Swayam

Introduction to Abstract and Linear Algebra

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Week 1: Basic Set Theory

Lecture 01:- Set Theory

Topics to be Covered: -

- Definition of Set,
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- Empty set,
- universal Set,
- Subset.

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Week – 03: Rings and Polynomial Rings

Week – 04: Field and Finite Field

Week – 05: Matrices and Determinants

Week – 06: Vector Spaces over Field

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Lecture 31:- Rank of a Matrix

Topics to be Covered: -

- Definition of Rank of a Matrix,
- Maximum Order of Non-Zero Minor, Elementary Row and Column Operations,
- Row reduced Echelon Form, Row Equivalence

Matrix is a rectangular arrangement of elements

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} ; \quad a_{ij} \in \mathbb{R}$$

Rank of $A = r$ means we have minor of size r whose Determinant $\neq 0$ (atleast 1)

$| \quad |_r \neq 0$ and every minor of order $> r$ must be 0

eg

$$A = \begin{pmatrix} 2 & 3 & -1 & 1 \\ 3 & 0 & 4 & 2 \\ 6 & 9 & -3 & 3 \end{pmatrix}_{3\times 4}$$

★ Rank of $A \leq \min\{m,n\}$

So here Rank of $A \leq 3$

So,

$$\begin{vmatrix} 2 & 3 & -1 \\ 3 & 0 & 4 \\ 6 & 9 & -3 \end{vmatrix} = 2[0-36] - 3[-9-24] - 1[27-0]$$
$$= -72 + 99 - 27 = 0$$

Similarly,

$$\begin{vmatrix} 3 & -1 & 1 \\ 0 & 4 & 2 \\ 9 & -3 & 3 \end{vmatrix} = 0$$

So all the minors of order 3 is 0.

Rank of 0 Matrix is 0

So, Rank < 3

let

$$\begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix} = -9 \neq 0$$

So Rank of $A = 0$.

If $A=(a_{ij})_{n\times n}$ and $\text{Rank}(A)=n$ then $|A|\neq 0$

If $|A| \neq$ then A is Non-Singular Matrix, and if A is non-singular then it's Inverse exists.

$$\Rightarrow A\cdot B = B\cdot A = I;\ \text{where } A^{-1}=B$$

$$\Rightarrow A\cdot(\text{adj}\,A) = (\text{adj}\,A)\cdot A = |A|\cdot I_n$$

$$\Rightarrow \boxed{A^{-1} = \frac{(\text{adj}\,A)}{|A|}} = B \quad \text{where } |A|\neq 0$$

eg $A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{pmatrix} \Rightarrow |A| = 2$

Computing it's Adjoint

$$\text{adj}\,A = \begin{pmatrix} \begin{vmatrix} 4 & 5 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & -4 \\ -2 & 2 & -2 \\ 1 & -3 & 4 \end{pmatrix}$$

$$\therefore \boxed{A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 3 & -4 \\ -2 & 2 & -2 \\ 1 & -3 & 4 \end{pmatrix}}$$

Elementary Row Operations (Same for Columns) to find Rank:-

- $R_i \longleftrightarrow R_j$
- $R_i \rightarrow R_i + R_j$
- $R_i \rightarrow k*R_i$, where k is any constant ($\neq 0$)

Lecture 32:- Rank of a Matrix (contd....)

Topics to be Covered: -

- Elementary Matrices,
- Row and Columns operations is same as multiplying with Elementary Matrices,
- Rank of a matrix is same as rank of the Row Reduced or Columns Reduced Matrix,
- Normal Form.

- **Row Equivalent** :- An m x n Matrix B is row equivalent to m x n matrix A iff B = PA for some non-singular matrix of order m

- **Column Equivalent** :- An m x n Matrix B is column equivalent to m x n matrix A iff B = AQ for some non-singular matrix of order n

- Equivalent Matrix :- An m x n Matrix B is equivalent to an m x n matrix iff B = PAQ where P, Q are non-singular matrices.
  In this case
  Rank(A) = Rank(B)
- If Rank(A) = r, then we can find non-singular matrices P, Q such that PAQ = $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, where I is an Identity martix

- The increase of non-singular matrix can be calculated by using elementary matrices. Let $|A| \neq 0$ and A is of order n. Then A is an equivalent to I_n.
- So for suitable elementary matrices E,
  $E_n\ E_{n-1} \ldots\ldots E_2\ E_1\ A = I_n$
  $\Rightarrow\ E_n\ E_{n-1} \ldots\ldots E_2\ E_1\ I_n = A^{-1}$
- Therefore, if a (finite) sequence of elementary row operation applied successfully on A reduces A to I_n, the same sequence of operation applied to I_n with reduce I_n to A^{-1}.
- This gives us technique to for finding A^{-1} described below by an example

