.16 Rectangular pulse function.

1.3.9 Triangular Pulse Function

The unit triangular pulse function $\Delta(t/\tau)$ shown in Figure 1.17 is defined as:

$$\Delta\left(\frac{t}{\tau}\right) = \begin{cases} 1 - (2|t|/\tau) & \text{for } |t| < (\tau/2) \\ 0 & \text{for } |t| > (\tau/2) \end{cases}$$

It is an even function of t.

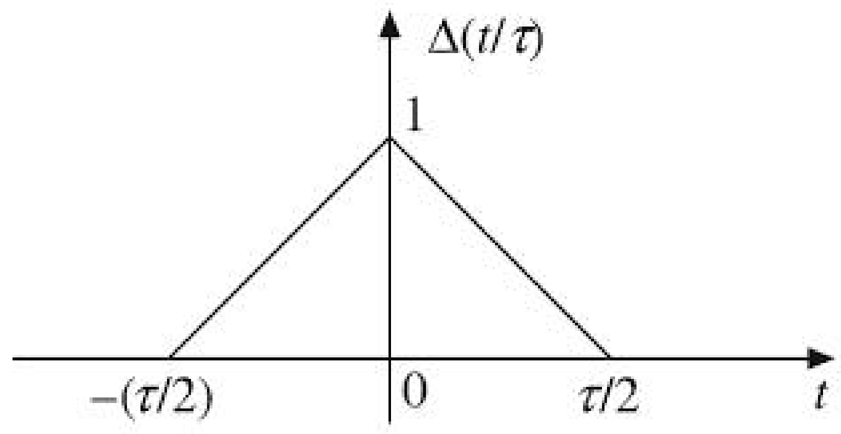


Figure 1.17 Triangular pulse function.

1.3.10 Signum Function

The unit signum function sgn (t) shown in Figure 1.18 is defined as:

$$\operatorname{sgn}(t) = \begin{cases} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{cases}$$

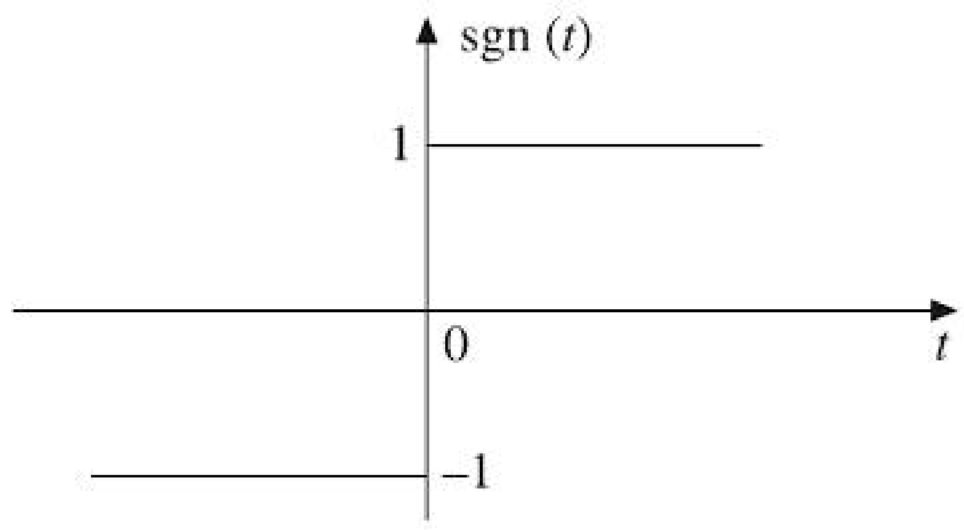


Figure 1.18 Signum function.

The signum function can be expressed in terms of unit step function as:

$$\operatorname{sgn}(t) = -1 + 2u(t)$$

1.3.11 Sinc Function

The sinc function sinc (t) shown in Figure 1.19 is defined as:

$$\operatorname{sinc}(t) = \frac{\sin t}{t} \quad \text{for } -\infty < t < \infty$$

The sinc function oscillates with period 2π and decays with increasing t. Its value is zero at $n\pi$, $n = \pm1, \pm2, \ldots$. It is an even function of t.

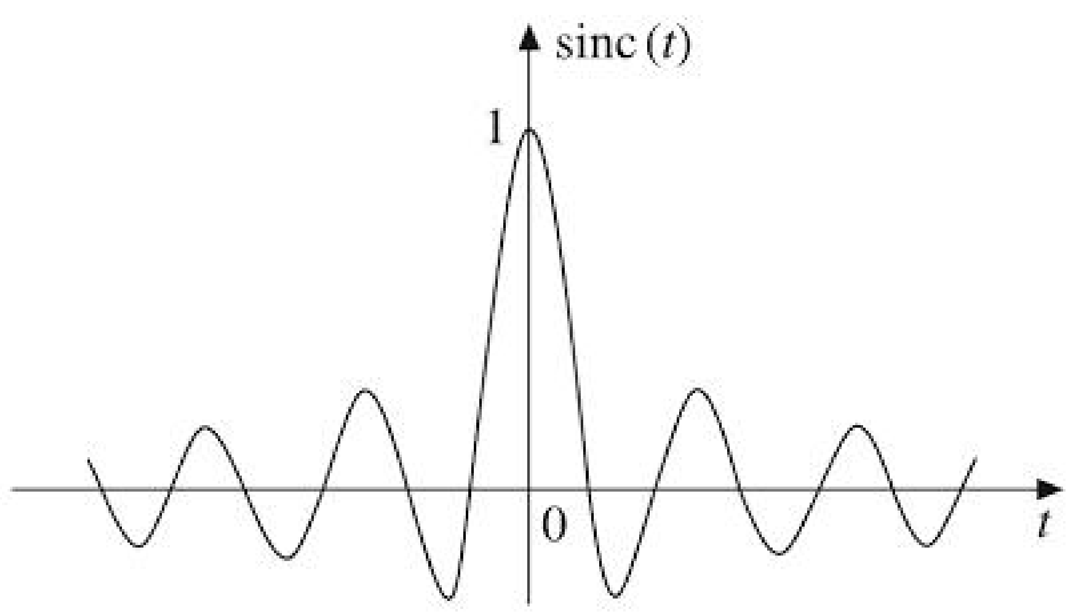


Figure 1.19 Sinc function.

1.3.12 Gaussian Function

The Gaussian function $g_a(t)$ shown in Figure 1.20 is defined as:

$$g_a(t) = e^{-at^2} \quad \text{for } -\infty < t < \infty$$

This function is extremely useful in probability theory.

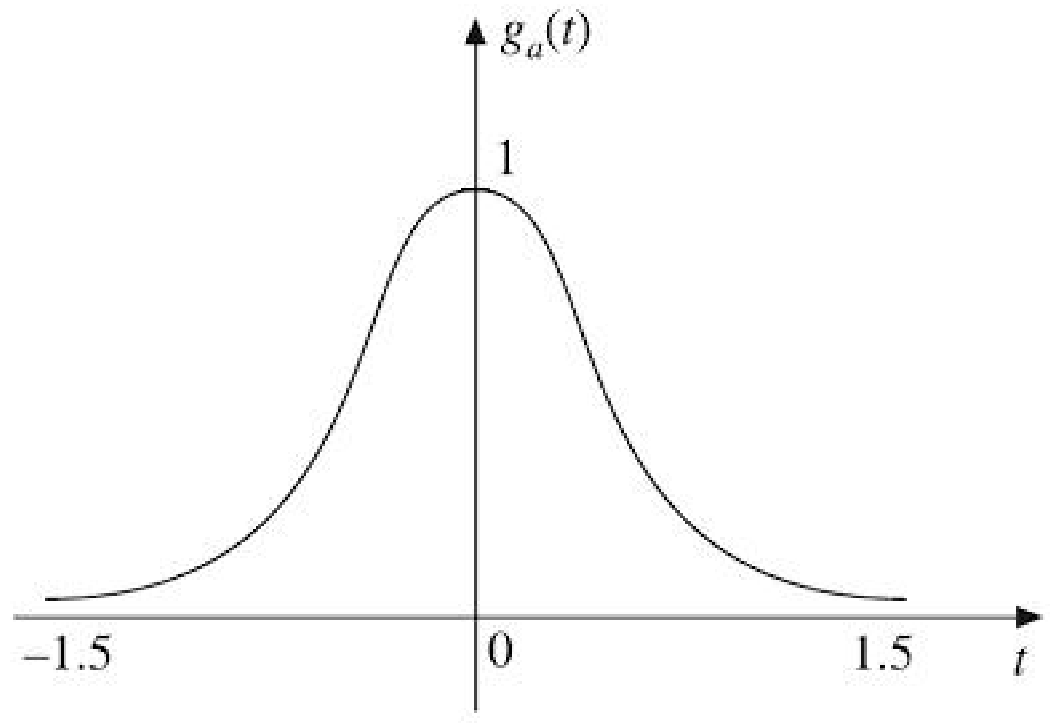


Figure 1.20 Gaussian function.

EXAMPLE 1.1 Evaluate the following integrals:

(a) $\int_{-\infty}^{\infty} e^{-at^2}\delta(t-5)\,dt$ (b) $\int_{0}^{\infty} t^2\delta(t-6)\,dt$

(c) $\int_{0}^{3} \delta(t)\sin 5\pi t\,dt$ (d) $\int_{-\infty}^{\infty} \delta(t+2)\,e^{-2t}\,dt$

(e) $\int_{-\infty}^{\infty} (t-2)^3\,\delta(t-2)\,dt$ (f) $\int_{-\infty}^{\infty} \delta(t)\,e^{-j\omega t}\,dt$

(g) $\int_{-\infty}^{\infty} [\delta(t)\cos 2t + \delta(t-2)\sin 2t]\,dt$

Solution:

(a) Given $$\int_{-\infty}^{\infty} e^{-at^2}\delta(t-5)\,dt$$

We know that $$\delta(t-5) = \begin{cases} 1 & \text{for } t = 5 \\ 0 & \text{elsewhere} \end{cases}$$

$$\therefore \quad \int_{-\infty}^{\infty} e^{-at^2}\delta(t-5)\,dt = \left[e^{-at^2}\right]_{t=5} = e^{-25a}$$

(b) Given $$\int_0^\infty t^2 \delta(t-6)\,dt$$

We know that $$\delta(t-6) = \begin{cases} 1 & \text{for } t = 6 \\ 0 & \text{elsewhere} \end{cases}$$

$\therefore$ $$\int_0^\infty t^2 \delta(t-6)\,dt = [t^2]_{t=6} = 36$$

(c) Given $$\int_0^3 \delta(t) \sin 5\pi t\,dt$$

We know that $$\delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{elsewhere} \end{cases}$$

$\therefore$ $$\int_0^3 \delta(t) \sin 5\pi t\,dt = [\sin 5\pi t]_{t=0} = 0$$

(d) Given $$\int_{-\infty}^\infty \delta(t+2) e^{-2t}\,dt$$

We know that $$\delta(t+2) = \begin{cases} 1 & \text{for } t = -2 \\ 0 & \text{elsewhere} \end{cases}$$

$\therefore$ $$\int_{-\infty}^\infty \delta(t+2)\, e^{-2t} dt = [e^{-2t}]_{t=-2} = e^4$$

(e) Given $$\int_{-\infty}^\infty (t-2)^3 \delta(t-2)\,dt$$

We know that $$\delta(t-2) = \begin{cases} 1 & \text{for } t = 2 \\ 0 & \text{elsewhere} \end{cases}$$

$\therefore$ $$\int_{-\infty}^\infty (t-2)^3\, \delta(t-2)\,dt = [(t-2)^3]_{t=2} = 0$$

(f) Given $$\int_{-\infty}^\infty \delta(t) e^{-j\omega t}\,dt$$

We know that $$\delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\therefore \qquad \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = [e^{-j\omega t}]_{t=0} = 1$$

(g) Given $$\int_{-\infty}^{\infty} [\delta(t) \cos 2t + \delta(t-2) \sin 2t]\, dt$$

We know that $\delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{elsewhere} \end{cases}$ and $\delta(t-2) = \begin{cases} 1 & \text{for } t = 2 \\ 0 & \text{elsewhere} \end{cases}$

$$\therefore \qquad \int_{-\infty}^{\infty} [\delta(t) \cos 2t + \delta(t-2) \sin 2t]\, dt = [\cos 2t]_{t=0} + [\sin 2t]_{t=2} = 1 + \sin 4t$$

EXAMPLE 1.2 Find the following summations

(a) $\sum_{n=-\infty}^{\infty} e^{3n}\, \delta(n-3)$ (b) $\sum_{n=-\infty}^{\infty} \delta(n-2) \cos 3n$

(c) $\sum_{n=-\infty}^{\infty} n^2 \delta(n+4)$ (d) $\sum_{n=-\infty}^{\infty} \delta(n-2)\, e^{n^2}$

(e) $\sum_{n=0}^{\infty} \delta(n+1)\, 4^n$

Solution:

(a) Given $$\sum_{n=-\infty}^{\infty} e^{3n}\, \delta(n-3)$$

We know that $$\delta(n-3) = \begin{cases} 1 & \text{for } n = 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$\therefore \qquad \sum_{n=-\infty}^{\infty} e^{3n} \delta(n-3) = [e^{3n}]_{n=3} = e^9$$

(b) Given $$\sum_{n=-\infty}^{\infty} \delta(n-2) \cos 3n$$

We know that $$\delta(n-2) = \begin{cases} 1 & \text{for } n = 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$\therefore \qquad \sum_{n=-\infty}^{\infty} \delta(n-2) \cos 3n = [\cos 3n]_{n=2} = \cos 6$$

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Proof: Let $x(t)$ be continuous at $t = t_0$ and let its value at $t = t_0$ be $x(t_0)$. We know that $\delta(t - t_0) = 1$ only for $t = t_0$. For all other time it is equal to zero. Therefore, the integration of the product term $x(t)\ \delta(t - t_0)$ from $-\infty$ to ∞ has a value only at $t = t_0$.

$$\therefore \qquad \int_{-\infty}^{\infty} x(t)\, \delta(t - t_0)\, dt = \int_{-\infty}^{\infty} x(t_0)\, \delta(t - t_0)\, dt = x(t_0) \int_{-\infty}^{\infty} \delta(t - t_0)\, dt = x(t_0)$$

Fourth property
$$\int_{-\infty}^{\infty} x(\tau)\, \delta(t - \tau)\, d\tau = x(t)$$

Proof: Let $x(\tau)$ be continuous at $\tau = t$. Let the value of $x(\tau)$ at $\tau = t$ be $x(t)$. We know that $\delta(t - \tau) = 1$ only at $\tau = t$. For all other τ it is zero. So the integration of the product $x(\tau)\ \delta(t - \tau)$ has a value only at $\tau = t$.

$$\therefore \qquad \int_{-\infty}^{\infty} x(\tau)\, \delta(t - \tau)\, d\tau = x(\tau)\big|_{\tau = t} = x(t)$$

This is the formula for convolution of $x(t)$ with $\delta(t)$. This property says that the convolution of any signal with an impulse results in the original signal itself.

Fifth property
$$\delta(at) = \frac{1}{|a|}\delta(t) \qquad \text{[Scaling property]}$$

Proof: Let $x(t)$ be some function. Consider the integral

$$\int_{-\infty}^{\infty} x(t)\, \delta(at)\, dt \qquad \text{for } a > 0$$

Let
$$at = \tau$$

$$\therefore \qquad t = \frac{\tau}{a} \quad \text{and} \quad dt = \frac{d\tau}{a}$$

If $a > 0$
$$\int_{-\infty}^{\infty} x(t)\, \delta(at)\, dt = \frac{1}{a} \int_{-\infty}^{\infty} x\left(\frac{\tau}{a}\right) \delta(\tau)\, d\tau$$

$$= \left[\frac{1}{a} x\left(\frac{\tau}{a}\right)\right]_{\tau = 0}$$

$$= \frac{1}{a} x(0)$$

Similarly, for $a < 0$
$$\int_{-\infty}^{\infty} x(t)\, \delta(at)\, dt = \frac{1}{-a} x(0)$$

$$\therefore \qquad \int_{-\infty}^{\infty} x(t)\, \delta(at)\, dt = \frac{1}{|a|} x(0)$$

Now, consider
$$\frac{1}{|a|}x(0)$$

We know that
$$x(0) = \int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt$$

$\therefore$
$$\frac{1}{|a|}x(0) = \frac{1}{|a|}\int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt$$
$$= \int_{-\infty}^{\infty} x(t)\,\frac{1}{|a|}\,\delta(t)\,dt$$

which indicates that
$$\delta(at) = \frac{1}{|a|}\delta(t)$$

Sixth property
$$\delta(t) = \delta(-t)$$

i.e. impulse function is an even function.

Proof: Consider the scaling property,
$$\delta(at) = \frac{1}{|a|}\delta(t)$$

Let
$$a = -1$$

$\therefore$
$$\delta(-t) = \frac{1}{|-1|}\delta(t) = \delta(t)$$

1.4 BASIC OPERATIONS ON SIGNALS

When we process a signal, this signal may undergo several manipulations involving the independent variable or the amplitude of the signal. The basic operations on signals are as follows:

1. Time shifting
2. Time reversal
3. Time scaling
4. Amplitude scaling
5. Signal addition
6. Signal multiplication

The first three operations correspond to transformation in independent variable t or n of a signal. The last three operations correspond to transformation on amplitude of a signal.

1.4.1 Time Shifting

Mathematically, the time shifting of a continuous-time signal $x(t)$ can be represented by
$$y(t) = x(t - T)$$

The time shifting of a signal may result in time delay or time advance. In the above equation if T is positive the shift is to the right and then the shifting delays the signal, and if T is negative the shift is to the left and then the shifting advances the signal. An arbitrary signal $x(t)$, its delayed version and advanced version are shown in Figure 1.21[(a), (b) and (c)]. Shifting a signal in time means that a signal may be either advanced in the time axis or delayed in the time axis.

