SAAKAAR

FOR IIT JAM 2025

Lecture- 01 | **Linear Algebra**

Introduction to Vector Spaces

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Topics *to be covered*

1. Motivation to Vector Spaces
2. What is a Field?
3. Definition of Vector space

Motivation to Vector Spaces

12^{th} Class

$$\boxed{\begin{aligned}\vec{a} &= a_1\hat{i} + a_2\hat{j} \\ \vec{b} &= b_1\hat{i} + b_2\hat{j}\end{aligned}}$$

Vector addition

$$\vec{a} + \vec{b} = (a_1+b_1)\hat{i} + (a_2+b_2)\hat{j}$$

Scalar multiplication:

If $\alpha \in \mathbb{R}$, $\alpha\vec{a} = (\alpha a_1)\hat{i} + (\alpha a_2)\hat{j}$

(I) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

(II) $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$

(III) There exist $\vec{0} = 0\hat{i} + 0\hat{j}$ such that $\vec{a} + \vec{0} = \vec{a}$

(IV) For all vector $\vec{a} = a_1\hat{i} + a_2\hat{j}$ there exist $(-\vec{a}) = (-a_1)\hat{i} + (-a_2)\hat{j}$

$$\vec{a} + (-\vec{a}) = 0\hat{i} + 0\hat{j} = \vec{0}$$

(v) $\alpha(\vec{a}+\vec{b})$

$= \alpha\vec{a} + \alpha\vec{b}$

(vi) $(\alpha+\beta)\vec{a}$

$= \alpha\vec{a} + \beta\vec{a}$

(vii) $(\alpha\beta)\vec{a} = \alpha(\beta\vec{a})$

(viii) $1\cdot\vec{a} = 1\cdot a_1\hat{i} + 1\cdot a_2\hat{j}$

$= \vec{a}$

1. "linear Algebra done right"

2. linear Algebra By Friedberg

3. linear Algebra

Vikas Bist

Vivek Sahai

What is a Field?

A $\boxed{\text{non-empty } \boxed{\text{set } F}}$

together with two binary operations

$+$ (called addition), $\cdot$ (called multiplication)

is said to be a field if following conditions are satisfied

(1) $a+b=b+a$ for all $a, b \in F$
$ab = ba$ for all $a, b \in F$ (Commutativity of addition and multiplication)

Associativity of + and ·

(II) $\forall\, a, b, c \in F$

$$a+(b+c) = (a+b)+c$$

$$a(bc) = (ab)c$$

(III) Existence of additive and multiplicative Identity

There exist $\boxed{0 \in F}$:

$$\boxed{a+0 = a \ \forall a \in F}$$

There exists $\boxed{1 \in F}$:

$$1a = a \ \forall a \in F$$

(IV) Existence of additive and multiplicative inverse.

for all $a \in F$ there exists $b \in F$ such that

$$\boxed{a+b = 0}$$

Note: b is denoted by $-a$.

also $\boxed{\forall a \in F, \boxed{a \neq 0}}$

there exists $\boxed{b \in F}$

Such that

$$\boxed{ab = 1}$$

<u>note</u>: b is denoted by a^{-1}

(v) <u>Distributivity</u>

$$a(b+c) = ab + ac$$

$$(a+b)c = ac + bc$$

for all $a, b, c \in F$.