Let

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 4 \\ 3 & 3 & 7 \end{pmatrix}$$

Consider,

$$(A|I_3) = \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 4 & 4 & 0 & 1 & 0 \\ 3 & 3 & 7 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \to \frac{1}{2}R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1/2 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & -1/2 & 0 \\ 0 & 1 & 0 & -1 & 1/2 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & -1/2 & -2 \\ 0 & 1 & 0 & -1 & 1/2 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right)$$

$$= (I_3 \,|\, A^{-1}) \Rightarrow A^{-1} = \begin{pmatrix} 8 & -1/2 & -2 \\ -1 & 1/2 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

Find the Rank of $A_{4 \times 4}$ Matrix?

$$A = \begin{pmatrix} 2 & 0 & 4 & 2 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

Applying $R_1 \to \frac{1}{2} R_1$, $A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix}$

Applying $R_2 \to R_2 - 3R_1$, and $R_3 \to R_3 - 5R_1$, $A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 5 \end{pmatrix}$

Applying $R_2 \Leftrightarrow R_4$ $A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 3 & 2 & 5 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \end{pmatrix}$

Applying $R_4 \to R_4 - R_3$ $A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 3 & 2 & 5 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Now A is in Row reduced Echelon Form. And we know that Rank of A and its Row Reduced Echelon Form is Same.

$\therefore \boxed{\rho(A) = 3}$ $A = \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 3 & 2 & 5 \\ 0 & 2 & 0 & 2 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$ Det $\neq 0$

★ Fully Reduced Normal Form:

If a matrix is in **Fully Reduced Normal Form**, then it is in

- Row Reduced Echelon Form
- Column Reduced Echelon From

Such that,

- No zero rows is followed by a non-zero row

- No zero column is followed by a non-zero column
- Leading 1 in each row is the only non-zero element in that row
- Leading 1 in each column is the only non-zero element in that column
- Leading 1 in k^{th} row is the leading 1 in k^{th} column

Eg Find the Fully Reduced Normal Form of $A_{4\times 5}$ Matrix?

$$A = \begin{pmatrix} 0 & 0 & 1 & 2 & 1 \\ 1 & 3 & 1 & 0 & 3 \\ 2 & 6 & 4 & 2 & 8 \\ 3 & 9 & 4 & 2 & 10 \end{pmatrix}$$

$$R_1 \longleftrightarrow R_2 \quad A = \begin{pmatrix} 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 2 & 6 & 4 & 2 & 8 \\ 3 & 9 & 4 & 2 & 10 \end{pmatrix}$$

$$\xrightarrow[R_4 \to R_4 - 4R_1]{R_3 \to R_3 - 2R_1} A = \begin{pmatrix} 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[\substack{R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - R_2}]{R_1 \to R_1 - R_2} A = \begin{pmatrix} 1 & 3 & 0 & -2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 \to \frac{-1}{2} R_3} A = \begin{pmatrix} 1 & 3 & 0 & -2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(Row Reduced Echelon Form)

$$\xrightarrow[C_5 \to C_5 - 2C_1]{C_2 \to C_2 - 3C_1} A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{C_5 \to C_5 - C_3} A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C_2 \longleftrightarrow C_3 \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C_3 \longleftrightarrow C_4 \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \left[\begin{array}{c|c} I_3 & 0 \\ \hline 0 & 0 \end{array}\right]$$

(Row Reduced Normal Form)

eg **Reduce the Matrix $A_{3\times 3}$ to Fully Reduced Normal Form and find non-singular matrices P, Q such that PAQ is the Fully Reduced Normal Form?**

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{C_3 \to C_3 - C_1} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) = R(Say)$$

$\Rightarrow$ Rank $(A) = 2 \neq 3 \Rightarrow A$ is Singular $(|A| = 0)$

So,

$$R = (C_3 - C_1)(R_3 - R_2)(R_3 - R_1)(R_2 - 2R_1)(R_1 \leftrightarrow R_2) A$$

$$= \underbrace{E_{32}(-1)\, E_{31}(-1)\, E_{21}(-2)\, E_{12}}_{P}\, A\, \underbrace{\{E_{31}(-1)\}^T}_{Q}$$

$= PAQ$

Now $E_{32}(-1) \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

Similarly $E_{31}(-1) \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

and $E_{21}(-2) \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $E_{12} \Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Also $Q = \{E_{31}(-1)\}^T = E_{13}(-1) \Rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

So $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 1 \\ -1 & 1 & 1 \end{pmatrix}$ (Check Calculation)

and $Q = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Therefore $\boxed{R = PAQ}$ where $P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ & $Q = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are non-singular matrices.