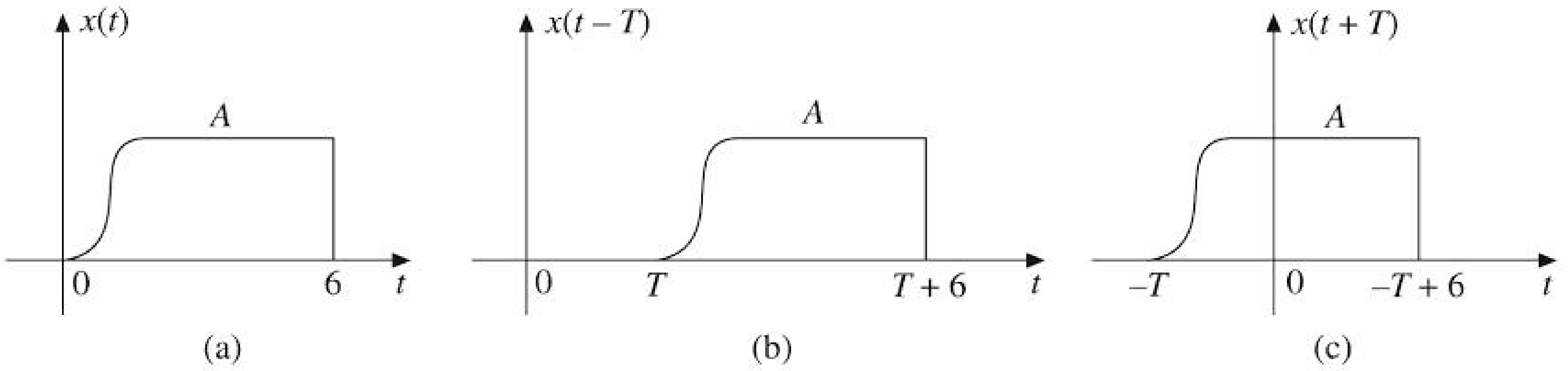

Figure 1.21 (a) Signal, (b) Its delayed version, (c) Its time advanced version.

Similarly, the time shifting operation of a discrete-time signal $x(n)$ can be represented by

$$y(n) = x(n - k)$$

This shows that the signal $y(n)$ can be obtained by time shifting the signal $x(n)$ by k units. If k is positive, it is delay and the shift is to the right, and if k is negative, it is advance and the shift is to the left.

An arbitrary signal $x(n)$ is shown in Figure 1.22(a). $x(n - 3)$ which is obtained by shifting $x(n)$ to the right by 3 units [i.e. delay $x(n)$ by 3 units] is shown in Figure 1.22(b). $x(n + 2)$ which is obtained by shifting $x(n)$ to the left by 2 units (i.e. advancing $x(n)$ by 2 units) is shown in Figure 1.22(c).

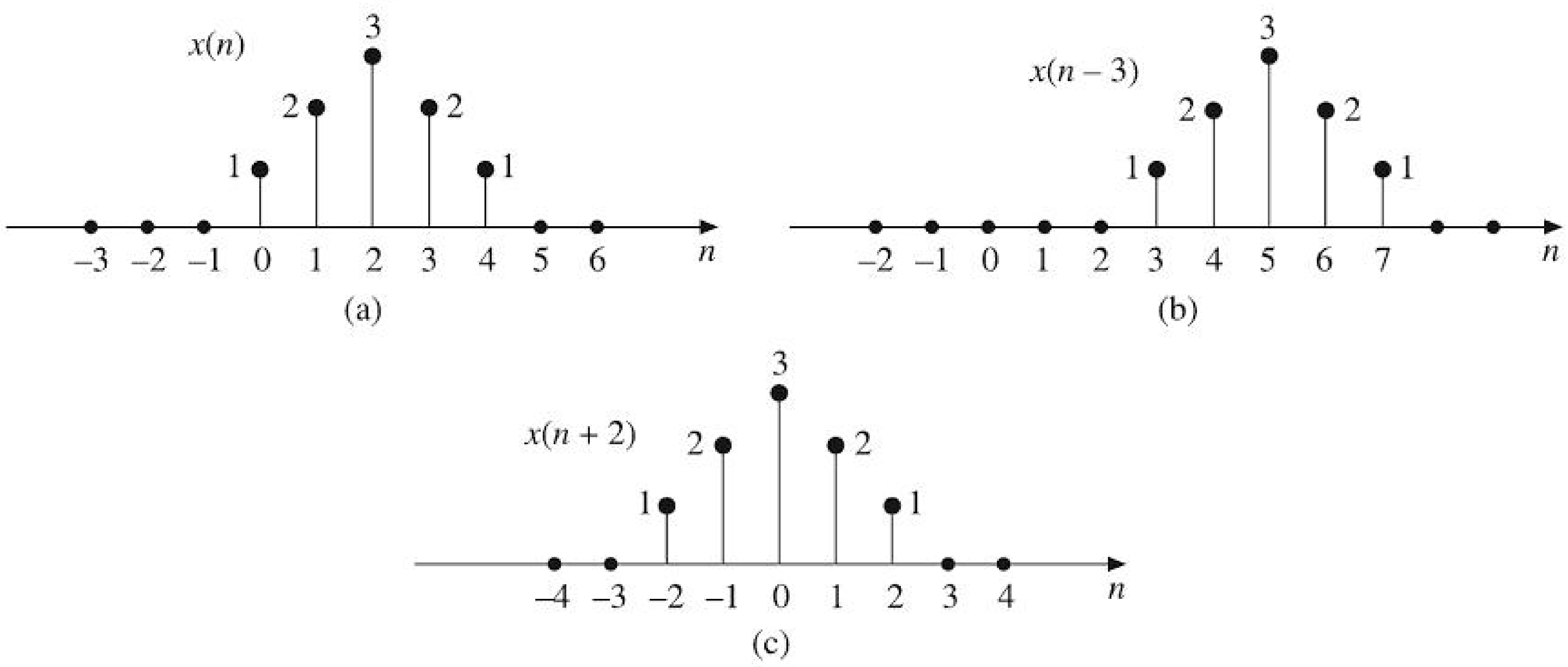

Figure 1.22 (a) Sequence $x(n)$, (b) $x(n - 3)$, (c) $x(n + 2)$.

1.4.2 Time Reversal

The time reversal, also called time folding of a signal $x(t)$ can be obtained by folding the signal about $t = 0$. This operation is very useful in convolution. It is denoted by $x(-t)$. It is obtained by replacing the independent variable t by $(-t)$. Folding is also called as the reflection of the signal about the time origin $t = 0$. Figure 1.23(a) shows an arbitrary signal $x(t)$, and Figure 1.23(b) shows its reflection $x(-t)$.

The signal $x(-t + 3)$ obtained by shifting the reversed signal $x(-t)$ to the right by 3 units (delay by 3 units) is shown in Figure 1.23(c). The signal $x(-t - 3)$ obtained by shifting the reversed signal $x(-t)$ to the left by 3 units (advance by 3 units) is shown in Figure 1.23(d).

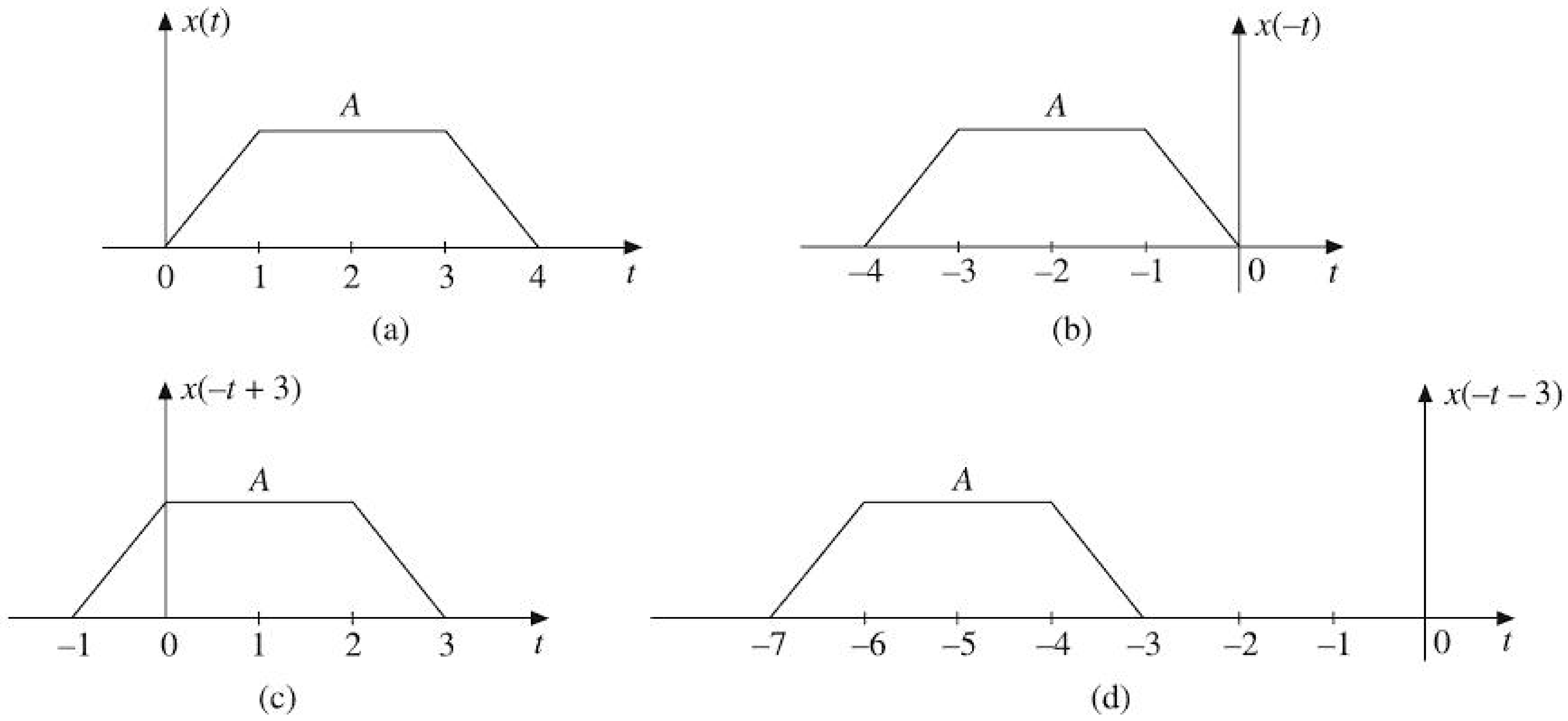

Figure 1.23 (a) An arbitrary signal $x(t)$, (b) Time reversed signal $x(-t)$, (c) Time reversed and delayed signal $x(-t + 3)$, (d) Time reversed and advanced signal $x(-t - 3)$.

Other examples for time reversal operation are shown in Figure 1.24.

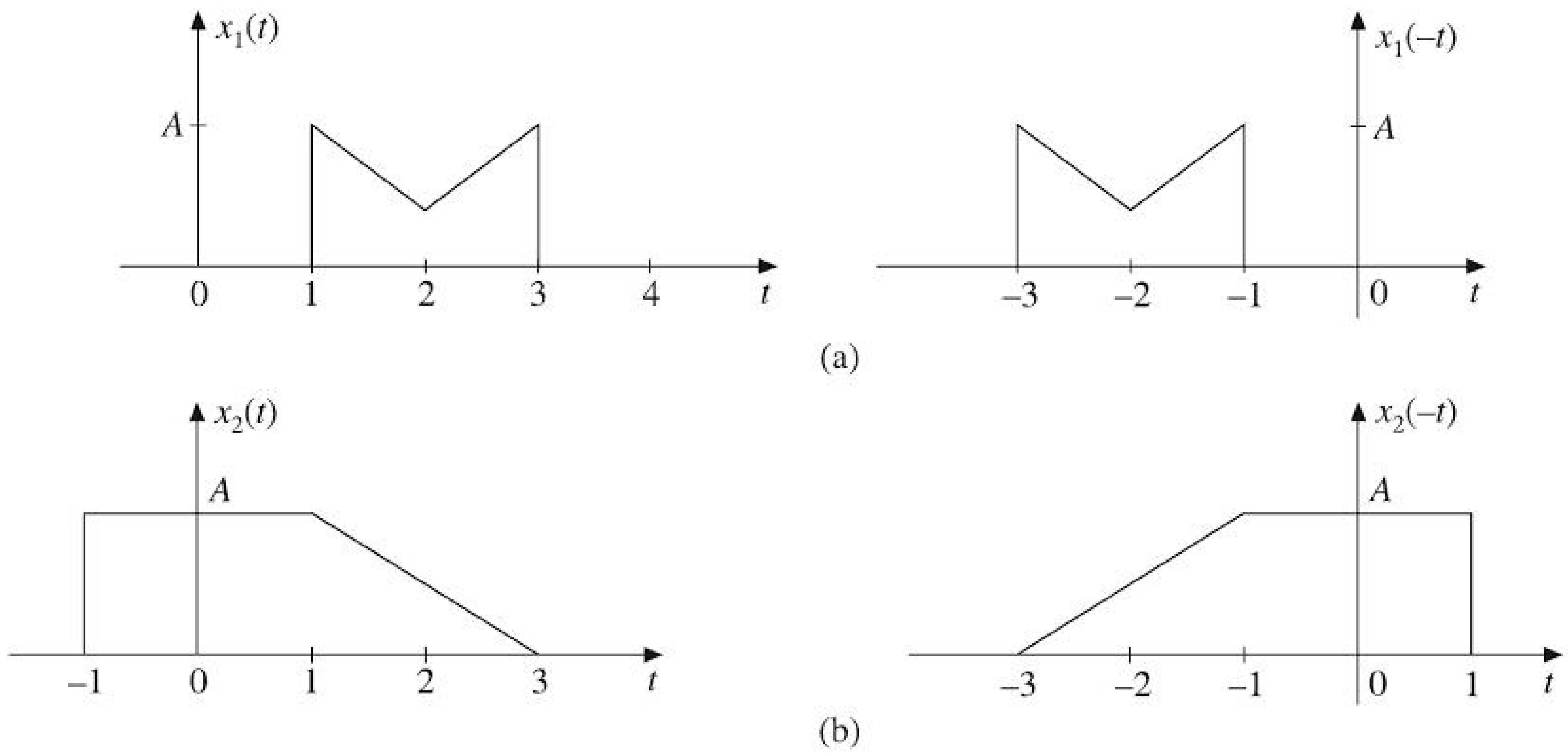

Figure 1.24 Time reversal operations.

The time reversal of a discrete-time signal $x(n)$ can be obtained by folding the sequence about $n = 0$. Figure 1.25(a) shows an arbitrary discrete-time signal $x(n)$ and its time reversed version $x(-n)$ is shown in Figure 1.25(b). Figure 1.25[(c) and (d)] shows the delayed and advanced versions of reversed signal $x(-n)$.

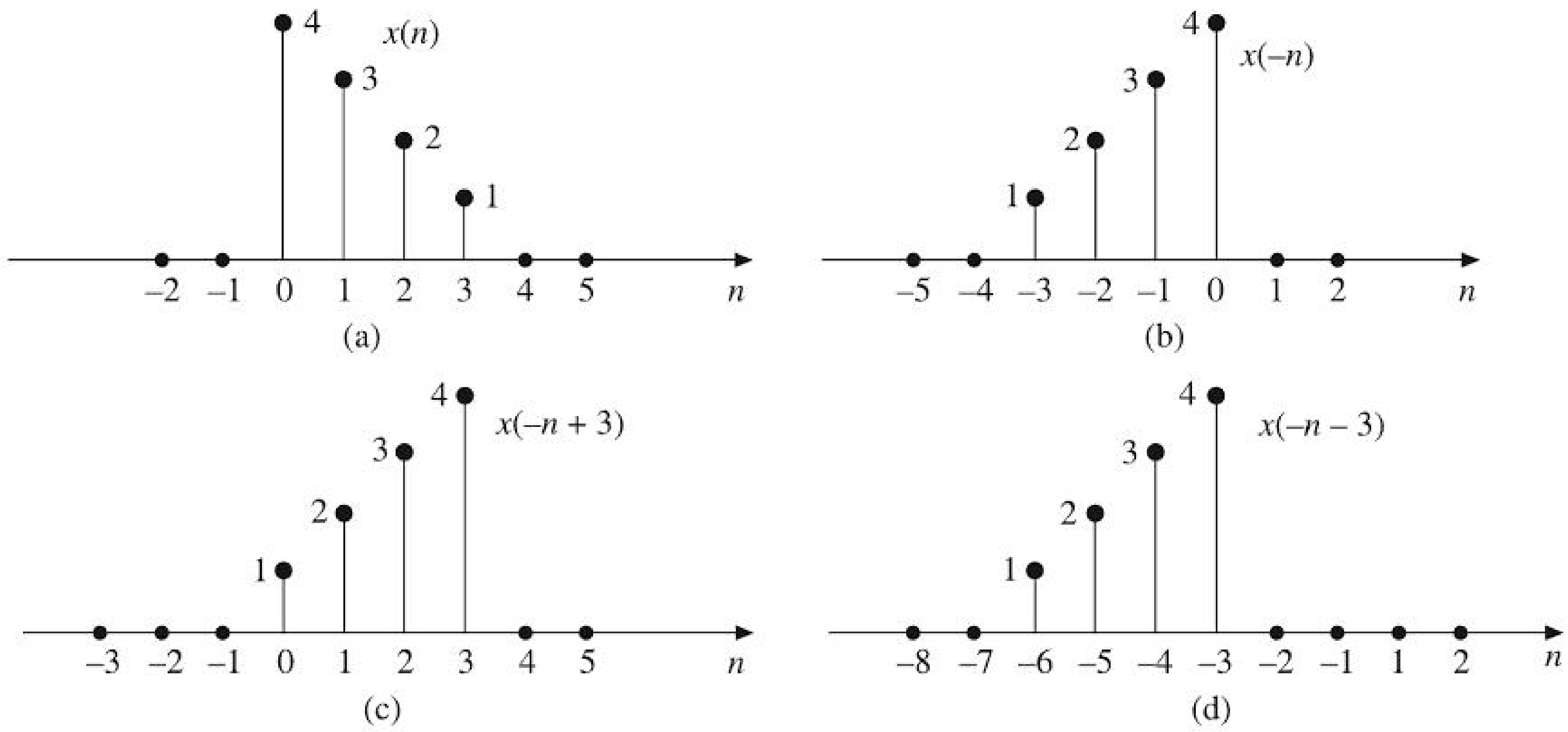

Figure 1.25 (a) Original signal $x(n)$, (b) Time reversed signal $x(-n)$, (c) Time reversed and delayed signal $x(-n + 3)$, (d) Time reversed and advanced signal $x(-n - 3)$.

