Equation of Hyperplane

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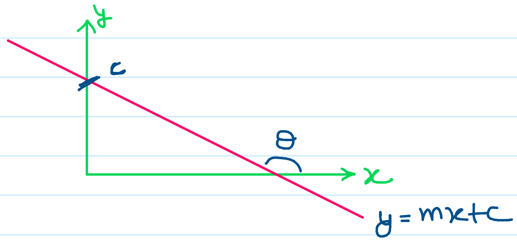
EQUATION OF LINE

$$\boxed{y = mx + c}$$



$y = mx + c$

c ← intercept wrt y

m ← slope wrt x

$m = \tan\theta$ {θ is the angle b/w line & x axis}

General eqn of a Line

$$\boxed{ax + by + c = 0}$$

Transforming it to $y = mx + c$, we will get

$$\boxed{y = -\frac{a}{b}x - \frac{c}{b}}$$

here, slope wrt $x = -\frac{a}{b}$

intercept wrt $y = -\frac{c}{b}$

Re-creating the eqn of a Line

$$ax + by + c = 0$$

UPDATES

- Replacing coefficients a, b and c with w_1, w_2 and w_0 respectively.
- Replacing x & y with x_1 & x_2 respectively.

RECREATING THE EQN

$$\boxed{w_1 x_1 + w_2 x_2 + w_0 = 0}$$

Representation in $y = mx + c$ format will give us slope & intercept.

$$\boxed{x_2 = -\frac{w_1}{w_2}x_1 - \frac{w_0}{w_2}}$$

here, slope wrt $x_1 = -\frac{w_1}{w_2}$

here, slope wrt $x_1 = -\frac{w_1}{w_2}$

intercept wrt $x_2 = -\frac{w_0}{w_2}$

Observe that, above eq^n can be represented as:

$\vec{w} \cdot \vec{x} + w_0 = 0$, where $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ & $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

As we already know the DOT PRODUCT formula, above eq^n can be written as:

$\rightarrow \quad w^T x + w_0 = 0$

OR

$\rightarrow \quad \sum_{i=1}^{2} w_i x_i + w_0 = 0$

OR

$\rightarrow \quad \|\vec{w}\| \, \|\vec{x}\| \cos\theta_{\vec{w},\vec{x}} + w_0 = 0$

Equation of LINE	Equation of PLANE	Equation of HYPERPLANE
$w_1x_1 + w_2x_2 + w_0 = 0$	$w_1x_1 + w_2x_2 + w_3x_3 + w_0 = 0$	$w_1x_1 + w_2x_2 \cdots + w_dx_d + w_0 = 0$
$\pi_2 : w^Tx + w_0 = 0$	$\pi_3 : w^Tx + w_0 = 0$	$\pi_d : w^Tx + w_0 = 0$
here, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ & $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	here, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ & $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$	here, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{bmatrix}$ & $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}$
$x_2 = \underbrace{-\frac{w_1}{w_2}}_{\text{slope wrt } x_1} x_1 \underbrace{-\frac{w_0}{w_2}}_{\text{intercept wrt } x_2}$	$x_2 = \underbrace{-\frac{w_1}{w_2}}_{\text{slope wrt } x_1} x_1 + \underbrace{\frac{w_3}{w_2}}_{\text{slope wrt } x_3} x_3 \underbrace{-\frac{w_0}{w_2}}_{\text{intercept wrt } x_2}$	$x_2 = \underbrace{-\frac{w_1}{w_2}}_{\text{slope wrt } x_1} x_1 \underbrace{-\frac{w_3}{w_2}}_{\text{slope wrt } x_3} x_3 \cdots \underbrace{-\frac{w_d}{w_2}}_{\text{slope wrt } x_d} x_d \underbrace{-\frac{w_0}{w_2}}_{\text{intercept wrt } x_2}$
# of slopes = 1 # of intercept = 1	# of slopes = 2 # of intercept = 1	# of slopes = d-1 # of intercept = 1

Note → Observe that if a line / plane / hyperplane passes through origin, then:

$$\boxed{-\frac{w_0}{w_2} = 0}$$ (i.e the intercept will be zero)

In order for $-\frac{w_0}{w_2}$ to be zero, notice that $w_2 \neq 0$. (i.e Zero Division Problem)

i.e $\boxed{w_0 = 0}$

<u>Understanding $\vec{w}$</u> : (Norm of the Hyperplane)

Assume that a plane is passing through origin,

$\Rightarrow w_0 = 0$

Eq^n of Hyperplane which is passing through origin:

$\Rightarrow w^Tx + \underbrace{w_0}_{0} = 0$

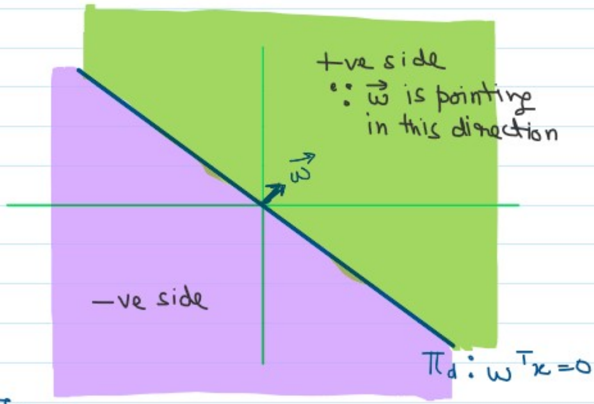
$\Rightarrow w^Tx = 0$

OR

$$\vec{w} \cdot \vec{x} = 0$$

We know, $\vec{a} \cdot \vec{b} = \|\vec{a}\| * \|\vec{b}\| * \cos\theta_{\vec{a},\vec{b}}$