H.W **Reduce the Matrix $A_{3\times 3}$ to Fully Reduced Normal Form and find non-singular matrices P, Q such that PAQ is the Fully Reduced Normal Form?**

$$A = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 4 & 6 \\ 3 & 0 & 7 & 9 \end{pmatrix}$$

Sol^n

$R = (I_3|0)$, $P = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(Check)

Lecture 33:- System of Linear Equations

Topics to be Covered: -

- augmented matrix, rank of augmented matrix,
- consistency, inconsistencies, no solution, unique solution and infinitely many solution

$$\left.\begin{aligned} x_1+x_2&=4\\ x_2-x_3&=1\\ 2x_1+x_2+4x_3&=7\end{aligned}\right\}$$ Can be written as

$$\begin{pmatrix}1&1&0\\0&1&-1\\2&1&4\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}4\\1\\7\end{pmatrix}$$

$(x_1 = 3, x_2 = 1, x_3 = 0)$

It is of the form $AX = B$

where $A = \begin{pmatrix}1&1&0\\0&1&-1\\2&1&4\end{pmatrix}$, $X = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ & $B = \begin{pmatrix}4\\1\\7\end{pmatrix}$

As A is a Square Matrix with $|A| = 3 \neq 0$

So A^{-1} exists and

$$A^{-1} = \begin{pmatrix}\frac{5}{3}&-\frac{4}{3}&-\frac{1}{3}\\-\frac{2}{3}&\frac{4}{3}&\frac{1}{3}\\-\frac{2}{3}&\frac{1}{3}&\frac{1}{3}\end{pmatrix}$$

Augmented Matrix $\bar{A} = (A|B) = \left(\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & -4 & 1 \\ 2 & 1 & 1 & 7 \end{array}\right)$

Reducing $\bar{A}$ to Row Reduced Echelon Form

$$R_3 \to R_3 - 2R_1 \quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & -4 & 1 \\ 0 & -1 & 1 & -1 \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 - R_2]{R_3 \to R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & 0 \end{array}\right) \xrightarrow{R_3 \to \frac{1}{3}R_3} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 + R_3} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1 \to R_1 - R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Here $\rho(A) = \rho(A|B) = 3$

From here we can say $x_1 = 3$, $x_2 = 1$ & $x_3 = 0$

Theorem :- The necessary & sufficient condition for a non-homogeneous system $AX = B$ to be consistent (that is, it has a solution) is

$$\boxed{\text{Rank}(A) = \text{Rank}(A|B)}$$

<table>
<tr><th>Consistent Solution</th><th>Inconsistent Solution</th></tr>
<tr><td>* Rank(A) = Rank(A : B)</td><td rowspan="3">No solution when
Rank(A) ≠ Rank(A : B)</td></tr>
<tr><td>(i) Unique Solution if
ⓐ $m = n$ ⓑ $m > n$
AND
$\text{Rank}(A) = \text{Rank}(A : B) = n$</td></tr>
<tr><td>(ii) Infinite Solution if
ⓐ $m = n$ & $R(A) = R(\bar{A}) < n$
ⓑ $m < n$ & $R(A) = R(\bar{A}) \leq m < n$
ⓒ $m > n$ & $R(A) = R(\bar{A}) < n$</td></tr>
</table>

<u>H.W</u> Consider the System of Equation

$$x_1 + 2x_2 - x_3 = 10$$
$$-x_1 + x_2 + 2x_3 = 2$$
$$2x_1 + x_2 - 3x_3 = 2$$

So

$$\underbrace{\begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & 1 & -3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 10 \\ 2 \\ 2 \end{pmatrix}}_{B}$$

$$\therefore \bar{A} = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 10 \\ -1 & 1 & 2 & 2 \\ 2 & 1 & -3 & 2 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 10 \\ 0 & 3 & 1 & 12 \\ 0 & -3 & 1 & -18 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 10 \\ 0 & 3 & 1 & 12 \\ 0 & 0 & 0 & -6 \end{array}\right)$$

$$\xrightarrow{R_2 \to \frac{1}{3} R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 10 \\ 0 & 1 & 1/3 & 4 \\ 0 & 0 & 0 & -6 \end{array}\right)$$

So $\rho(A) = 2$ & $\rho(A : B) = 3$

Hence System has no solution.

H.W Consider the system

$$x + 2y + z = 1$$
$$3x + y + 2z = 3$$
$$x + 7y + 2z = 1$$

So $\bar{A} = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & 1 & 2 & 3 \\ 1 & 7 & 2 & 1 \end{array}\right)$

$\sim \left(\begin{array}{ccc|c} 1 & 0 & 3/5 & 1 \\ 0 & 1 & 1/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

The equivalent System is

$\left.\begin{array}{l} x_1 + \frac{3}{5}z = 1 \\ y + \frac{1}{5}z = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x = 1 - \frac{3}{5}z \\ y = -\frac{1}{5}z \end{array}$

for any $z \in \mathbb{R}$ is a Solution

Hence the System has Infinite Solution.