The signal $x(-n + 3)$ is obtained by delaying (shifting to the right) the time reversed signal $x(-n)$ by 3 units of time. The signal $x(-n - 3)$ is obtained by advancing (shifting to the left) the time reversed signal $x(-n)$ by 3 units of time.

Figure 1.26 shows other examples for time reversal of signals.

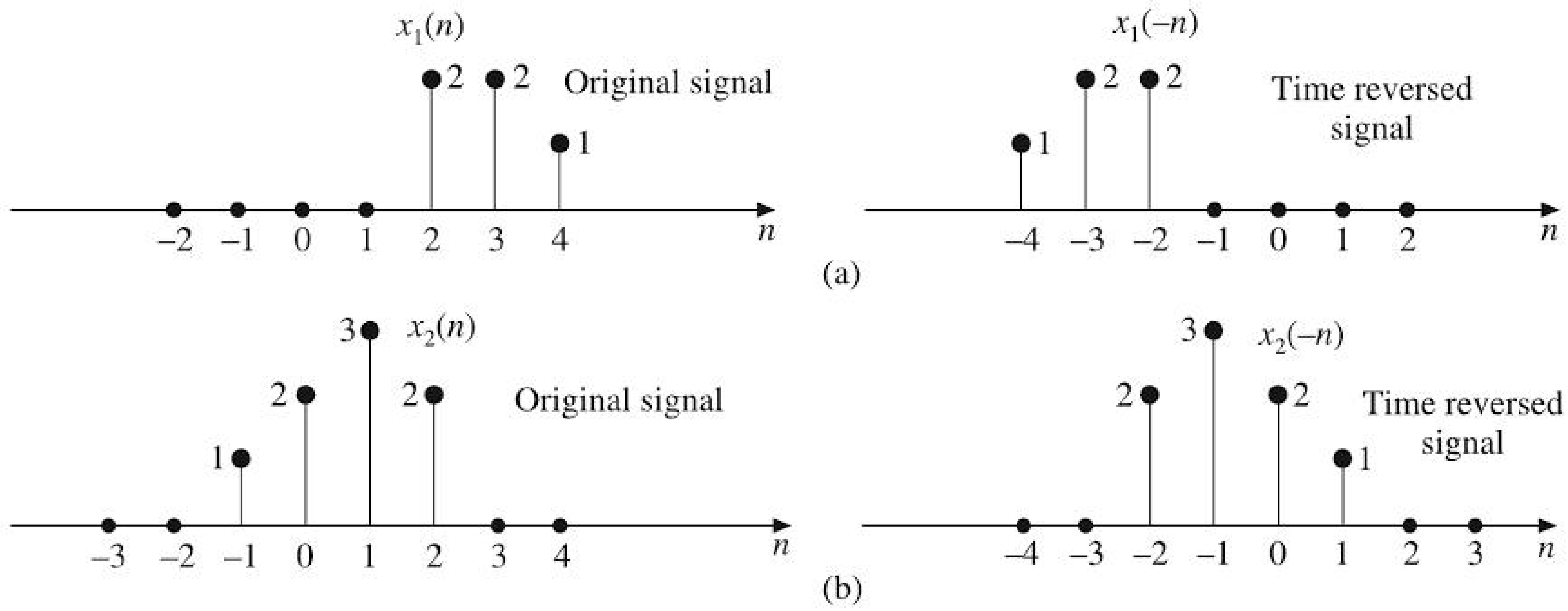

Figure 1.26 Time reversal operations.

EXAMPLE 1.4 Sketch the following signals

(a) $u(-t + 2)$ (b) $-2u(t + 2)$

(c) $-4r(t)$ (d) $2r(t - 2)$

(e) $r(-t + 3)$ (f) $\Pi(t - 2)$

Solution:

(a) Given $x(t) = u(-t + 2)$

The signal $u(-t + 2)$ can be obtained by first drawing the unit step signal $u(t)$ as shown in Figure 1.27(a), then time reversing the signal $u(t)$ about $t = 0$ to obtain

$u(-t)$ as shown in Figure 1.27(b), and then shifting the reversed signal to the right by 2 units of time to obtain $u(-t + 2)$ as shown in Figure 1.27(c).

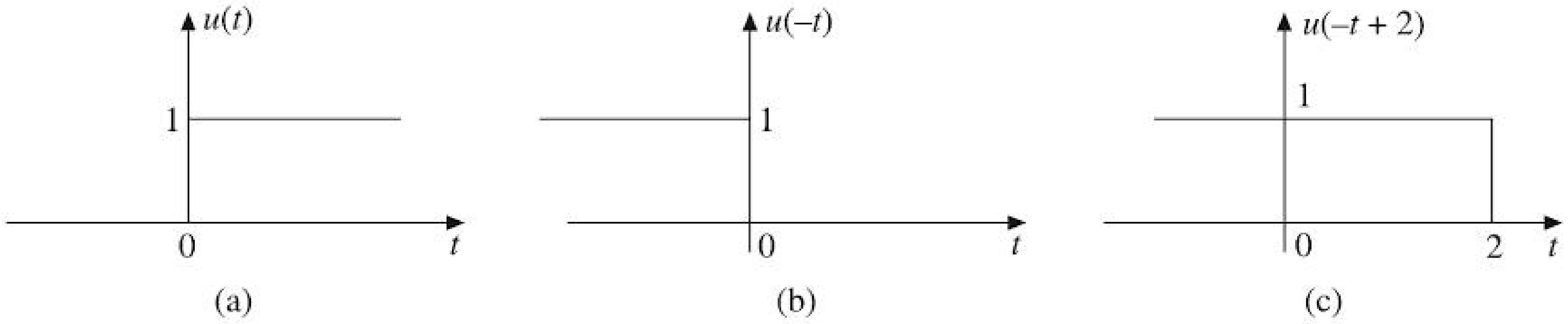

Figure 1.27 (a) Unit step signal, (b) Folded unit step signal, (c) Delayed folded signal.

(b) Given $$x(t) = -2u(t + 2)$$

The signal $-2u(t + 2)$ can be obtained by first drawing the unit step signal $u(t)$ as shown in Figure 1.28(a), then shifting the signal $u(t)$ to the left by 2 units of time to obtain $u(t + 2)$ as shown in Figure 1.28(b), and then multiplying that signal $u(t + 2)$ by -2 to obtain $-2u(t + 2)$ as shown in Figure 1.28(c).

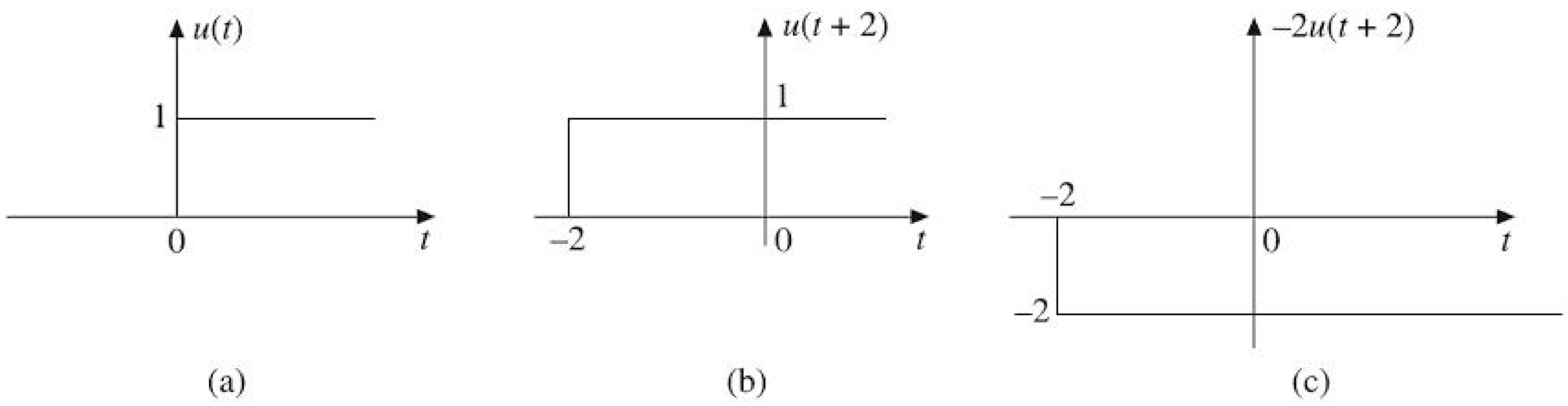

Figure 1.28 (a) Unit step signal $u(t)$, (b) Shifted signal $u(t + 2)$, (c) Scaled signal $-2u(t + 2)$.

(c) Given $$x(t) = -4r(t)$$

The signal $x(t)$ is a ramp signal with a slope of -4 as shown in Figure 1.29.

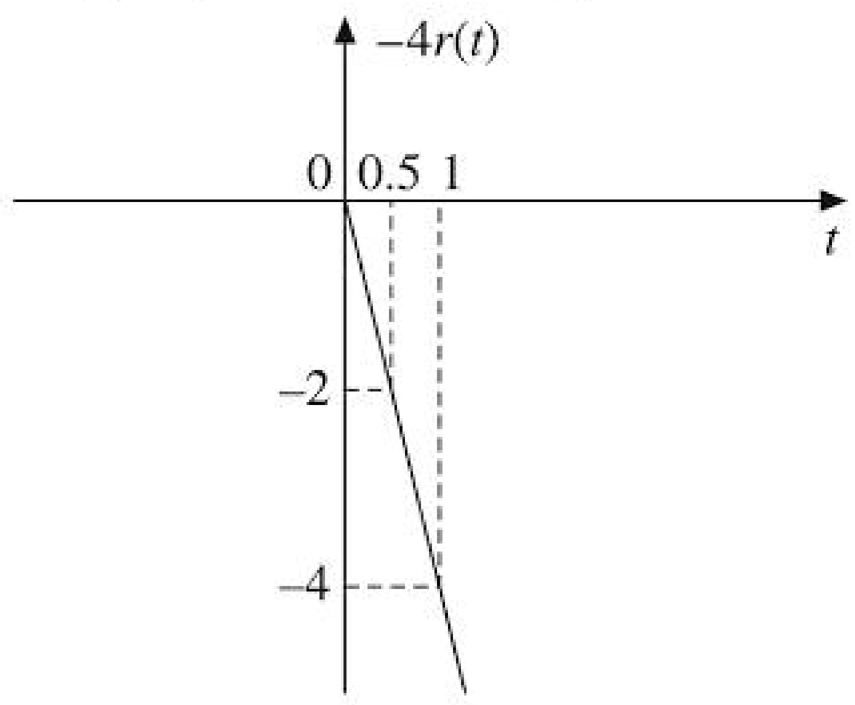

Figure 1.29 Ramp signal $x(t) = -4r(t)$.

(d) Given $$x(t) = 2r(t - 2)$$

The signal $2r(t - 2)$ can be obtained by first drawing the ramp signal $2r(t)$ with slope of 2 as shown in Figure 1.30(a), and then shifting it to the right by 2 units to obtain $2r(t - 2)$ as shown in Figure 1.30(b).

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For $1 \le t \le 3$ $\quad x_1(t) = 1$ and $x_2(t) = 1$

Hence $x_1(t) - x_2(t) = 1 - 1 = 0$

For $3 \le t \le 4$ $\quad x_1(t) = 2$ and $x_2(t)$ is falling linearly from 1 to 0.

Hence $x_1(t) - x_2(t)$ rises linearly from $2 - 1 = 1$ to $2 - 0 = 2$.

The difference $x_1(t) - x_2(t)$ is as shown in Figure 1.39(d).

In discrete-time domain, the sum of two signals $x_1(n)$ and $x_2(n)$ can be obtained by adding the corresponding sample values and the subtraction of $x_2(n)$ from $x_1(n)$ can be obtained by subtracting each sample of $x_2(n)$ from the corresponding sample of $x_1(n)$ as illustrated below.

If $\quad x_1(n) = \{1, 2, 3, 1, 5\}$

and $\quad x_2(n) = \{2, 3, 4, 1, -2\}$

Then $\quad x_1(n) + x_2(n) = \{1 + 2, 2 + 3, 3 + 4, 1 + 1, 5 - 2\} = \{3, 5, 7, 2, 3\}$

and $\quad x_1(n) - x_2(n) = \{1 - 2, 2 - 3, 3 - 4, 1 - 1, 5 + 2\} = \{-1, -1, -1, 0, 7\}$

1.4.6 Signal Multiplication

The multiplication of two continuous-time signals can be performed by multiplying their values at every instant. Two continuous-time signals $x_1(t)$ and $x_2(t)$ shown in Figure 1.40[(a) and (b)] are multiplied as shown below to obtain $x_1(t)\, x_2(t)$ shown in Figure 1.40(c).

For $0 \le t \le 1$ $\quad x_1(t) = 2$ and $x_2(t) = 1$

Hence $x_1(t)\, x_2(t) = 2 \times 1 = 2$

For $1 \le t \le 2$ $\quad x_1(t) = 1$ and $x_2(t) = 1 + (t - 1)$

Hence $x_1(t)\, x_2(t) = (1)[1 + (t - 1)] = 1 + (t - 1)$

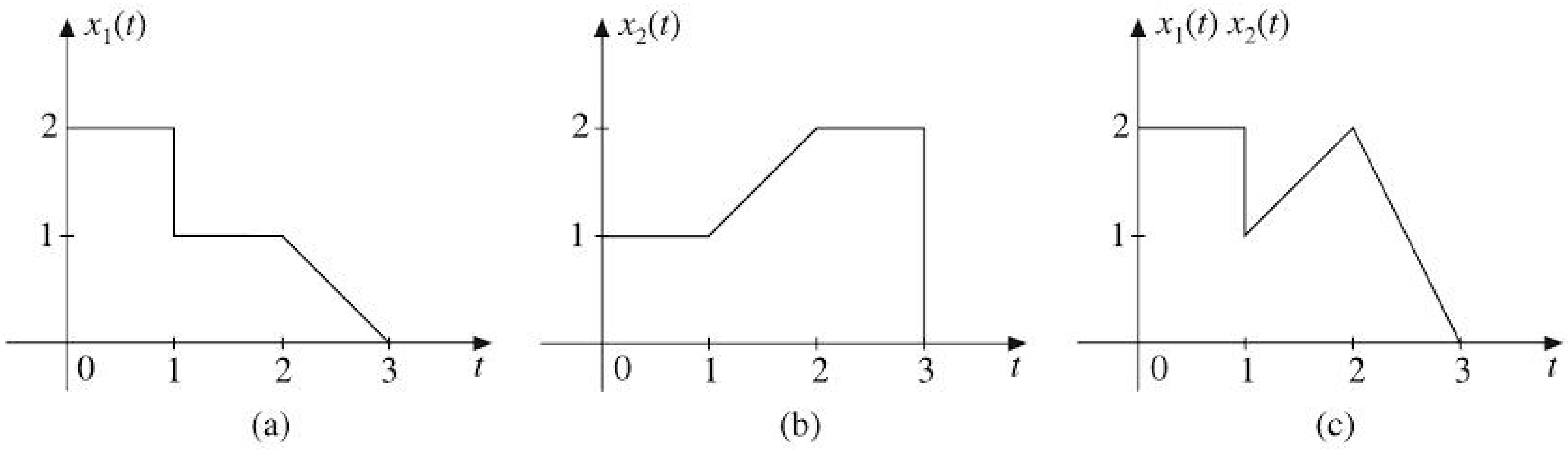

Figure 1.40 Multiplication of continuous-time signals.

For $2 \le t \le 3$ $\quad x_1(t) = 1 - (t - 2)$ and $x_2(t) = 2$

Hence $x_1(t)\, x_2(t) = [1 - (t - 2)]\, 2 = 2 - 2(t - 2)$

Multiplication of two discrete-time sequences can be performed by multiplying their values at the sampling instants.

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The output due to weighted sum of inputs is:

$$y_3(t) = T[ax_1(t) + bx_2(t)] = e^{[ax_1(t)+bx_2(t)]}$$

$$y_3(t) \neq ay_1(t) + by_2(t)$$

The weighted sum of outputs is not equal to the output due to weighted sum of inputs. The superposition principle is not satisfied. Therefore the system is non-linear.

(h) Given $$y(n) = n^2 x(n)$$

$$y(n) = T[x(n)] = n^2 x(n)$$

Let an input $x_1(n)$ produce an output $y_1(n)$.

$$\therefore \quad y_1(n) = T[x_1(n)] = n^2 x_1(n)$$

Let an input $x_2(n)$ produce an output $y_2(n)$.

$$\therefore \quad y_2(n) = T[x_2(n)] = n^2 x_2(n)$$

The weighted sum of outputs is:

$$ay_1(n) + by_2(n) = a[n^2 x_1(n)] + b[n^2 x_2(n)] = n^2[ax_1(n) + bx_2(n)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = n^2[ax_1(n) + bx_2(n)]$$

$$y_3(n) = ay_1(n) + by_2(n)$$

The weighted sum of outputs is equal to the output due to weighted sum of inputs. The superposition principle is satisfied. Therefore, the given system is linear.

