

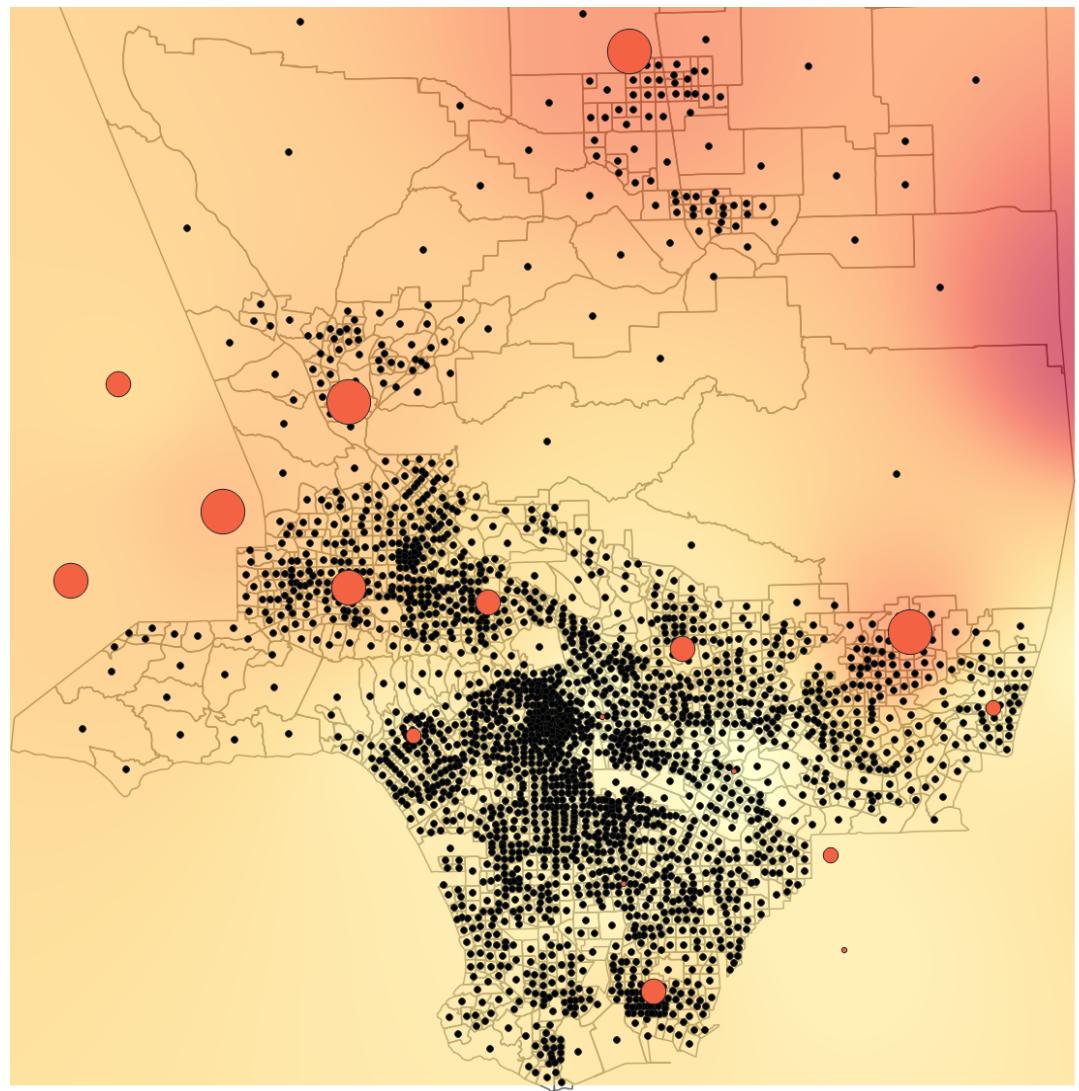
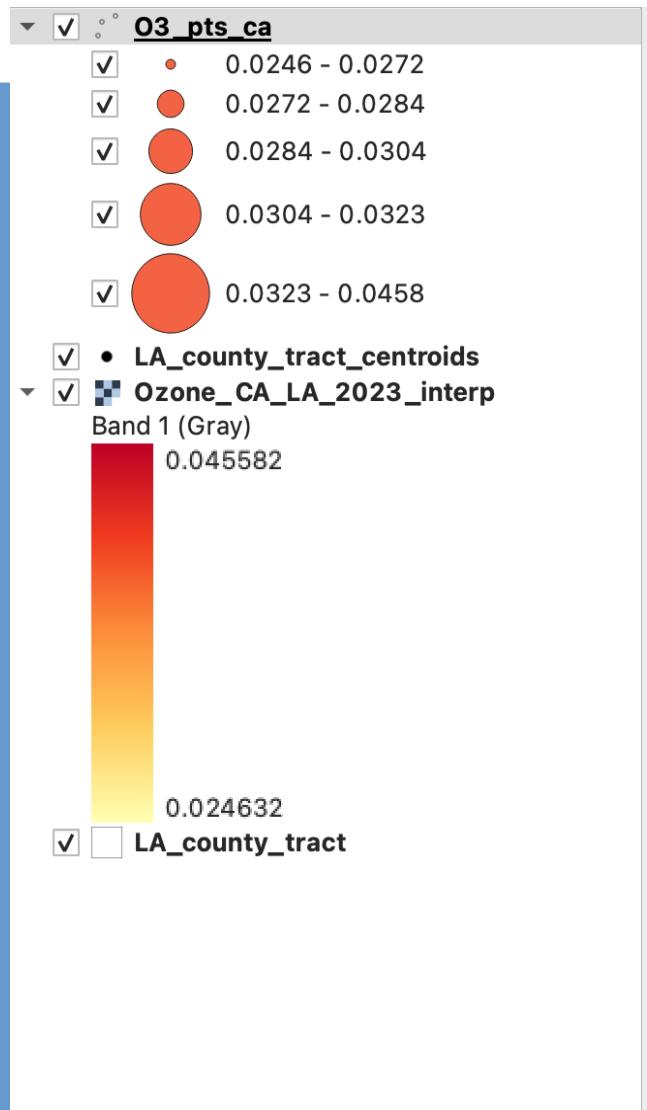
Spatial Interpolation

Lecture #20 | GEOG 510
GIS & Spatial Analysis in Public Health

Varun Goel

Outline

- Interpolation
 - Point-based methods
 - Nearest neighbor
 - Trend surface
 - IDW
 - Kriging (classical, EB)



What is the missing value?

10



20



?



40



50



What is the missing value?

10



20



30



40



50



What is the missing value?

2	4	6	?
			

What is the missing value?

2



4



6



10



What is the missing value?

2



4



6



4



?



What is the missing value?

2



4



6



4



2



10



20

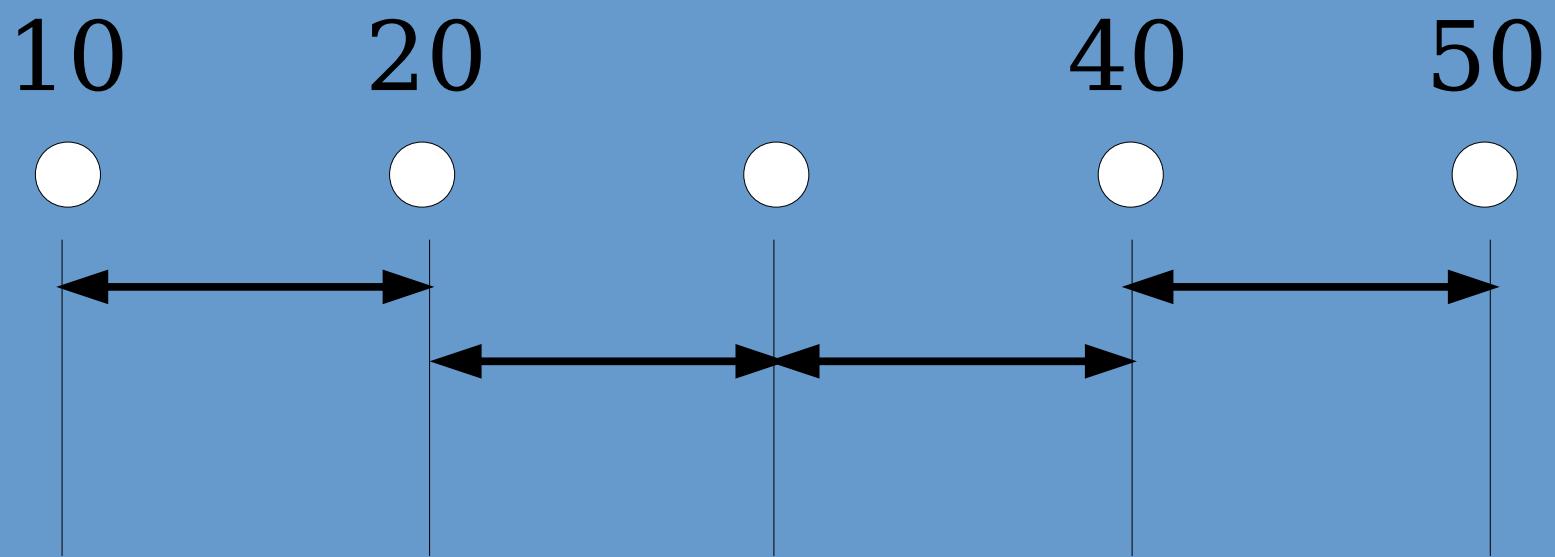


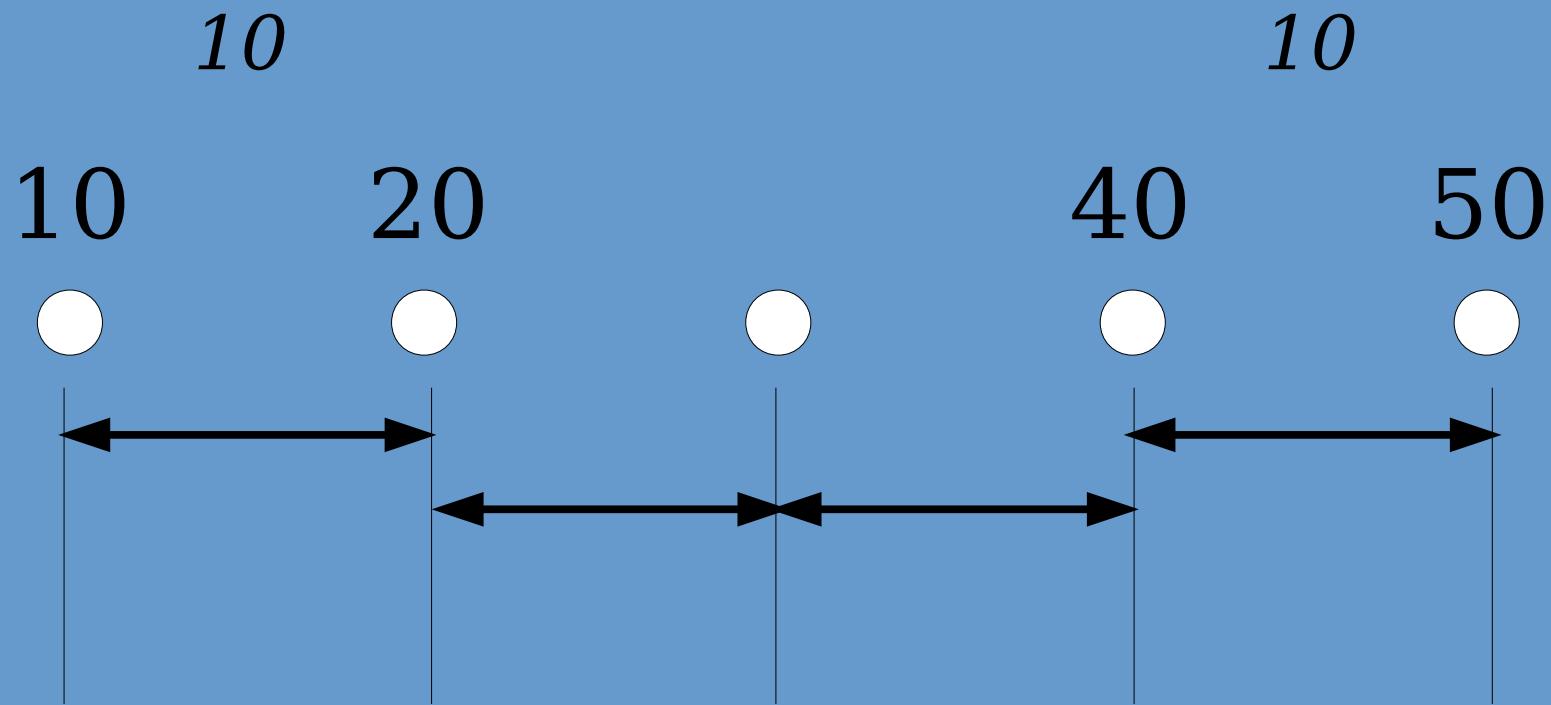
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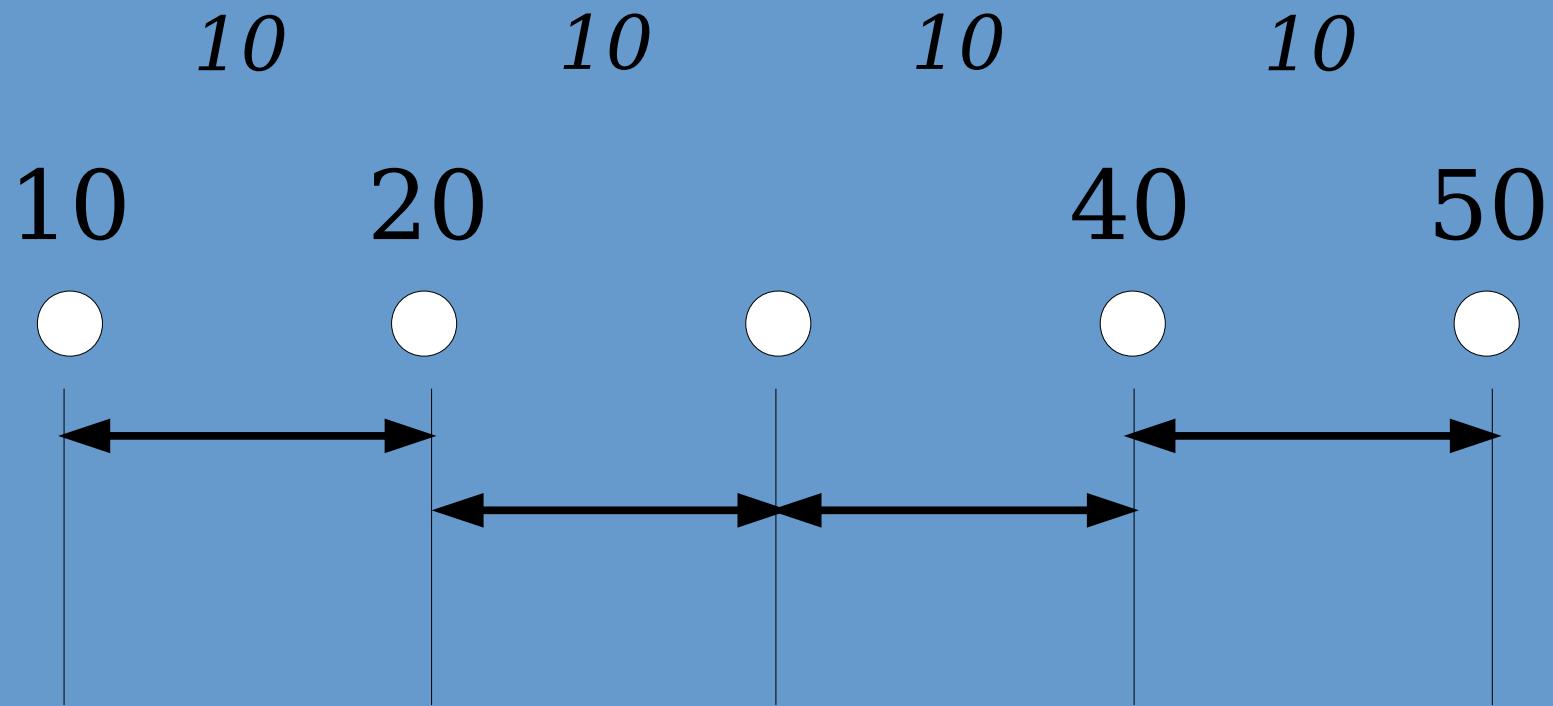