H.W Determine the Condition for which the following System has

(i) Only one Solution

(ii) No Solution

(iii) Infinite Solution

$$x + y + z = 1$$
$$x + 2y - z = b$$
$$5x + 7y + az = b^2$$

Sol^n (i) $a \neq 1$ (ii) $a = 1, b \neq -1, 3$ (iii) $a = 1, b = -1, 3$

Lecture 34:- Row Rank and Column Rank

Topics to Be Covered: -

- Row space and Column space of a matrix, Row Rank, Column Rank

$A = (a_{ij})_{m \times n}$ $a_{ij} \in F$ Where $(F, +, \cdot)$ is a Field

$$= \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \\ \vdots \\ \to R_m \end{array}$$
$$\begin{array}{llll} \downarrow & \downarrow & & \downarrow \\ C_1 & C_2 & \cdots & C_n \end{array}$$

So i^{th} Row is

$$R_i = (a_{i1} \; a_{i2} \; \cdots \; a_{in})$$
$i = 1, 2, \ldots, m$

and $R_i \in F^n = F \times F \times \cdots F$

Now if we take this collection of Rows

$$R = \{R_1, R_2, \ldots, R_m\} \subseteq F^n$$

Now if we consider the Space which is generated by the vectors (called Linear Span)

$$L(R) = L(\{R_1, R_2, \ldots, R_m\})$$

It is a Sub-space

$$= \{\alpha_1 R_1 + \alpha_2 R_2 + \ldots + \alpha_m R_m\} \; \alpha \in F$$

* $L(R)$ is a Sub-Space of F^n where $L(R)$ is the set of all vectors of F^n which are Linear Combination of Row Vectors
* $L(R)$ is called Row Space of A.

* Similarly, $L(C) = L(\{C_1, C_2, \ldots, C_n\})$ is a Sub-Space of F^m and $L(C)$ is called the Column Space of A.

Dimension

* Denote, $R(A) = L(R) \subseteq F^n \Rightarrow \text{Dim}(R(A)) \leq n$
* Define, Row Rank of $A = \text{Dim}(R(A))$
* Denote $C(A) = L(C) \subseteq F^m \Rightarrow \dim(C(A)) \leq m$
* Define, Column Rank of $A = \dim(C(A))$

~~eg~~

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \end{array}$

$R(A) = \{R_1, R_2, R_3\} = \{(1,0,0), (0,1,0), (0,0,1)\}$
$= R^3$

$\text{Dim}(R(A)) = 3$

~~eg~~ Let $A = \begin{pmatrix} 2 & 1 & 4 & 3 \\ 3 & 2 & 6 & 9 \\ 1 & 1 & 2 & 6 \end{pmatrix}$

So $R(A) = \{(2,1,4,3), (3,2,6,9), (1,1,2,6)\}$

Applying E. Row Op^n on A

$A \xrightarrow{R_1 \leftrightarrow R_3} \xrightarrow[R_3 \rightarrow R_3 - 2R_1]{R_2 \rightarrow R_2 - 3R_1} \xrightarrow{R_3 \rightarrow R_3 - R_2} \begin{pmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & 0 & -9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\xrightarrow[R_2 \rightarrow (-1)R_2]{R_1 \rightarrow R_1 + R_2} \begin{pmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Now $R(A) = \{(1,0,2,-3), (0,1,0,9)\}$

$\text{Dim}(R(A)) = 2$

$\Rightarrow$ Row Rank of $A = 2$

To get the column rank of A, consider A^T and apply elementary row operation on A^T.

$$\therefore A^T = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ 4 & 6 & 2 \\ 3 & 9 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{C}(A) = \{(1,0,-1), (0,1,1)\}$$

$\Rightarrow$ Dim $(\mathcal{C}(A)) = 2$ $\Rightarrow$ Column Rank of $A = 2$

Some Results on $A = (a_{ij})_{m \times n}$:-

1. Row Rank of $A \leq n$
2. Column Rank of $A \leq m$
3. Row Rank = Column Rank = Rank of A
4. $\text{Rank}(AB) \leq \min\{\text{Rank}(A), \text{Rank}(B)\}$
5. $\text{Rank}(A+B) \leq \text{Rank}(A) + \text{Rank}(B)$

Lecture 35:- Eigen Value of a Matrix

Topics to be Covered: -

- Characteristic equation,
- Cayley-Hamilton theorem,
- Eigen Value of a Matrix, Product of Eigen Values

Characteristics Equation

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_{n\times n}$ ← Square Matrix

Now $|A - x I_{n\times n}| = \begin{vmatrix} a_{11}-x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-x & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-x \end{vmatrix} = \Psi_A(x)$

