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Lecture- 03 | **Linear Algebra**

Examples of Vector Spaces, and Subspaces

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Recap

of previous lecture

1. Definition of Vector space
2. Examples of vector spaces

Topics

to be covered

1. Examples of vector spaces
2. Properties of vector spaces
3. Subspace
4. Properties of Subspace

V(F) is a Vector Space

→ $V \neq \phi$

→ F is a field

→ There exist

1. vector addition

$\forall\, x, y \in V,\ x+y \in V$ (unique)

2. Scalar multiplication

$\forall\, \alpha \in F,\ x \in V,\ \alpha x \in V$ (unique)

1. $x + y = y + x \quad \forall\, x, y \in V$
2. $x + (y + z) = (x + y) + z \quad \forall\, x, y, z \in V$
3. There exist $0 \in V$: $x + 0 = x \ \forall\, x \in V$
4. $\forall\, x \in V\ \exists\, y \in V : x + y = 0$

5. $\alpha(x + y) = \alpha x + \alpha y \quad \forall\, \alpha \in F,\ \forall\, x, y \in V$
6. $(\alpha + \beta) x = \alpha x + \beta x \quad \forall\, \alpha, \beta \in F,\ \forall\, x \in V$
7. $(\alpha\beta) x = \alpha(\beta x) \quad \forall\, \alpha, \beta \in F\ \forall\, x \in V$
8. $1 \cdot x = x \quad \forall\, x \in V$

Properties of vector space $\rightarrow$:

let $V(F)$ be a v.s.

(i) $0 \cdot x = 0 \quad \forall\, x \in V$

(ii) $\alpha \cdot 0 = 0 \quad \forall\, \alpha \in F$

$$\left.\begin{array}{ll} \text{(iii)} & \alpha(x-y) = \alpha x - \alpha y \\ \text{(iv)} & (\alpha-\beta)x = \alpha x - \beta x \end{array}\right\} \begin{array}{l} \forall\, \alpha, \beta \in F \\ \forall\, x, y \in V \end{array}$$

(v) $\alpha(-x) = (-\alpha)x = -(\alpha x) \quad \forall\, x \in V, \ \forall\, \alpha \in F$

Subspaces

let $V(F)$ be any vector space

, and W be a non-empty subset of V

then W is said to be a subspace of V

if

(I) $0 \in W$ (0 is 0 vector)

(II) $\forall\, x, y \in W \Rightarrow x - y \in W$

(III) $\forall\, \alpha \in F, \forall\, x \in W \Rightarrow \alpha x \in W$

Ex $V = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{R}\}$

$F = \mathbb{R}$

Vector addition

$(a_1, a_2) + (b_1, b_2) = (a_1+b_1, a_2+b_2)$

Scalar multiplication

$\alpha(a_1, a_2) = (\alpha a_1, \alpha a_2)$

$V(F)$ is a vector space

$0 = (0,0)$

$W_1 = \{(a_1, a_2) \mid a_1 = a_2\}$

$W_1 = \{(k,k) \mid k \in \mathbb{R}\}$

(I) $0 = (0,0) \in W_1$

(II) $x = (k,k),\ y = (k', k')$

$x - y = (k-k', k-k') \in W_1$

(III) $\alpha \in \mathbb{R},\ x \in W$

$\alpha x = \alpha(k,k) = (\alpha k, \alpha k) \in W_1$

W_1 is a subspace of V.

Result - I

Let $V(F)$ be a v.s and $\phi \neq W \subseteq V$.

Then W is a subspace of V

$\iff$ $W(F)$ is a vector space w.r.t vector addition, and scalar multiplication that made $V(F)$ a vector space.

See all proofs in Friedberg

Subspace Test - I

let $V(F)$ be a vector space

and $\phi \neq W \subseteq V$, then W is a subspace

of $V \iff$

(I) $\forall x, y \in W \Rightarrow x + y \in W$

(II) $\forall \alpha \in F \; \forall x \in W \Rightarrow \alpha x \in W$

Subspace Test - II

let $V(F)$ be a vector space

, and $\phi \neq W \subseteq V$, then

W is a subspace of V

$\Longleftrightarrow \forall \alpha, \beta \in F \ \forall x, y \in W$

$\Longrightarrow \alpha x + \beta y \in W$

Subfield →

Let $(F,+,\cdot)$ be any field a non-empty subset K of F is said to be a subfield of F if