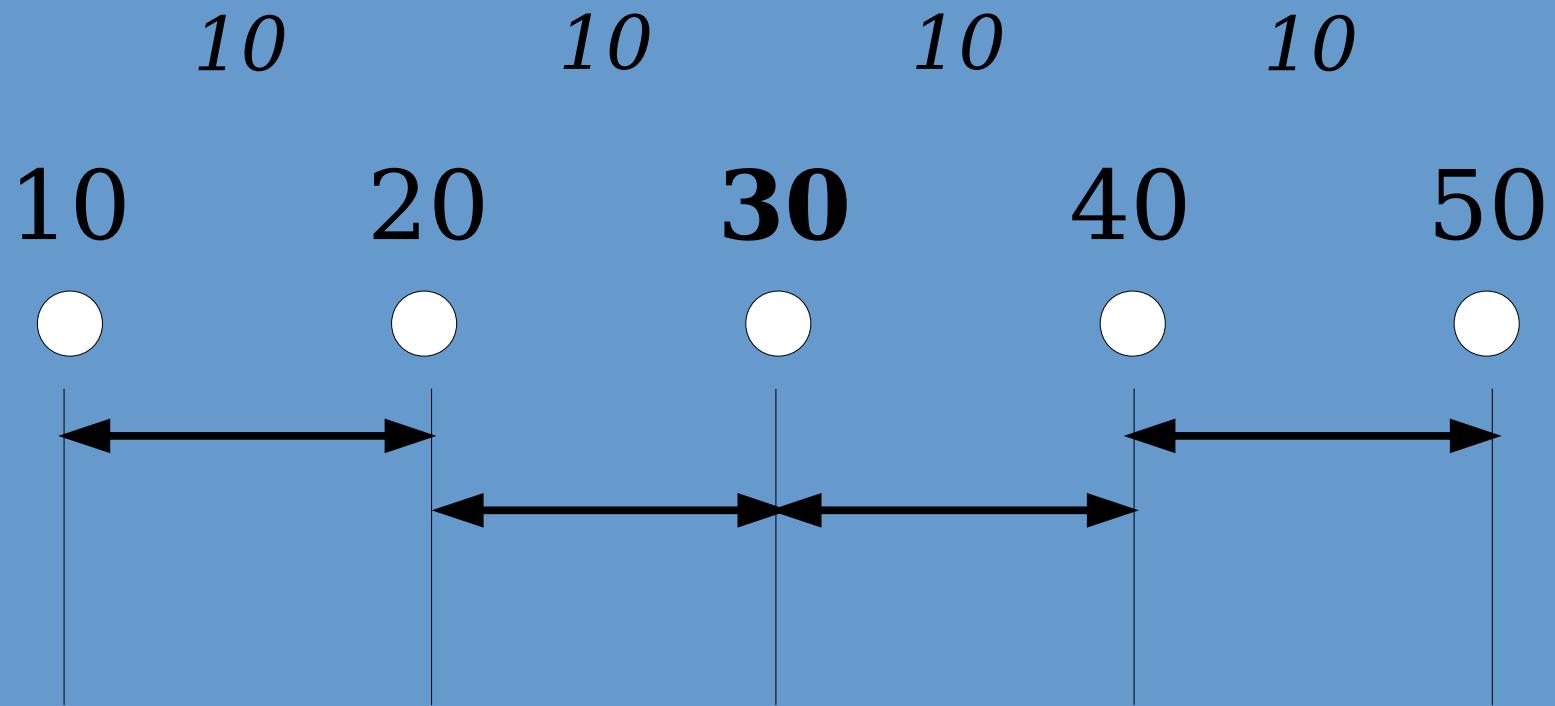
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What is the missing value?

10



20



?

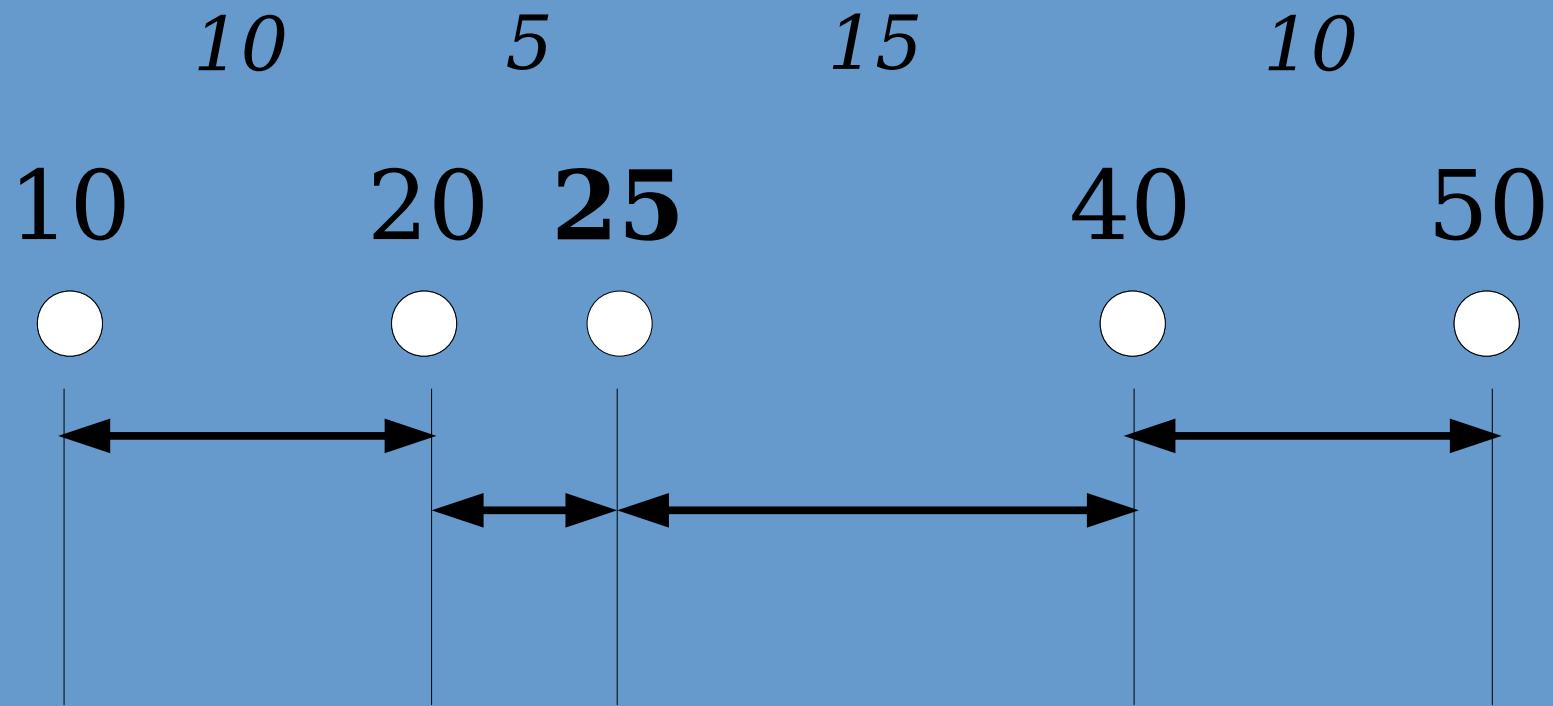


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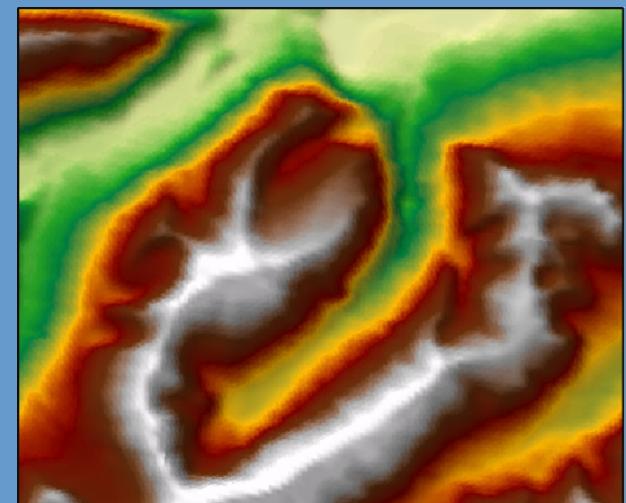
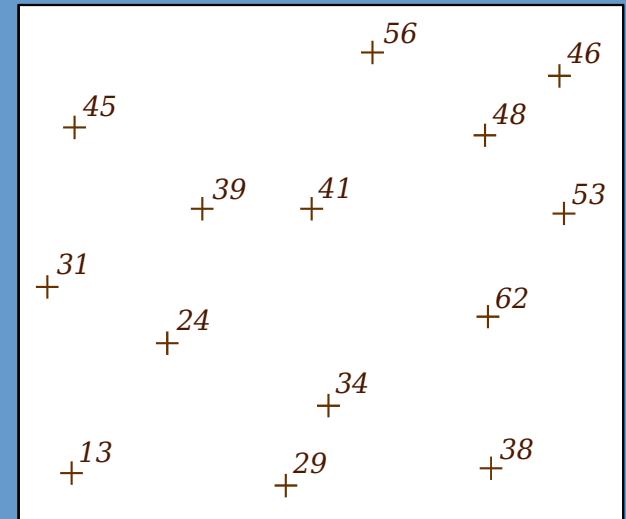


What are the missing values?

?	?	?	?	?	?	?	?	?	56	?	?	?	?		
?	?	?	?	?	?	?	?	+	?	?	?	?	46?		
?	+	45	?	?	?	?	?	?	?	?	48	?	?		
?	?	?	?	?	?	?	?	?	?	?	?	?	?		
?	?	?	?	?	+	39	?	+	41	?	?	?	?	+	53
?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	
+	31	?	?	?	?	?	?	?	?	?	?	?	?	?	
?	?	?	+	24	?	?	?	?	?	?	+	62	?	?	
?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	
?	?	?	?	?	?	?	?	34	?	?	?	?	?	?	
?	?	?	?	?	?	?	+	?	?	?	?	?	?	?	
?	+	13	?	?	?	?	?	29	?	?	?	+	38	?	?
?	?	?	?	?	?	?	+	?	?	?	?	?	?	?	