$= C_0 x^n + C_1 x^{n-1} + \dots + C_n$

(A Polynomial of degree n)

* $\Psi_A(x)$ is Called the Characteristics Polynomial of A.
* $C_r = (-1)^{n-r}$ [Sum of Principle Minor of A of order n]

So, $C_0 = (-1)^n$

$C_1 = (-1)^{n-1}(a_{11} + a_{22} + \dots + a_{nn})$

$\boxed{C_1 = (-1)^{n-1}(\text{trace}(A))}$

* $C_n = |A|$

eg Let $A = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$

$\Psi_A(x) = \det(A - x I_{n\times n}) = \begin{vmatrix} 2-x & 1 \\ 3 & 5-x \end{vmatrix} = 0$

$\Rightarrow (2-x)(5-x) - 3 = 0 \Rightarrow x^2 - (5+2)x + 10 - 3 = 0$

$\Rightarrow x^2 - 7x + 7 = 0 \quad (\lambda^2 - (\text{trace}(A))\lambda + |A| = 0)$

So, $\Psi_A(x) = 0$ is called the Characteristics Equation of A.

★ Cayley-Hamilton Theorem

Every Square Matrix satisfies its own characteristic's equation

$$\Psi_A(A) = 0$$

★ **From Characteristics equation we can get the Inverse of a matrix**

$$A^2 - 7A + 7I_{2\times2} = 0$$

$$\Rightarrow A(A - 7I_{2\times2}) + 7I_{2\times2} = 0 \Rightarrow A(A - 7I_2) = -7I$$

$$\Rightarrow A\left(\frac{-1}{7}(A - 7I_2)\right) = I$$

As we know that $AB = I$ then $A^{-1} = B$

$$\Rightarrow A^{-1} = \frac{-1}{7}(A - 7I_2) = \frac{-1}{7}\begin{pmatrix} 2-7 & 1 \\ 3 & 5-7 \end{pmatrix} = \frac{-1}{7}\begin{pmatrix} -5 & 1 \\ 3 & -2 \end{pmatrix}$$

★ **Using Cayley-Hamilton Theorem we can also find A^n.**

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let's calculate A^{50}.

$$\Psi_A(x) = x^2 - 2x + 1$$

By Cayley-Hamilton Theorem

$$A^2 - 2A + I = 0 \Rightarrow A^2 - A = A - I$$

$$\Rightarrow A^3 - A^2 = A^2 - A = A - I$$

$$\Rightarrow A^4 - A^3 = A^3 - A^2 = A - I$$

$$\Rightarrow A^5 - A^4 = A^4 - A^3 = A - I$$

$$\dots \quad \dots \quad \dots$$

$$\Rightarrow A^{50} - A^{49} = A^{49} - A^{48} = A - I$$

Adding

$$\Rightarrow A^{50} - A = 49(A - I)$$

$$\Rightarrow A^{50} = 49(A - I) + A = 50A - 49I_2$$

$$\Rightarrow A^{50} = \begin{pmatrix} 50 & 50 \\ 0 & 50 \end{pmatrix} - \begin{pmatrix} 49 & 0 \\ 0 & 49 \end{pmatrix} = \begin{pmatrix} 1 & 50 \\ 0 & 1 \end{pmatrix}$$

Eigen Value of a Matrix

* Let $A = (a_{ij})_{n\times n}$. Then the roots of $\psi_A(x)$ are the Eigen Value of A.
* If λ is an E. Value of A then we say λ has Algebraic Multiplicity r.
* If $\psi_A(x) = (x-\lambda)^r \phi(x)$ where $\phi(x) \neq 0$ we say λ an r-fold E. Value of A.

eg Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

It is an example where matrix is from Real Field and E. Value is from Complex Field.

$$\psi_A(x) = \begin{vmatrix} 0-x & 1 \\ 1 & 0-x \end{vmatrix} = x^2 + 1$$

As $\psi_A(x) = 0 \Rightarrow x^2 + 1 = 0 \Rightarrow x^2 = \pm\sqrt{-1} = \pm i$

So A has Complex Eigen Values.

NOTES:-

1. If $A_{n\times n}$ is Real Symmetric Matrix then Eigen Value of A are all Real Number.
2. If $\lambda_1, \lambda_2, \dots, \lambda_n$ are E. Value of A, then $|A| = \lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_n$
   If $|A| = 0 \Rightarrow \lambda_i = 0$ for some $i = 1, 2, \dots, n$
3. If A is a Diagonal Matrix, then $\psi_A(x) = (a_{11}-x)(a_{22}-x)\dots(a_{nn}-x)$
   as $\psi_A(x) = 0 \Rightarrow x = a_{11}, a_{22}, \dots, a_{nn}$
   $\therefore$ E. Value of Diagonal Matrix are Diagonal Elements.