$$\Rightarrow \|\vec{w}\| * \|\vec{x}\| * \cos\theta_{\vec{w},\vec{x}} = 0$$

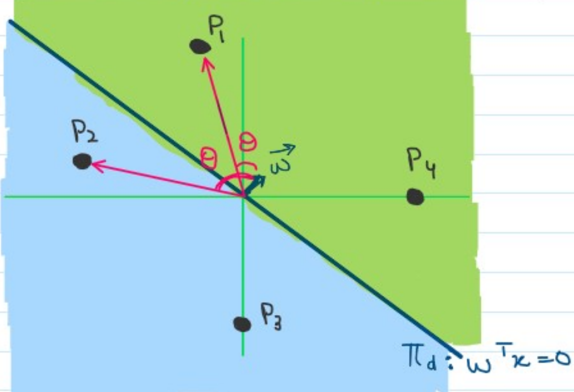


Observe that, inorder for dot product to be equal to zero, the two vectors should be ORTHOGONAL OR PERPENDICULAR.

$$\Rightarrow \boxed{\vec{w} \perp \vec{x} \quad \text{OR} \quad \vec{w} \perp \pi_d}$$

$\vec{w}$ is called as NORM of the Hyperplane π_d.
$\vec{w}$ helps you identify the direction in which the Hyperplane is facing.

How to find the position of a POINT wrt HYPERPLANE



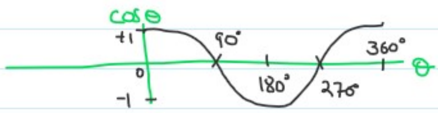
For P_1, Vizually P_1 lies on the +ve side of π_d.
Mathematically we can compute the DOT PRODUCT between the POINT & the $\vec{w}$ of π_d and it will help us determine the position of a point.

$$\vec{w} \cdot \vec{P_1} = \underbrace{\|\vec{w}\|}_{+ve} * \underbrace{\|\vec{P_1}\|}_{+ve} * \underbrace{\cos\theta_{\vec{w},\vec{P_1}}}_{\text{Depends on the angle b/w } \vec{w} \text{ \& } \vec{P_1}}$$

Observe that $\theta_{\vec{w},\vec{P_1}}$ lies in the range of 0 to 90°

$$\Rightarrow \cos\theta_{\vec{w},\vec{P_1}} = +ve \text{ value}$$

$\therefore \vec{w} \cdot \vec{P_1} = +ve$ value (i.e P_1 lies on +ve side of π_d)



$\cos\theta = +ve$ if $0 \le \theta \le 90°$, $270° \le \theta \le 360°$

$\cos\theta = -ve$ if $90° < \theta < 270°$

For P_2, vizually it lies on the −ve side (or opposite side) of π_d. Lets check it mathematically:

$$\vec{w} \cdot \vec{P_2} = \underbrace{\|\vec{w}\|}_{+ve} * \underbrace{\|\vec{P_2}\|}_{+ve} * \underbrace{\cos\theta_{\vec{w},\vec{P_2}}}_{\text{Depends on the angle b/w } \vec{w} \text{ \& } \vec{P_2}}$$

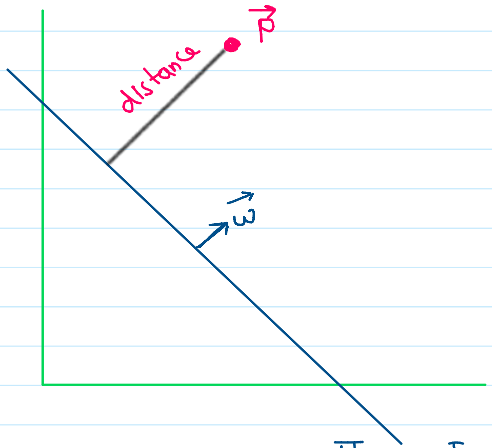
Observe that $\theta_{\vec{w},\vec{P_2}}$ lies in the range of 90° to 180°

$$\Rightarrow \cos\theta_{\vec{w},\vec{P_2}} = -ve \text{ value}$$

$\therefore \vec{w} \cdot \vec{P_2} = -ve$ value (i.e P_2 lies on −ve side of π_d)

Can you try same for P_3 & P_4?

Distance of a POINT from a Hyperplane :



$\pi_d : w^T x + w_0 = 0$

Given a Hyperplane (π_d) & a point $(\vec{p})$, distance b/w π_d & $\vec{p}$ can be represented by :

$$\boxed{d = \frac{w_1 p_1 + w_2 p_2 + \cdots + w_d p_d + w_0}{\sqrt{w_1^2 + w_2^2 + \cdots + w_d^2}}}$$

$\pi_d : w^T x + w_0 = 0$

$\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{bmatrix}$ & $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_d \end{bmatrix}$

Observe that,

- if π_d passes through origin $\Rightarrow w_0 = 0$
- if $\vec{w}$ of π_d is a unit vector $\Rightarrow \|\vec{w}\| = 1$

$$\Rightarrow \quad d = \frac{w_1 p_1 + w_2 p_2 + \cdots + w_d p_d + \underbrace{w_0}_{0}}{\underbrace{\sqrt{\sum_{i=1}^{d} w_i^2}}_{\|\vec{w}\|}} = w_1 p_1 + w_2 p_2 \cdots + w_d p_d$$

$$\boxed{d = \vec{w} \cdot \vec{p}}$$