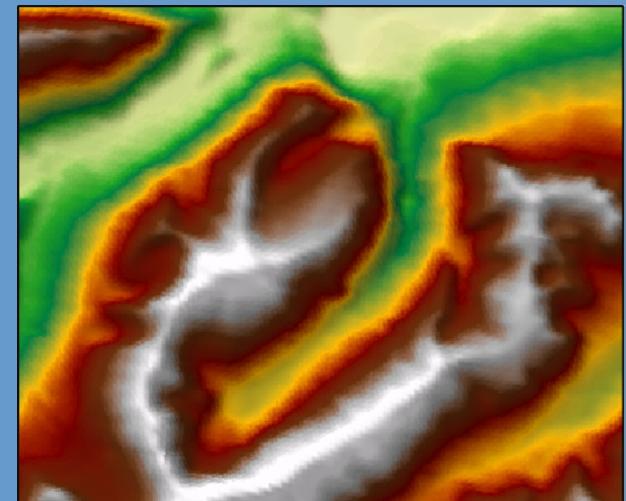
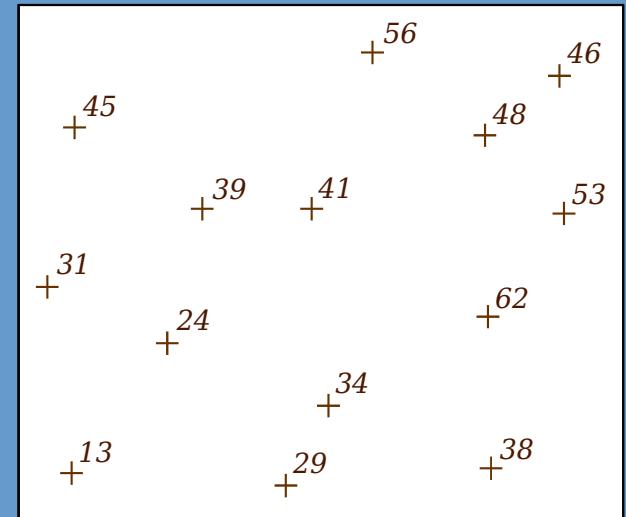
Interpolation

- *Interpolation* is the process of estimating attribute values at unmeasured locations
 - Especially useful when we only have *samples* distributed throughout space
 - Works for a single variable
 - Known values at specific locations
 - Interpolation “fills in the blanks”

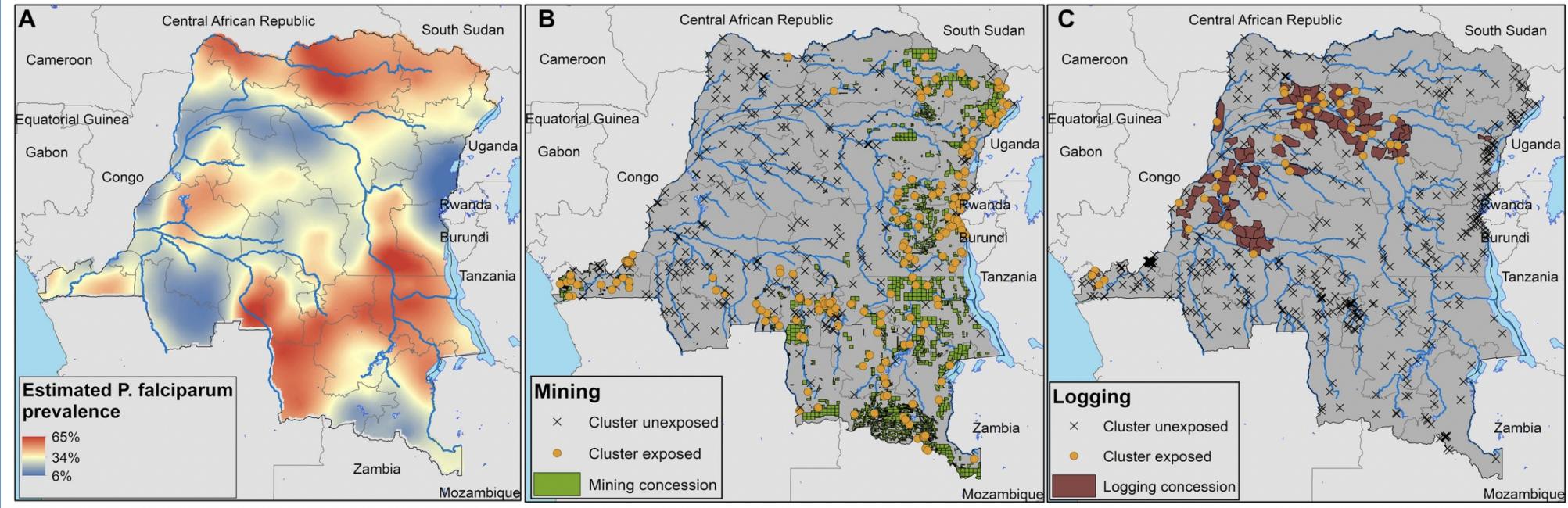


Interpolation

- *Interpolation* is the process of estimating attribute values at unmeasured locations
 - Takes advantage of our understanding of how values near to each other tend to be similar
 - Spatial autocorrelation



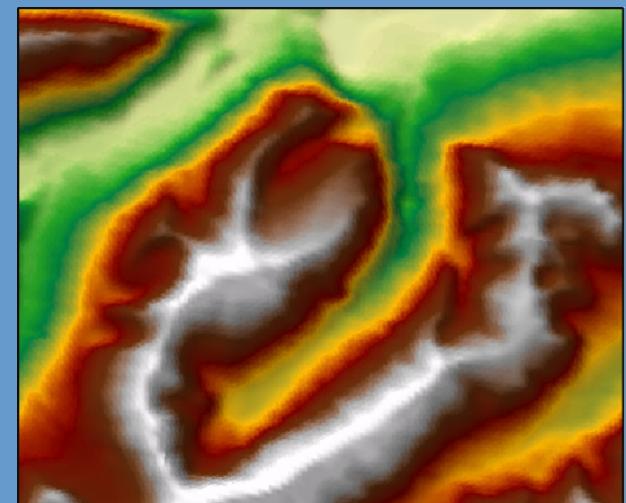
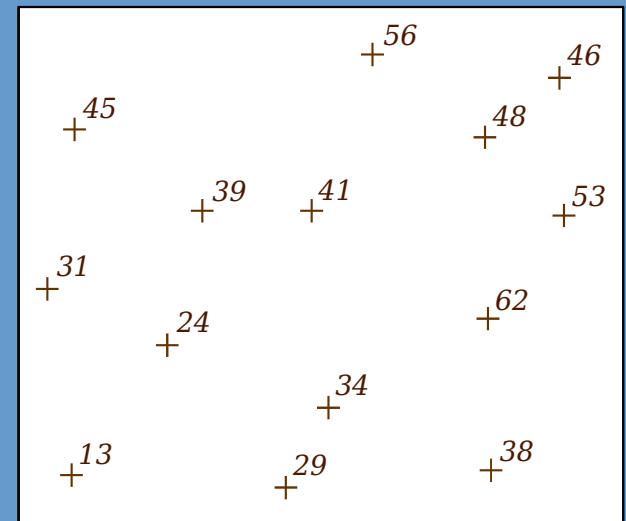
From: [Impact of extractive industries on malaria prevalence in the Democratic Republic of the Congo: a population-based cross-sectional study](#)



Mitchell et al., 2022

Interpolation

- *Interpolation* is the process of estimating attribute values at unmeasured locations
 - Assumes the underlying values are measured on a continuous scale
 - Assumes the underlying distribution is continuous across space



Cow Test!

Thanks to Dr. Tim Leslie



✗

?

✗



✗

Cow Test!

Thanks to Dr. Tim Leslie



✗



✗

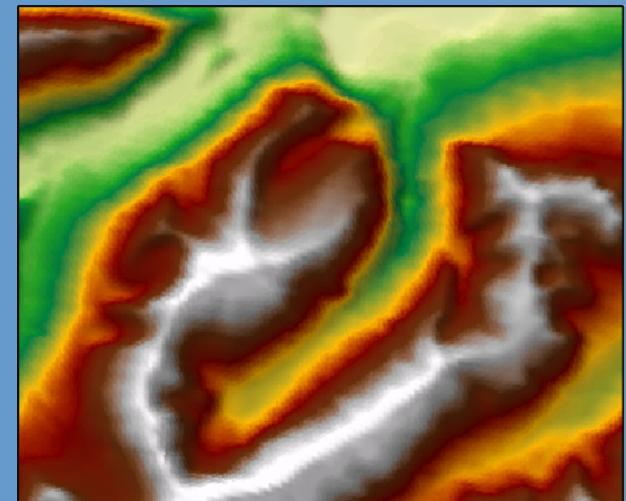
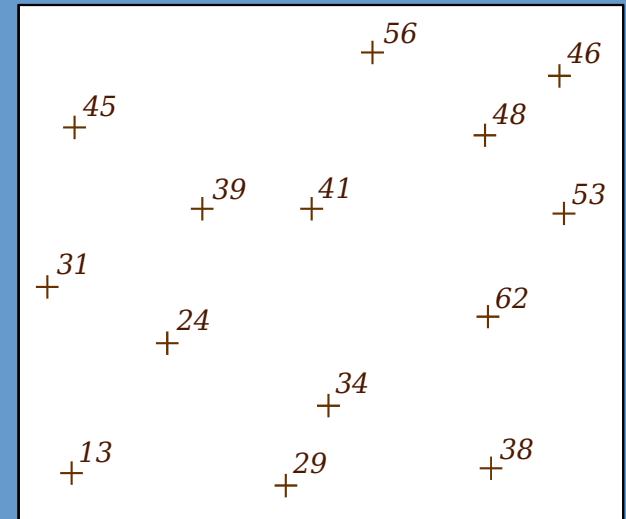


✗

Does 1.5 cows make sense?

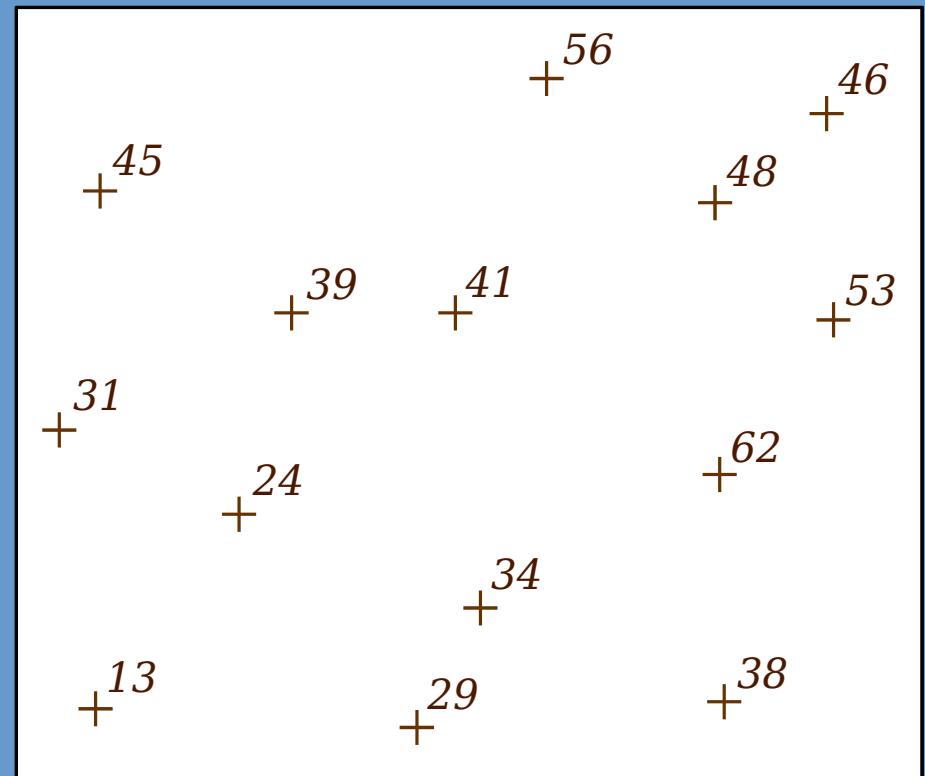
Interpolation

- *Interpolation* is the process of estimating attribute values at unmeasured locations
 - Useful for estimating the value of hazards at unknown locations
 - e.g., air pollution
 - Collection is only at discrete locations (points) but we want to estimate at non-sampled locations



Interpolation

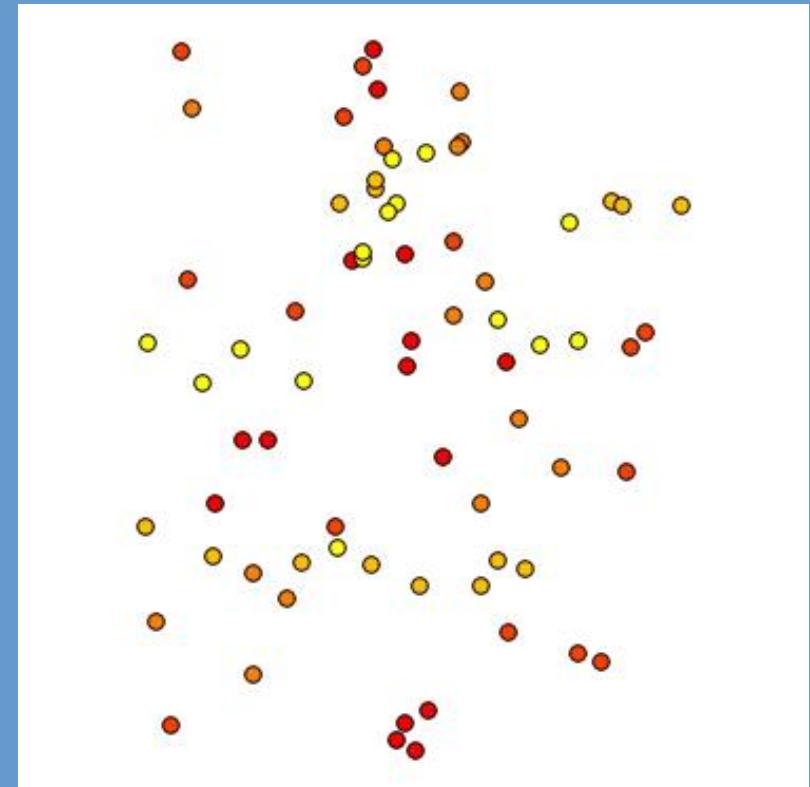
- Point-based approaches
 - Nearest neighbor
 - Trend surface
 - Inverse distance weighted (IDW)
 - Kriging



Input: set of point observations with measured attribute values
Output: continuous distribution of attribute values throughout a region