Test Questions

2019

1. Rank of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ is ______.

A. 3
B. 1
C. 2
D. 0

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

So $\rho = 2$ $\therefore$ C is the correct answer

2. Find the value of k such that rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & k \end{bmatrix}$ is 1.

A. 3
B. 6
C. 2
D. 9

$$\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & k-9 \end{bmatrix}$$

for ρ to be 1

$k - 9 = 0 \Rightarrow \boxed{k = 9}$

$\therefore$ D is the Correct answer

3] Let A be a non-singular matrix of order 10×11 and B be another matrix of order 11×12. If the rank of B is 5 then the rank of AB is equal to______.

A. 10
B. 12
C. 5
D. 11

$|A_{10 \times 11}| \neq 0 \Rightarrow \rho(A) = 10$ & $\rho(B_{11 \times 12}) = 5$

$\rho(AB) \leq \text{rank}\{\rho(A), \rho(B)\}$

So $\rho(AB) \leq 5$ $\therefore$ C is the correct answer.

4] Column rank of the matrix $A \to \begin{bmatrix} 2 & 1 & 4 & 3 \\ 3 & 2 & 6 & 9 \\ 1 & 1 & 2 & 6 \end{bmatrix}$ is ______.

A. 2
B. 1
C. 3
D. 0

$$\xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 2 & 6 \\ 3 & 2 & 6 & 9 \\ 2 & 1 & 4 & 3 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & 0 & -9 \\ 0 & -1 & 0 & -9 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ A is the correct answer

$\therefore \rho(A) = 2$, as Row Rank = Column Rank

$\therefore$ Col. Rank $(A) = 2$

5] Let A be a $m \times n$ matrix over real field $\mathbb{R}$. Let us consider the following statements

(i) The solutions of homogeneous system $AX = 0$ form a subspace of $\mathbb{R}^n$.

(ii) The solutions of non-homogeneous system $AX = Y$ form a subspace of $\mathbb{R}^n$.

Select the correct option from below.

A. Only (i) is true.

B. Only (ii) is true.

C. (i) and (ii) both are true.

D. (i) and (ii) both are false.

A is the Correct answer

6] The following system of equations has ______.

$$x + 3y + z = 0$$
$$2x - y + z = 0$$

A. No solution

B. Unique solution

C. Infinite Solutions

D. Finite but more than one solution

$$\rho(A:B) = \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 2 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -7 & -1 & 0 \end{array}\right]$$

$$\underset{2\times3}{\begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 1 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = 0$$

Here $m < n$ and

Here $m < n$ & $\underbrace{R(A)}_{2} = \underbrace{R(\bar{A})}_{2} \lesssim 2 < 3$

$\therefore$ Infinite Solution

$\therefore$ C is the Correct answer.

7] The following system of equations has ______.

$$x + 2y + z - 3w = 1$$
$$2x + 4y + 3z + w = 3$$
$$3x + 6y + 4z - 2w = 5$$

A. No solution

B. Unique solution

C. Infinite Solutions

D. Finite but more than one solution

$$\underset{3\times4}{\begin{bmatrix} 1 & 2 & 1 & -3 \\ 2 & 4 & 3 & 1 \\ 3 & 6 & 4 & -2 \end{bmatrix}} \underset{4\times1}{\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}} = \underset{3\times1}{\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}}$$

$$[A:B] = \left[\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 1 \\ 2 & 4 & 3 & 1 & 3 \\ 3 & 6 & 4 & -2 & 5 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 1 \\ 0 & 0 & 1 & 7 & 1 \\ 0 & 0 & 1 & 7 & 2 \end{array}\right]$$

$\therefore$ A is the Correct answer.

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 1 \\ 0 & 0 & 1 & 7 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Here $\rho(A) \neq \rho(A:B)$. Therefore No solution

8] Solution of the following system of equations is ______.

$$x + 2y + 3z = 1$$
$$x + 3y + 6z = 3$$
$$2x + 6y + 13z = 5$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 6 \\ 2 & 6 & 13 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$(x = -6, y = 5, z = -1)$

A. $x = 6, y = 1, z = 3$
B. $x = 1, y = 3, z = 0$
C. $x = -6, y = 5, z = -1$
D. $x = 0, y = 5, z = -1$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 3 & 6 & 3 \\ 2 & 6 & 13 & 5 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 3 & 6 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & -3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[R_2 \to R_2 - 3R_3]{R_1 \to R_1 + 3R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$\therefore x = -6,\ y = 5,\ \&\ z = -1$ $\quad\therefore$ C is the Correct answer.

9] Let A be a matrix of order 3×3 and 2,-3,0 be the eigen value of A. Then determinant of A is equal to______.

A. -1
B. 2
C. -3
D. 0

$|A|$ = Product of it's Eigen Values

$= 2 \times (-3) \times 0 = 0$

$\therefore$ D is the Correct answer.

