SAAKAAR

FOR IIT JAM 2025

Lecture- 04 | **Linear Algebra**

Subspace and Properties Part-01

By- Sanjeev sir

Recap *of previous lecture*

1. Examples of vector spaces
2. Properties of vector spaces
3. Subspace

Topics *to be covered*

1. Examples of vector spaces
2. Subspace
3. Properties of Subspace

Space of Polynomials

let F be any field, and K be a subfield of F. Define $F[x] = \left\{ a_0 + a_1 x + a_2 x^2 + \cdots \;\middle|\; \begin{array}{l} a_0, a_1, \ldots \in F \\ \text{all } a_i = 0 \\ \text{except finitely many} \end{array} \right\}$.

Then $F[x]$ is a vector space over K, w.r.t

(I) vector addition

$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

$g(x) = b_0 + b_1 x + b_2 x^2 + \cdots$;

$$\boxed{f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots}$$

(II) Scalar multiplication:

for $\alpha \in K$, $\alpha f(x) = (\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2 + \cdots$

zero polynomial → $0(x) = 0 + 0x + 0x^2 + 0x^3 \cdots$

Ex (i) $F = \mathbb{R}, K = \mathbb{R}$

$\mathbb{R}[x](\mathbb{R})$ is a v.s

$F = \mathbb{R}, K = \mathbb{Q}$

$\mathbb{R}[x](\mathbb{Q})$ is a v-s

$F = \mathbb{R}, K = \mathbb{Q}(\sqrt{2})$

$\mathbb{R}[x](\mathbb{Q}(\sqrt{2}))$ is a v-s

Ex (ii) $F = \mathbb{Q}, K = \mathbb{Q}$

$\mathbb{Q}[x](\mathbb{Q})$ is a v-s

$F = \mathbb{Q}(\sqrt{2}), K = \mathbb{Q}$

$\mathbb{Q}(\sqrt{2})[x](\mathbb{Q})$ is a v.s

Ex (iii) $\mathbb{C}[x](\mathbb{R})$ is a v.s

$\mathbb{C}[x](\mathbb{Q})$ is a v.s

$\mathbb{C}[x](\mathbb{Q}(\sqrt{2}))$ is a v.s

$\mathbb{C}[x](\mathbb{Q}(i))$ is a v.s

$\mathbb{C}[x](\mathbb{C})$ " " " "

Space of Functions

Let F be any field, and K be a subfield of F.
Suppose $S \neq \phi$, Define $V = F^S = \{ f \mid f : S \to F \}$ → Collection of all functions from S to F.