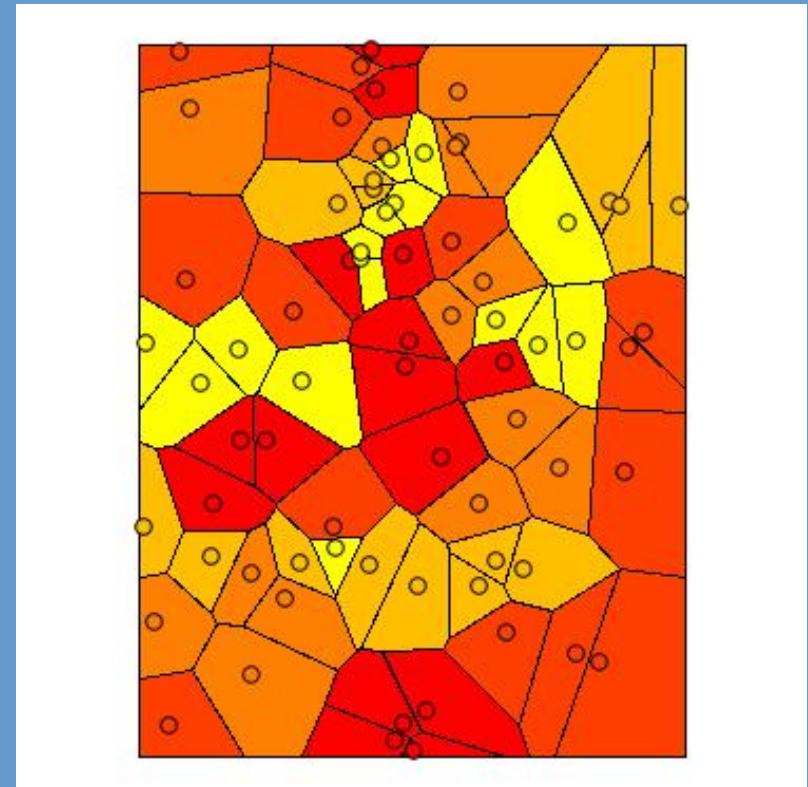
Nearest Neighbor

- Assigns each location to be interpolated the same value as the nearest sample point
 - Gives a stair-step shape to the surface (discontinuities)
- *Thiessen Polygons* are created when each location on the map is assigned to the nearest sample point



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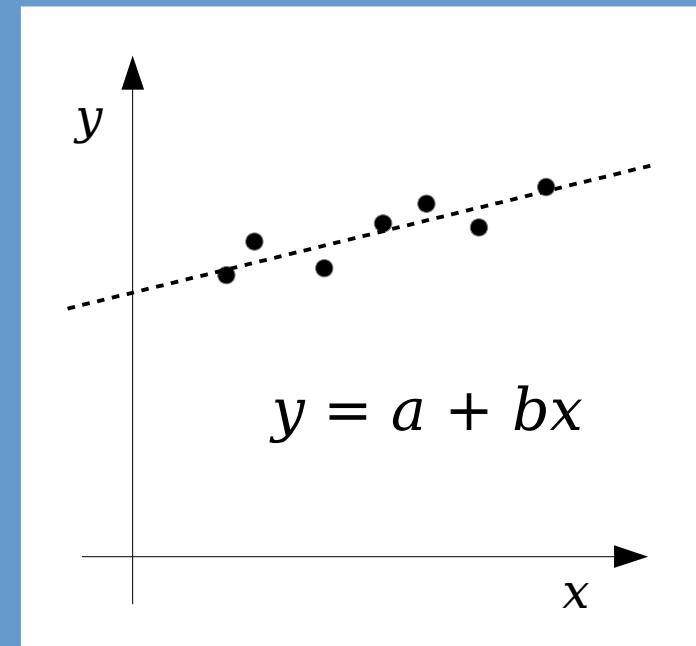


Nearest Neighbor

- Output features: polygons
- Estimated values are constrained to the input values
- At sample locations, estimated value will always equal sample value
- Results in a “shelved” surface

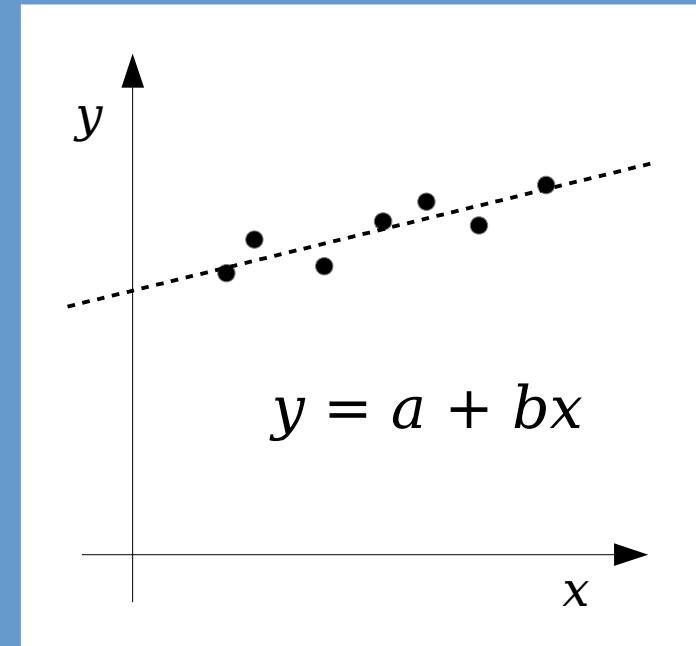
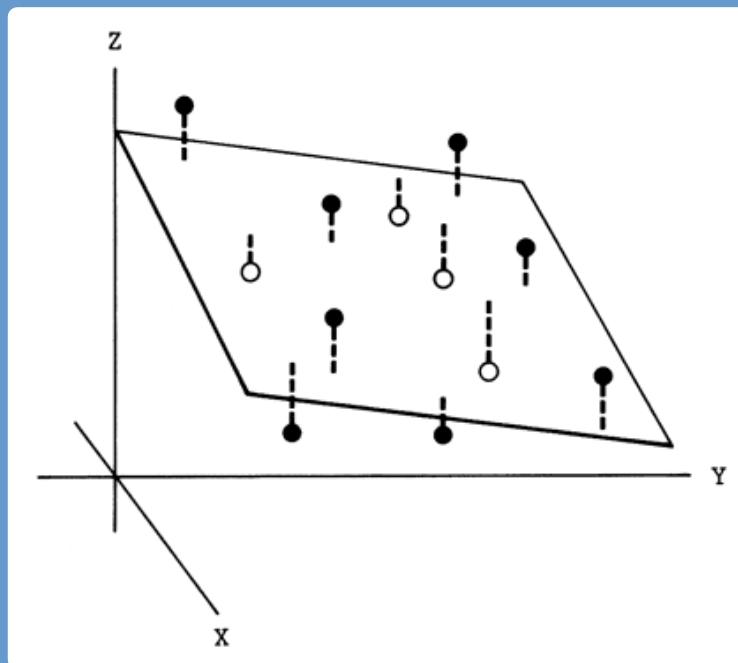
Trend Surface Analysis

- Fit a plane or other surface through the observed points
 - Similar to fitting a regression line to a set of observations



Trend Surface Analysis

- Fit a plane or other surface through the observed points
 - Similar to fitting a regression line to a set of observations

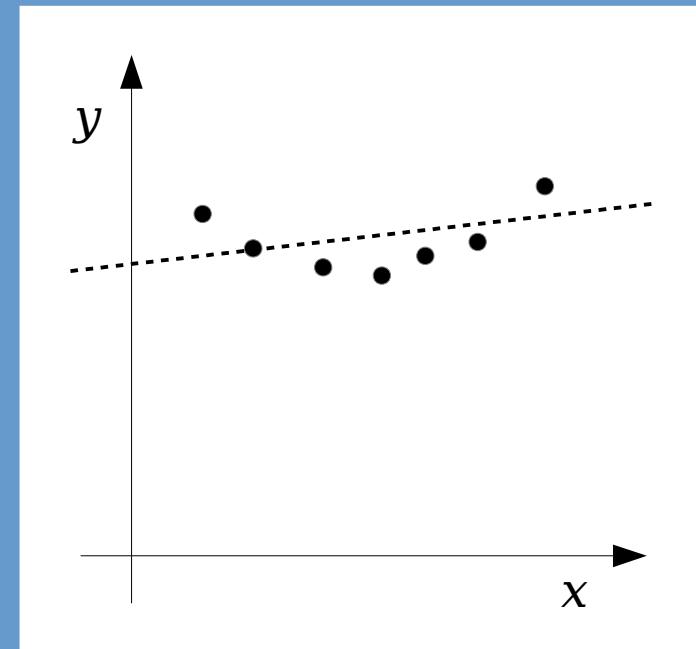


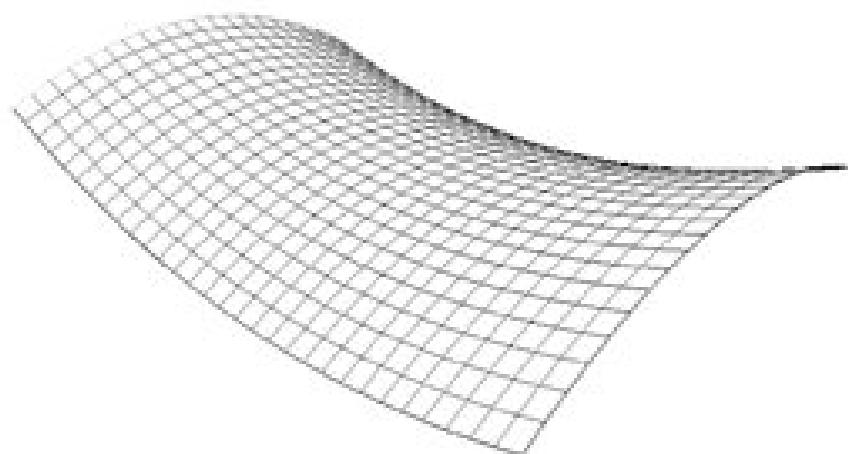
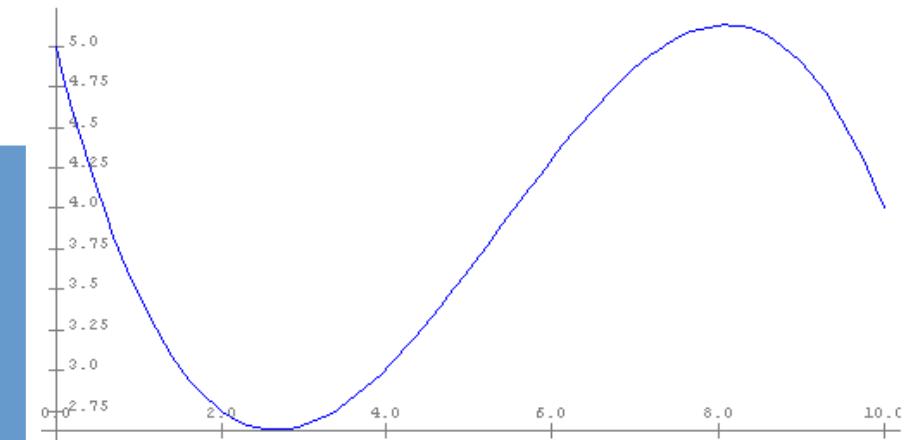
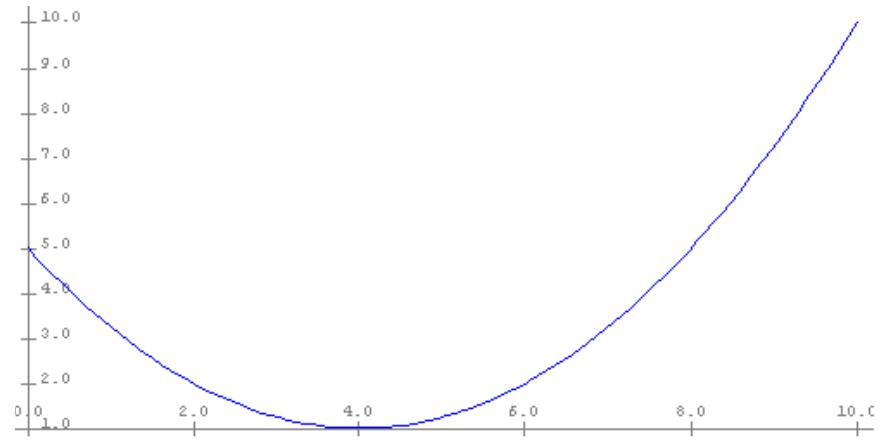
- First order polynomial equation

$$z(x,y) = a + bx + cy$$

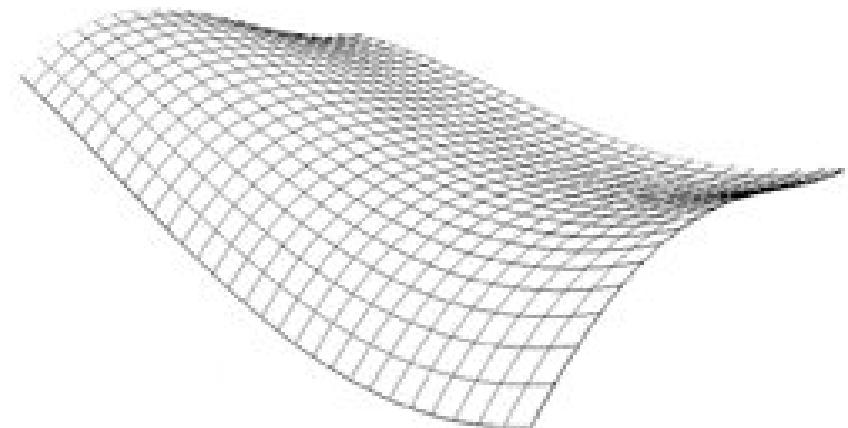
Trend Surface Analysis

- Fit a plane or other surface through the observed points
 - Similar to fitting a regression line to a set of observations





d) 2nd order trend surface



e) 3rd order trend surface

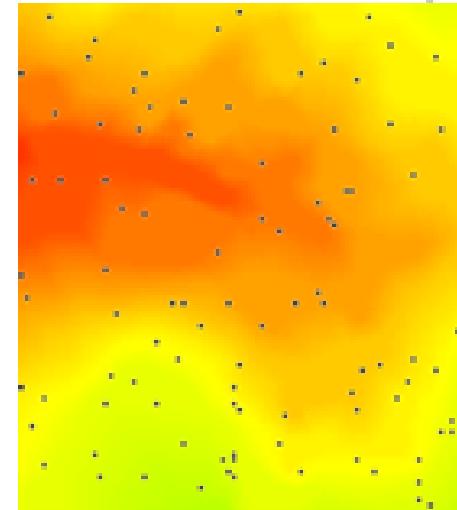
- Second order polynomial equation

$$z(x,y) = a + bx + cy + dxy + ex^2 + fy^2$$

Trend Surface Analysis

- Fitting a trend surface to observations
- Order of the equation and sample points
 - Limit on the number of coefficients: you cannot have more than the number of sample points
 - Just because you can doesn't mean you should!
 - Surface fit and model residuals
 - Regression diagnostics
 - R^2 , f , residual plot

