SAAKAAR

FOR IIT JAM 2025

Lecture- 07 | **Linear Algebra**

Linear Combination, and Linear Span Part-01

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Recap

of previous lecture

1. Direct Sum of Subspaces
2. Examples of Some Direct sums

Topics

to be covered

1. Quotient space
2. Linear Span
3. Properties of Linear Span

Direct sum of subspaces

$$V = W_1 \oplus W_2$$

if

(i) W_1, W_2 are subspace of V

(ii) $V = W_1 + W_2$

(iii) $W_1 \cap W_2 = \{0\}$

Theorem →:

let $V(F)$ be a v.s, and W_1, W_2 be subspaces of V.

Then $V = W_1 \oplus W_2$

$\Longleftrightarrow$ $\forall\, x \in V$

$x = u + v$

for unique vectors $u \in W_1$ & $v \in W_2$

$\mathbb{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$

$W_1 = \{(x_1, 0) \mid x_1 \in \mathbb{R}\}$

$W_2 = \{(0, x_2) \mid x_2 \in \mathbb{R}$

$\mathbb{R}^2 = W_1 \oplus W_2$

$\forall (x_1, x_2) \in \mathbb{R}^2$

$(x_1, x_2) = (x_1, 0) + (0, x_2)$

$(1, 3) = (1, 0) + (0, 3)$

Definition →:

$$V = W_1 \oplus W_2 \oplus W_3$$

$\Longleftrightarrow$

(I) W_1, W_2, W_3 subspaces of V

(II) $V = W_1 + W_2 + W_3$

(i.e. $\forall x \in V,\ x = u_1 + u_2 + u_3;\ u_i \in W_i\ \forall i = 1, 2, 3$)

(III) $W_1 \cap W_2 = \{0\}$

$(W_1 + W_2) \cap W_3 = \{0\}$

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$$

(I) $W_1, W_2, \ldots W_n$ are subspaces of V

(II) $V = W_1 + W_2 + \cdots + W_n$

(III) $(W_1 + W_2 + \cdots + W_{i-1}) \cap W_i = \{0\}$

$\forall i = 2, 3, \ldots, n$

$W_1 \cap W_2 = \{0\}$

$(W_1 + W_2) \cap W_3 = \{0\}$

$(W_1 + W_2 + W_3) \cap W_4 = \{0\}$

Theorem :-

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_n$$

$(\Leftarrow)$ $\forall x \in V$

$\Rightarrow x = u_1 + u_2 + \cdots + u_n$

$u_i \in W_i \ \forall i = 1, 2, \ldots, n$

Ex

$$V = \mathbb{R}^3 = \{(a,b,c) \mid a,b,c \in \mathbb{R}\}$$

$$W_1 = \{(x,0,0) \mid x \in \mathbb{R}\}$$

$$W_2 = \{(0,y,0) \mid y \in \mathbb{R}\}$$

$$W_3 = \{(0,0,z) \mid z \in \mathbb{R}\}$$

(i) W_1, W_2, W_3 are subspaces of V

(ii) $\forall \ (a,b,c) \in V$

$$(a,b,c) = (a,0,0) + (0,b,0) + (0,0,c)$$

$$\Rightarrow V = W_1 + W_2 + W_3$$

(iii)

$$W_1 \cap W_2 = \{(0,0,0)\}$$

$$W_1 + W_2 = \{(x,y,0) \mid x,y \in \mathbb{R}\}$$

$$(W_1 + W_2) \cap W_3 = \{(0,0,0)\}$$

$$\boxed{V = W_1 \oplus W_2 \oplus W_3}$$

Home-work →:

$$V = \mathbb{R}^{2\times 2}(\mathbb{R})$$

$$V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{R} \right\}$$

$$W_1 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| \, a,b \in \mathbb{R} \right\}$$

$$W_2 = \left\{ \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \middle| \, c,d \in \mathbb{R} \right\}$$

is $V = W_1 \oplus W_2$ — ①

let $W_3 = \left\{ \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \middle| \, c \in \mathbb{R} \right\}$

$W_4 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \middle| \, d \in \mathbb{R} \right\}$

is $\boxed{V = W_1 \oplus W_3 \oplus W_4}$

??

