The Basis problems

Coding the Matrix, 2015

For auto-graded problems, edit the file The_Basis_problems.py to include your solution.

Span of vectors over $\mathbb{R}$

Problem 1: Let $\mathcal{V} = \text{Span}\ \{[2,0,4,0],[0,1,0,1],[0,0,-1,-1]\}$. For each of the following vectors, show it belongs to $\mathcal{V}$ by writing it as a linear combination of the generators of $\mathcal{V}$.

(a) $[2,1,4,1]$

(b) $[1,1,1,0]$

(c) $[0,1,1,2]$

Problem 2: Let $\mathcal{V} = \text{Span}\ \{[0,0,1],[2,0,1],[4,1,2]\}$. For each of the following vectors, show it belongs to $\mathcal{V}$ by writing it as a linear combination of the generators of $\mathcal{V}$.

(a) $[2,1,4]$

(b) $[1,1,1]$

(c) $[5,4,3]$

(d) $[0,1,1]$

Span of Vectors over $GF(2)$

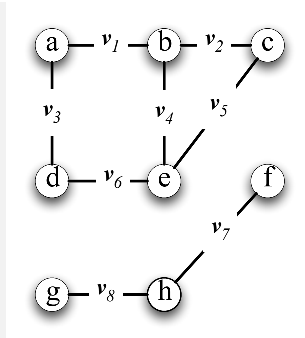
Problem 3: Let $\mathcal{V} = \text{Span}\ \{[0,1,0,1],[0,0,1,0],[1,0,0,1],[1,1,1,1]\}$ where the vectors are over $GF(2)$. For each of the following vectors over $GF(2)$, show it belongs to $\mathcal{V}$ by writing it as a linear combination of the generators of $\mathcal{V}$.

(a) $[1,1,0,0]$

(b) $[1,0,1,0]$

(c) $[1,0,0,0]$

Problem 4: The vectors over $GF(2)$ representing the graph



are

	a	b	c	d	e	f	g	h
$\boldsymbol{v}_1$	1	1						
$\boldsymbol{v}_2$		1	1					
$\boldsymbol{v}_3$	1			1				
$\boldsymbol{v}_4$		1			1			
$\boldsymbol{v}_5$			1		1			
$\boldsymbol{v}_6$				1	1			
$\boldsymbol{v}_7$						1		1
$\boldsymbol{v}_8$							1	1

For each of the following vectors over $GF(2)$, show it belongs to the span of the above vectors by writing it as a linear combination of the above vectors.

(a) $[0,0,1,1,0,0,0,0]$

(b) $[0,0,0,0,0,1,1,0]$

(c) $[1,0,0,0,1,0,0,0]$

(d) $[0,1,0,1,0,0,0,0]$

Linear Dependence over $\mathbb{R}$

Problem 5: For each of the parts below, show the given vectors over $\mathbb{R}$ are linearly dependent by writing the zero vector as a nontrivial linear combination of the vectors.

(a) $[1,2,0],[2,4,1],[0,0,-1]$

(b) $[2,4,0],[8,16,4],[0,0,7]$

(c) $[0,0,5],[1,34,2],[123,456,789],[-3,-6,0],[1,2,0.5]$

Problem 6: For each of the parts below, show the given vectors over $\mathbb{R}$ are linearly dependent by writing the zero vector as a nontrivial linear combination of the vectors. You can use `sqrt( )` and `pi`.

(a) $[1,2,3],[4,5,6],[1,1,1]$

(b) $[0,-1,0,-1],[\pi,\pi,\pi,\pi],[-\sqrt{2},\sqrt{2},-\sqrt{2},\sqrt{2}]$

(c) $[1,-1,0,0,0],[0,1,-1,0,0],[0,0,1,-1,0],[0,0,0,1,-1],[-1,0,0,0,1]$

Problem 7: Show that one of the vectors is superfluous by expressing it as a linear combination of the other two.

$$\begin{aligned}\boldsymbol{u} &= [3, 9, 6, 5, 5] \\ \boldsymbol{v} &= [4, 10, 6, 6, 8] \\ \boldsymbol{w} &= [1, 1, 0, 1, 3]\end{aligned}$$

Problem 8: Give four vectors that are linearly dependent but such that any three are linearly independent.

Linear Dependence over $GF(2)$

Problem 9: For each of the subproblems, show the given vectors over $GF(2)$ are linearly dependent by writing the zero vector as a nontrivial linear combination of the vectors.

(a) $[one, one, one, one], [one, 0, one, 0], [0, one, one, 0], [0, one, 0, one]$

(b) $[0, 0, 0, one], [0, 0, one, 0], [one, one, 0, one], [one, one, one, one]$

(c) $[one, one, 0, one, one], [0, 0, one, 0, 0], [0, 0, one, one, one], [one, 0, one, one, one], [one, one, one, one, one]$

Problem 10: Each of the subproblems specifies some of the vectors over $GF(2)$ specified in Problem 4. For each subproblem, show that the vectors are linearly dependent by giving the coefficients of a nontrivial linear combination whose sum is the zero vector. (Hint: Looking at the graph will help.)

(a) $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5$

(b) $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5, \boldsymbol{v}_7, \boldsymbol{v}_8$

(c) $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_6$

(d) $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_5, \boldsymbol{v}_6, \boldsymbol{v}_7, \boldsymbol{v}_8$

Problem 11: Let $S = \{[1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]\}$, and let $A = \{[1, 0, 0, 0, 0], [0, 1, 0, 0, 0]\}$. For each of the following vectors $\boldsymbol{z}$, find a vector $\boldsymbol{w}$ in $S - A$ such that Span $S =$ Span $(S \cup \{\boldsymbol{z}\} - \{\boldsymbol{w}\})$.