Then F^S is a vector space over K w.r.t

(I) vector addition

for $f, g \in V$, $(f+g)(x) = f(x) + g(x) \ \forall x \in S$

(II) Scalar multiplication

for $\alpha \in K$, $f \in V$, $(\alpha f)(x) = \alpha f(x) \ \forall x \in S$

Zero vector

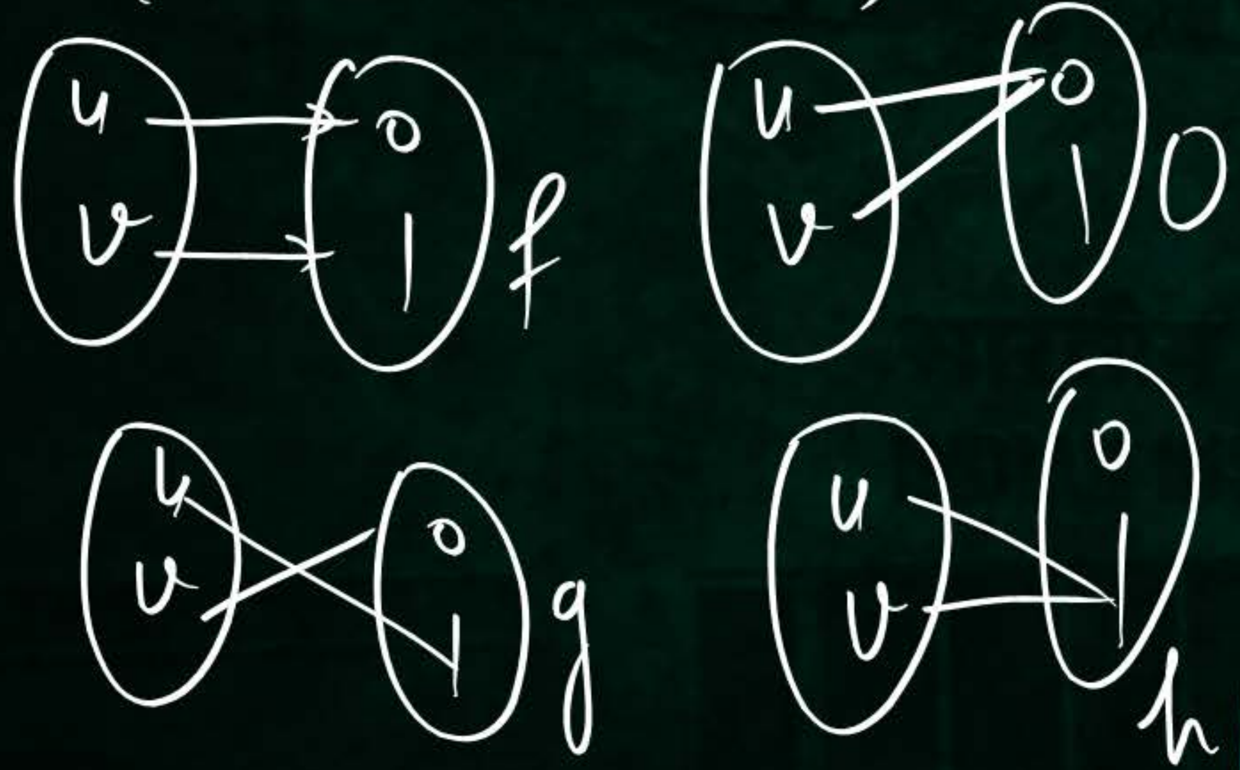
$0 : S \to F$

$0(x) = 0 \ \forall x \in S$

Ex $S = \{u, v\}$

$F = \mathbb{Z}_2 = \{0, 1\}$

$F^S = \{ f \mid f : S \to \mathbb{Z}_2 \}$



$F^S = \{0, f, g, h\}$

$\boxed{0(u) = 0 = 0(v)}$

$f(u) = 0, \quad f(v) = 1$

$g(u) = 1 \quad g(v) = 0$

$h(u) = 1 = h(v)$

$\boxed{F^S(F) \text{ is a vector space.}}$

Space of Sequences

Let F be any field, and K be a subfield of F. Define $V = F^{\infty} = \{\langle a_n \rangle \mid a_n \in F \ \forall n \in \mathbb{N}\}$

$(\langle a_n \rangle = \langle a_1, a_2, a_3, a_4, a_5, \ldots \rangle)$

Then V is a vector space over K w.r.t

(I) vector addition $\langle a_n \rangle + \langle b_n \rangle = \langle a_n + b_n \rangle$

(II) Scalar Multiplication $\alpha \in K$, $\alpha \langle a_n \rangle = \langle \alpha a_n \rangle$

Zero-vector

$\langle 0 \rangle = \langle 0, 0, 0, 0, \ldots \rangle$

Ex 6

$F = \mathbb{R}$

$K = \mathbb{R}$

$$\underbrace{\langle a_n \rangle}_{} + \underbrace{\langle 0 \rangle}_{}$$

$$= \langle a_1, a_2, \text{---} \rangle + \langle 0, 0, 0, \text{---} \rangle$$

$$= \langle a_1 + 0, a_2 + 0, \text{---} \rangle = \langle a_1, a_2, \text{---} \rangle = \underbrace{\langle a_n \rangle}_{}$$

$$\mathbb{R}^\infty = \{ \langle a_n \rangle = \langle a_1, a_2, a_3, \text{---} \rangle \mid a_1, a_2 \text{---} \in \mathbb{R} \}$$

$$\langle a_n \rangle = \langle a_1, a_2, a_3, \text{---} \rangle$$

$$\langle b_n \rangle = \langle b_1, b_2, b_3 \text{---} \rangle$$

$$\langle a_n \rangle + \langle b_n \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3, \text{---} \rangle$$

$$\alpha \langle a_n \rangle = \langle \alpha a_1, \alpha a_2, \alpha a_3 \text{---} \rangle$$

Examples of Subspaces

W is subspace of V

$(\Leftarrow)$ (i) $0 \in W$

(ii) If $x, y \in W \Rightarrow x - y \in W$

(iii) If $\alpha \in F$, $x \in W$ $\Rightarrow$ $\alpha x \in W$

$V = \mathbb{R}^n, \quad F = \mathbb{R}$

$\sum_{i=1}^{n} i a_i = 0$

✓ $W_1 = \{(a_1, a_2, \ldots a_n) \in V \mid 1.a_1 + 2a_2 + 3a_3 + \cdots + na_n = 0\}$

✗ $W_2 = \{(a_1, a_2, \ldots a_n) \in V \mid \boxed{a_1 a_2 = 0}\} \rightarrow 0 = (0, 0 \ldots 0) \in W_2$

✗ $W_3 = \{(a_1, a_2, \ldots a_n) \in V \mid \boxed{a_1 + a_2 = 1}\} \rightarrow 0 = (0, 0 \ldots 0) \notin W_3$

✓ $W_4 = \{(a_1, a_2, \ldots a_n) \in V \mid a_1 = 2a_2\}$ (Home-work)

