SAAKAAR

FOR IIT JAM 2025

Lecture- 08 | **Linear Algebra**

Linear Combination, and Linear Span Part-02

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Recap

of previous lecture

1. Direct Sum of Subspaces
2. Quotient space

Topics

to be covered

1. Linear Span
2. Properties of Linear Span

linear combination →:

V(F) → vector space.

$x_1, x_2, — x_k \in V$

$\alpha_1, \alpha_2, — \alpha_k \in F$.

$$\boxed{x = \boxed{\alpha_1} x_1 + \boxed{\alpha_2} x_2 + — + \boxed{\alpha_k} x_k \in V}$$

→ x is a (L.C) of $x_1, x_2, — x_k$

Ex: $V = \mathbb{R}^3$, $F = \mathbb{R}$

$$\begin{cases} x = (1,2,3) \\ x_1 = (1,1,0) \\ x_2 = (0,1,3) \end{cases}$$

$(1,2,3) = (1,1,0) + (0,1,3)$

$$x = \underset{\alpha_1}{\underset{\shortparallel}{1}} x_1 + \underset{\alpha_2}{\underset{\shortparallel}{1}} x_2$$

$$\boxed{x = \alpha_1 x_1 + \alpha_2 x_2}$$

x is a L.C of x_1 & x_2

Ex $V = \mathbb{R}^3, \ F = \mathbb{R}$

$$x_1 = (1,1,0)$$

$$x_2 = (0,1,1)$$

$$x = (1,2,3)$$

is x a linear combination of x_1, x_2 ?

$$x = \boxed{\alpha_1} x_1 + \boxed{\alpha_2} x_2$$

$$\Rightarrow (1,2,3) = \alpha_1(1,1,0) + \alpha_2(0,1,1)$$

$$\Rightarrow (1,2,3) = (\underline{\alpha_1}, \underline{\alpha_1+\alpha_2}, \underline{\alpha_2})$$

$$\Rightarrow \alpha_1 = 1$$

$$\alpha_1 + \alpha_2 = 2$$

$$\alpha_2 = 3$$

$$\Rightarrow 1+3 = 2$$

$$\Rightarrow \boxed{4 = 2} \ \%$$

x is not a L.C of x_1, x_2

H.W $V = P_3(\mathbb{R}),\ F = \mathbb{R}$

$$x_1 = 1 + 2x + 3x^2$$

$$x_2 = x + x^3$$

$$x = 1 + x + x^2$$

Is x a L.C of x_1 & x_2?

$$x = \alpha_1 x_1 + \alpha_2 x_2$$

$$1 + x + x^2 = \alpha_1(1 + 2x + 3x^2) + \alpha_2(x + x^3)$$

$$= \underbrace{\alpha_1} + \underbrace{(2\alpha_1 + \alpha_2)}x + 3\alpha_1 x^2 + \alpha_2 x^3$$

$$\boxed{\alpha_1 = 1}$$

$$2\alpha_1 + \alpha_2 = 1$$

$$\boxed{3\alpha_1 = 1} \rightarrow \boxed{3 = 1}\ \times$$

$$\boxed{\alpha_2 = 0}$$

Linear span

Let $V(F)$ be a vector space.

, and let S be any subset of V.

Then we denote the linear span of S by $L(S)$ or $\text{Span}(S)$ or $\langle S \rangle$

, and it is defined as

$$L(S) = \begin{cases} \{0\} & \text{if } S = \phi \\ \left\{ \sum_{i=1}^{n} \alpha_i x_i \;\middle|\; \begin{array}{l} n \in \mathbb{N} \\ \alpha_i \in F \ \forall i \\ x_i \in S \ \forall i \end{array} \right\} & \text{if } S \neq \phi \end{cases}$$

If $S = \{x_1, x_2, \ldots, x_K\}$

$$L(S) = \{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_K x_K \mid \alpha_1, \alpha_2, \ldots, \alpha_K \in F\}$$

Ex →: $S = \{\overbrace{(1,1,0)}^{x_1}, \overbrace{(0,1,1)}^{x_2}\} \subseteq \mathbb{R}^3, \; F = \mathbb{R}.$

$$L(S) = \{\alpha_1 x_1 + \alpha_2 x_2 \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$$

$$= \{\alpha_1 (1,1,0) + \alpha_2 (0,1,1) \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$$

$$= \{(\alpha_1, \alpha_1, 0) + (0, \alpha_2, \alpha_3) \mid \alpha_1, \alpha_2 \in \mathbb{R}\} = \{(\alpha_1, \alpha_1 + \alpha_2, \alpha_2) \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$$

Eg: $V = \mathbb{R}^{2\times 3}$, $F = \mathbb{R}$

$$S = \left\{ \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{x_1}, \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{x_2}, \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{x_3} \right\}$$

Find $L(S)$.