$(F, +, \cdot) \rightarrow$ field

(A1) $\forall\, a, b \in F \Rightarrow a+b \in F$
also $\boxed{a+b = b+a}$

(A2) $\forall\, a, b, c \in F$
$\Rightarrow a+(b+c) = (a+b)+c$

(A3) There exists $\boxed{0 \in F}$:
$a+0 = a \;\forall\, a \in F$

(A4) $\forall\, a \in F$
there exists $b \in F$:
$\boxed{a+b = 0}$

(M1) $\forall\, a, b \in F \Rightarrow ab \in F$
also $\boxed{ab = ba}$

(M2) $\forall\, a, b, c \in F$
$\Rightarrow a(bc) = (ab)c$

(M3) There exists $1 \in F$:
$1 \cdot a = a \;\forall\, a \in F$

(M4) $\forall\, a \in F$, $a \neq 0$, there exists $b \in F$:
$\boxed{ab = 1}$

(D) Distributivety

$$a(b+c) = ab + ac$$

$$(a+b)c = ac + bc$$

$$\forall \ a, b, c \in F.$$

Ex. $(\mathbb{R}, +, \cdot)$

is a field

Ex. $(\mathbb{Q}, +, \cdot)$

is a field

Ex. $(\mathbb{C}, +, \cdot)$

is a field

Ex. $\boxed{\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}}$ F

If $x = a_1 + b_1\sqrt{2}$, $y = a_2 + b_2\sqrt{2}$

Define

$\boxed{x+y} = (a_1 + a_2) + (b_1 + b_2)\sqrt{2}$

$\boxed{xy} = (a_1 a_2 + 2b_1 b_2) + \sqrt{2}(a_1 b_2 + b_1 a_2)$

$(\mathbb{Q}(\sqrt{2}), +, \cdot)$ is a field.

$0 = 0 + \sqrt{2} \cdot 0$

$1 = 1 + \sqrt{2} \cdot 0$

Ex $(\mathbb{N}, +, \cdot)$ is not field

Since $0 \notin \mathbb{N}$

(i.e. has no additive Identity)

Ex $(\mathbb{Z}, +, \cdot)$ is not field

for $2 \in \mathbb{Z}, 2 \neq 0$

But $2b = 1 \Rightarrow \boxed{b = \frac{1}{2} \notin \mathbb{Z}}$

Example of a finite field →:

Let p be prime.

Define $\mathbb{Z}_p = \{0, 1, 2, 3, 4, \ldots, p-1\}$

and Binary operations.

(i) $a+b$ = least non-negative remainder when $a+b$ is divided by 'p'

$(a+b) = (a+b) \bmod p$

(ii) ab = least non-negative remainder when ab is divided by p.

$ab = (ab) \bmod p$

Ex:

If $\boxed{p=5}$

$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$

If $p = 3$

$\mathbb{Z}_3 = \{0, 1, 2\}$

$(3 \cdot 4 \bmod 5) = 2$

$(3+4) \bmod 5 = 2$

Eg $\mathbb{Z}_3 = \{0, 1, 2\}$

$(a+b) = (a+b) \bmod 3$

$(ab) = (ab) \bmod 3$

+	0	1	$\boxed{2}$
0	0	1	2
1	1	2	0
2	2	0	1

$-1 = 2, \; -2 = 1$

$-0 = 0$

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	$\boxed{1}$

$\boxed{1^{-1} = 1}$ $\boxed{2^{-1} = 2}$

$2 \cdot 2$

$= (2 \cdot 2 \bmod 3)$

$= 4 \bmod 3$

$= 1$

Fields:

(I) $(\mathbb{R}, +, \cdot)$

(II) $(\mathbb{Q}, +, \cdot)$

(III) $(\mathbb{C}, +, \cdot)$

(IV) $\boxed{(\mathbb{Z}_p, +, \cdot)}$

p is prime

$(a+b) = (a+b) \bmod p, \quad ab = (ab) \bmod p$

in $(\mathbb{Z}_p, +, \cdot)$

$$\boxed{-a = p - a}$$

Ex in $(\mathbb{Z}_7, +, \cdot)$

$-2 = 7 - 2 = 5$

$-5 = 7 - 5 = 2$

$-3 = 7 - 3 = 4$

$-4 = 7 - 4 = 3$

$\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}\}$

(I) $(a_1+b_1 i) + (a_2+b_2 i)$

$= (a_1+a_2) + i(b_1+b_2)$

(II) $(a_1+b_1 i)(a_2+b_2 i)$

$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$

2 Mins Summary

1. Motivation to Vector Spaces
2. What is a Field?

THANK YOU