SAAKAAR

FOR IIT JAM 2025

Lecture- 02 | **Linear Algebra**

Properties of Vector Space, and Subspace

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Recap
of previous lecture

1. Motivation to Vector Spaces
2. What is a Field?

Topics *to be covered*

1. Definition of Vector space
2. Examples of vector spaces
3. Subspace
4. Properties of vector space

"काक चेष्टा बको ध्यानं, श्वान निद्रा तथैव च।
अल्पहारी गृह त्यागी, विद्यार्थी पंच लक्षणं ॥"

Student's qualities according to chankya

1. Listening (shravana)

2. Retention (Grahan).

3. Understanding (Dhyana)

4. Contemblation (Dharana)

5. Application (Tatva)

Field →:

$(F, +, \cdot)$ field

(A1) $\forall\, a, b \in F \Rightarrow a+b \in F$

(A2) $\forall\, a, b \in F \Rightarrow a+b = b+a$

(A3) $\forall\, a, b, c \in F$

$a+(b+c) = (a+b)+c$

(A4) there exists $0 \in F$:

$a+0 = a \;\forall\, a \in F$

(A5) $\forall\, a \in F$ there is $b \in F$: $a+b = 0$

(M1) $\forall\, a, b \in F \Rightarrow a \cdot b \in F$

(M2) $\forall\, a, b \in F \Rightarrow a \cdot b = b \cdot a$

(M3) $\forall\, a, b, c \in F$

$a(bc) = (ab)c$

(M4) there exist $1 \in F$

$a \cdot 1 = a \;\forall\, a \in F$

(M5) $\forall\, a \in F,\; a \neq 0$

there is $b \in F$:

$ab = 1$

(D1) $\forall\, a, b, c \in F$

$a(b+c) = ab+ac$

(D2) $\forall\, a, b, c \in F$

$(a+b)c$

$= ac+bc$

Important field →

(i) $(\mathbb{R}, +, \cdot)$

(ii) $(\mathbb{C}, +, \cdot)$

(iii) $(\mathbb{Q}, +, \cdot)$

(iv) $(\mathbb{Z}_p, \oplus_p, \odot_p)$

$a \oplus_p b = (a+b) \bmod p$

$a \odot_p b = (ab) \bmod p$

$\boxed{\begin{array}{l} \text{note: in } \mathbb{Z}_p \\ -a = (p-a) \bmod p \end{array}}$

(v) $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$

(vi) $\mathbb{Q}(i) = \{a + bi \mid a, b \in \mathbb{Q}\}$

(vii) if p is prime

$\mathbb{Q}(\sqrt{p}) = \{a + b\sqrt{p} \mid a, b \in \mathbb{Q}\}$

Definition of vector space

Let F be any field and V be a non-empty set.

We say that $\boxed{V \text{ is a vector space over } F}$

If we can define operations

1. Vector addition:

   For $x, y \in V$, $x + y \in V$ (unique element)

2. Scalar multiplication:

   For $\alpha \in F$ and $x \in V$, $\alpha \cdot x \in V$ (unique Element)

Such that following Conditions are Satisfied

(V-1) $x+y=y+x \ \forall x,y \in V$

(V-2) $\forall x,y,z \in V$

$x+(y+z)=(x+y)+z$

(V-3) there exists $\boxed{0 \in V}$:

$\boxed{x+0=x \ \forall x \in V}$

(V-4) $\forall \boxed{x \in V}$ there exists $\boxed{y \in V}$: $x+y=0$

, and y is denoted by $-x$

$\forall x,y \in V$ and $\alpha, \beta \in F$.

(S-1) $\alpha \cdot (x+y) = \alpha \cdot x + \alpha \cdot y$

(S-2) $(\alpha+\beta) \cdot x = \alpha x + \beta x$

(S-3) $(\alpha \beta) x = \alpha \cdot (\beta \cdot x)$

(S-4) $1 \cdot x = x$

Ex let $V = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{R}\}$

$F = (\mathbb{R}, +, \cdot)$

1. Vector addition:

$x = (a_1, a_2),\ y = (b_1, b_2) \in V$

$$\boxed{x + y = (a_1 + b_1,\ a_2 + b_2)}$$

2. Scalar multiplication

for $\alpha \in F$, $x = (a_1, a_2) \in V$

$$\boxed{\alpha x = (\alpha a_1, \alpha a_2)}$$

(V-I)