✗ $W_5 = \{(a_1, a_2, \ldots a_n) \in V \mid \frac{a_1}{a_2} = 2\} \rightarrow 0 = (0, 0, \ldots, 0) \in W_5$

($\frac{0}{0}$ is not defined)

$$W_1 = \{(a_1, a_2, \ldots a_n) \in V \mid \underbrace{\sum_{i=1}^{n} i a_i = 0}\}$$

(i) $0 = (0, 0, \ldots 0) \in W_1$

$a_1 \quad a_2 \quad a_n$

$1 \cdot a_1 + 2 \cdot a_2 + \cdots + n a_n$

$= 1 \cdot 0 + 2 \cdot 0 + \cdots + n \cdot 0$

$= 0 + 0 + \cdots + 0$

$= 0$

(ii) $x = (a_1, a_2, \ldots a_n) \in W_1 \Rightarrow 1 \cdot a_1 + 2 a_2 + \cdots + n a_n = 0$

$y = (b_1, b_2, \ldots b_n) \in W_1 \Rightarrow 1 \cdot b_1 + 2 \cdot b_2 + \cdots + n b_n = 0$

$x - y = (\underbrace{a_1 - b_1}_{d_1}, \underbrace{a_2 - b_2}_{d_2}, \ldots, \underbrace{a_n - b_n}_{d_n}) = (d_1, d_2, \ldots d_n)$

$1 \cdot d_1 + 2 \cdot d_2 + \cdots + n d_n$

$= 1 \cdot (a_1 - b_1) + 2 \cdot (a_2 - b_2) + \cdots + n(a_n - b_n)$

$= (1 \cdot a_1 + 2 \cdot a_2 + \cdots + n a_n) - (b_1 + 2 b_2 + \cdots + n b_n)$

$= 0 - 0 = 0$

(iii) let $\alpha \in \mathbb{R}$

$x = (a_1, a_2, \ldots, a_n) \in W_1$

$1 \cdot a_1 + 2 \cdot a_2 + \cdots + n a_n = 0$

now

$\alpha x = \alpha(a_1, a_2 \ldots a_n)$

$= (\underbrace{\alpha a_1}_{d_1}, \underbrace{\alpha a_2}_{d_2}, \ldots \underbrace{\alpha a_n}_{d_n})$

$1 \cdot d_1 + 2 \cdot d_2 + \cdots + n d_n$

$= 1 \cdot (\alpha a_1) + 2(\alpha a_2) + \cdots + n(\alpha a_n)$

$= \alpha(1 \cdot a_1 + 2 a_2 + \cdots + n a_n)$

$= \alpha \cdot 0 = 0$

$\alpha x \in W_1$

$\Rightarrow$ $\boxed{W_1 \text{ is a subspace}}$

$$W_2 = \{(a_1, a_2, \ldots a_n) \in \mathbb{R}^n \mid a_1 a_2 = 0\}$$

$x = (a_1, a_2 \ldots a_n)$, $a_1 a_2 = 0$

$y = (b_1, b_2 \ldots b_n)$ $b_1 b_2 = 0$

$$x - y = (\underbrace{a_1 - b_1}, \underbrace{a_2 - b_2}, \ldots, a_n - b_n)$$

$$(a_1 - b_1)(a_2 - b_2)$$

$$= a_1 a_2 - a_1 b_2 - b_1 a_2 + b_1 b_2$$

$$= 0 - (a_1 b_2 + b_1 a_2) + 0$$

$$= -(a_1 b_2 + b_1 a_2)$$

$x = (1, 0, \ldots 0)$

$y = (0, 1, 0 \ldots)$

$x - y = (1, -1, 0 \ldots 0) \notin W_2$

$1 \times (-1) = -1 \neq 0$

$\therefore$ W_2 is not a subspace

FACT:

let F be any field, K be any subfield of F.

If $\alpha_1, \alpha_2, \ldots, \alpha_n \in K$ (are fixed)

$$\sum_{i=1}^{n} \alpha_i a_i = 0$$

Then $W = \{(a_1, a_2, \ldots, a_n) \in F^n \mid \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n = 0\}$

is a subspace of $F^n(K)$

Ex $V = \mathbb{R}^3, \ F = \mathbb{R}$

subspaces of $\mathbb{R}^3(\mathbb{R})$

$$W_1 = \{(a_1, a_2, a_3) \in V \mid a_1 + a_2 + a_3 = 0\}$$

$$W_2 = \{(a_1, a_2, a_3) \in V \mid 2a_1 + 3a_2 + 4a_3 = 0\}$$

$$W_3 = \{(a_1, a_2, a_3) \in V \mid \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{5} = 0\}$$

2 Mins Summary

1. Examples of vector spaces
2. Subspace
3. Properties of Subspace

THANK YOU