Source surface with sample points

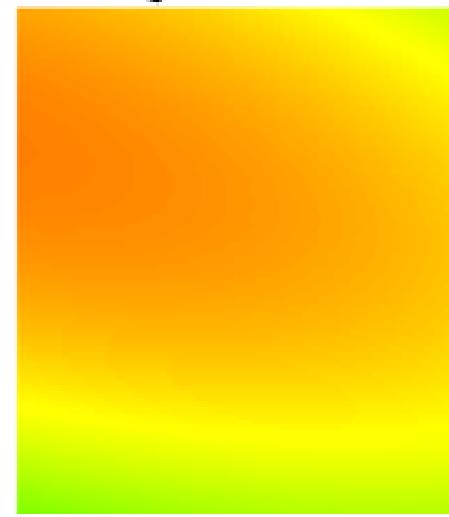


Linear



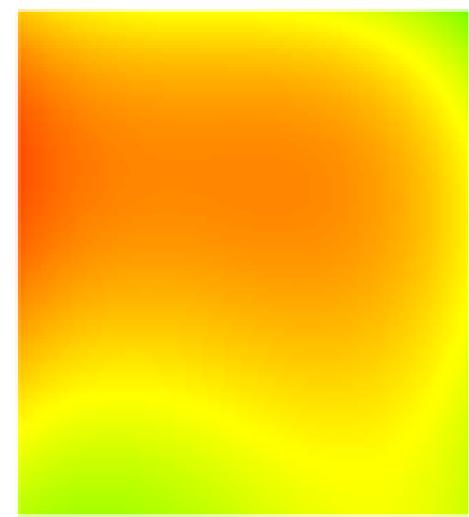
Goodness of fit (R^2) = 45.42 %

Quadratic



Goodness of fit (R^2) = 82.11 %

Cubic



Goodness of fit (R^2) = 92.72 %

Trend Surface Analysis

- Output features: raster (in reality, a function)
- Estimated values are not constrained to the input values
- At sample locations, estimated value does not always equal sample value
- Results in a very smooth surface

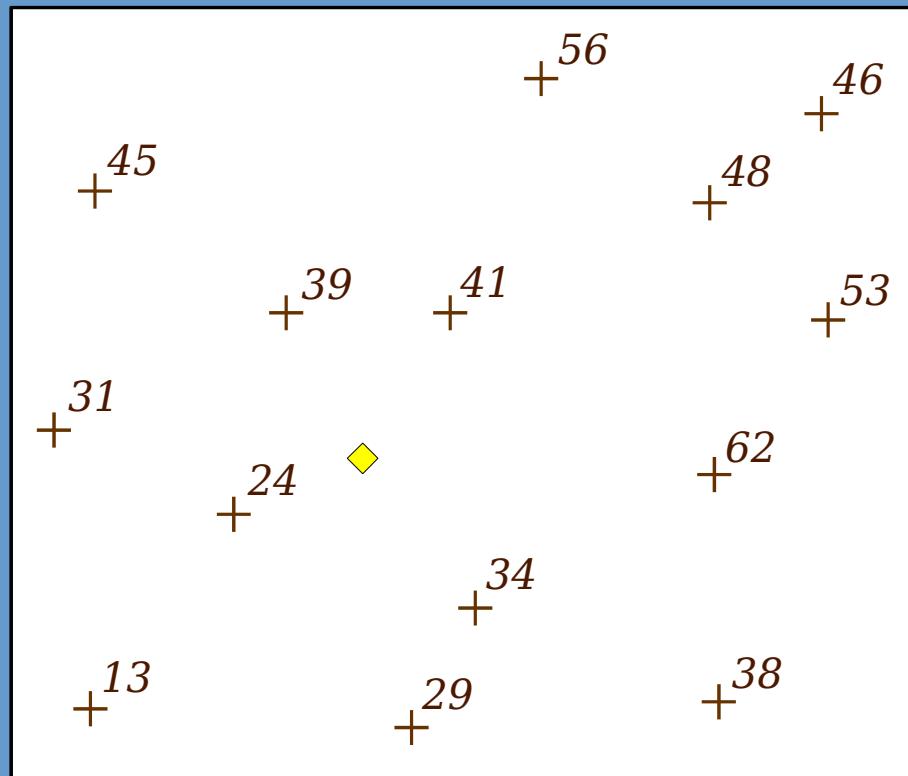
IDW

- Inverse Distance Weighted interpolation
 - Value for non-sampled locations is calculated as the weighted average of nearby values
 - Weighted average often takes the form:
 - Requires: neighborhood to be defined (n)
 - Specific number of points or specified d (radius)
 - Key assumption: the value at a point is assumed to be related to the value at other points such that the similarity declines at a rate of $1/d^x$
 - $1/d^x$ is only a guess at how nearby values are related ($1/d^2$ is commonly used)

$$z_j = \frac{\sum_{i=1}^n \frac{z_i}{d_i^2}}{\sum_{i=1}^n \frac{1}{d_i^2}}$$

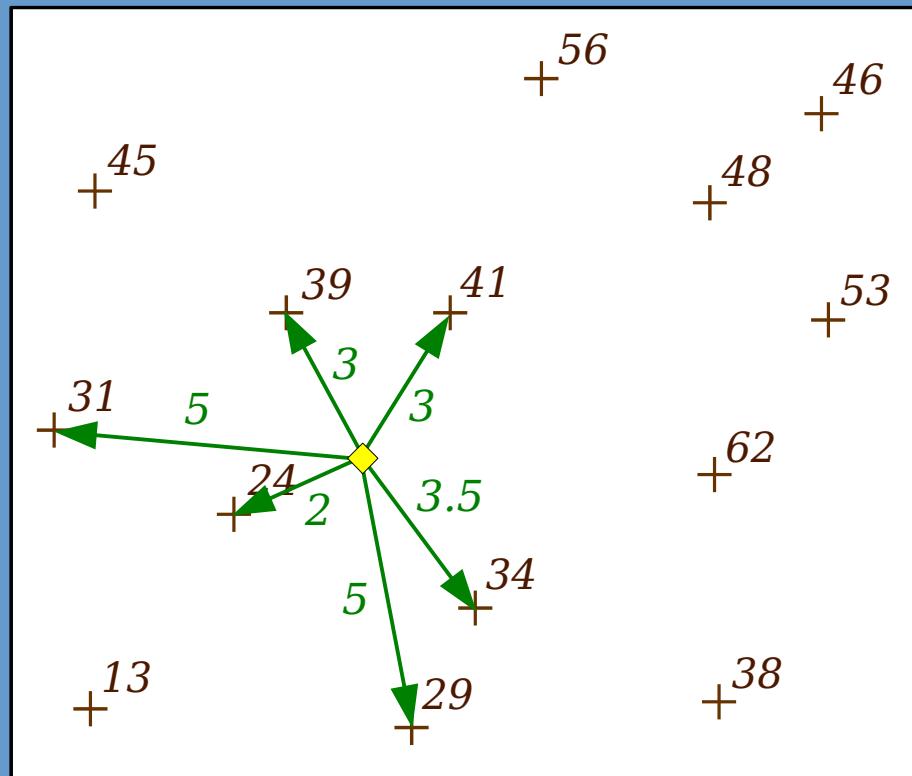
IDW Calculation

- Calculate value at the yellow diamond
 - $n = 6$



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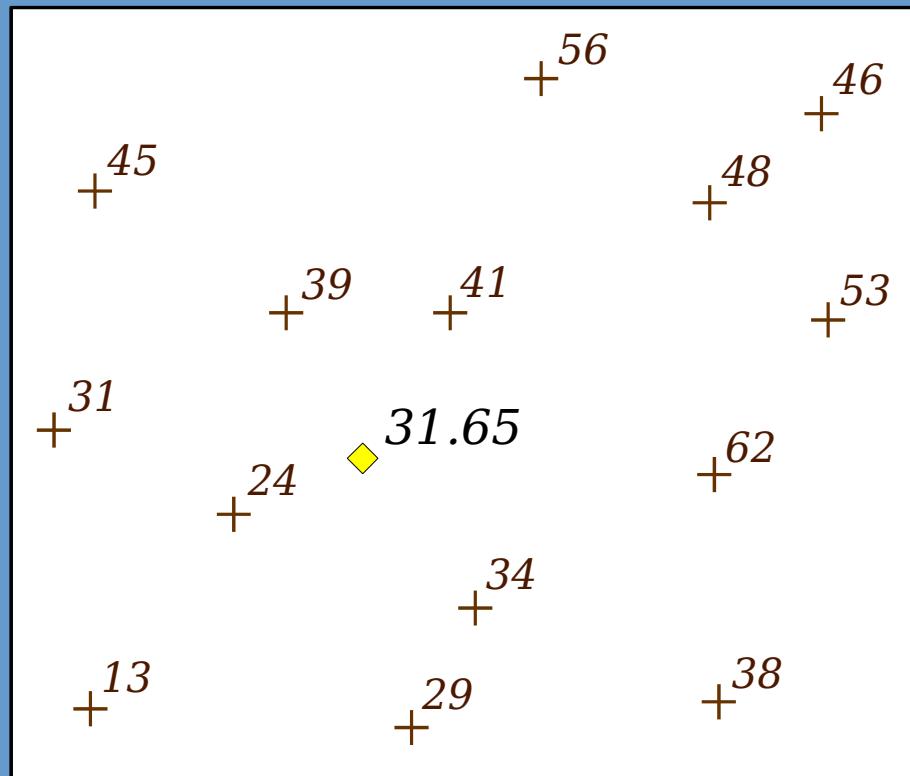
$$z_j = \frac{\frac{24}{2^2} + \frac{39}{3^2} + \frac{41}{3^2} + \frac{34}{3.5^2} + \frac{31}{5^2} + \frac{29}{5^2}}{\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{3.5^2} + \frac{1}{5^2} + \frac{1}{5^2}}$$

$$z_j = \frac{20.0644}{0.6339}$$

$$z_j = 31.6546$$

IDW Calculation

- Calculate value at the yellow diamond
 - $n = 6$



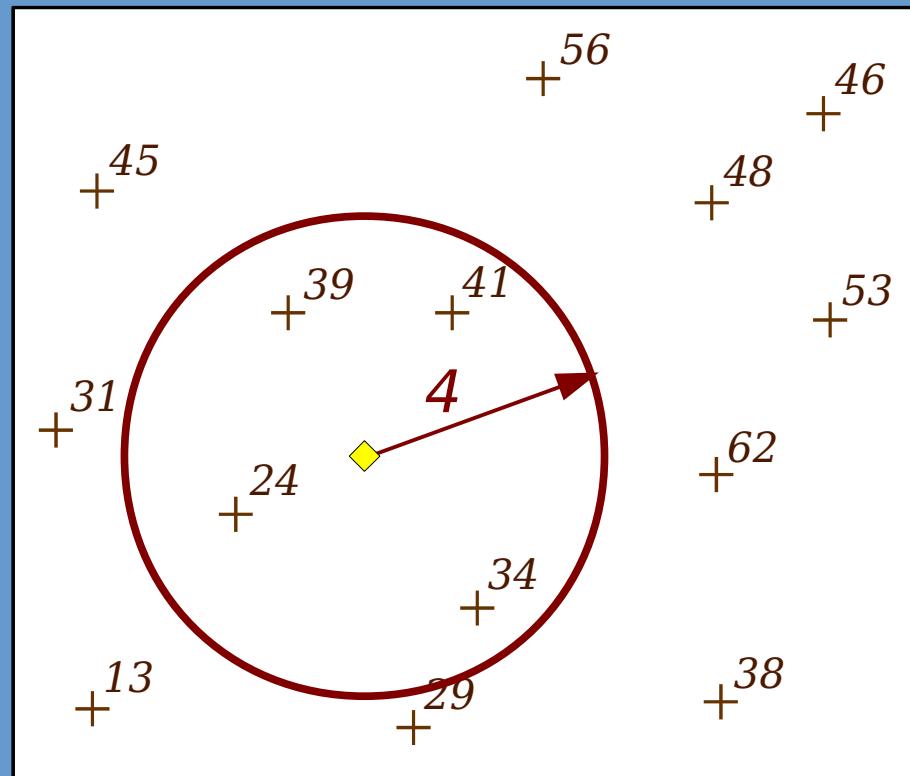
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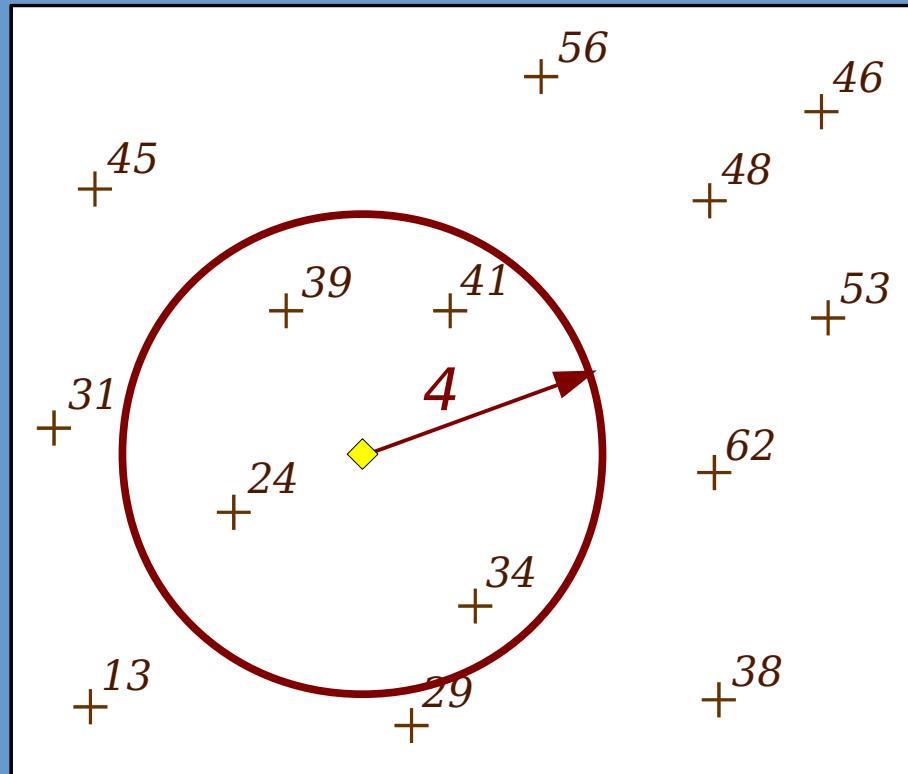
IDW Calculation