(i) Given $$y(n) = x(n) + \frac{1}{2x(n-2)}$$

$$y(n) = T[x(n)] = x(n) + \frac{1}{2x(n-2)}$$

For an input $x_1(n)$,

$$y_1(n) = T[x_1(n)] = x_1(n) + \frac{1}{2x_1(n-2)}$$

For an input $x_2(n)$,

$$y_2(n) = T[x_2(n)] = x_2(n) + \frac{1}{2x_2(n-2)}$$

The weighted sum of outputs is given by

$$ay_1(n) + by_2(n) = a\left[x_1(n) + \frac{1}{2x_1(n-2)}\right] + b\left[x_2(n) + \frac{1}{2x_2(n-2)}\right]$$

$$= [ax_1(n) + bx_2(n)] + \frac{a}{2x_1(n-2)} + \frac{b}{2x_2(n-2)}$$

The systems satisfying both linearity and time-invariance properties are popularly known as linear time-invariant or simply LTI systems.

EXAMPLE 2.4 Determine whether the following systems are time-invariant or not:

(a) $y(t) = t^2 x(t)$ (b) $y(t) = x(t) \sin 10\pi t$

(c) $y(t) = x(t^2)$ (d) $y(t) = x(-2t)$

(e) $y(t) = e^{2x(t)}$ (f) $y(n) = x(n/2)$

(g) $y(n) = x(n)$ (h) $y(n) = x^2(n-2)$

(i) $y(n) = x(n) + nx(n-2)$

Solution:

(a) Given $$y(t) = t^2 x(t)$$

$$y(t) = T[x(t)] = t^2 x(t)$$

The output due to input delayed by T sec is:

$$y(t, T) = T[x(t-T)] = y(t)\Big|_{x(t)=x(t-T)} = t^2 x(t-T)$$

The output delayed by T sec is:

$$y(t-T) = y(t)\Big|_{t=t-T} = (t-T)^2\, x(t-T)$$

$$y(t, T) \neq y(t-T)$$

i.e. the delayed output is not equal to the output due to delayed input. Therefore, the system is time-variant.

(b) Given $$y(t) = x(t) \sin 10\pi t$$

$$y(t) = T[x(t)] = x(t) \sin 10\pi t$$

The output due to input delayed by T sec is:

$$y(t, T) = T[x(t-T)] = y(t)\Big|_{x(t)=x(t-T)} = x(t-T) \sin 10\pi t$$

The output delayed by T sec is:

$$y(t-T) = y(t)\Big|_{t=t-T} = x(t-T) \sin 10\pi(t-T)$$

$$y(t, T) \neq y(t-T)$$

i.e. the delayed output is not equal to the output due to delayed input. Therefore, the system is time-variant.

(c) Given $$y(t) = x(t^2)$$

$$y(t) = T[x(t)] = x(t^2)$$

The output due to input delayed by T sec is:

$$y(t, T) = T[x(t-T)] = y(t)\Big|_{x(t)=x(t-T)} = x(t^2 - T)$$

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The weighted sum of outputs is:

$$ay_1(t) + by_2(t) = a[x_1(t) + x_1(t-2)] + b[x_2(t) + x_2(t-2)]$$

The output due to weighted sum of inputs is:

$$y_3(t) = T[ax_1(t) + bx_2(t)] = [ax_1(t) + bx_2(t)] + ax_1(t-2) + bx_2(t-2)$$

$$y_3(t) = ay_1(t) + by_2(t)$$

So the system is linear.

$$y(t, T) = y(t)\big|_{x(t)=x(t-T)} = x(t-T) + x(t-2-T)$$

$$y(t-T) = y(t)\big|_{t=t-T} = x(t-T) + x(t-T-2)$$

$$y(t, T) = y(t-T)$$

So the system is time-invariant. Hence the given system is linear time-invariant.

EXAMPLE 2.6 Check whether the following systems are:

1. Static or dynamic
2. Linear or non-linear
3. Causal or non-causal
4. Time-invariant or time-variant

(a) $\dfrac{d^3y(t)}{dt^3} + 2\dfrac{d^2y(t)}{dt^2} + 4\dfrac{dy(t)}{dt} + 3y^2(t) = x(t+1)$

(b) $\dfrac{d^2y(t)}{dt^2} + 2y(t)\dfrac{dy(t)}{dt} + 3t\,y(t) = x(t)$

(c) $y(t) = ev\{x(t)\}$

(d) $y(t) = at^2x(t) + bt\,x(t-4)$

(e) $y(n) = x(n)\,x(n-2)$

(f) $y(n) = \log_{10}|x(n)|$

(g) $y(n) = a^nu(n)$

(h) $y(n) = x^2(n) + \dfrac{1}{x^2(n-1)}$

Solution:

(a) Given $$\frac{d^3y(t)}{dt^3} + 2\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y^2(t) = x(t+1)$$

1. The system is described by a differential equation. Hence the system is dynamic.
2. There is a square term of output [i.e. $y^2(t)$]. So the system is non-linear. This can be proved.

   Let an input $x_1(t)$ produce an output $y_1(t)$. So the differential equation becomes

$$\frac{d^3y_1(t)}{dt^3} + 2\frac{d^2y_1(t)}{dt^2} + 4\frac{dy_1(t)}{dt} + 3y_1^2(t) = x_1(t+1)$$

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4. Given $$y(t) = T[x(t)] = at^2 x(t) + bt\,x(t-4)$$

The output due to input delayed by T sec is:

$$y(t, T) = T[x(t-T)] = y(t)\big|_{x(t)=x(t-T)} = at^2 x(t-T) + bt\,x(t-4-T)$$

The output delayed by T sec is:

$$y(t-T) = y(t)\big|_{t=t-T} = a(t-T)^2 x(t-T) + b(t-T)x(t-T-4)$$

$$y(t, T) \neq y(t-T)$$

Hence the system is time-variant.

So the given system is dynamic, linear, causal and time-variant.

(e) Given $$y(n) = x(n)\,x(n-2)$$

1. The output depends on past values of input. So it requires memory. Hence the system is dynamic.
2. The only term contains the product of input and delayed input. So the system is non-linear. This can be proved.

   Let an input $x_1(n)$ produce an output $y_1(n)$.

   Then $$y_1(n) = x_1(n)\,x_1(n-2)$$

   Let an input $x_2(n)$ produce an output $y_2(n)$.

   Then $$y_2(n) = x_2(n)\,x_2(n-2)$$

   The weighted sum of outputs is:

   $$ay_1(n) + by_2(n) = ax_1(n)\,x_1(n-2) + bx_2(n)\,x_2(n-2)$$

   The output due to weighted sum of inputs is:

   $$y_3(n) = T[ax_1(n) + bx_2(n)] = [ax_1(n) + bx_2(n)]\,[ax_1(n-2) + bx_2(n-2)]$$

   $$y_3(n) \neq ay_1(n) + by_2(n)$$

   Hence the system is non-linear.
3. The output depends only on the present and past values of input. It does not depend on future values of input. So the system is causal.
4. Given $$y(n) = x(n)\,x(n-2)$$

   The output due to input delayed by k units is:

   $$y(n, k) = y(n)\big|_{x(n)=x(n-k)} = x(n-k)\,x(n-2-k)$$

   The output delayed by k units is:

   $$y(n-k) = y(n)\big|_{n=n-k} = x(n-k)\;x(n-k-2)$$

   $$y(n, k) = y(n-k)$$

   Hence the system is time-invariant.

   So the given system is dynamic, non-linear, causal and time-invariant.

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The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = [ax_1(n) + bx_2(n)]^2 + \frac{1}{[ax_1(n-1) + bx_2(n-1)]^2}$$

$$y_3(n) \neq ay_1(n) + by_2(n)$$

Hence the system is non-linear.

3. The output does not depend on future values of input. Hence the system is causal.

4. Given $$y(n) = T[x(n)] = x^2(n) + \frac{1}{x^2(n-1)}$$

The output due to input delayed by k units is:

$$y(n, k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = x^2(n-k) + \frac{1}{x^2(n-1-k)}$$

The output delayed by k units is:

$$y(n-k) = y(n)\big|_{n=n-k} = x^2(n-k) + \frac{1}{x^2(n-k-1)}$$

$$y(n, k) = y(n-k)$$

Hence the system is time-invariant.

So the given system is dynamic, non-linear, causal and time-invariant.

2.2.6 Stable and Unstable Systems

A bounded signal is a signal whose magnitude is always a finite value. For example, a sinewave is a bounded signal. A system is said to be bounded-input, bounded-output (BIBO) stable, if and only if every bounded input produces a bounded output. The output of a stable system does not diverge or does not grow unreasonably large.

Let the input signal $x(t)$ be bounded (finite), i.e.

$$|x(t)| \leq M_x < \infty \text{ for all } t$$

where M_x is a positive real number.

If $$|y(t)| \leq M_y < \infty$$

i.e. if $y(t)$ is also bounded, then the system is BIBO stable. Otherwise, the system is unstable. That is, we say that a system is unstable even if one bounded input produces an unbounded output.

It is very important to know about the stability of the system. Stability indicates the usefulness of the system. The stability can be found from the impulse response of the system which is nothing but the output of the system for a unit impulse input. If the impulse response is absolutely integrable for a continuous-time system or absolutely summable for a discrete-time system, then the system is stable.

BIBO stability criterion

The necessary and sufficient condition for a system to be BIBO stable is given by the expression

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

where $h(t)$ is the impulse response of the system. This is called BIBO stability criterion.

Proof: Consider a linear time-invariant system with $x(t)$ as input and $y(t)$ as output. The input and output of the system are related by the convolution integral.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Taking absolute values on both sides, we have

$$|y(t)| = \left| \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right|$$

Using the fact that the absolute value of the integral of the product of two terms is always less than or equal to the integral of the product of their absolute values, we have

$$\left| \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right| \le \int_{-\infty}^{\infty} |x(\tau)||h(t-\tau)|\, d\tau$$

If the input $x(\tau)$ is bounded, i.e. there exists a finite number M_x such that,

$$|x(\tau)| \le M_x < \infty$$

$$|y(t)| \le M_x \int_{-\infty}^{\infty} |h(t-\tau)|\, d\tau$$

Changing the variables by $m = t - \tau$, the output is bounded if

$$\int_{-\infty}^{\infty} |h(m)|\, dm < \infty$$

Replacing m by t, we have $$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

which is the necessary and sufficient condition for a system to be BIBO stable.

Figure 2.5 shows bounded and unbounded signals.

A system is stable if its impulse response $h(n)$ is absolutely summable.

i.e. $$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

In this case,

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} a\delta(n-7) = a \quad \text{for } n=7$$

Hence the given system is stable if the value of a is finite.

(b) Given $$y(n) = x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-2)$$

Let $$x(n) = \delta(n)$$

Then $$y(n) = h(n)$$

$\therefore$ $$h(n) = \delta(n) + \frac{1}{2}\delta(n-1) + \frac{1}{4}\delta(n-2)$$

A discrete-time system is stable if

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

The given $h(n)$ has a value only at $n = 0$, $n = 1$, and $n = 2$. For all other values of n from $-\infty$ to ∞, $h(n) = 0$.

At $n = 0$, $h(0) = \delta(0) + \frac{1}{2}\delta(0-1) + \frac{1}{4}\delta(0-2) = \delta(0) + \frac{1}{2}\delta(-1) + \frac{1}{4}\delta(-2) = 1$

At $n = 1$, $h(1) = \delta(1) + \frac{1}{2}\delta(1-1) + \frac{1}{4}\delta(1-2) = \delta(1) + \frac{1}{2}\delta(0) + \frac{1}{4}\delta(-1) = \frac{1}{2}$

At $n = 2$, $h(2) = \delta(2) + \frac{1}{2}\delta(2-1) + \frac{1}{4}\delta(2-2) = \delta(2) + \frac{1}{2}\delta(1) + \frac{1}{4}\delta(0) = \frac{1}{4}$

$\therefore$ $$\sum_{n=-\infty}^{\infty} |h(n)| = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} < \infty \text{ (a finite value)}$$

Hence the system is stable.

(c) Given $$h(n) = a^n \quad \text{for } 0 < n < 11$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} |a^n| = \sum_{n=0}^{11} a^n = \frac{1-a^{12}}{1-a}$$

This value is finite for finite value of a. Hence the system is stable if a is finite.

(d) Given $$h(n) = 2^n u(n)$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} |2^n u(n)| = \sum_{n=0}^{\infty} 2^n = \infty$$

The impulse response is not absolutely summable. Hence this system is unstable.

(e) Given $$h(n) = u(n)$$

For stability,

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

In this case,

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} 1 = 1 + 1 + 1 + \cdots = \infty$$

So the output is not bounded and the system is unstable.

EXAMPLE 2.9 Check whether the following digital systems are BIBO stable or not:

(a) $y(n) = ax(n+1) + bx(n-1)$

(b) $y(n) = \text{maximum of } [x(n), x(n-1), x(n-2)]$

(c) $y(n) = ax(n) + b$

(d) $y(n) = e^{-x(n)}$

(e) $y(n) = ax(n) + bx^2(n-1)$

Solution:

(a) Given $$y(n) = ax(n+1) + bx(n-1)$$

If $$x(n) = \delta(n)$$

Then $$y(n) = h(n)$$

Hence the impulse response is:

$$h(n) = a\delta(n+1) + b\delta(n-1)$$

When $n = 0, h(0) = a\delta(1) + b\delta(-1) = 0$

When $n = 1, h(1) = a\delta(2) + b\delta(0) = b$

When $n = 2, h(2) = a\delta(3) + b\delta(1) = 0$

In general, $$h(n) = \begin{cases} b & \text{for } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$\therefore$ $$\sum_{n=-\infty}^{\infty} |h(n)| = b$$

The necessary and sufficient condition for BIBO stability is:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

So the system is BIBO stable if $|b| < \infty$.

(b) Given $y(n) = \text{maximum of } [x(n), x(n-1), x(n-2)]$

If $x(n) = \delta(n)$

Then $y(n) = h(n)$

$\therefore$ $h(n) = \text{maximum of } [\delta(n), \delta(n-1), \delta(n-2)]$

$$h(0) = \text{maximum of } [\delta(0), \delta(-1), \delta(-2)] = 1$$

$$h(1) = \text{maximum of } [\delta(1), \delta(0), \delta(-1)] = 1$$

$$h(2) = \text{maximum of } [\delta(2), \delta(1), \delta(0)] = 1$$

$$h(3) = \text{maximum of } [\delta(3), \delta(2), \delta(1)] = 0$$

Similarly $h(4) = 0 = h(5) = h(6) \ldots$

$\therefore$
$$\sum_{n=-\infty}^{\infty} |h(n)| = |h(0)| + |h(1)| + |h(2)| + \cdots$$
$$= 1 + 1 + 1 + 0 + 0 + \cdots = 3$$

So the given system is BIBO stable.

(c) Given $y(n) = ax(n) + b$

If $x(n) = \delta(n)$

Then $y(n) = h(n)$

Hence the impulse response is:

$$h(n) = a\delta(n) + b$$

When $n = 0,\ h(0) = a\delta(0) + b = a + b$

When $n = 1,\ h(1) = a\delta(1) + b = b$

Here, $h(1) = h(2) = \ldots = h(n) = b$

Therefore,

$$h(n) = \begin{cases} a+b & \text{when } n = 0 \\ b & \text{when } n \neq 0 \end{cases}$$

The necessary and sufficient condition for BIBO stability is:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Therefore,
$$\sum_{n=-\infty}^{\infty} |h(n)| = |h(0)| + |h(1)| + |h(2)| + \cdots + |h(n)| + \cdots + \cdots$$
$$= |a+b| + |b| + |b| + \cdots + |b| + \cdots$$

This series never converges since the ratio between the successive terms is one. Hence the given system is BIBO unstable.

(d) Given $y(n) = e^{-x(n)}$

If $x(n) = \delta(n)$

Then $y(n) = h(n)$

Hence the impulse response is:

$$h(n) = e^{-\delta(n)}$$

When $n = 0$, $h(0) = e^{-\delta(0)} = e^{-1}$

When $n = 1$, $h(1) = e^{-\delta(1)} = e^{0} = 1$

In general,

$$h(n) = \begin{cases} e^{-1} & \text{when } n = 0 \\ 1 & \text{when } n \neq 0 \end{cases}$$

The necessary and sufficient condition for BIBO stability is:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Therefore,
$$\sum_{n=-\infty}^{\infty} |h(n)| = |h(0)| + |h(1)| + |h(2)| + \cdots + |h(n)| + \cdots$$
$$= e^{-1} + 1 + 1 + 1 + \cdots + 1 + \cdots$$

Since the given sequence never converges, it is BIBO unstable.

(e) Given $\quad y(n) = ax(n) + bx^2(n-1)$

If $\quad x(n) = \delta(n)$

Then $\quad y(n) = h(n)$

Hence the impulse response is:

$$h(n) = a\delta(n) + b\delta^2(n-1)$$

When $n = 0$, $h(0) = a\delta(0) + b\delta^2(-1) = a$

When $n = 1$, $h(1) = a\delta(1) + b\delta^2(0) = b$

When $n = 2$, $h(2) = a\delta(2) + b\delta^2(1) = 0$

Hence,
$$\sum_{n=-\infty}^{\infty} |h(n)| = |h(0)| + |h(1)| + |h(2)| + \cdots + |h(n)| + \cdots$$
$$= |a| + |b| + 0 + 0 + \cdots$$

Hence, the given system is BIBO stable if $|a| + |b| < \infty$.

EXAMPLE 2.10 Determine whether each of the system with impulse response/output listed below is (i) causal (ii) stable.