Let $V(F)$ be any vector space, and let W be a subspace of V.

Define a relation "$\sim$" for $a, b \in V$

$$\boxed{a \sim b \iff a - b \in W}$$

then $\sim$ is an equivalence relation

Equivalence classes →:

$$\bar{x} = \{y \in V \mid y \sim x\}$$

$$= \{y \in V \mid y - x \in W\}$$

$$= \{y \in V \mid y - x = \omega;\ \omega \in W\}$$

$$y = x + \omega;\ \omega \in W$$

$$= \{x + \omega \mid \omega \in W\}$$

$$\boxed{\bar{x} = x + W}$$

Now

$$\overline{a} = \overline{b}$$

$$\Longleftrightarrow \quad a \sim b$$

$$\Longleftrightarrow \quad a - b \in W$$

i.e. $a + W = b + W$

$$\Longleftrightarrow \quad a - b \in W$$

Note: $\boxed{a + W = 0 + W = W}$

$$\Longleftrightarrow \quad \overline{a} = \overline{0}$$

$$\Longleftrightarrow \quad a \sim 0$$

$$\Longleftrightarrow \quad a - 0 \in W$$

$$\Longleftrightarrow \quad \boxed{a \in W}$$

$$\boxed{a + W = W \iff a \in W}$$

Quotient space

Let $V(F)$ be any vector space, and W be a subspace of V.
Then the set

$$\boxed{\frac{V}{W} = \{ \bar{x} = x + W \mid x \in V \} \text{ is a vector space over } F.}$$

w.r.t

1. Vector addition: $\bar{x} = x + W$; $\bar{y} = y + W$; $\boxed{\bar{x} + \bar{y} = (x+y) + W}$
2. Scalar Multiplication $\boxed{\alpha \bar{x} = \alpha x + W}$

Zero vector

$\bar{x} + \boxed{\bar{y}} = \bar{x}$

$\bar{x} = x + W$, $\boxed{\bar{0} = 0 + W = W}$

$\bar{x} + \bar{0} = (x+0) + W = x + W = \bar{x}$

$\bar{x} + \bar{0} = \bar{x} \quad \forall \bar{x} \in V/W$

Ex

$$V = \mathbb{R}^{2\times 3} = \left\{ \begin{pmatrix} a & b & c \\ e & f & g \end{pmatrix} \,\middle|\, a,b,c,e,f,g \in \mathbb{R} \right\}$$

$$W = \left\{ \begin{pmatrix} 0 & 0 & w \\ u & v & 0 \end{pmatrix} \,\middle|\, w,u,v \in \mathbb{R} \right\}$$

$$\frac{V}{W} = \left\{ x + W \,\middle|\, x \in V \right\}$$

$$= \left\{ \begin{pmatrix} a & b & c \\ e & f & g \end{pmatrix} + W \,\middle|\, a,b,c,e,f,g \in \mathbb{R} \right\}$$

$$\begin{pmatrix} a & b & c \\ e & f & g \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ 0 & 0 & g \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 0 & c \\ e & f & 0 \end{pmatrix}}_{\in W}$$

$$\begin{pmatrix} a & b & c \\ e & f & g \end{pmatrix} + W = \begin{pmatrix} a & b & 0 \\ 0 & 0 & g \end{pmatrix} + \underbrace{\underbrace{\begin{pmatrix} 0 & 0 & c \\ e & f & 0 \end{pmatrix}}_{\in W} + W}$$

$$\begin{pmatrix} a & b & c \\ e & f & g \end{pmatrix} + W = \begin{pmatrix} a & b & 0 \\ 0 & 0 & g \end{pmatrix} + W$$

$$\frac{V}{W} = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & 0 & g \end{pmatrix} + W \,\middle|\, a, b \in \mathbb{R} \right\}$$