10] Let $A = \begin{bmatrix} 1 & -1 & -0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$ then ______(Note: $\mathbf{0}$ is the zero matrix of order 3×3 and I is the identity matrix of order 3×3).

A. $A^3 - 2A^2 - A + I = \mathbf{0}_{3,3}$
B. $A^3 - A^2 - A + I = \mathbf{0}_{3,3}$
C. $A^3 - A^2 - 3A + I = \mathbf{0}_{3,3}$
D. $A^3 - A^2 - A + 3I = \mathbf{0}_{3,3}$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 & 0 \\ 1 & 2-\lambda & -1 \\ 3 & 2 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(2-\lambda)(-2-\lambda)+2\right] + 1\left[-2-\lambda+3\right] = 0$$
$$\Rightarrow (1-\lambda)\left[\lambda^2 - 2\right] + \left[1-\lambda\right] = 0$$
$$\Rightarrow \lambda^2 - 2 - \lambda^3 + 2\lambda + 1 - \lambda = 0$$
$$\Rightarrow -\lambda^3 + \lambda^2 + \lambda - 1 = 0$$
$$\boxed{\Rightarrow \lambda^3 - \lambda^2 - \lambda + 1 = 0}$$

$\therefore$ B is the correct answer.

2023

1]

Rank of the matrix $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{bmatrix}$ is ______.

(a) 3

(b) 1

(c) 2

(d) 0

C is the Correct answer

2]

Let A be a matrix of order $m \times n$ over the field F. Then rank of A is less or equal to______.

(a) $\max\{m, n\}$.

(b) $\min\{m, n\}$.

(c) m.

(d) n.

b is the Correct answer

3]

The following system of equations has ______.

$$\begin{aligned} x + y - z &= 0 \\ 2x - 3y + z &= 0 \\ x - 4y + 2z &= 0 \end{aligned}$$

(a) No solution

(b) Unique solution

(c) Infinite Solutions

(d) Finite but more than one solution

C is the Correct answer

4] Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

then _____(Note: $\mathbf{0}_{3,3}$ is a zero matrix of order 3×3).

(a) $A^3 - 2A^2 - A + 1 = \mathbf{0}_{3,3}$

(b) $A^3 - 5A^2 + 5A - 1 = \mathbf{0}_{3,3}$

(c) $A^3 - A^2 - 3A + 1 = \mathbf{0}_{3,3}$

(d) $A^3 - A^2 - A + 3 = \mathbf{0}_{3,3}$

b is the correct answer.

5] Let A be a matrix of order 3×3 and 3, 4, 5 be the eigenvalues of A. Then determinant of A is equal to _____.

(a) 60

(b) 12

(c) 3

(d) 0

a is the correct answer

6] Let A be a matrix of order 3×3 and 1, -4, 5 be the eigen value of A. Then trace of A is equal to_____.

(a) -20

(b) 20

(c) 3

(d) 2

d is the correct answer

7] Find the value of k such that rank of the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & k \\ 5 & 7 & 1 & k^2 \end{bmatrix}$ is 2.

(a) 3
(b) 0
(c) 5
(d) 6

a is the correct answer

8] The following system of equations has ______.

$$2x - y = 1$$
$$6x - 3y = 12$$

(a) Infinite Solutions
(b) Unique solution
(c) No solution
(d) Finite but more than one solution

c is the correct answer

9] If $M = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ satisfies the matrix equation $M^2 - \lambda M + 2I = O$, then the value of λ is ....

(a) 1
(b) -1
(c) 2
(d) -2

a is the correct answer

10] Given that, the rank of matrix $A_{7\times5}$ is 5 and the rank of matrix $B_{5\times7}$ is 3, then the rank of AB is at most ......

(a) 1

(b) 3

(c) 5

(d) 7

b is the correct answer

2024

1] Let A and B be two matrices over the same field F such that $A+B$ is defined. Let us consider the following statements

(i) $rank(A+B) \leq rank(A) + rank(B)$

(ii) $rank(A+B) \geq rank(A) + rank(B)$

(a) (i) and (ii) both are false.

(b) Only (ii) is true.

(c) Only (i) is true.

(d) (i) and (ii) both are true .

c is the correct answer

2] The rank of the matrix $\begin{bmatrix} 1 & 4 & 8 & 7 \\ 2 & 10 & 22 & 20 \\ 0 & 4 & 12 & 12 \end{bmatrix}$ is ....

(a) 1

(b) 2

(c) 3

(d) None of these

b is the correct answer

3] Find the value of a and b so that the rank of the matrix

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 2 & a & b \end{bmatrix}$$

is 2.