Sol^n

$$L(S) = \{ \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \}$$

$$= \left\{ \alpha_1 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} \alpha_1 + \alpha_2 & \alpha_2 + \alpha_3 & \alpha_1 + \alpha_2 \\ 0 & 0 & \alpha_1 + \alpha_3 \end{pmatrix} \middle| \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}.$$

Properties Linear Span

1 let $V(F)$ be any V.S, $S \subseteq V$

Then

① $L(S)$ is subspace of V.

⑪ $S \subseteq L(S)$

(III) If W is subspace of V such that $S \subseteq W$ then $L(S) \subseteq W$

"$L(S)$ is the smallest subspace of V that contains S"

#2 $L(S) = S$

$\Longleftrightarrow$ $\underline{S}$ is subspace of V.

#3 If $S_1 \subseteq S_2 \subseteq V$

then $L(S_1) \subseteq L(S_2)$

#4 If $S_1, S_2 \subseteq V$

then $L(S_1 \cup S_2) = L(S_1) + L(S_2)$

#5 $L(L(S)) = L(S)$

$L(S) = S$

$\Leftrightarrow$ S is a subspace

$(\Rightarrow)$ let $S = \boxed{L(S)} \rightarrow$ subspace

$\rightarrow$ S is subspace.

$(\Leftarrow)$ let S is subspace

also $S \subseteq S$

$L(S) \subseteq S$

But $S \subseteq L(S)$

$\Rightarrow L(S) = S$

Ex $V = \mathbb{R}^2$

$S_1 = \{(1,0)\}$

$S_2 = \{(0,1)\}$

$L(S_1) = \{\alpha_1(1,0) \mid \alpha_1 \in \mathbb{R}\}$

$= \{(\alpha_1, 0) \mid \alpha_1 \in \mathbb{R}\}$

$L(S_2) = \{\alpha_2(0,1) \mid \alpha_2 \in \mathbb{R}\} = \{(0,\alpha_2) \mid \alpha_2 \in \mathbb{R}\}$

$S_1 \cup S_2 = \{\overbrace{(1,0)}, \overbrace{(0,1)}\}$

$L(S_1 \cup S_2) = \{\alpha_1(1,0) + \alpha_2(0,1) \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$

$= \{(\alpha_1, \alpha_2) \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$

also

$L(S_1) + L(S_2) = \{(\alpha_1, 0) + (0, \alpha_2) \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$

$= \{(\alpha_1, \alpha_2) \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$

Ex→: $V = \mathbb{R}^{2\times 2}$

$F = \mathbb{R}$.

$$W = \left\{ \underbrace{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}} \;\middle|\; a,b,c \in \mathbb{R} \right\}$$

W is span of which subset of V?

i.e. $S = ?$, $\boxed{S \subseteq V}$

and $\boxed{L(S) = W.}$

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = a\overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} + b\overbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} + c\overbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$W = L\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Ex $V=\mathbb{R}^{2\times3}$

$$W=\left\{\begin{pmatrix}a+b & a & b\\ 0 & a+c & c\end{pmatrix} \middle| \ a,b,c\in\mathbb{R}\right\}$$

Find a $S\subseteq V$: $L(S)=W$.

Sol^n $\Leftrightarrow$

$$\begin{pmatrix}a+b & a & b\\ 0 & a+c & c\end{pmatrix}=a\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\end{pmatrix}+b\begin{pmatrix}1 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}+c\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 1\end{pmatrix}$$

$$W=L\left\{\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\end{pmatrix},\begin{pmatrix}1 & 0 & 1\\ 0 & 0 & 0\end{pmatrix},\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 1\end{pmatrix}\right\}$$

Definition →:

let $V(F)$ be any vector space and W be a subspace of V.