(a) $h(n) = 3^n u(-n)$

(b) $h(n) = \cos \dfrac{n\pi}{2}$

(c) $h(n) = \delta(n) + \cos n\pi$

(d) $h(n) = e^{3n}u(n-2)$

(e) $y(n) = \cos x(n)$

(f) $y(n) = \sum_{k=-\infty}^{n+5} x(k)$

(g) $y(n) = \log|x(n)|$ (h) $h(n) = [u(n) - u(n-15)]\, 2^n$

(i) $h(n) = 4^n u(2-n)$ (j) $h(n) = e^{-5|n|}$

Solution:

(a) Given $$h(n) = 3^n u(-n)$$

$u(-n)$ exists for $-\infty < n \le 0$. Hence $h(n) \ne 0$ for $n < 0$. So the system is non-causal.

For stability $$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$\sum_{n=-\infty}^{\infty} 3^n u(-n) = \sum_{n=-\infty}^{0} 3^n = \sum_{n=0}^{\infty} 3^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$$

$$= 1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots$$

$$= \left(1 - \frac{1}{3}\right)^{-1} = \frac{1}{1-(1/3)} = \frac{3}{2} < \infty$$

So the system is stable.

(b) Given $$h(n) = \cos\frac{n\pi}{2}$$

$\cos(n\pi/2)$ exists for $-\infty < n < \infty$. So $h(n) \ne 0$ for $n < 0$. Therefore, the system is non-causal.

For stability $$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$\sum_{n=-\infty}^{\infty} \left|\cos\frac{n\pi}{2}\right| = \infty$$

because for odd values of n, $\left|\cos\frac{n\pi}{2}\right| = 0$ and for even values of n, $\left|\cos\frac{n\pi}{2}\right| = 1$.

So the system is unstable.

(c) Given $$h(n) = \delta(n) + \cos n\pi$$

$\delta(n) = 1$ for $n = 0$ and $\delta(n) = 0$ for $n \ne 0$. $|\cos n\pi| = 1$ for all values of n.

For stability $$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = |h(-\infty)| + \cdots + |h(-1)| + |h(0)| + |h(1)| + \cdots + |h(\infty)|$$

$$= 1 + \cdots + 1 + 2 + 1 + \cdots + 1 = \infty$$

Therefore, the system is unstable.

(j) Given $$h(n) = e^{-5|n|}$$

The system is non-causal since $h(n) \neq 0$ for $n < 0$.

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} e^{-5n} = \sum_{n=-\infty}^{-1} e^{5n} + \sum_{n=0}^{\infty} e^{-5n} = \sum_{n=1}^{\infty} e^{-5n} + \sum_{n=0}^{\infty} e^{-5n}$$

$$= \frac{e^{-5}}{1-e^{-5}} + \frac{1}{1-e^{-5}} = \frac{1+e^{-5}}{1-e^{-5}} < \infty$$

Therefore, the system is stable.

EXAMPLE 2.11 Comment about the linearity, stability, time invariance and causality for the following filter:

$$y(n) = 2x(n+1) + [x(n-1)]^2$$

Solution: Given $$y(n) = 2x(n+1) + [x(n-1)]^2$$

1. There is a square term of delayed input [i.e., $x(n-1)^2$] in the difference equation. So the system is nonlinear.
2. The output depends on the future value of input [i.e., $2x(n+1)$]. So the system is non-causal.
3. For $x(n) = \delta(n)$, $y(n) = h(n)$

$$\therefore \qquad h(n) = 2\delta(n+1) + \{\delta(n-1)\}^2$$

$$h(0) = 2\delta(1) + \{\delta(-1)\}^2 = 0 + 0 = 0$$
$$h(1) = 2\delta(2) + \{\delta(0)\}^2 = 0 + 1 = 1$$
$$h(-1) = 2\delta(0) + \{\delta(-2)\}^2 = 2 + 0 = 2$$
$$h(-2) = 2\delta(-1) + \{\delta(-3)\}^2 = 0 + 0 = 0$$
$$h(n) = 0, \text{ for any other } n$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = 0 + 1 + 2 + 0 + 0 + \cdots = 3 < \infty$$

Impulse response is absolutely summable. So the system is stable. Also we can say that since the output depends only on the delayed and advanced inputs, if the input is bounded, the output is bounded. So the system is BIBO stable.

4. The output due to delayed input is given by

$$y(n, k) = 2x(n+1-k) + \{x(n-1-k)\}^2$$

The delayed output is

$$y(n-k) = 2x(n+1-k) + \{x(n-k-1)\}^2$$

$$\therefore \qquad y(n, k) = y(n-k)$$

Therefore, the system is time-invariant.

Also, we can say that since the system is described by constant coefficient difference equation, the system is time-invariant. So the given system is non-linear, stable, time-invariant and non-causal.

EXAMPLE 2.12 State whether the following system is linear, causal, time-invariant and stable:

$$y(n) + y(n-1) = x(n) + x(n-2)$$

Solution: Given $\quad y(n) = -y(n-1) + x(n) + x(n-2)$

1. Let an input $x_1(n)$ produce an output $y_1(n)$ and an input $x_2(n)$ produce an output $y_2(n)$. Therefore, the weighted sum of outputs is:

$$ay_1(n) + by_2(n) = -[ay_1(n-1) + by_2(n-1)] + [ax_1(n) + bx_2(n)] \\ + [ax_1(n-2) + bx_2(n-2)]$$

   The output due to weighted sum of inputs is:

$$y_3(n) = -\{ay_1(n-1) + by_2(n-1)\} + \{ax_1(n) + bx_2(n)\} + \{ax_1(n-2) + bx_2(n-2)\}$$
$$y_3(n) = ay_1(n) + by_2(n)$$

   So the system is linear.

2. The output depends only on the present and past inputs and past outputs. So the system is causal.
3. All the coefficients of the differential equation are constants. So the system is time-invariant.
4. For $x(n) = \delta(n)$, $y(n) = h(n)$

$$\therefore \quad h(n) = -h(n-1) + \delta(n) + \delta(n-2)$$
$$h(0) = -h(-1) + \delta(0) + \delta(-2) = 1$$
$$h(1) = -h(0) + \delta(1) + \delta(-1) = -1$$
$$h(2) = -h(1) + \delta(2) + \delta(0) = 1 + 0 + 1 = 2$$
$$h(3) = -h(2) + \delta(3) + \delta(1) = -2 + 0 + 0 = -2$$
$$\sum_{n=-\infty}^{\infty} |h(n)| = 1 + 1 + 2 + 2 + \cdots = \infty$$

   i.e., the impulse response is not absolutely summable. So the system is unstable. Therefore, the given system is non-linear, causal, time-variant and unstable.

EXAMPLE 2.13 Determine whether the following system is linear, stable, causal and time-invariant using appropriate tests:

$$y(n) = nx(n) + x(n+2) + y(n-2)$$

Solution: Given $\quad y(n) = nx(n) + x(n+2) + y(n-2)$

1. Let an input $x_1(n)$ produce an output $y_1(n)$ and an input $x_2(n)$ produce an output $y_2(n)$. Then the weighted sum of outputs is:

$$ay_1(n) + by_2(n) = n[ax_1(n) + bx_2(n)] + [ax_1(n+2) + bx_2(n+2)] \\ +[ay_1(n-2) + by_2(n-2)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = n\{ax_1(n) + bx_2(n)\} + \{ax_1(n+2) + bx_2(n+2)\} + \{ay_1(n-2) + by_2(n-2)\}$$

$$y_3(n) = ay_1(n) + by_2(n)$$

So the system is linear.

2. For $x(n) = \delta(n)$, $y(n) = h(n)$

$$\therefore \quad h(n) = n\delta(n) + \delta(n+2) + h(n-2)$$

$$h(-2) = -2\delta(-2) + \delta(0) + h(-4) = 1$$

$$h(0) = 0\delta(0) + \delta(2) + h(-2) = 0 + 0 + 1 = 1$$

$$h(1) = 1\delta(1) + \delta(3) + h(-1) = 0$$

$$h(2) = 2\delta(2) + \delta(4) + h(0) = 1$$

$$h(3) = 3\delta(3) + \delta(5) + h(1) = 0$$

$$h(4) = 4\delta(4) + \delta(6) + h(2) = 1$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = 1 + 0 + 1 + 0 + \cdots = \infty$$

So the system is unstable.

3. $y(2) = 2x(2) + x(4) + y(0)$

The output depends on future inputs. So the system is non-causal.

4. The coefficient of $x(n)$ is a function of time. So it is a time-varying system. Therefore, the given system is linear, unstable, non-causal and time-varying.

EXAMPLE 2.14 Find the linearity, invariance, causality of the following systems:

(a) $y(n) = -ax(n-1) + x(n)$

(b) $y(n) = x(n^2) + x(-n)$

Solution:

(a) Given $\quad y(n) = -ax(n-1) + x(n)$

1. Let an input $x_1(n)$ produce an output $y_1(n)$ and an input $x_2(n)$ produce an output $y_2(n)$. Then the weighted sum of outputs is:

$$py_1(n) + qy_2(n) = -a[px_1(n-1) + qx_2(n-1)] + [px_1(n) + qx_2(n)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = -a[px_1(n-1) + qx_2(n-1)] + [px_1(n) + qx_2(n)]$$

$$y_3(n) = py_1(n) + qy_2(n)$$

So the system is linear.

2. The output depends only on the present and past inputs. So the system is causal.
3. The output due to delayed input is:

$$y(n, k) = -ax(n-1-k) + x(n-k)$$

The delayed output is:

$$y(n-k) = -ax(n-1-k) + x(n-k)$$

$$y(n, k) = y(n - k)$$

So the system is time-invariant. Therefore, the given system is linear, causal and time invariant.

(b) Given $$y(n) = x(n^2) + x(-n)$$

1. Let an input $x_1(n)$ produce an output $y_1(n)$ and an input $x_2(n)$ produce an output $y_2(n)$. Then the weighted sum of outputs is:

$$ay_1(n) + by_2(n) = [ax_1(n^2) + bx_2(n^2)] + [ax_1(-n) + bx_2(-n)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = \{ax_1(n^2) + bx_2(n^2)\} + \{ax_1(-n) + bx_2(-n)\}$$
$$y_3(n) = ay_1(n) + by_2(n)\}$$

So the system is linear.

2. $y(-2) = x(4) + x(2)$

$y(2) = x(4) + x(-2)$

The output depends upon future inputs. So the system is non-causal.

3. The output due to delayed input is:

$$y(n, k) = x(n^2 - k) + x(-n - k)$$

The delayed output is:

$$y(n - k) = x\{(n - k)^2\} + x\{-(n - k)\}$$
$$y(n, k) \neq y(n - k)$$

So the system is time-variant. Therefore, the system is linear, non-causal and time variant.

EXAMPLE 2.15 Test the causality and stability of the following system:

$$y(n) = x(n) - x(-n - 1) + x(n - 1)$$

Solution: Given $$y(n) = x(n) - x(-n - 1) + x(n - 1)$$

1. $y(-2) = x(-2) - x(1) + x(-3)$

For negative values of n, the output depends on future values of input. So the system is non-causal.

2. For $x(n) = \delta(n)$, $y(n) = h(n)$

$\therefore$
$$h(n) = \delta(n) - \delta(-n - 1) + \delta(n - 1)$$
$$h(0) = \delta(0) - \delta(-1) + \delta(-1) = 1 - 0 + 0 = 1$$
$$h(1) = \delta(1) - \delta(-2) + \delta(0) = 0 - 0 + 1 = 1$$
$$h(-1) = \delta(-1) - \delta(0) + \delta(-2) = 0 - 1 + 0 = -1$$
$$h(n) = 0 \text{ for any other value of } n$$
$$\sum_{n=-\infty}^{\infty} |h(n)| = 1 + 1 + 1 + 0 + 0 + \cdots = 3 < \infty$$

i.e., the impulse response is absolutely summable. So the system is stable.

EXAMPLE 2.16 If a system is represented by the following difference equation:

$$y(n) = 3y^2(n-1) - nx(n) + 4x(n-1) - x(n+1),\ n \geq 0$$

(a) Is the system linear? Explain.
(b) Is the system shift-invariant? Explain.
(c) Is the system causal? Why or why not?

Solution: Given $y(n) = 3y^2(n-1) - nx(n) + 4x(n-1) - x(n+1),\ n \geq 0$

(a) No, the system is non-linear because there is a square term of delayed output in the difference equation.
(b) No, the system is shift variant because the coefficient of $x(n)$ is not a constant. It is a function of time.
(c) No, the system is non-causal because the output depends on future inputs.

EXAMPLE 2.17 Test the following systems for linearity, time invariance, stability and causality:

(a) $y(n) = a^{\{x(n)\}}$ (b) $y(n) = \sin\left\{\dfrac{2\pi bfn}{F}\right\} x(n)$

Solution:

(a) Given $$y(n) = a^{\{x(n)\}}$$

1. Let an input $x_1(n)$ produce an output $y_1(n)$ and an input $x_2(n)$ produce an output $y_2(n)$. Then the weighted sum of outputs is:

$$py_1(n) + qy_2(n) = pa^{\{x_1(n)\}} + qa^{\{x_2(n)\}}$$

The output due to weighted sum of inputs is:

$$y_3(n) = a^{\{px_1(n) + qx_2(n)\}}$$
$$y_3(n) \neq py_1(n) + qy_2(n)$$

So the system is non-linear.

2. The output due to delayed input is:

$$y(n, k) = a^{\{x(n-k)\}}$$

The delayed output is:

$$y(n-k) = a^{\{x(n-k)\}}$$
$$y(n, k) = y(n-k)$$

So the system is shift-invariant.

3. When input $x(n) = \delta(n)$, $y(n) = h(n)$

$$h(n) = a^{\{\delta(n)\}}$$
$$h(0) = a^{\{\delta(0)\}} = a$$
$$h(n) = a^0 = 1 \quad \text{(for any other } n\text{)}$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = 1 + 1 + \cdots + a + 1 + 1 + \cdots + = \infty$$

The impulse response is not absolutely summable. So the system is unstable.

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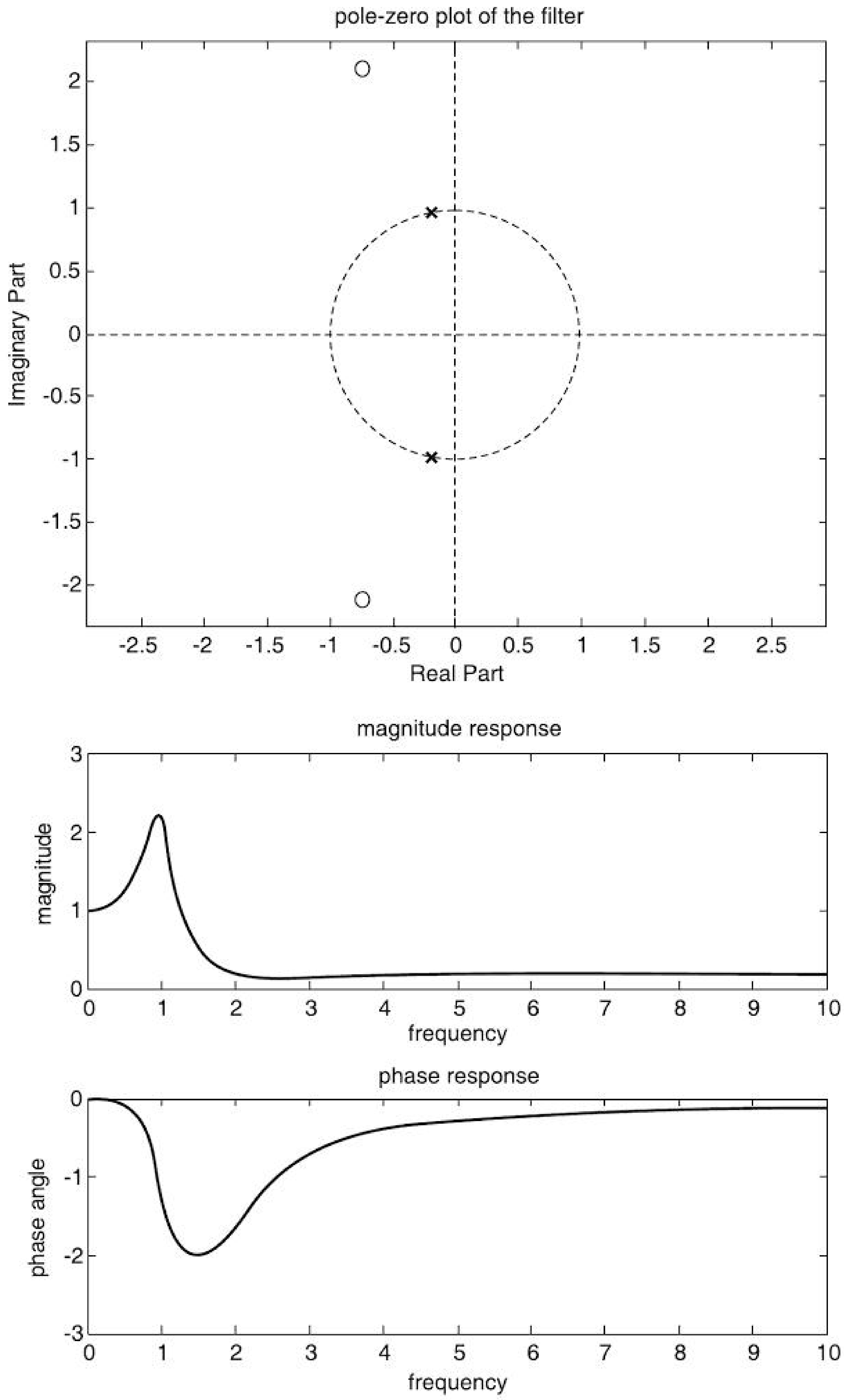
```
xlabel('frequency');ylabel('magnitude');
title('magnitude response');
Ha=angle(H);
subplot(2,1,2);plot(w,Ha);
xlabel('frequency');ylabel('phase angle');
title('phase response');
```