(a) $a = 4, b = 2$

(b) $a = 2, b = 4$

(c) $a = 1, b = 2$

(d) $a = 2, b = 1$

b is the Correct answer

4] If A and B are two 3×3 matrices and rank(AB)=1, then rank(BA) cannot be

(a) 0

(b) 2

(c) 3

(d) 1

c is the correct answer

5] A system of linear equations is said to be consistent if

(a) It has at least one solution.

(b) It has no solution.

(c) It has exactly one solution.

(d) It has infinitely many solutions.

a is the Correct answer

6] The rank of a matrix A is m. If a column is added to A, the rank of the new matrix is

(a) Always $m + 1$

(b) m or $m + 1$

(c) Always m.

(d) m or $m - 1$.

b is the correct answer

7] For a 3×3 matrix A, if $\det(A) = 12$ and all eigenvalues are distinct positive integers, what is the sum of the eigenvalues?

(a) 12

(b) 7

(c) 8

(d) 3

c is the correct answer

8] Solution of the following system of equations is ____.

$$x + 2y - z = 3$$
$$x + 3y + z = 5$$
$$3x + 8y + 4z = 11$$

(a) $x = -\frac{13}{3}, y = \frac{10}{3}, z = -\frac{2}{3}$

(b) $x = -\frac{2}{3}, y = \frac{17}{3}, z = \frac{4}{3}$

(c) $x = \frac{17}{3}, y = \frac{4}{3}, z = -\frac{2}{3}$

(d) $x = \frac{4}{3}, y = -\frac{2}{3}, z = \frac{17}{3}$

a is the correct answer

9] Given a 4×4 matrix A with eigenvalues 1, 2, 3, and 4, what is trace($2A$)?

(a) 24

(b) 20

(c) 10

(d) 5

b is the correct answer

10] If A is an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, what is the eigenvalue of $A + kI$ where I is the identity matrix and k is a scalar?

(a) $\lambda_1 + k, \lambda_2 + k, \ldots, \lambda_n + k$

(b) $\lambda_1, \lambda_2, \ldots, \lambda_n$

(c) $k\lambda_1, k\lambda_2, \ldots, k\lambda_n$

(d) $\lambda_1/k, \lambda_2/k, \ldots, \lambda_n/k$

a is the correct answer

Week – 08: Linear Equations

Lecture 36:- Eigen Vector

Topics to be Covered: -

- Eigen value of a non-singular matrix,
- Eigen Vector,
- Eigen Vector corresponding to unique Eigen Value, independence of eigen vectors

★ **Theorem:-** If λ is an Eigen Value of a non-Singular matrix A, then λ^{-1} is an Eigen Value of A^{-1}.

Proof :- Let A be a non-Singular matrix of order $n \times n$

A is non-Singular $\Rightarrow A^{-1}$ exists and λ^{-1} exists

$$|A - \lambda I_n| = 0$$

Now, $|A^{-1} - \lambda^{-1} I_n| = |A^{-1} - \lambda^{-1} A^{-1} A| = |A^{-1} - A^{-1} \lambda^{-1} A|$

$$= |A^{-1}(I_n - \lambda^{-1} A)| = |A^{-1}| \cdot |I_n - \lambda^{-1} A|$$

$$= |A^{-1}| \cdot (\lambda^{-1})^n \, |\lambda I_n - A|$$

$$= (|A|)^{-1} (\lambda^{-1})^n (-1)^n \, |A - \lambda I_n| = 0$$

$\Rightarrow \lambda^{-1}$ is the Eigen Value of A^{-1}.

★ **Theorem:-** If A & P are both $n \times n$ matrices and P be a non-singular, then A and $P^{-1}AP$ have the same Eigen Value.

<u>Proof</u>:- The Characteristics Polynomial of $P^{-1}AP$ is $|P^{-1}AP - xI_n|$

So, $|P^{-1}AP - xI_n| = |P^{-1}AP - P^{-1}(xI_n)P|$ since $P^{-1}(xI_n)P = xI_n$

$= |P^{-1}(A - xI_n)P| = |P^{-1}| \cdot |A - xI_n| \cdot |P|$

$= |A - xI_n| \cdot |P \cdot P^{-1}| = |A - xI_n| \cdot |I_n|$

$\Rightarrow \boxed{|P^{-1}AP - xI_n| = |A - xI_n|}$

So, the Matrix $P^{-1}AP$ and A have same Characteristics Polynomial hence so they have same Eigen Value.

Test Papers of 2025

Week – 01:

To be Continued….

Week – 07:

To be Continued….

Important Links

Current: - https://onlinecourses.nptel.ac.in/noc25_ma78/preview

2019: - https://onlinecourses.nptel.ac.in/noc19_ma23/preview

2023: - https://onlinecourses.nptel.ac.in/noc23_ma87/preview

2024: - https://onlinecourses.nptel.ac.in/noc24_ma58/preview