Ex $V = \mathbb{R}^2 = \{(a,b) \mid a,b \in \mathbb{R}\}$

$W_1 = \{(a,0) \mid a \in \mathbb{R}\}$

$\frac{V}{W_1} = \{x + W_1 \mid x \in V\}$

$= \{(a,b) + W_1 \mid a,b \in \mathbb{R}\}$

$= \{(0,b) + \underbrace{(a,0) + W_1} \mid a,b \in \mathbb{R}\}$

$\frac{V}{W_1} = \{(0,b) + W_1 \mid b \in \mathbb{R}\}$

$W_2 = \{\underbrace{(0,b)} \mid b \in \mathbb{R}\}$

$\frac{V}{W_2} = \{x + W_2 \mid x \in V\}$

$= \{(a,b) + W_2 \mid a,b \in \mathbb{R}\}$

$= \{(a,0) + \underbrace{(0,b) + W_2} \mid a,b \in \mathbb{R}\}$

$= \{(a,0) + W_2 \mid a \in \mathbb{R}\}$

Ex $V = P_3(\mathbb{R}) = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\}$

$W_1 = \{p(x) \in V \mid p(x) = p(-x)\}$

$W_2 = \{p(x) \in V \mid p(-x) = -p(x)\}$

Find $\frac{V}{W_1}$ & $\frac{V}{W_2}$

Sol^n

$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \in W_1$

$\Rightarrow p(-x) = p(x)$

$\Rightarrow a_0 - a_1 x + a_2 x^2 - a_3 x^3 = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$\Rightarrow a_0 = a_0, \boxed{-a_1 = a_1}, a_2 = a_2, \boxed{-a_3 = a_3}$

$-a_1 = a_1$

$\Rightarrow 2a_1 = 0$

$\Rightarrow a_1 = 0$

$-a_3 = -a_3$

$\Rightarrow 2a_3 = 0$

$\Rightarrow a_3 = 0$

$W_1 = \{ \underbrace{a_0 + a_2 x^2} \mid a_0, a_2 \in \mathbb{R} \}$

IIrly

$W_2 = \{ \underbrace{a_1 x + a_3 x^3} \mid a_1, a_3 \in \mathbb{R} \}$

$$\frac{V}{W_1} = \{ p(x) + W_1 \mid p(x) \in V \}$$

$$= \{ \underbrace{a_0 + a_1 x + a_2 x^2 + a_3 x^3} + W_1 \mid a_0, a_1, a_2, a_3 \in \mathbb{R} \}$$

$$= \{ (a_1 x + a_3 x^3) + \underbrace{(a_0 + a_2 x^2) + W_1} \mid a_0, a_1, a_2, a_3 \in \mathbb{R} \}$$

$$= \{ a_1 x + a_3 x^3 + W_1 \mid a_1, a_3 \in \mathbb{R} \}$$

H.W

IIrly

$$\frac{V}{W_2} = \{ a_0 + a_2 x^2 + W_2 \mid a_0, a_2 \in \mathbb{R} \}$$

linear combination →:

let $V(F)$ be a vector space,

and
$$\boxed{\begin{array}{l} x_1, x_2, \ldots, x_k \in V \\ \alpha_1, \alpha_2, \ldots \alpha_k \in F \end{array}}$$

then
$$\boxed{\alpha_1} x_1 + \boxed{\alpha_2} x_2 + \ldots + \boxed{\alpha_k} x_k$$

is called a linear combination of $x_1, x_2, \ldots, x_k$.

Ex: $V = \mathbb{R}^3, F = \mathbb{R}$

$x_1 = (1,0,0)$

$x_2 = (0,1,1)$

$x_3 = (0,0,1)$

$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$

$= \alpha_1(1,0,0) + \alpha_2(0,1,1) + \alpha_3(0,0,1)$

$= (\alpha_1,0,0) + (0,\alpha_2,\alpha_2) + (0,0,\alpha_3)$

$= (\alpha_1, \alpha_2, \alpha_2 + \alpha_3)$

$$\boxed{\alpha_1 = 1,\ \alpha_2 = 0,\ \alpha_3 = 1}$$

$$\boxed{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3} = (\alpha_1, \alpha_2, \alpha_2 + \alpha_3)$$

$$= (1, 0, 1)$$

$$\boxed{\alpha_1 = 0,\ \alpha_2 = 1,\ \alpha_3 = 1}$$

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$$

$$= (0, 1, 2)$$

2 Mins Summary

1. Quotient space
2. Linear Span
3. Properties of Linear Span

THANK YOU