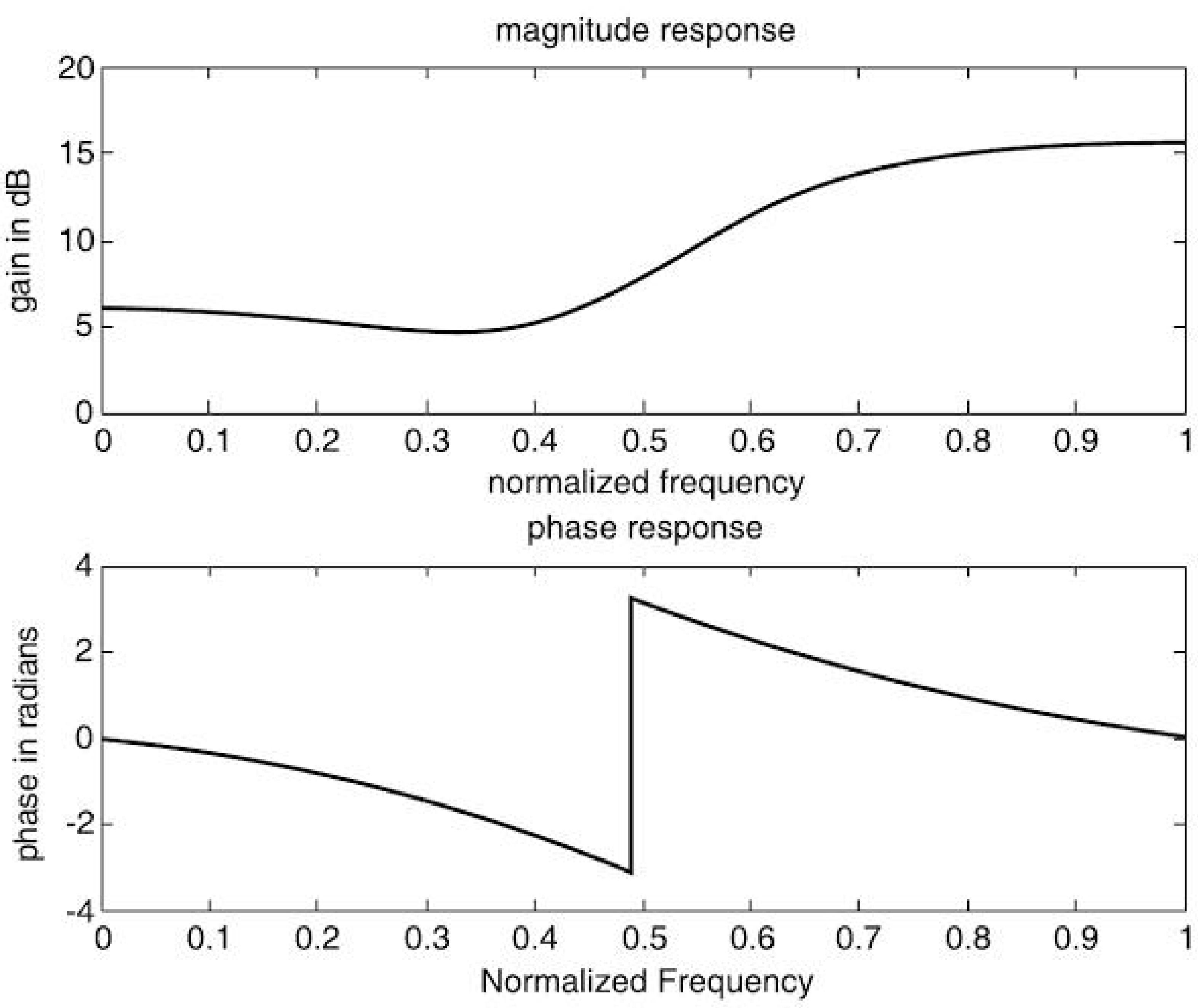
Output:



Program 2.3

```
%Frequency response of the given system
clc; clear all; close all;
num=[1 -1 3];
den=[1 1/3 1/6];
[h,om]=freqz(num,den);
subplot(2,1,1);plot(om/pi,20*log10(abs(h)));
xlabel('normalized frequency')
ylabel('gain in dB')
title('magnitude response')
subplot(2,1,2);plot(om/pi,angle(h));
xlabel('Normalized Frequency')
ylabel('phase in radians')
title('phase response')
```

Output:

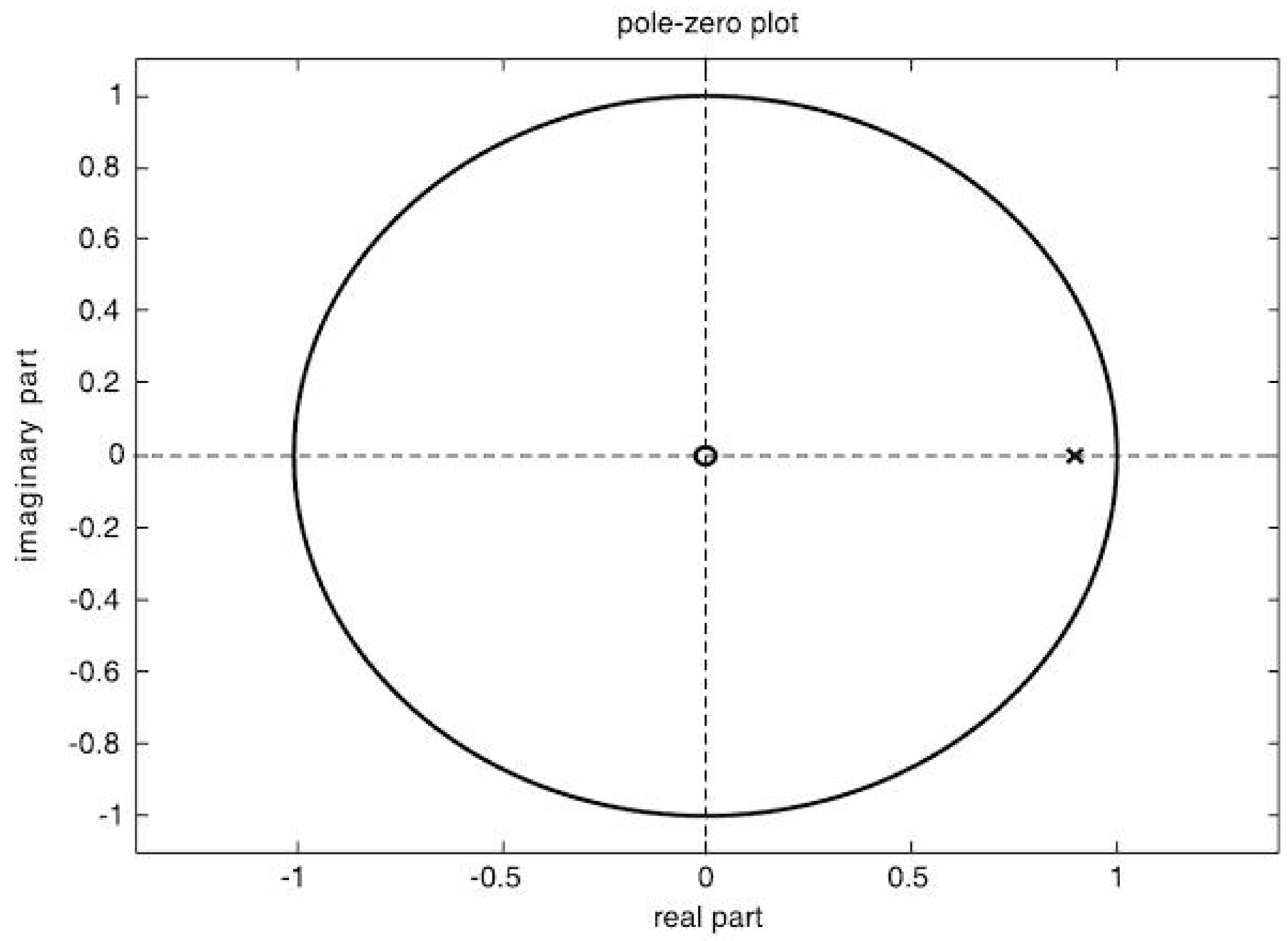


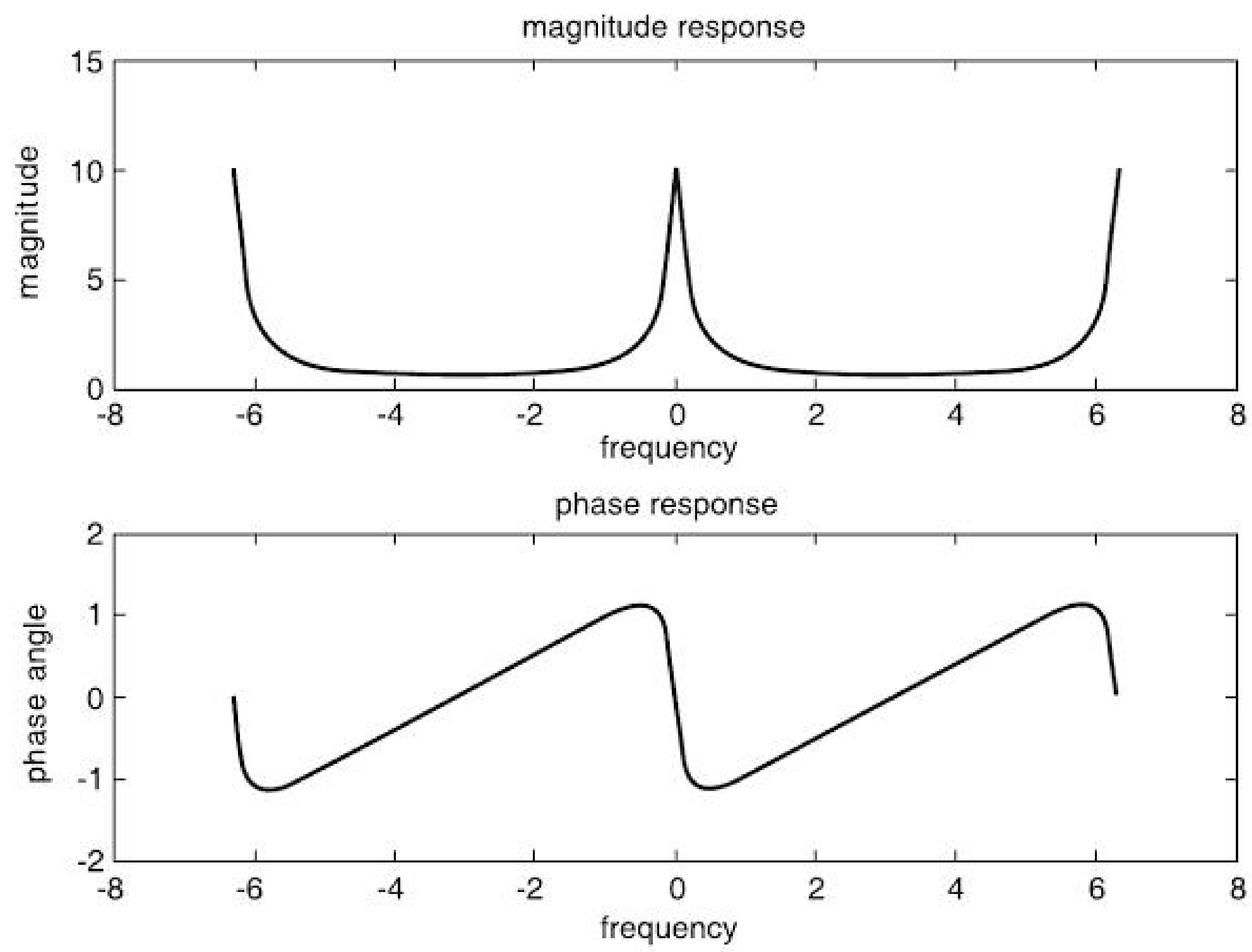
Program 2.4

```
% Frequency response of a discrete-time system
% y(n)=0.9y(n-1)+x(n)
clc;
% pole-zero plot
num=[1];den=[1 -0.9];
zplane(num,den);
xlabel('real part');ylabel('imaginary part');
title('pole-zero plot');
% frequency response
w=-2*pi:pi/100:2*pi;
[H,w]=freqz(num,den,w);
```

```
% magnitude response
Hm=abs(H);
figure;
subplot(2,1,1);plot(w,Hm);
xlabel('frequency');ylabel('magnitude');
title('magnitude response');
% phase response
Ha=angle(H);
subplot(2,1,2);plot(w,Ha);
xlabel('frequency');ylabel('phase angle');
title('phase response');
```

Output:



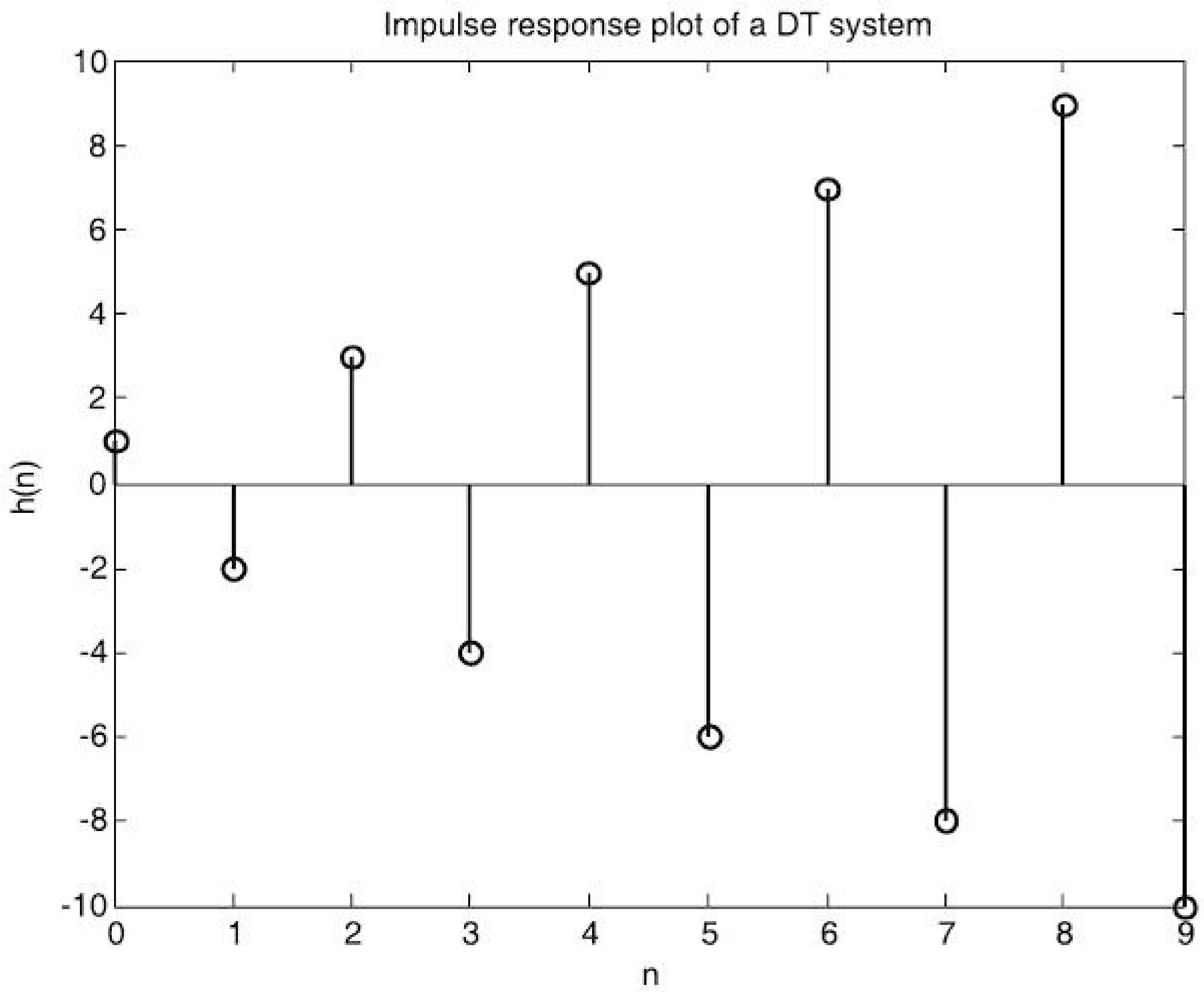


Program 2.5

```
% Impulse response of a discrete-time system with transfer function H(z)
clc;clear all;close all;
num=input('enter coefficients of numerator polynomial');
den=input('enter coefficients of denominator polynomial');
[h,n]=impz(num,den);
stem(n,h);
xlabel('n');ylabel('h(n)');
title('Impulse response plot of a DT system');
disp('Impulse response of a DT system');
disp(h)
```

Output:

```
enter coefficients of numerator polynomial      [1]
enter coefficients of denominator polynomial    [1 2 1]
Impulse response of a DT system
 [ 1  -2  3 -4  5 -6  7 -8  9 -10]
```