(a) $\boldsymbol{z} = [1, 1, 1, 1, 1]$

(b) $\boldsymbol{z} = [0, 1, 0, 1, 0]$

(c) $\boldsymbol{z} = [1, 0, 1, 0, 1]$

Problem 12: We refer in this problem to the vectors over $GF(2)$ specified in Problem 4.

Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$. Each of the following parts specifies a subset A of S and a vector $\boldsymbol{z}$ such that $A \cup \{\boldsymbol{z}\}$ is linearly independent. For each part, specify a vector $\boldsymbol{w}$ in $S - A$ such that Span S = Span $(S \cup \{\boldsymbol{z}\} - \{\boldsymbol{w}\})$. (Hint: Drawing subgraphs of the graph will help.)

(a) $A = \{\boldsymbol{v}_1, \boldsymbol{v}_4\}$ and $\boldsymbol{z}$ is

a	b	c	d	e	f	g	h
			1	1			

(b) $A = \{\boldsymbol{v}_2, \boldsymbol{v}_3\}$ and $\boldsymbol{z}$ is

a	b	c	d	e	f	g	h
		1	1				

(c) $A = \{\boldsymbol{v}_2, \boldsymbol{v}_3\}$ and $\boldsymbol{z}$ is

a	b	c	d	e	f	g	h
1				1			

Problem 13: Write and test a procedure `rep2vec(u, veclist)` with the following spec:

- *input:* a vector $\boldsymbol{u}$ and a list *veclist* of `Vecs` $[\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{n-1}]$. The domain of $\boldsymbol{u}$ should be $\{0, 1, 2, n-1\}$ where n is the length of `veclist`.
- *output:* the vector $\boldsymbol{v}$ such that $\boldsymbol{u}$ is the coordinate representation of $\boldsymbol{v}$ with respect to $\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{n-1}$, where entry i of $\boldsymbol{u}$ is the coefficient of $\boldsymbol{a}_i$ for $i = 0, 1, 2 \ldots, n-1$.

Your procedure should not use any loops or comprehensions but of course can use the operations on instances of `Mat` and `Vec` and can also use procedures from the `matutil` module. Note that the procedures `coldict2mat` and `rowdict2mat` (defined in `matutil`) can accept lists, not just dictionaries.

Problem 14: Write and test a procedure `vec2rep(veclist, v)` with the following spec:

- *input:* a list `veclist` of vectors $[\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{n-1}]$, and a vector $\boldsymbol{v}$ with the domain $\{0, 1, 2, \ldots, n-1\}$ where n is the length of `veclist`. You can assume $\boldsymbol{v}$ is in Span $\{\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{n-1}\}$.
- *output:* the vector $\boldsymbol{u}$ whose coordinate representation in tems of $\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{n-1}$ is $\boldsymbol{v}$.

As in Problem 13, your procedure should use no loops or comprehensions directly but can use procedures defined in `matutil` and can use the procedure `solve(A, b)` defined in the `solver` module.

Problem 15: Write and test a procedure `is_superfluous(S, v)` with the following spec:

- *input:* a set S of vectors, and a Vec $\boldsymbol{v}$ in S
- *output:* `True` if the span of the vectors in S equals the span of the set of vectors in S that are not equal to $\boldsymbol{v}$

Your procedure should not use loops or comprehensions but can use procedures defined in the module `matutil` and can use the procedure `solve(A,b)` defined in `solver` module. Your procedure will most likely need a special case for the case where len(S) is 1.

Note that the `solve(A,b)` always returns a vector $\boldsymbol{u}$. It is up to you to check that $\boldsymbol{u}$ is in fact a solution to the equation $A\boldsymbol{x} = \boldsymbol{b}$. Moreover, over $\mathbb{R}$, even if a solution exists, the solution returned by `solve` is approximate due to roundoff error. To check whether the vector $\boldsymbol{u}$ returned is a solution, you should compute the residual $\boldsymbol{b} - A * \boldsymbol{u}$, and test if it is close to the zero vector:

```
>>> (b - A*u).is_almost_zero()
True
```

For a vector over the real numbers, the is_almost_zero() method checks whether the dot-product of the vector with itself is less than 10^{-20}. This isn't a very principled test but it will work for this course. A vector over $GF(2)$ is of course considered "almost zero" only if it is truly a zero vector.

Problem 16: Write and test a procedure is_independent(S) with the following spec:

- *input:* a set S of vectors
- *output:* True if the set is linearly independent

Your algorithm for this procedure should be based on the Span Lemma. You can use as a subroutine any one of the following:

- the procedure is_superfluous(S, v) from the previous problem or
- the solve(A,b) procedure from the solver module (but see the provisos in the previous problem).

You will need a loop or comprehension for this procedure.

Note: This problem illustrates one way of thinking about linear dependence but don't make the mistake of making it your only or even your main way of thinking about it. For most purposes, it is best to go back to the *definition* of linear dependence.

Also, the intended algorithm for this problem is *not* a particularly efficient algorithm for testing linear dependence. We'll see some better methods later.

Problem 17: Write and test a procedure exchange(S, A, z) with the following spec:

- *input:* A set S of vectors, a set A of vectors that are all in S (such that len$(A) <$ len(S)), and a vector $\boldsymbol{z}$ such that $A \cup \{z\}$ is linearly independent
- *output:* a vector $\boldsymbol{w}$ in S but not in A such that

$$\text{Span } S = \text{Span } (\{\boldsymbol{z}\} \cup S - \{\boldsymbol{w}\})$$

Your procedure should follow the proof of the Exchange Lemma. You should use the solver module or the procedure vec2rep(veclist, u) from a previous problem. You can test whether a vector is in a list or set C using the expression v in C. Note that S need not be linearly independent.