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 - $d = 4$



IDW Calculation

- Calculate value at the yellow diamond
 - $d = 4$



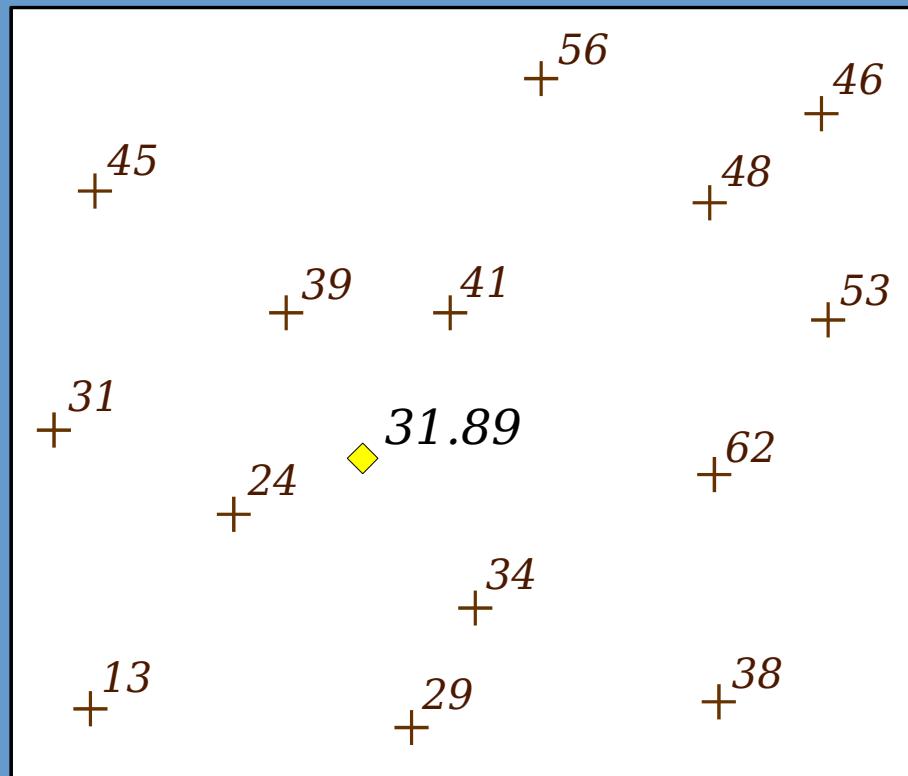
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$$z_j = \frac{17.6644}{0.5539}$$

$$z_j = 31.8936$$

IDW Calculation

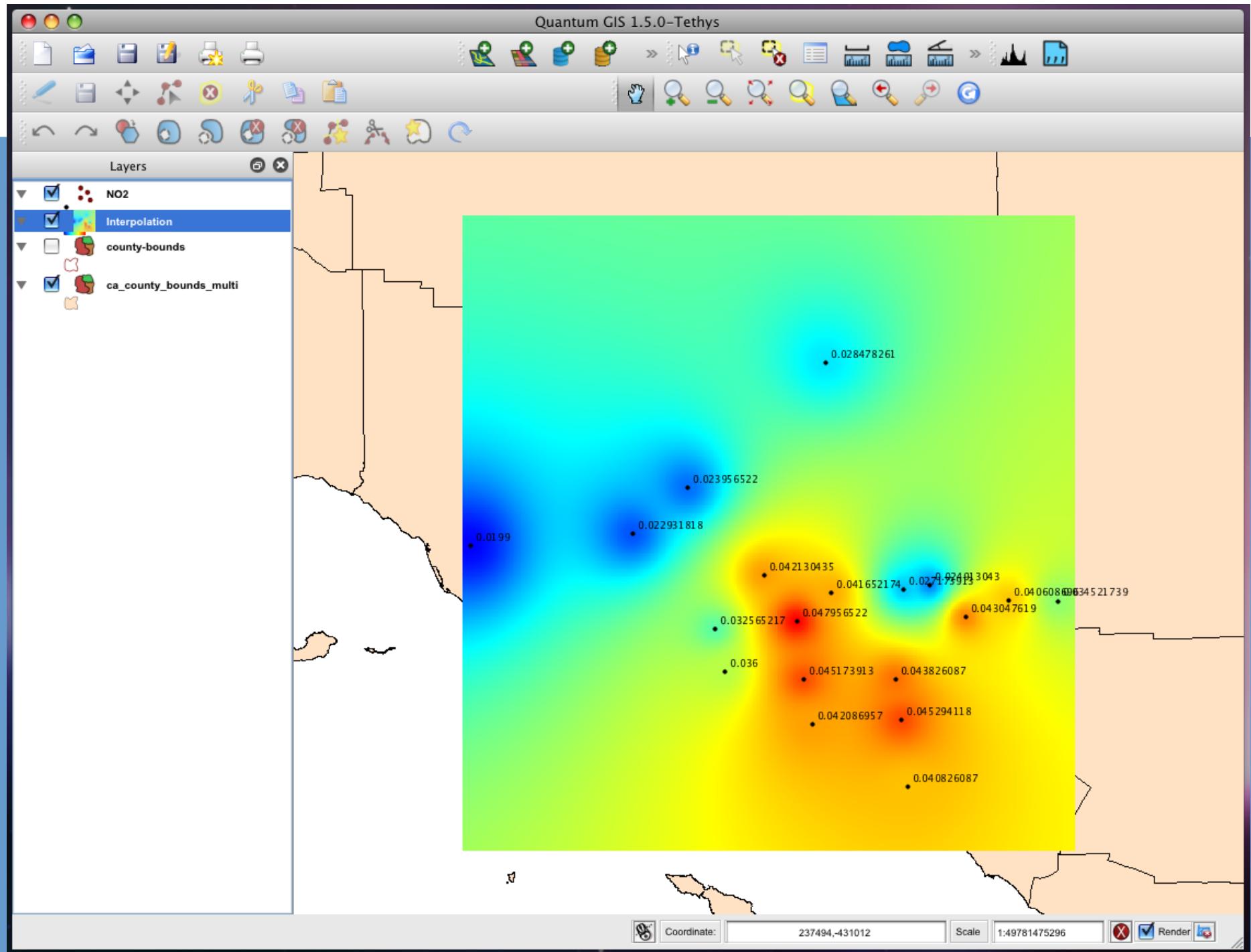
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 - $d = 4$



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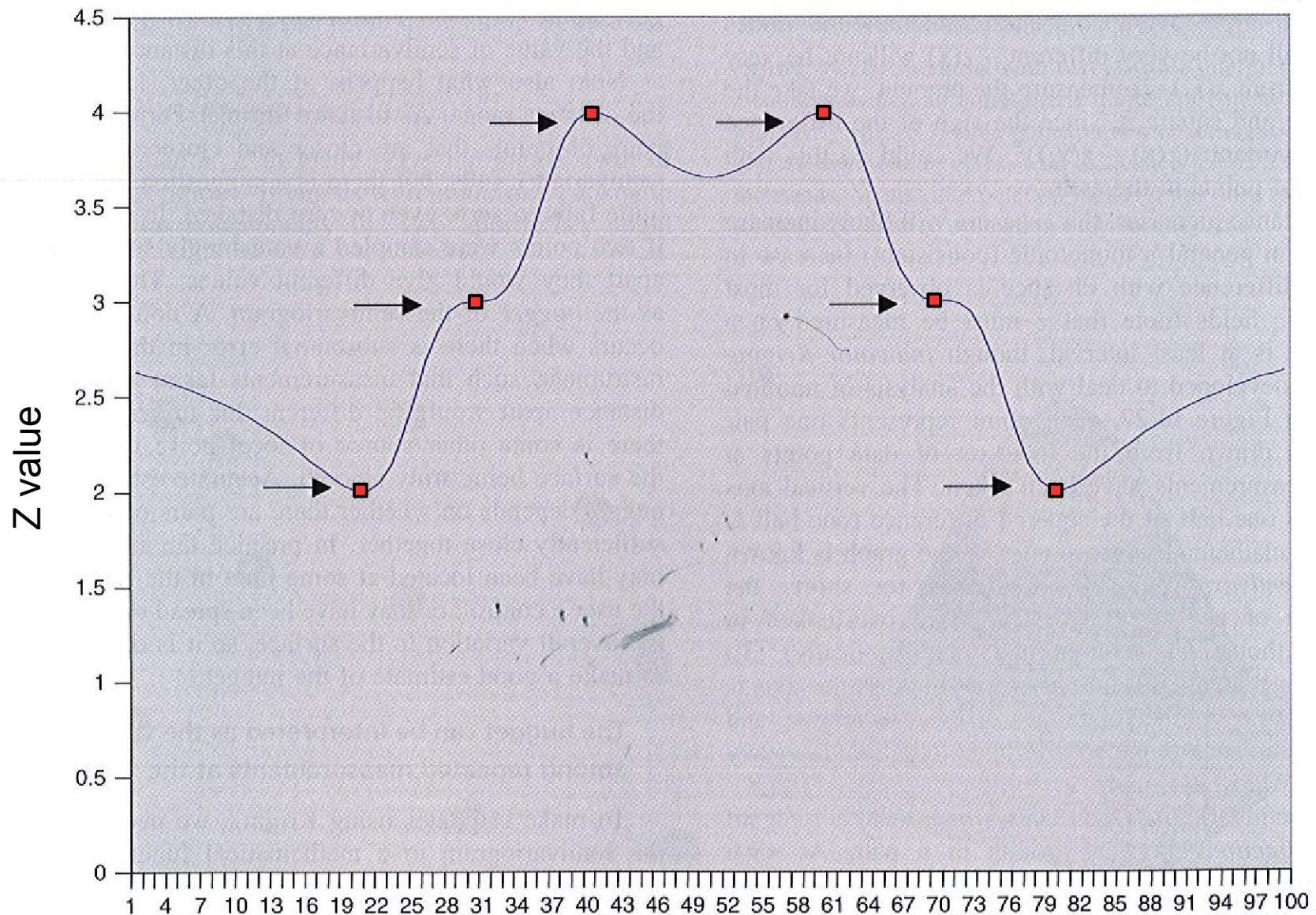


Figure 14.26 Potentially undesirable characteristics of IDW interpolation. Data points located at 20, 30, 40, 60, 70, and 80 have measured values of 2, 3, 4, 4, 3, and 2 respectively. The interpolated profile shows a pit between the two highest values, and regression to the overall mean value of 3 outside the area covered by the data