(I) $\forall\, a,b \in K \Rightarrow a-b \in K$

(II) $\forall\, a,b \in K,\ b \neq 0 \Rightarrow a b^{-1} \in K$

Ex

$Q(\sqrt{2}) = \{a + b\sqrt{2} \mid a,b \in Q\}$

$x = a_1 + b_1\sqrt{2}$, $a_1, b_1 \in Q$

$y = \boxed{a_2 + b_2\sqrt{2}}$; $a_2, b_2 \in Q$

(I) $x - y = (a_1 - a_2) + (b_1 - b_2)\sqrt{2} \in Q(\sqrt{2})$

(II) $y \neq 0 \Rightarrow a_2 \neq 0$ or $b_2 \neq 0$

$$x y^{-1} = x/y = \left(\frac{a_1 + b_1\sqrt{2}}{a_2 + b_2\sqrt{2}}\right) \times \frac{a_2 - b_2\sqrt{2}}{a_2 - b_2\sqrt{2}}$$

$$= \frac{(a_1 a_2 - b_1 b_2 \cdot 2) + \sqrt{2}(a_2 b_1 - a_1 b_2)}{(a_2^2 - 2 b_2^2)}$$

$\in Q(\sqrt{2})$

Ex

$\underline{\mathbb{R}}, \underline{\mathbb{Q}}, \underline{\mathbb{Q}(\sqrt{2})}$ subfields of $\mathbb{C}$

$\mathbb{Q}$ is subfield of $\mathbb{R}$

$\mathbb{Q}$ is subfield of $\mathbb{Q}(\sqrt{2})$

$\mathbb{Q}(i)$ is subfield of $\mathbb{C}$

$\mathbb{Q}$ is subfield of $\mathbb{Q}(i)$

$\mathbb{Q}(\sqrt{2})$ is a subfield of $\mathbb{R}$.

$\mathbb{Q}(i) = \{a+bi \mid a, b \in \mathbb{Q}\}$

(i) $(a+ib)+(c+id)$
$= (a+c)+i(b+d)$

(ii) $(a+ib)(c+id)$
$= (ac-bd)+i(ad+bc)$

FACT →:

Every field is a subfield of itself.

$\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}\}$

$\mathbb{Q}(i) = \{a+bi \mid a, b \in \mathbb{Q}\}$

Example ⇒

If F is a field

, and K is a subfield of F

then $F(K)$ is a vector space

Ex $\mathbb{R}(\mathbb{Q})$ is vector space

$\mathbb{R}(\mathbb{Q}(\sqrt{2}))$, , , , , ,

$\mathbb{Q}(\sqrt{2})(\mathbb{Q})$, , , , , ,

$\mathbb{C}(\mathbb{R})$ is a vector space

$\mathbb{C}(\mathbb{Q})$ " " " "

$\mathbb{C}(\mathbb{Q}(\sqrt{2}))$ " " " "

$\mathbb{C}(\mathbb{Q}(i))$ " " " "

$\mathbb{R}(\mathbb{R})$ " " "

$\mathbb{C}(\mathbb{C})$ " " "

$\mathbb{Q}(\mathbb{Q})$ " " "

$\mathbb{Q}(\sqrt{2})(\mathbb{Q}(\sqrt{2}))$ " " " "

Space of n Tuples

Let F be any field, and K be a subfield of F.

Define $V = F^n = \{(a_1, a_2, a_3, \ldots, a_n) \mid a_1, a_2, \ldots a_n \in F\}$

Then $V(K)$ is a vector space w.r.t

1. Vector addition $(a_1, a_2, a_3, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$
2. Scalar multiplication $\alpha \in K$, $\alpha(a_1, a_2, \ldots, a_n) = (\alpha a_1, \alpha a_2, \ldots \alpha a_n)$

Zero-vector: $0 = (0, 0, 0, \ldots, 0)$

If F is a field
K is subfield of F
then $F^n(K)$ is a vector space

Ex $F = \mathbb{R}$, $K = \mathbb{R}$

$\mathbb{R}^n(\mathbb{R})$ is a vector space

$\mathbb{R}(\mathbb{R})$ - - - -

$\mathbb{R}^2(\mathbb{R})$ - - ______

$\mathbb{R}^3(\mathbb{R})$ ______

$\mathbb{R}^n = \{(a_1, a_2, \ldots a_n) \mid a_1, a_2, \ldots a_n \in \mathbb{R}\}$