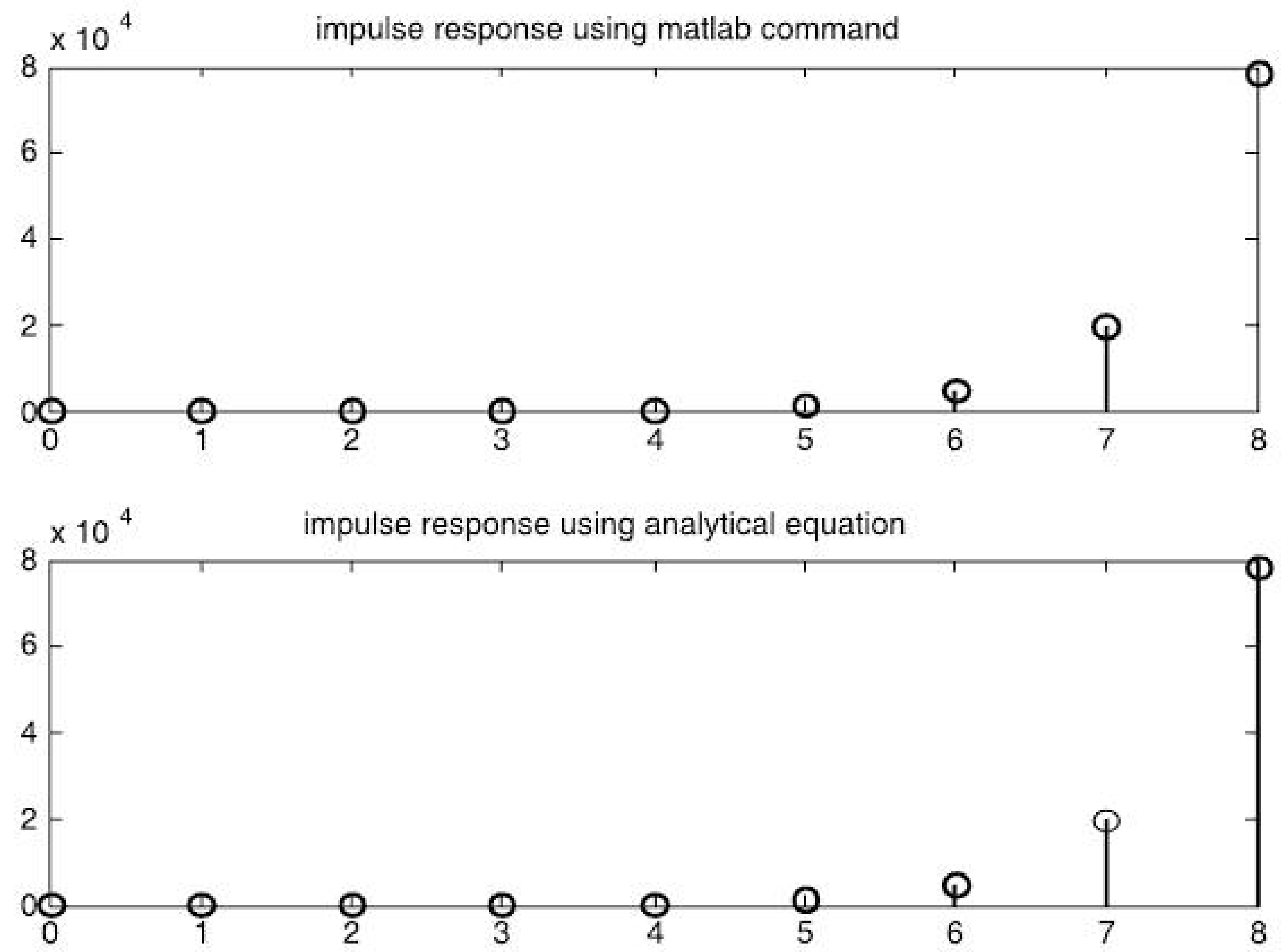


Program 2.6

```
% Impulse response of a discrete-time system
% y(n)-3y(n-1)-4y(n-2)=x(n)+2x(n-1)
clc;close all;clear all;
```

```
syms n z
num=[1 2];
den=[1 -3 -4];
% Transfer function
H=tf('z');
Ts=[];
H=tf(num,den,Ts)
% Impulse response
%h=iztrans(H,z,n)
[h,n]=impz(num,den)
subplot(2,1,1);stem(n,h);
title('impulse response using matlab command');
% analytical equation of impulse response
ha=1.2*4.^n-0.2*(-1).^n;
subplot(2,1,2);stem(n,ha)
title('impulse response using analytical equation');
```

Output:

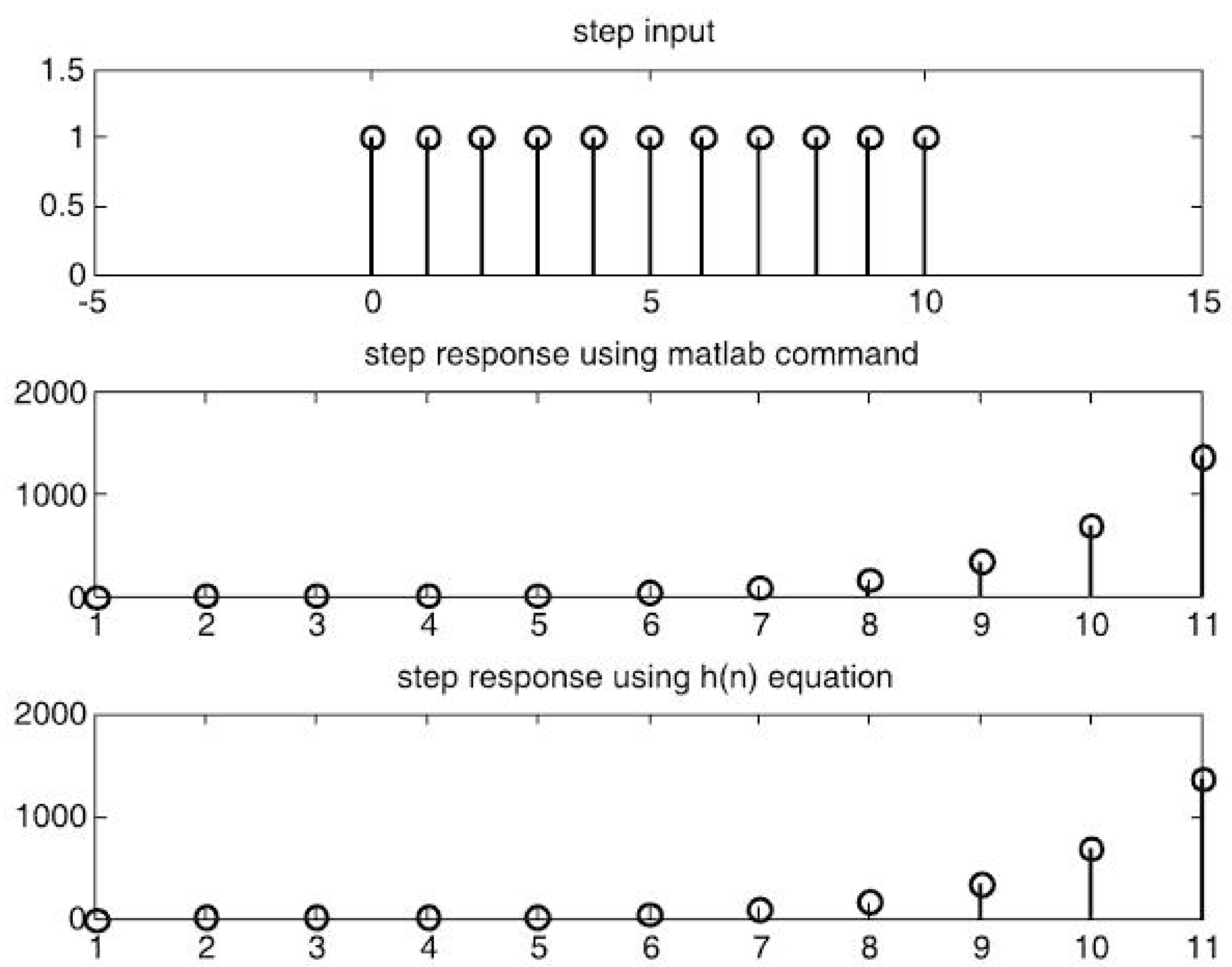


Program 2.7

```
% Step response of a discrete-time system
clc;close all;clear all;
% step input
n=0:1:10;
x=ones(1,length(n));
subplot(3,1,1);stem(n,x);
axis([-5 15 0 1.5]);
```

```
title('step input');
% System specifications
num=[0 1 2];
den=[1 -1 -2];
% step response using matlab command
y=filter(num,den,x);
subplot(3,1,2);stem(y);
title('step response using matlab command');
% step response using analytical equation
y=(4/3)*(2).^n+(1/6)*(-1).^n-(3/2);
subplot(3,1,3);stem(y);
title('step response using h(n) equation');
```

Output:

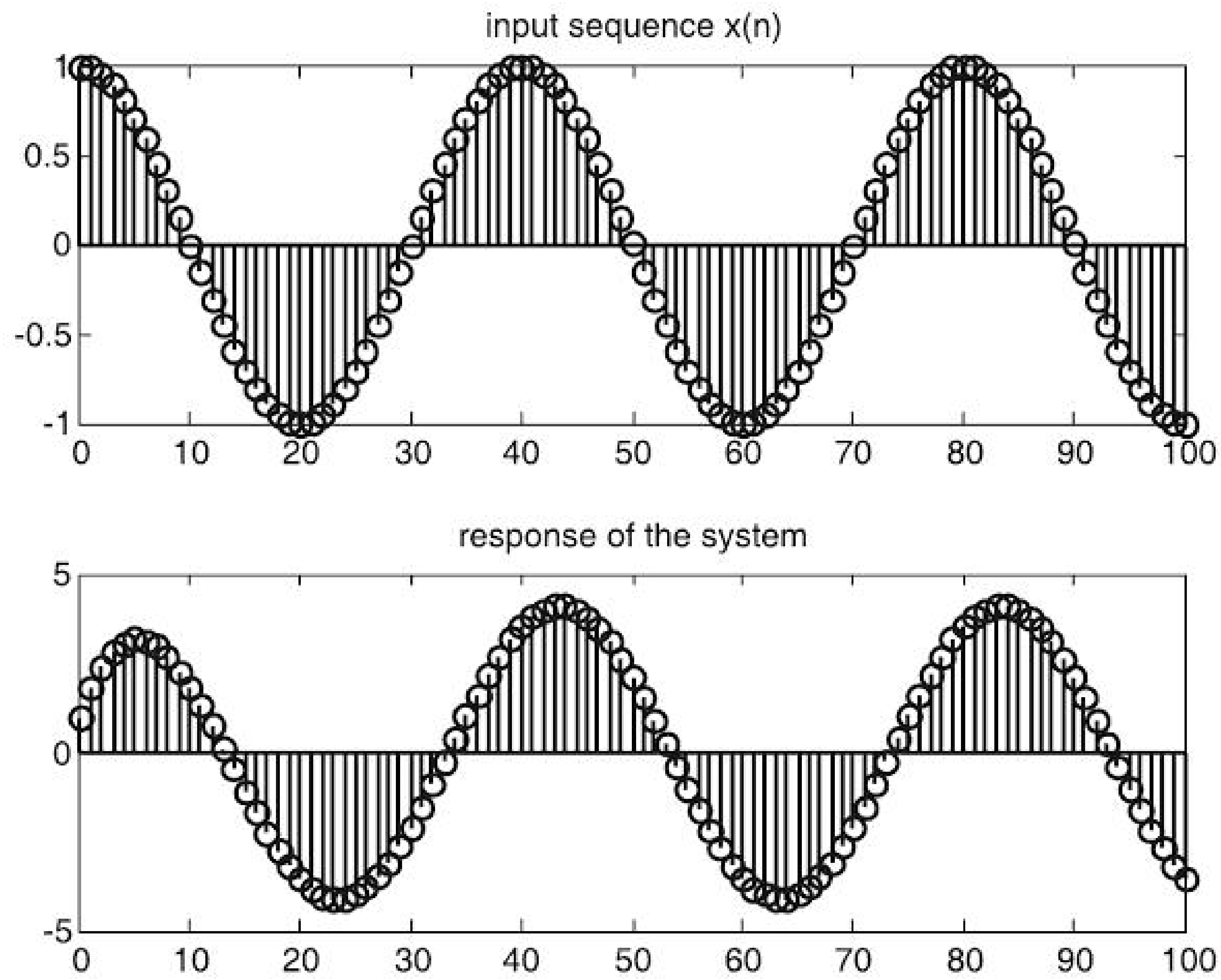


Program 2.8

```
% Response of a discrete-time system
% y(n)=0.8y(n-1)+x(n) for the input x(n)=cos(0.05*pi*n)u(n)
clc;close all;clear all;
% input x(n)=cos(0.05*pi*n)u(n)
n=0:1:100;
x=cos(0.05*pi*n);
subplot(2,1,1);stem(n,x);
title('input sequence x(n)');
% System specifications
num=[1];
```

```
den=[ 1 -0.8 ];
% response of the system
y=filter(num,den,x);
subplot(2,1,2);stem(n,y);
title('response of the system');
```

Output:



SHORT QUESTIONS WITH ANSWERS

1. Define a system.

   Ans. A system is defined as a physical device, that generates a response or output signal for a given input signal.

2. How are systems classified?

   Ans. The systems are classified as follows:

   1. Continuous-time and discrete-time systems
   2. Lumped parameter and distributed parameter systems.
   3. Static (memoryless) and dynamic (memory) systems
   4. Causal and non-causal systems
   5. Linear and non-linear systems
   6. Time-invariant and time varying systems
   7. Stable and unstable systems.
   8. Invertible and non-invertible systems

3. Define a continuous-time system.

   Ans. A continuous-time system is a system which transforms continuous-time input signals into continuous-time output signals.

4. Define a discrete-time system.

   Ans. A discrete-time system is a system which transforms discrete-time input signals into discrete-time output signals.

5. Define a lumped parameter system.

   Ans. A lumped parameter system is a system in which each component is lumped at one point in space. These systems are described by ordinary differential equations.

6. Define a distributed parameter system.

   Ans. A distributed parameter system is a system in which the signals are functions of space as well as time. These systems are described by partial differential equations.

7. Define a static system.

   Ans. A static or memoryless system is a system in which the response at any instant is due to present input alone, i.e. for a static or memoryless system, the output at any instant t (or n) depends only on the input applied at that instant t (or n) but not on the past or future values of input.

8. Define a dynamic system.

   Ans. A dynamic or memory system is a system in which the response at any instant depends upon past or future inputs.

9. Define a causal system.

   Ans. A causal (or non-anticipative) system is a system whose output at any time t depends only on the present and past values of the input but not on future inputs.

10. Define a non-causal system.

    Ans. A non-causal (anticipative) system is a system whose output at any time t depends on future inputs.

11. What is homogeneity property?

    Ans. Homogeneity property means a system which produces an output $y(t)$ for an input $x(t)$ must produce an output $ay(t)$ for an input $ax(t)$.

12. What is superposition property?

    Ans. Superposition property means a system which produces an output $y_1(t)$ for an input $x_1(t)$ and an output $y_2(t)$ for an input $x_2(t)$ must produce an output $y_1(t) + y_2(t)$ for an input $x_1(t) + x_2(t)$.

13. Define a linear system.

    Ans. A linear system is a system which obeys the principle of superposition and principle of homogeneity.

14. Define a non-linear system.

    Ans. A non-linear system is a system which does not obey the principle of superposition and principle of homogeneity.

15. Define a time-invariant system.

    Ans. A time-invariant (or shift-invariant) system is a system whose input/output characteristics do not change with time, i.e. a system for which a time shift in the input results in a corresponding time shift in the output.

16. Define a time-variant system.

    Ans. A time-variant (or shift-variant) system is a system whose input/output characteristics change with time, i.e. a system for which a time shift in the input does not result in a corresponding time shift in the output.

17. Define a bounded input-bounded output stable system.

    Ans. A bounded input-bounded output stable system is a system which produces a bounded output for every bounded input.

18. Define an unstable system.

    Ans. An unstable system is a system which produces an unbounded output for a bounded input.

19. What is an invertible system?

    Ans. An invertible system is a system which has a unique relation between its input and output.

20. What is a non-invertible system?

    Ans. A non-invertible system is a system which does not have a unique relation between its input and output.

REVIEW QUESTIONS

1. Define a system. How are systems classified? Define each one of them.
2. Distinguish between
    1. Continuous-time and discrete-time systems
    2. Lumped parameter and distributed parameter systems
    3. Static (memoryless) and dynamic (memory) systems
    4. Causal and non-causal systems
    5. Linear and non-linear systems
    6. Time-invariant and time varying systems
    7. Stable and unstable systems

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1. _________ systems are described by ordinary differential equations.
2. _________ systems are described by partial differential equations.
3. For a static system, the output does not depend on the _________ values of input.
4. For a dynamic system, the output depends on the _________ and/or _________ values of input.

5. Static systems are also called _________ systems.
6. Dynamic systems are also called _________ systems.
7. A causal system is one whose output depends on _________ values of input.
8. A non-causal system is one whose output depends on _________ values of input.
9. A causal system is also known as a _________ system.
10. A non-causal system is also known as an _________ system.
11. A _________ system is definitely a dynamic system.
12. A _________ system obeys the principle of superposition.
13. A _________ system does not obey the principle of superposition.
14. An LTI system is one which satisfies the properties of _________ and _________.
15. For a time-invariant system, its _________ do not change with time.
16. For a time-variant system, its _________ change with time.
17. For a _________ system, the coefficients of the differential equation are constants.
18. For a _________ system, the coefficients of the differential equation are functions of time.
19. A system is said to be stable if every _________ input produces a bounded output.
20. For a continuous-time system to be stable, its impulse response must be _________.
21. For a discrete-time system to be stable, its impulse response must be _________.
22. A system which has a unique relation between its input and output is called _________.
23. A system which does not have a unique relation between its input and output is called _________.