Suppose $S \subseteq V$, then we say that "W is spanned by S" or "S spans W" if

$$\boxed{W = L(S)}$$

Ex $\boxed{V = F^3(F)}$

$$V = \{(a_1, a_2, a_3) \mid a_1, a_2, a_3 \in F\}$$

$$(a_1, a_2, a_3) = a_1 \underbrace{(1, 0, 0)} + a_2 \underbrace{(0, 1, 0)} + a_3 \underbrace{(0, 0, 1)}$$

$$\Rightarrow \boxed{V = L\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}}$$

Ex $V = F^n(F)$

$$V = \{(a_1, a_2, \ldots a_n) \mid a_1, a_2, \ldots a_n \in F\}$$

$$\boxed{V = L\{\overset{=e_1}{(1,0,0,\ldots 0)}, \overset{=e_2}{(0,1,0,\ldots 0)}, \ldots \overset{=e_n}{(0,0,\ldots,0,1)}\}}$$

Home-work →:

1. $V = \mathbb{R}^{2\times 3}$

find S such that $V = L(S)$

2. $V = \mathbb{R}^{m\times n}$

find S such that

$V = L(S)$

Ex $P_n(F) = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_0, a_1 \ldots a_n \in F\}$

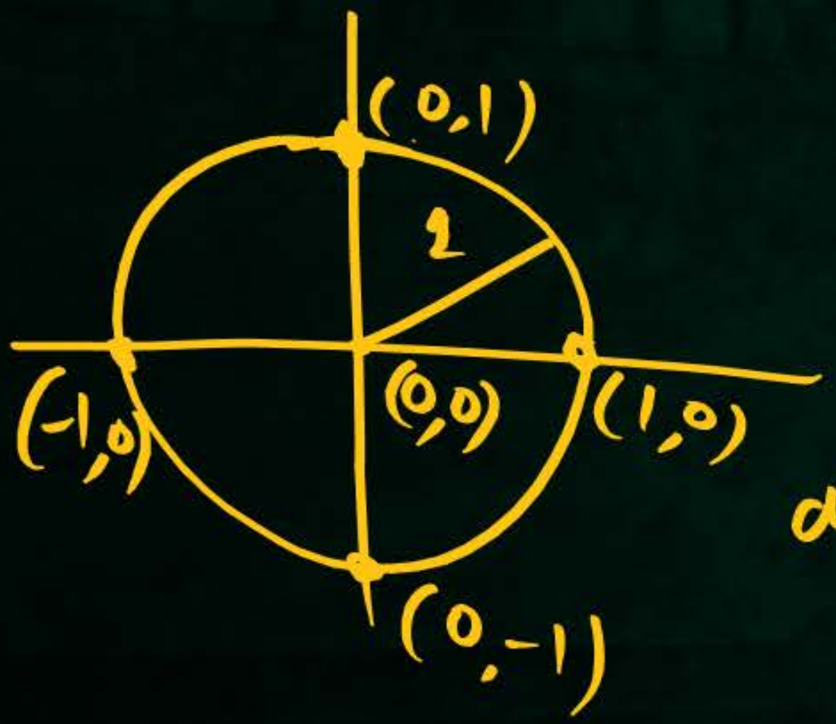
$= L\{1, x, x^2, \text{---}\, x^n\}$

$$\mathbb{R}^2 = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{R}\}$$

Ex

$$S = \{(x, y) \mid x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$$

What is $L(S)$



$$S_1 = \{(1,0), (0,1)\} \subseteq S$$

$$\Rightarrow L(S_1) \subseteq L(S) \quad - ①$$

also $L(S_1) = \{\alpha_1(1,0) + \alpha_2(0,1) \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$

$$= \{(\alpha_1, \alpha_2) \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$$

$$= \mathbb{R}^2.$$

$$L(S_1) = \mathbb{R}^2$$

By ①

$$\mathbb{R}^2 \subseteq L(S) \subseteq \mathbb{R}^2$$

$$\Rightarrow \boxed{L(S) = \mathbb{R}^2}$$

FACT:

If $V(F)$ is a v.s

and $S_1 \subseteq S \subseteq V$

Such that

$$L(S_1) = V$$

then $\boxed{L(S) = V}$

H.W

$S = \{(x,y) \mid |x|+|y|=1\} \subseteq \mathbb{R}^2$

then

Ⓐ L(S) is proper subspace of $\mathbb{R}^2$

Ⓑ $L(S) = \mathbb{R}^2$

Ⓒ $L(S) = S$

Ⓓ $L(S) \neq S$

2 Mins Summary

1. Linear Span
2. Properties of Linear Span

THANK YOU