$x + y = (a_1 + b_1,\ a_2 + b_2)$

$= (b_1 + a_1,\ b_2 + a_2)$

$= (b_1, b_2) + (a_1, a_2)$

$$\boxed{x + y = y + x}$$

V-2 let $x = (a_1, a_2)$

$y = (b_1, b_2)$

$z = (c_1, c_2)$

$x + (y + z) = (a_1, a_2) + (b_1 + c_1,\ b_2 + c_2)$

$= (a_1 + (b_1 + c_1),\ a_2 + (b_2 + c_2))$

$$x+(y+z)$$

$$= (a_1+(b_1+c_1),\ a_2+(b_2+c_2))$$

$$= ((a_1+b_1)+c_1,\ (a_2+b_2)+c_2)$$

$$= (\underline{a_1+b_1},\ \underline{a_2+b_2}) + (\underline{c_1, c_2})$$

$$= (x+y)+z$$

V-3 $\quad 0 = (0,0) \in V$

$\forall x \in V;\ x+0 = (a_1,a_2)+(0,0)$

$$= (a_1+0, a_2+0) = (a_1,a_2) = x$$

V-4 $\quad \forall x = (a_1,a_2) \in V$

Define $y = (-a_1,-a_2) \in V$

$$x+y = (a_1,a_2)+(-a_1,-a_2)$$

$$= (a_1-a_1,\ a_2-a_2)$$

$$x+y = (0,0) = 0$$

S-I let $x=(a_1,a_2)$

$y=(b_1,b_2)$

$$\alpha(x+y)=\alpha(a_1+b_1,a_2+b_2)$$
$$=(\alpha(a_1+b_1),\alpha(a_2+b_2))$$
$$=(\alpha a_1+\alpha b_1,\alpha a_2+\alpha b_2)$$
$$=(\alpha a_1,\alpha a_2)+(\alpha b_1,\alpha b_2)$$
$$=\alpha(a_1,a_2)+\alpha(b_1,b_2)$$
$$=\alpha x+\alpha y$$

S-2 $(\alpha+\beta)x$

$$=(\alpha+\beta)(a_1,a_2)$$
$$=((\alpha+\beta)a_1,(\alpha+\beta)a_2)$$
$$=(\alpha a_1+\beta a_1,\alpha a_2+\beta a_2)$$
$$=(\alpha a_1,\alpha a_2)+(\beta a_1,\beta a_2)$$
$$=\alpha(a_1,a_2)+\beta(a_1,a_2)$$
$$=\alpha x+\beta x$$

(S-3)

$(\alpha\beta)x$

$= (\alpha\beta)(a_1, a_2)$

$= ((\alpha\beta)a_1, (\alpha\beta)a_2)$

$= (\alpha(\beta a_1), \alpha(\beta a_2))$

$= \alpha(\beta a_1, \beta a_2)$

$= \alpha(\beta(a_1, a_2))$

$= \alpha(\beta x)$

(S-4)

$1 \cdot x = 1 \cdot (a_1, a_2)$

$= (1 \cdot a_1, 1 \cdot a_2)$

$= (a_1, a_2)$

$= x$

Ex→ $V = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{R}\}$

$F = \mathbb{R}$

1 → $x = (a_1, a_2),\ y = (b_1, b_2)$

$$\boxed{x + y = (a_1 b_1,\ a_2 + b_2)}$$

2 → $$\boxed{\alpha \cdot x = (\alpha a_1,\ \alpha^2 a_2)}$$

is V a vector space over R. ?

V-1
$x + y = (a_1 b_1,\ a_2 + b_2)$
$= (b_1 a_1,\ b_2 + a_2)$
$= \underbrace{(b_1, b_2)} + \underbrace{(a_1, a_2)}$
$= y + x$

V-2 ✓

V-3 $\boxed{0 = (1, 0)}$

$x + 0 = (a_1, a_2) + (1, 0)$
$= (a_1 \cdot 1,\ a_2 + 0) = (a_1, a_2)$
$= x$

V-4 let $x = (a_1, a_2)$

<u>To find</u> $y = (b_1, b_2)$:

$$x + y = 0 = (1, 0)$$

$$\Rightarrow (a_1 b_1, a_2 + b_2) = (1, 0)$$

$$\Rightarrow \boxed{a_1 \boxed{b_1} = 1} \Rightarrow \boxed{b_1 = \frac{1}{a_1}} \; a_1 \neq 0$$

$$a_2 + b_2 = 0$$

$$\Rightarrow \boxed{b_2 = -a_2}$$

$$x = (0, 2)$$

$$x + y = 0 = (1, 0)$$

$$\Rightarrow (0 \cdot b_1, 2 + b_2) = (1, 0)$$

$$\Rightarrow (\boxed{0}, 2 + b_2) = (\boxed{1}, 0)$$

$$0 = 1 \quad \times$$

$\boxed{\text{Home-work} \rightarrow :}$

Check properties

(S-1) to (S-7)

Note →:

If V is a vector space over F

we write "$V(F)$ is a vector space".

Note -: If $V(F)$ is a vector space

then $\boxed{\text{elements of } V}$ are called $\boxed{\text{vectors}}$

, and $\boxed{\text{Elements of } F}$ are called $\boxed{\text{Scalars}}$.

2 Mins Summary

1. Definition of Vector space
2. Examples of vector spaces

THANK YOU