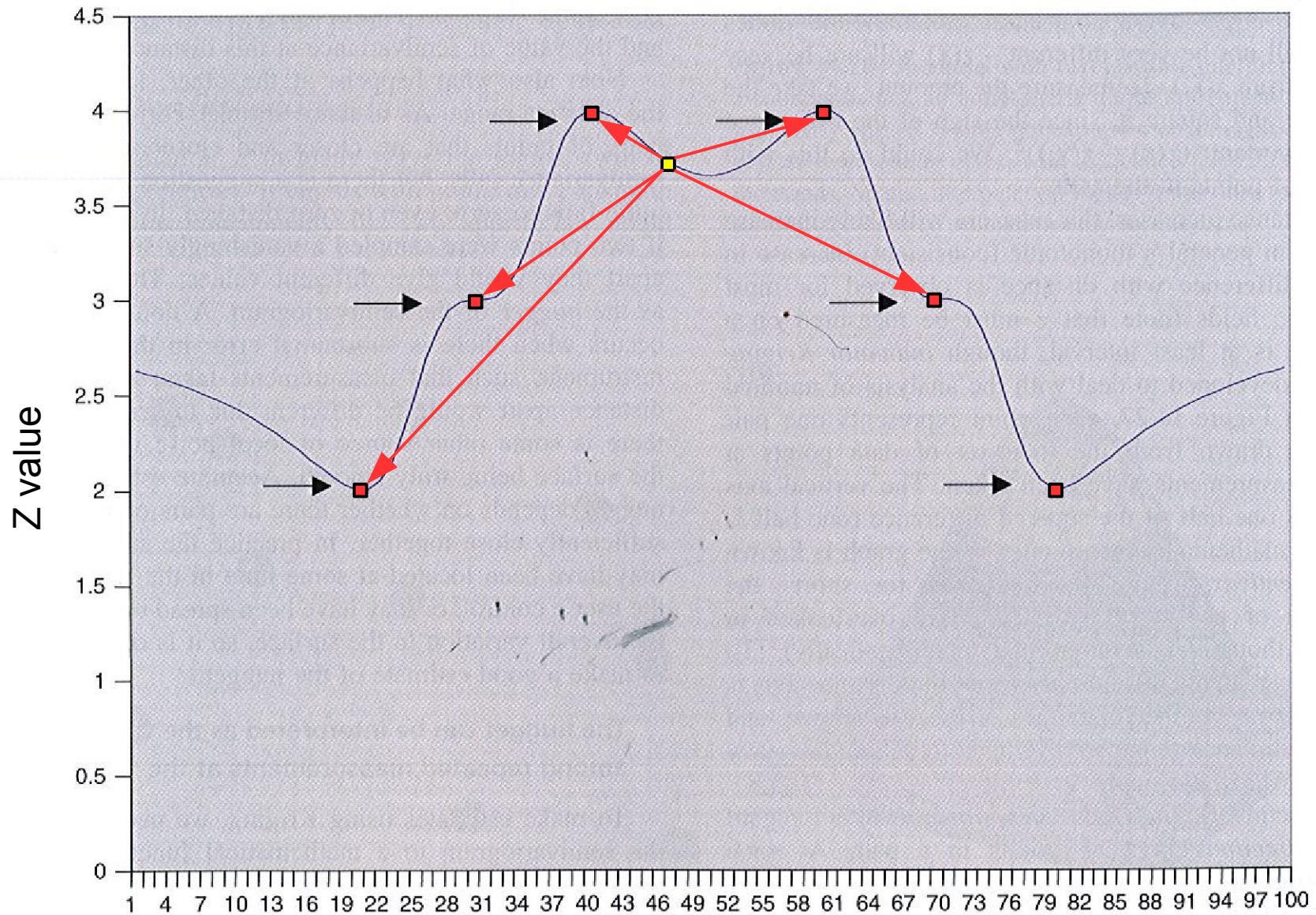
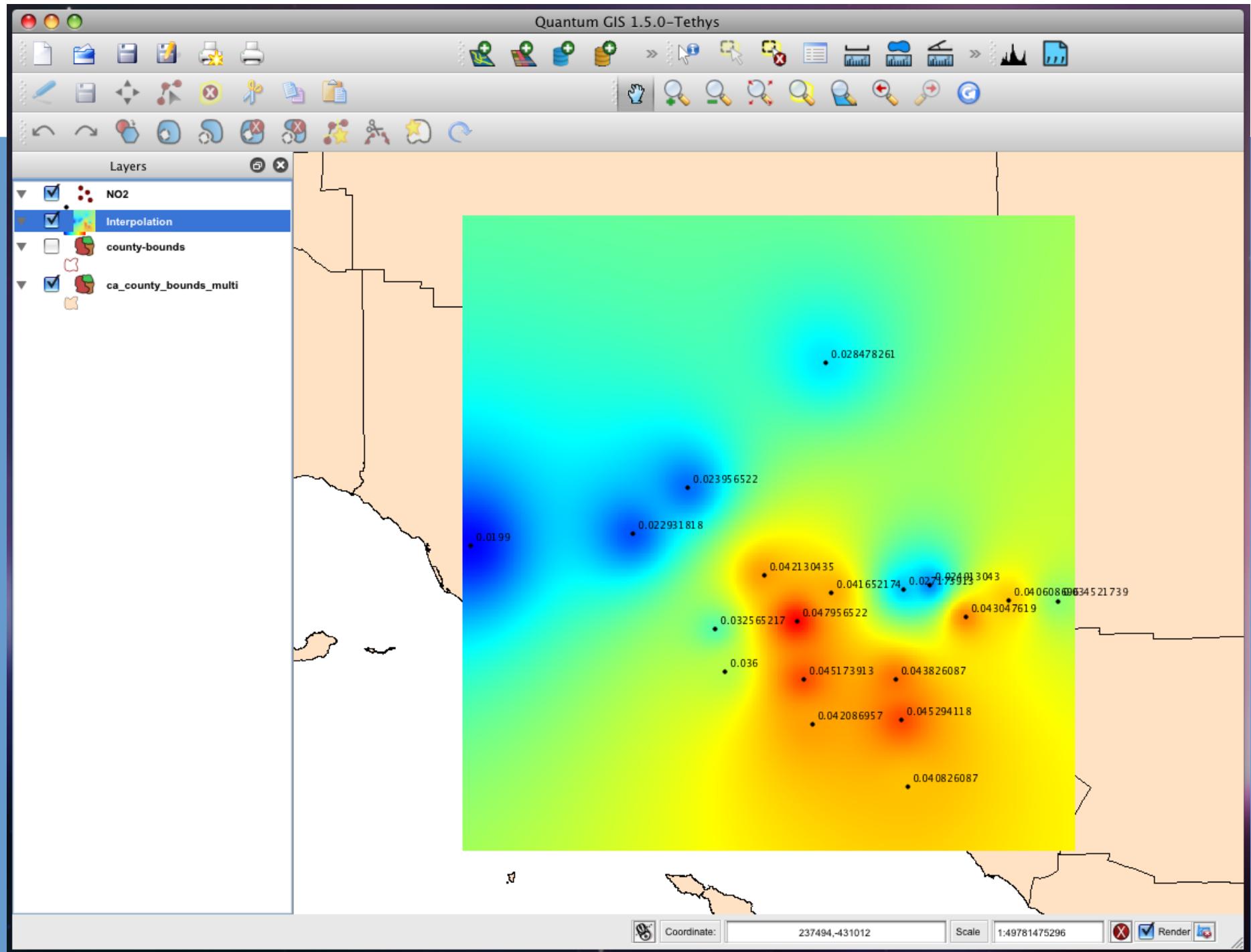


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IDW

- Output features: raster
- Estimated values are constrained to the range of the input values
- At sample locations, estimated value always equals sample value
- Results in a pitted surface (local highs and lows)

Kriging

- Uses a *distance weighted average* method to calculate the value at unsampled locations
 - Weights are estimated from the data using the **semivariogram**
 - *Rather than guessing at the relationship between similarity of values and distance (like we do when we guess at the exponent in IDW)*

Kriging

- Called a *geostatistical* method
 - Parameters based on input data
 - Observed similarity as a function of distance
 - Statistical, not mathematical
 - Provide a measure of uncertainty

Semivariance

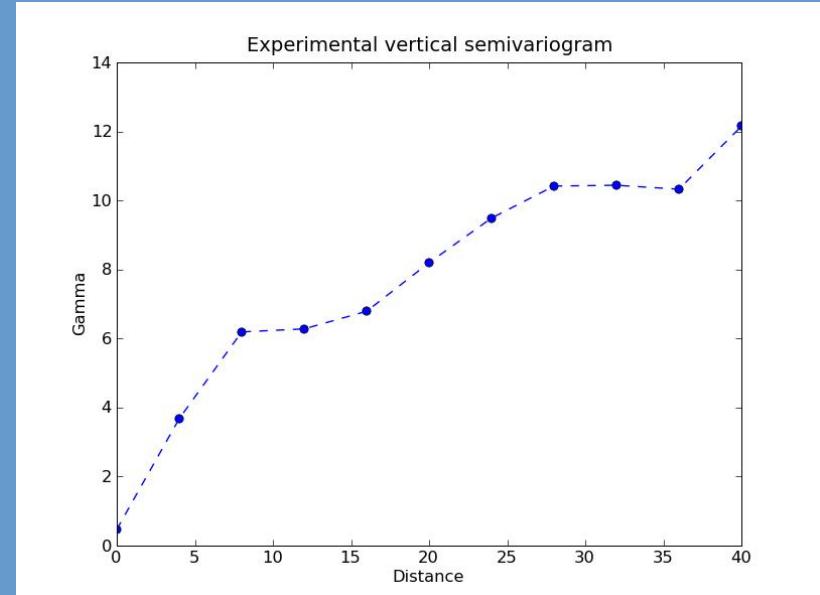
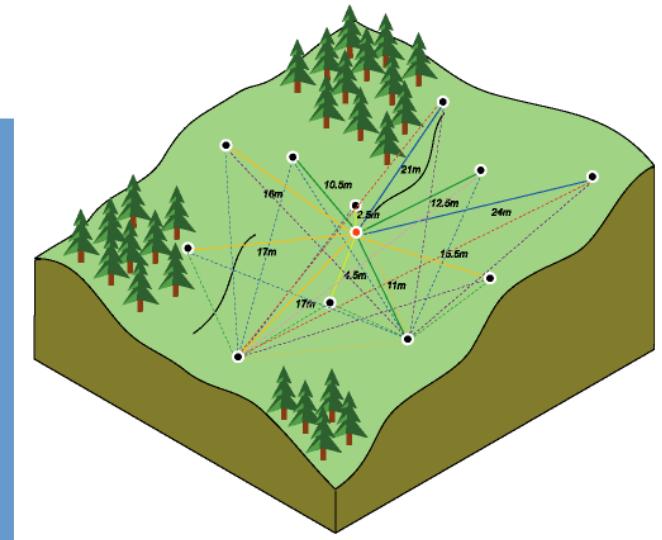
- *Semivariance*: the observed average variation between values measured at some separation distance (h)

$$\gamma(h) = \frac{1}{2n} \sum_{i=1}^n [z(x_i) - z(x_i + h)]^2$$

- *Semivariogram*: the observed relationship between semivariance and distance (h)
 - Provides a description of spatial structure
 - Called “omnidirectional” when calculated regardless of any directional effects

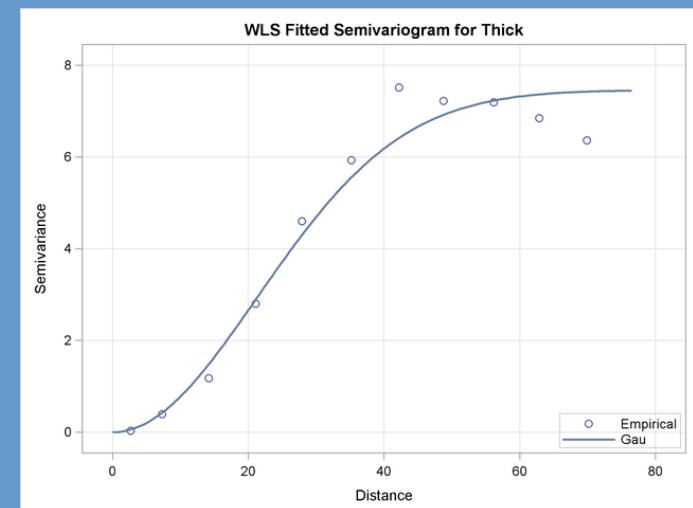
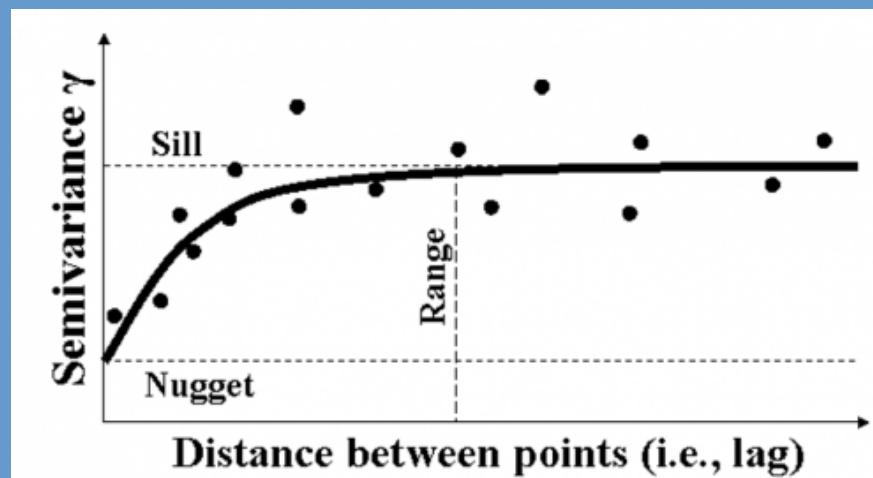
Semivariogram

- In order to know how similarity changes with distance we calculate the empirical semivariogram from the sample data
 - Its shape tells us something about the general form of the surface (or continuous field)
 - Typically shows that difference between values increases with distance



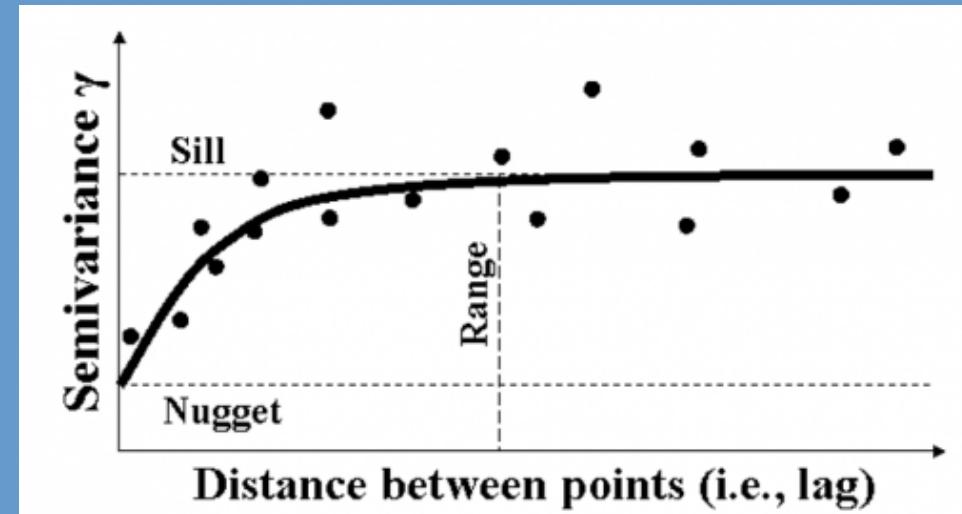
Semivariogram

- Modeling the semivariogram
 - Describe the relationship between difference of values (variation) and distance using a mathematical function
 - Several different types of functions can be used, each with a different form to the distance function



Modeling the Semivariogram

- Described quantitatively by sill, range, and nugget and using different function forms