OBJECTIVE TYPE QUESTIONS

1. A system is a
   (a) physical device (b) mathematical model
   (c) linear model (d) ideal device
2. A lumped parameter system is described by
   (a) ordinary differential equations (b) partial differential equations
   (c) both (a) and (b) (d) none of the above
3. A distributed parameter system is described by
   (a) ordinary differential equations (b) partial differential equations
   (c) both (a) and (b) (d) none of the above
4. A system whose output depends on future inputs is a
   (a) static system (b) dynamic system
   (c) non-causal system (d) both (b) and (c)
5. A non-anticipative system is a
   (a) static system (b) dynamic system
   (c) causal system (d) non-causal system

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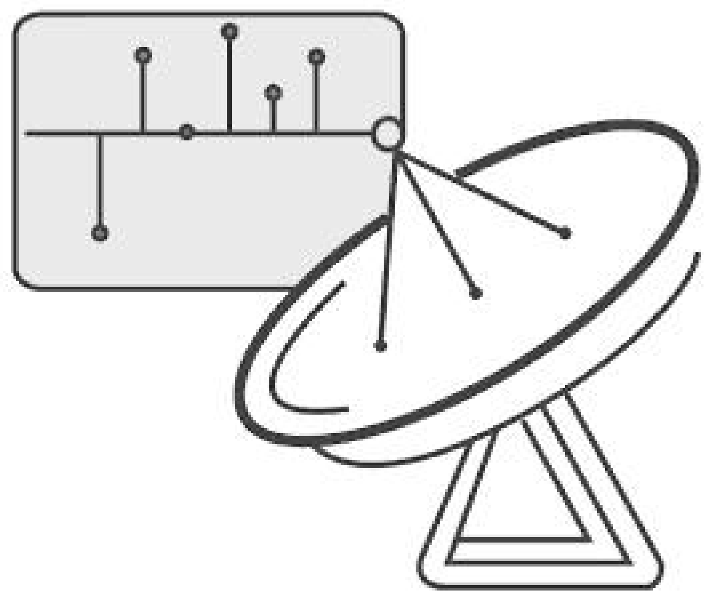
Signal Analysis



3.1 INTRODUCTION

It is very easy to understand or remember a new problem when it is associated with a familiar phenomenon. So while studying a new problem, we always search for analogies between that problem and a familiar phenomenon. Some insight into the new problem can be gained from the knowledge of the analogous phenomenon. For example, there is a perfect analogy between electrical and mechanical systems. If we are familiar with electrical systems, mechanical systems can be easily understood by using the analogy between them. We are familiar with vectors and in this book we want to discuss signal analysis. There is a perfect analogy between vectors and signals. In this chapter, using the analogy between vectors and signals, we try to understand signal analysis in a better way.

3.2 ANALOGY BETWEEN VECTORS AND SIGNALS

Vectors

A vector is specified by magnitude and direction. We shall denote all vectors by boldface type and their magnitudes by light face type. Consider two vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ as shown in Figure 3.1. The vector $\mathbf{V}_1$ can be expressed in terms of vector $\mathbf{V}_2$ in infinite ways by drawing a line from the end of $\mathbf{V}_1$ on to $\mathbf{V}_2$. In each representation, $\mathbf{V}_1$ is represented in terms of $\mathbf{V}_2$ plus another vector which will be called the *error vector*. In Figure 3.1(a), $\mathbf{V}_1$ is approximated by $C_1\mathbf{V}_2$ with an error $\mathbf{V}_{e_1}$, i.e. $\mathbf{V}_1 = C_1\mathbf{V}_2 + \mathbf{V}_{e_1}$. In Figure 3.1(b), $\mathbf{V}_1$ is approximated by $C_2\mathbf{V}_2$ with an error $\mathbf{V}_{e_2}$ as $\mathbf{V}_1 = C_2\mathbf{V}_2 + \mathbf{V}_{e_2}$. In Figure 3.1(c), $\mathbf{V}_1$ is approximated by $C_{12}\mathbf{V}_2$ with an error $\mathbf{V}_e$ as $\mathbf{V}_1 = C_{12}\mathbf{V}_2 + \mathbf{V}_e$. Geometrically the component of a vector $\mathbf{V}_1$ along vector $\mathbf{V}_2$ is obtained by drawing a perpendicular from the end of $\mathbf{V}_1$ onto $\mathbf{V}_2$. In this case, error vector $\mathbf{V}_e$ is the minimum. The component of a vector $\mathbf{V}_1$ along another vector $\mathbf{V}_2$ is given by $C_{12}\mathbf{V}_2$, where C_{12} is chosen such that the error vector is minimum. The magnitude of C_{12} is an indication of the similarity of the two vectors. If C_{12} is zero, then vector $\mathbf{V}_1$ has no component along the other vector $\mathbf{V}_2$, and hence two vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ are mutually perpendicular. The vectors which are mutually perpendicular to each other are called *orthogonal vectors*. Orthogonal vectors are thus independent vectors. If the vectors are orthogonal, then the parameter C_{12} is zero.

The component $$\mathbf{V}_1 \text{ along } \mathbf{V}_2 = \frac{\mathbf{V}_1 \cdot \mathbf{V}_2}{V_2} = C_{12}V_2$$

Therefore, $$C_{12} = \frac{\mathbf{V}_1 \cdot \mathbf{V}_2}{V_2^2} = \frac{\mathbf{V}_1 \cdot \mathbf{V}_2}{\mathbf{V}_2 \cdot \mathbf{V}_2}$$

When vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ are orthogonal, their dot product

$$\mathbf{V}_1 \cdot \mathbf{V}_2 = 0$$

and $$C_{12} = 0$$

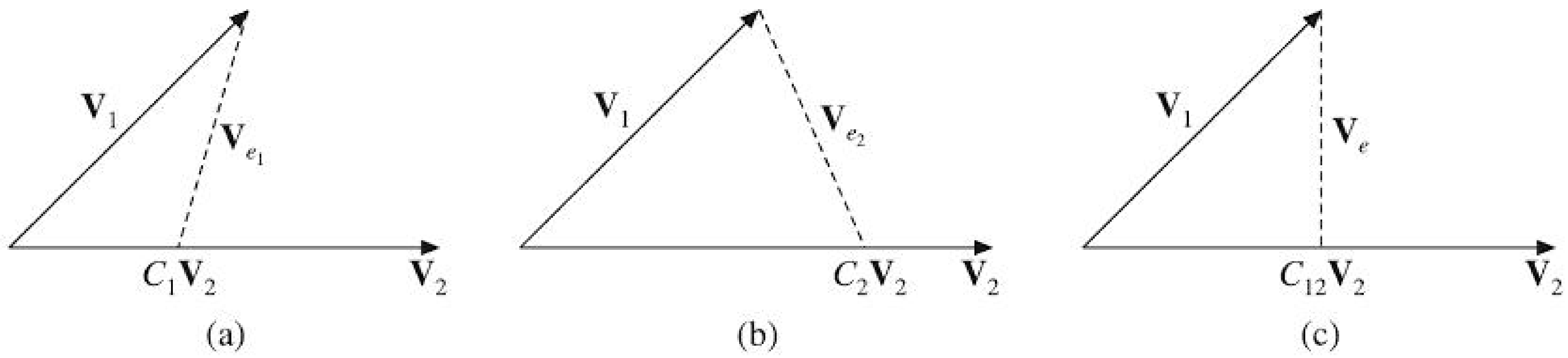


Figure 3.1 One vector in terms of another.

Signals

The concept of vector comparison and orthogonality can be extended to signals. Consider two signals $x_1(t)$ and $x_2(t)$. We can approximate $x_1(t)$ in terms of $x_2(t)$ over a certain interval $(t_1 < t < t_2)$ as follows:

$$x_1(t) \approx C_{12}x_2(t) \qquad \text{for } (t_1 < t < t_2)$$

[This is similar to $\mathbf{V}_1 \approx C_{12}\mathbf{V}_2$ in the case of vectors.]

Like in vectors, the main criterion in selecting C_{12} is to minimise the error between the actual function and the approximated function over the interval $(t_1 < t < t_2)$.

The error function $x_e(t)$ is defined as:

$$x_e(t) = x_1(t) - C_{12}x_2(t)$$

If we choose to minimise the error $x_e(t)$ over the interval t_1 to t_2 by minimising the average value of $x_e(t)$ over this interval, i.e. by minimising

$$\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} x_e(t)\,dt, \quad \text{i.e. } \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \left[x_1(t) - C_{12}x_2(t)\right] dt$$

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$$\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt = 0$$

[Two vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ are orthogonal if $\mathbf{V}_1 \cdot \mathbf{V}_2 = 0$]
Thus, we can conclude that in the case of vectors,

$$C_{12} = \frac{\mathbf{V}_1 \cdot \mathbf{V}_2}{V_2^2}$$

and in the case of signals,

$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt}{\int_{t_1}^{t_2} x_2^2(t)\, dt}$$

In both cases for orthogonality, $C_{12} = 0$.

Two vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ are orthogonal if their dot product is zero, i.e. $\mathbf{V}_1 \cdot \mathbf{V}_2 = 0$. Two signals $x_1(t)$ and $x_2(t)$ are orthogonal if the integral of the product of those signals is zero, i.e.

$$\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt = 0$$

In the analogy of vectors and signals, the dot product of two vectors is analogous to the integral of the product of two signals, that is

$$\mathbf{A} \cdot \mathbf{B} \sim \int_{t_1}^{t_2} x_{\text{A}}(t)\, x_{\text{B}}(t)\, dt$$

The square of the length A of vector $\mathbf{A}$ is analogous to the integral of the square of a function, that is

$$\mathbf{A} \cdot \mathbf{A} = A^2 \sim \int_{t_1}^{t_2} x_{\text{A}}^2(t)\, dt$$

We can easily show that the functions sin $n\omega_0 t$, sin $m\omega_0 t$; cos $n\omega_0 t$, cos $m\omega_0 t$; and sin $n\omega_0 t$, cos $m\omega_0 t$ are orthogonal over any interval $[t_0, t_0 + (2\pi/\omega_0)]$ for integral values of m and n.

EXAMPLE 3.1 Show that the following signals are orthogonal over an interval [0, 1]:

$$x_1(t) = 2$$

$$x_2(t) = \sqrt{3}(1 - 2t)$$

Solution: We know that two signals $x_1(t)$ and $x_2(t)$ are orthogonal over an interval (t_1, t_2), if

$$\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt = 0$$

We know that the signals $x_p(t)$ and $x_q(t)$ are orthogonal over the period (0 to T), if

$$\int_0^T x_p(t)\, x_q(t)\, dt = 0$$

In this case,

$$x_p(t)\, x_q(t) = \frac{1}{\sqrt{T}}[\cos p\omega_0 t + \sin p\omega_0 t]\frac{1}{\sqrt{T}}[\cos q\omega_0 t + \sin q\omega_0 t]$$

$$= \frac{1}{T}[\cos p\omega_0 t \cos q\omega_0 t + \sin p\omega_0 t \sin q\omega_0 t + \cos p\omega_0 t \sin q\omega_0 t + \sin p\omega_0 t \cos q\omega_0 t]$$

$$= \frac{1}{T}[\cos(p-q)\,\omega_0 t + \sin(p+q)\,\omega_0 t]$$

$$\therefore \qquad I = \int_{t_0}^{t_0+T} x_p(t)\, x_q(t)\, dt$$

$$= \int_0^T \frac{1}{T}[\cos(p-q)\,\omega_0 t + \sin(p+q)\,\omega_0 t]\, dt$$

$$= \frac{1}{T}\left[\frac{\sin(p-q)\,\omega_0 t}{(p-q)\,\omega_0} - \frac{\cos(p+q)\,\omega_0 t}{(p+q)\,\omega_0}\right]_0^T$$

$$= \frac{1}{T}\left[\frac{\sin(p-q)\,\omega_0 T - \sin 0}{(p-q)\,\omega_0} - \frac{\cos(p+q)\,\omega_0 T - \cos 0}{(p+q)\,\omega_0}\right]$$

$$= \frac{1}{T}\left[\frac{\sin(p-q)\,\omega_0(2\pi/\omega_0) - \sin 0}{(p-q)\,\omega_0} - \frac{\cos(p+q)\,\omega_0(2\pi/\omega_0) - \cos 0}{(p+q)\,\omega_0}\right]$$

$$= \frac{1}{T}\left[\frac{\sin(p-q)2\pi - \sin 0}{(p-q)\omega_0} - \frac{\cos(p+q)2\pi - \cos 0}{(p+q)\omega_0}\right]$$

$$= \frac{1}{T}\left[\frac{0-0}{(p-q)\omega_0} - \frac{1-1}{(p+q)\omega_0}\right]$$

$$= 0$$

$$\therefore \qquad \int_0^T x_p(t)\, x_q(t) = 0$$

Thus, the functions $x_p(t)$ and $x_q(t)$ are orthogonal over the period [0 to T].

EXAMPLE 3.5 Prove that the signals $x(t)$ and $y(t)$ shown in Figure 3.2 are orthogonal over the interval [0, 4].

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3.5.1 Approximation of a Function by a Set of Mutually Orthogonal Functions

Consider a set of n functions $g_1(t)$, $g_2(t)$, …, $g_n(t)$ which are orthogonal to one another over an interval t_1 to t_2, that is

$$\int_{t_1}^{t_2} g_j(t)\, g_k(t)\, dt = 0 \qquad j \neq k$$

and let

$$\int_{t_1}^{t_2} g_j^2(t)\, dt = K_j$$

Let an arbitrary signal $x(t)$ be approximated over an interval (t_1, t_2) by a linear combination of these n mutually orthogonal signals.

$$x(t) \approx C_1 g_1(t) + C_2 g_2(t) + \cdots + C_k g_k(t) + \cdots + C_n g_n(t)$$

$$= \sum_{r=1}^{n} C_r g_r(t)$$

For the best approximation we must find the proper values of constants C_1, C_2, …, C_n such that ε, the mean square of error $x_e(t)$ is minimised.
By definition,

$$x_e(t) = x(t) - \sum_{r=1}^{n} C_r g_r(t)$$

and

$$\varepsilon = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \left[x(t) - \sum_{r=1}^{n} C_r g_r(t) \right]^2 dt$$

This equation shows that ε is a function of C_1, C_2, …, C_n and to minimise ε, we must have

$$\frac{\partial \varepsilon}{\partial C_1} = \frac{\partial \varepsilon}{\partial C_2} = \cdots = \frac{\partial \varepsilon}{\partial C_j} = \cdots = \frac{\partial \varepsilon}{\partial C_n} = 0$$

Consider the equation,

$$\frac{\partial \varepsilon}{\partial C_j} = 0$$

Since $(t_2 - t_1)$ is a constant, we can write

$$\frac{\partial}{\partial C_j} \left\{ \int_{t_1}^{t_2} \left[x(t) - \sum_{r=1}^{n} C_r g_r(t) \right]^2 dt \right\} = 0$$

i.e.

$$\frac{\partial}{\partial C_j} \left\{ \int_{t_1}^{t_2} \left[x^2(t) - 2 \sum_{r=1}^{n} x(t)\, C_r g_r(t) + \sum_{r=1}^{n} C_r^2 g_r^2(t) \right] dt \right\} = 0$$

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CHAPTER 12

Answers to Fill in the Blanks

1. $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$
2. $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$
3. signal spectrum
4. 2π
5. integration, summation
6. $-\infty$ to ∞, $-\pi$ to π
7. $x(n) = \dfrac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$
8. unit circle
9. $X(\omega) = X(z)\big|_{z=e^{j\omega}}$
10. continuous and periodic
11. impulse response
12. frequency response
13. frequency response, transfer function
14. continuous
15. magnitude function, phase function
16. symmetric, antisymmetric
17. real, imaginary
18. $x(n) * h(n)$

Answers to Objective Type Questions

1. (b)
2. (b)
3. (a)
4. (a)
5. (c)
6. (a)
7. (d)
8. (c)
9. (b)
10. (a)
11. (b)
12. (b)
13. (a)
14. (d)
15. (a)

Answers to Problems

1. (a) $X(\omega) = 2 - e^{-j\omega} + 3e^{-j2\omega} + 2e^{-j3\omega}$

   (b) $X(\omega) = 16 \dfrac{e^{j2\omega}}{1 - (1/4)\, e^{-j\omega}}$

   (c) $X(\omega) = \dfrac{1}{1 - 0.2e^{-j\omega}} + \dfrac{1}{1 - 2e^{-j\omega}}$

   (d) $X(\omega) = \dfrac{1 - a \cos \omega_0 e^{-j\omega}}{1 - 2a \cos \omega_0 e^{-j\omega} + a^2 e^{-j2\omega}}$

2. (a) $X(\omega) = e^{-j3\omega} \dfrac{3/4}{(5/4) - \cos \omega}$

   (b) $X(\omega) = e^{-j4\omega} \dfrac{1}{1 - (1/2)e^{-j\omega}}$

   (c) $X(\omega) = -\dfrac{e^{j\omega}}{(1 - e^{j\omega})^2}$

   (d) $X(\omega) = \dfrac{1}{1 - e^{-j(\omega-2)}}$

   (e) $X(\omega) = -\dfrac{2e^{j\omega}}{(1 - 2e^{j\omega})^2}$

3. $y(n) = \{1, 4, 4, 10, 7, -5, -1\}$
4. $x(n) = \{1, 2, \underset{\uparrow}{3}, 2, 1\}$
5. $X(\omega) = (1 + 4 \cos \omega)\, e^{-j\omega}$

6. $y(n) = \frac{1}{3}[a^{n+1}u(n+1) + a^n u(n) + a^{n-1}u(n-1)]$

7. $H(\omega) = \frac{1-\cos\omega}{2}$

8. $H(\omega) = \frac{1 + 0.81e^{-j\omega} + 0.81e^{-j2\omega}}{1 + 0.45e^{-j2\omega}}$, $|H(\omega)| = \left[\frac{2.31 + 2.94\cos\omega + 1.62\cos 2\omega}{1.2 + 0.9\cos 2\omega}\right]^{1/2}$,

$\angle H(\omega) = \tan^{-1}\left[\frac{-0.45\sin\omega - 0.36\sin 2\omega}{1.36 + 1.17\cos\omega + 1.26\cos 2\omega}\right]$

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