$\mathbb{R}^2 = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{R}\}$

$\mathbb{R}^3 = \{(a_1, a_2, a_3) \mid a_1, a_2, a_3 \in \mathbb{R}\}$

$\mathbb{R}^4 = \{(a_1, a_2, a_3, a_4) \mid a_1, a_2, a_3, a_4 \in \mathbb{R}\}$

$F = \mathbb{R}, \quad K = \mathbb{Q}$

$\mathbb{R}^n(\mathbb{Q})$ is a v-s

$\mathbb{R}(\mathbb{Q})$,, ,, ,,

$\mathbb{R}^2(\mathbb{Q})$,, ,, ,,

$\mathbb{R}^3(\mathbb{Q})$. ____

$\mathbb{R}^4(\mathbb{Q})$ ____

$F = \mathbb{R}, \quad K = \mathbb{Q}(\sqrt{2})$

$\mathbb{R}^n(\mathbb{Q}(\sqrt{2}))$ is a v-s

$\mathbb{R}(\mathbb{Q}(\sqrt{2}))$ is a v-s

$\mathbb{R}^2(\mathbb{Q}(\sqrt{2}))$ is a v-s

$\mathbb{R}^3(\mathbb{Q}(\sqrt{2}))$ is a v s

$F = \mathbb{C}, \quad K = \mathbb{C}$	$F = \mathbb{C}, K = \mathbb{R}$	$F = \mathbb{C}, K = \mathbb{Q}$
$\mathbb{C}^n(\mathbb{C})$ is a v.s		
$\mathbb{C}(\mathbb{C})$ " " "	Home-work	Home-work
$\mathbb{C}^2(\mathbb{C})$ " " "		
$\mathbb{C}^3(\mathbb{C})$ — —		

$$\mathbb{C} = \{a+ib \mid a, b \in \mathbb{R}\}$$

$$\mathbb{C}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{C}\}$$

$$= \{(a_1+ib_1, a_2+ib_2) \mid a_1, a_2, b_1, b_2 \in \mathbb{R}\}$$

$$\mathbb{C}^3 = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{C}\}$$

$$= \{(a_1+ib_1, a_2+ib_2, a_3+ib_3) \mid a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}\}$$

Space of Matrices

Let F be any field, and K be a subfield of F.

for $m, n \in \mathbb{N}$, define $\boxed{V = F^{m\times n} = \left\{ A = (a_{ij})_{m\times n} \,\middle|\, \begin{array}{l} a_{ij} \in F \\ \forall\, i = 1, 2, \dots m \\ \quad j = 1, 2, \dots n \end{array} \right\}}$

Then V is a vector space over K w.r.t

1. vector addition

$$(a_{ij})_{m\times n} + (b_{ij})_{m\times n} = (a_{ij} + b_{ij})_{m\times n}$$

2. Scalar multiplication $\alpha (a_{ij})_{m\times n} = (\alpha a_{ij})_{m\times n}$

Zero-vector $O = (0)_{m\times n} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}_{m\times n}$

$F = \mathbb{R}, \quad K = \mathbb{R}$

$\mathbb{R}^{m\times n}(\mathbb{R})$ is a v.s

$\mathbb{R}^{2\times 3}(\mathbb{R})$ is a v.s

$$\mathbb{R}^{2\times 3} = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \middle| \begin{array}{l} a_{ij} \in \mathbb{R} \\ i = 1, 2 \\ j = 1, 2, 3 \end{array} \right\}$$

$\mathbb{R}^{n\times n}(\mathbb{R})$ is a v.s

$F = \mathbb{R}, \quad K = \mathbb{Q}$

$F = \mathbb{R}, \quad K = \mathbb{Q}(\sqrt{2})$

$F = \mathbb{C},\ K = \mathbb{C}$

$\mathbb{C}^{m\times n}(\mathbb{C})$ is a v.s

$\mathbb{C}^{n\times n}(\mathbb{C})$ is a v.s

$F = \mathbb{C},\ K = \mathbb{R}$

$\mathbb{C}^{m\times n}(\mathbb{R})$ is a v.s

$\mathbb{C}^{n\times n}(\mathbb{R})$ is a v.s

$F = \mathbb{C},\ K = \mathbb{Q}(i)$

$\mathbb{C}^{m\times n}(\mathbb{Q}(i))$ is a v.s

$\mathbb{C}^{n\times n}(\mathbb{Q}(i))$ is a v.s

2 Mins Summary

1. Examples of vector spaces
2. Properties of vector spaces
3. Subspace
4. Properties of Subspace

THANK YOU