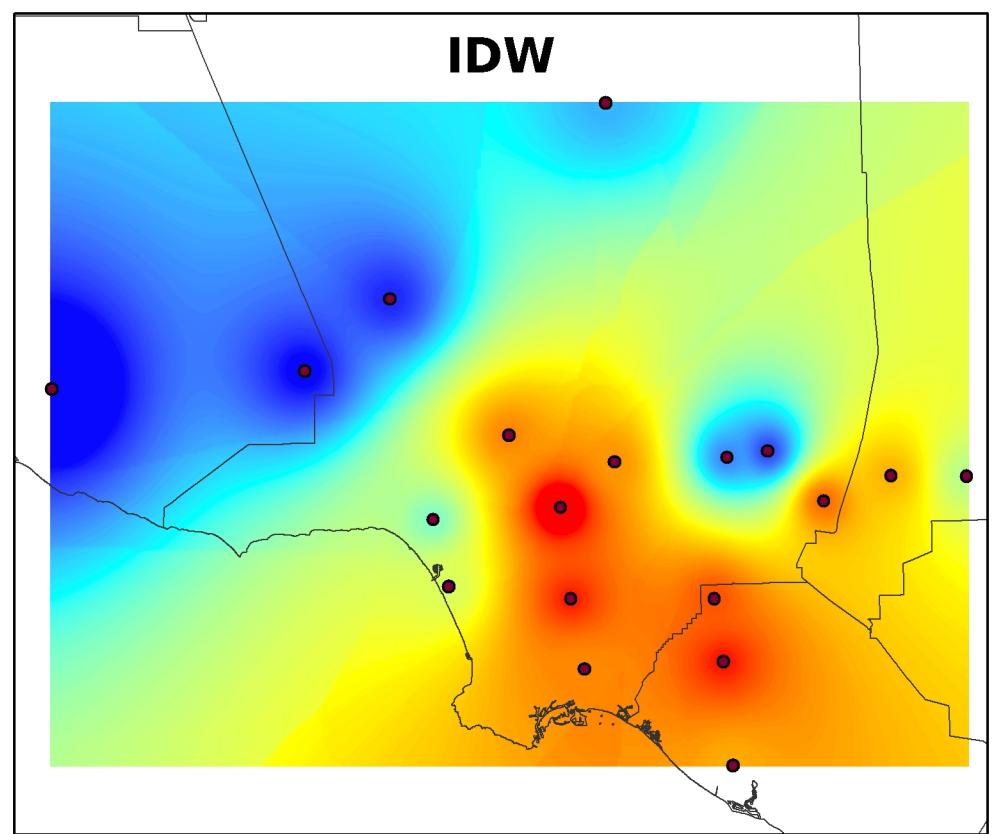
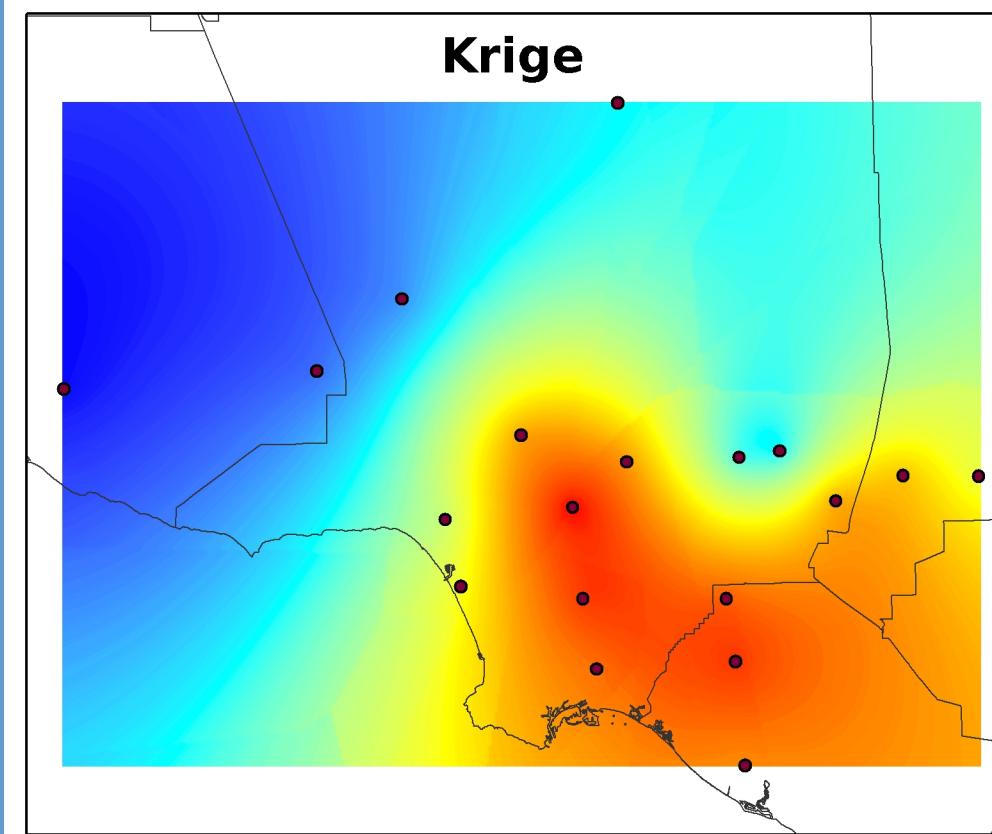


- Range*: distance beyond which values are not spatially autocorrelated
- Sill*: the true variance of the variable, observed between variables separated by a distance $>$ range
- Nugget*: the level of variance observed when the semivariogram curve is projected back to zero distance

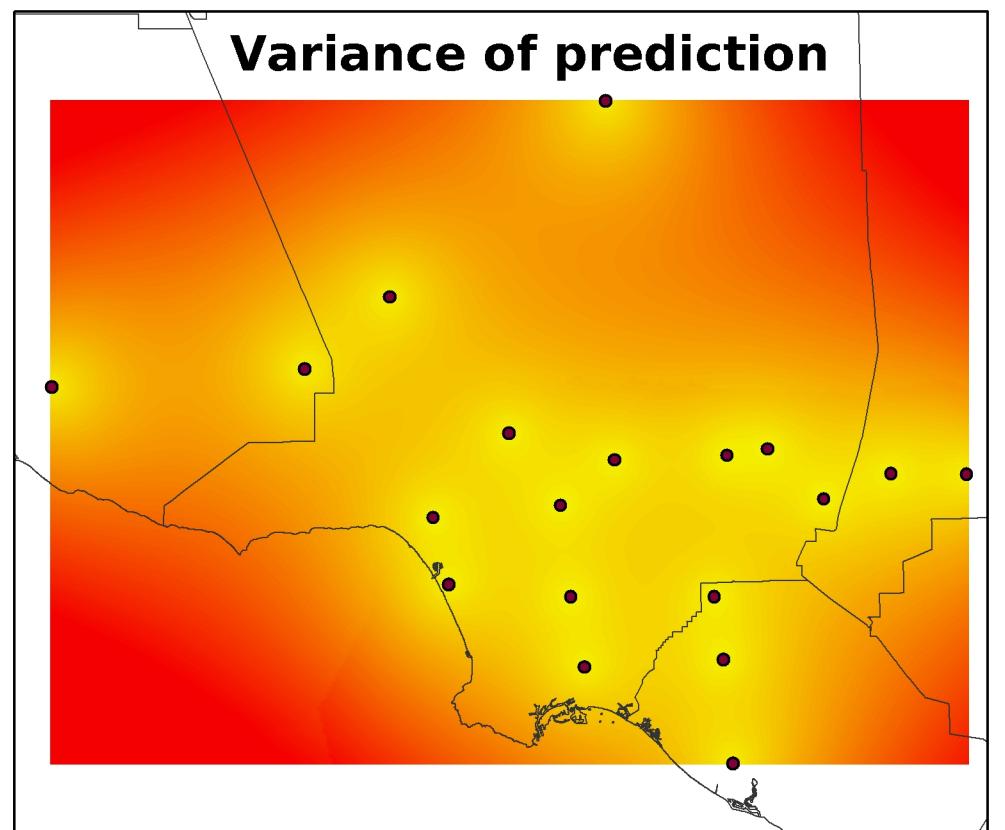
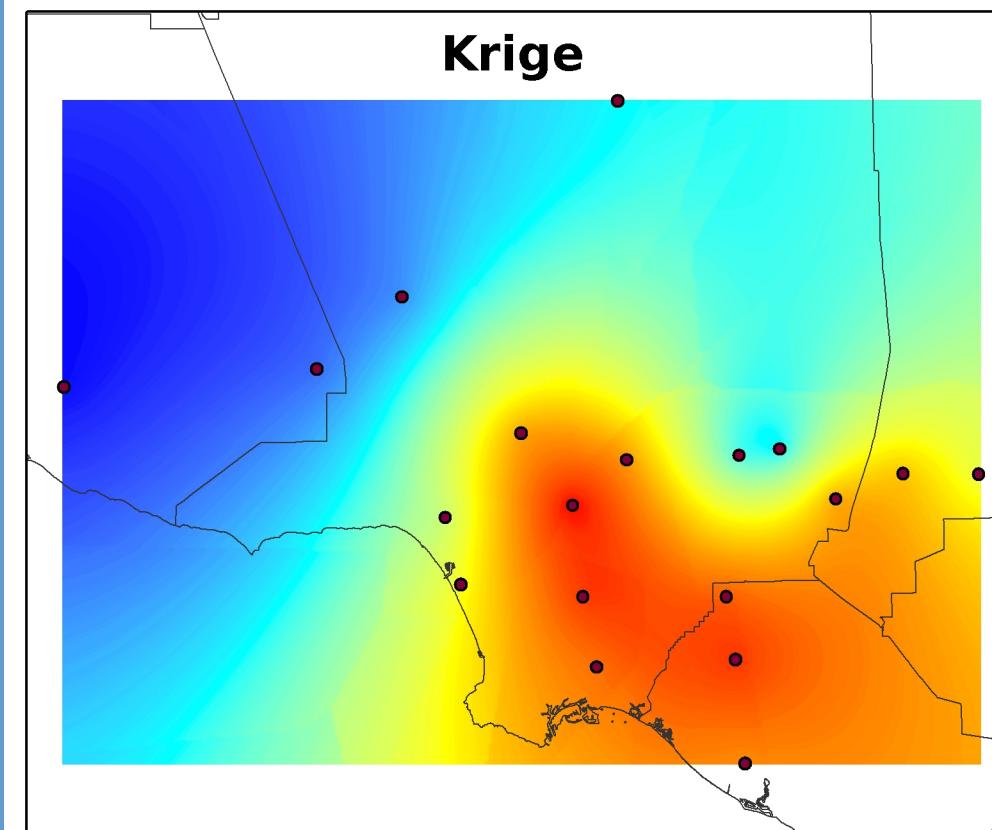
Kriging

- Estimates the weights based on distance
 - Using mathematical function derived from the semivariogram
- Calculates the weighted average of neighboring values for each location
 - Neighbors values as modified by distance and weight (associated with distance)
 - Kriging can be very computationally intensive
- *Provides interpolated surface plus an surface of uncertainty of prediction*

Kriging



Kriging



Kriging

- Output features: raster (estimated value and uncertainty)
- Estimated values are not constrained to the range of the input values
- At sample locations, estimated value not always equal to sample value
- Results in a smoother surface (than IDW)

Non-stationarity

- Thus, far the methods assume that there is a global spatial autocorrelation structure in the data
 - e.g., $1 / d$ applied to all places in IDW
 - e.g., global semivariogram function in Kriging
- This may not be appropriate for variables/phenomena that have a highly variable spatial structure
 - e.g., Think about how temperature varies in plains vs mountains (orographic effects)

Empirical Bayesian Kriging

- Does not assume a stationary process
- Accounts for error introduced by estimating a global semivariogram model
 - Estimates numerous local semivariogram models
 - Captures variation in the spatial dependence from place to place in the study region

Empirical Bayesian Kriging

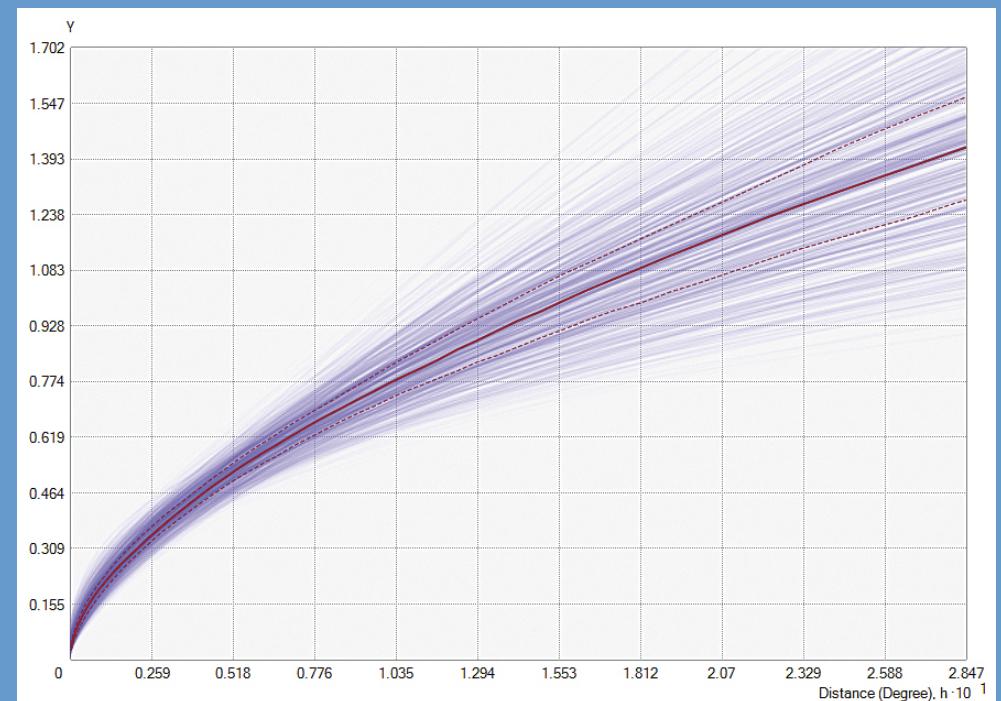
- Subset observed data to local region
 - Estimate initial semivariogram model from observed data values
 - Use initial semivariogram model to simulate a value at each original data point
 - Estimate new semivariogram model from simulated data values
 - Calculate the likelihood that the original data values were generated from the new semivariogram and assign weight (Bayes' rule)

*Repeated
Simulations*

Empirical Bayesian Kriging

- Creates a large number of semivariograms for each subset (local region)
 - When plotted together, the result is a distribution of semivariograms

Example of distribution of Semivariograms:
The darker the blue color, the more semivariogram models are present



Empirical Bayesian Kriging

- Prediction: for each unmeasured location, calculate a new semivariogram distribution using likelihood-based sampling of individual semivariograms
 - Using weight values
 - Can be drawn from multiple spatial subsets
 - For example, if a prediction location has neighbors with observed values in two different subsets, the prediction will be calculated using simulated semivariograms from both of the subsets
 - Semivariograms are chosen based on their likelihood values

EB Kriging

- Output features: raster (estimated value and uncertainty)
- Estimated values are not constrained to the range of the input values
- At sample locations, estimated value not always equal to sample value
- Results in a different surface (than classical kriging), depending on whether non-stationary

Keywords

- Interpolation
- Samples
- Unmeasured locations
- Cow test
- Point-based
- Thiessen polygons
- Trend surface
- IDW
- Kriging
- Semivariogram
- Geostatistical
- Uncertainty