SAAKAAR

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Lecture- 06 | **Linear Algebra**

Subspaces and Properties Part- 02

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Recap

of previous lecture

1. Examples of vector spaces
2. Subspace
3. Properties of Subspace

Topics *to be covered*

1. Direct Sum of Subspaces
2. Examples of Some Direct sums
3. Quotient space

Let $V = F^{m \times n}$, F - field

Consider $V(F)$ as a V.S.

$$(a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \text{---} & a_{1n} \\ a_{21} & a_{22} & \text{---} & a_{2n} \\ a_{31} & a_{32} & \text{---} & a_{3n} \\ \hline a_{m1} & a_{m2} & \text{---} & a_{mn} \end{pmatrix}$$

Define

$$W_1 = \left\{ (a_{ij}) \in V \;\middle|\; \begin{array}{l} \sum_{j=1}^{n} a_{1j} = 0 \\ \sum_{j=1}^{n} a_{2j} = 0 \\ \hline \\ \sum_{j=1}^{n} a_{mj} = 0 \end{array} \right\} = W_1 = \left\{ (a_{ij}) \in V \;\middle|\; \sum_{j=1}^{n} a_{ij} = 0 \;\; \forall i = 1, 2 \text{---} m \right\}$$

$$W_2 = \left\{ (a_{ij}) \in V \;\middle|\; \sum_{i=1}^{m} a_{ij} = 0 \;\; \forall j = 1, 2, \text{--} n \right\}$$

Then W_1 & W_2 are subspaces of V.

✗ $W_1 \subseteq W_2$ $\left(\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \in W_1 \text{ but } \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \notin W_2, \text{ for } m = n = 2 \right)$

✗ $W_2 \subseteq W_1$ $\left(\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \in W_2 \text{ but } \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \notin W_1 \right)$

✗ $W_1 \cap W_2 = \{0\}$ $\left(\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \neq 0, \ \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \in W_1 \cap W_2 \right)$

(i) $W_1 \cap W_2$ is a subspace

(ii) is $\underbrace{W_1} \cup \underbrace{W_2}$ a subspace

$$\left(\begin{array}{l} x, y \in W_1 \cup W_2 \\ x+y \text{ or } x-y \in W_1 \cup W_2 \end{array} \right)$$

$$x = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \in W_1 \subseteq W_1 \cup W_2$$

$$y = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \in W_2 \subseteq W_1 \cup W_2$$

$$x+y = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \begin{array}{l} \notin W_1 \\ \notin W_2 \end{array}$$

Sum of two subspaces

let $V(F)$ be any vector space, and

Suppose W_1 and W_2 are subspaces of V.

Then the set

$\boxed{W_1 + W_2 = \{x + y \mid x \in W_1,\ y \in W_2\}}$ is called

the sum of W_1 and W_2.

Theorem:

Let $V(F)$ be any vector space, and let W_1, W_2 be subspaces of V. Then

(i) $W_1 + W_2$ is a subspace of V

(ii) $W_1 \subseteq W_1 + W_2$, $W_2 \subseteq W_1 + W_2$

(iii) If $\boxed{W \text{ is a subspace of } V}$ such that $\boxed{W_1 \subseteq W \ \& \ W_2 \subseteq W}$ then $\boxed{W_1 + W_2 \subseteq W}$

i.e.

"Sum of two subspaces is the smallest subspace containing both the subspaces"

Proof

(1) $W_1 + W_2 = \{x+y \mid x \in W_1, y \in W_2\}$

(i) $0 \in W_1,\ 0 \in W_2$

$0 = \underbrace{0}_{} + \underbrace{0}_{} \in W_1 + W_2$

(ii) let $\alpha, \beta \in F$, $\boxed{u, v \in W_1 + W_2}$

To show

$\alpha u + \beta v \in W_1 + W_2$

now $\boxed{u = x_1 + y_1}$, $x_1 \in W_1$, $y_1 \in W_2$

$\boxed{v = x_2 + y_2}$, $x_2 \in W_1$, $y_2 \in W_2$

$\alpha u + \beta v$

$= \alpha(x_1 + y_1) + \beta(x_2 + y_2)$

$= \underbrace{\alpha x_1 + \beta x_2}_{\in W_1} + \underbrace{\alpha y_1 + \beta y_2}_{\in W_2}$

$\Rightarrow \alpha u + \beta v \in W_1 + W_2$

$\Rightarrow W_1 + W_2$ is a subspace

(ii) let $x \in W_1$

$\Rightarrow x = \underbrace{x}_{\in W_1} + \underbrace{0}_{\in W_2} \in W_1 + W_2$

$\Rightarrow x \in W_1 + W_2$

$\Rightarrow \boxed{W_1 \subseteq W_1 + W_2}$

$y \in W_2$

$\Rightarrow y = \underbrace{0}_{\in W_1} + \underbrace{y}_{\in W_2} \in W_1 + W_2$

$\Rightarrow y \in W_1 + W_2$

$\therefore \boxed{W_2 \subseteq W_1 + W_2}$

(iii) let W is a subspace of V

and $W_1 \subseteq W,\ W_2 \subseteq W$

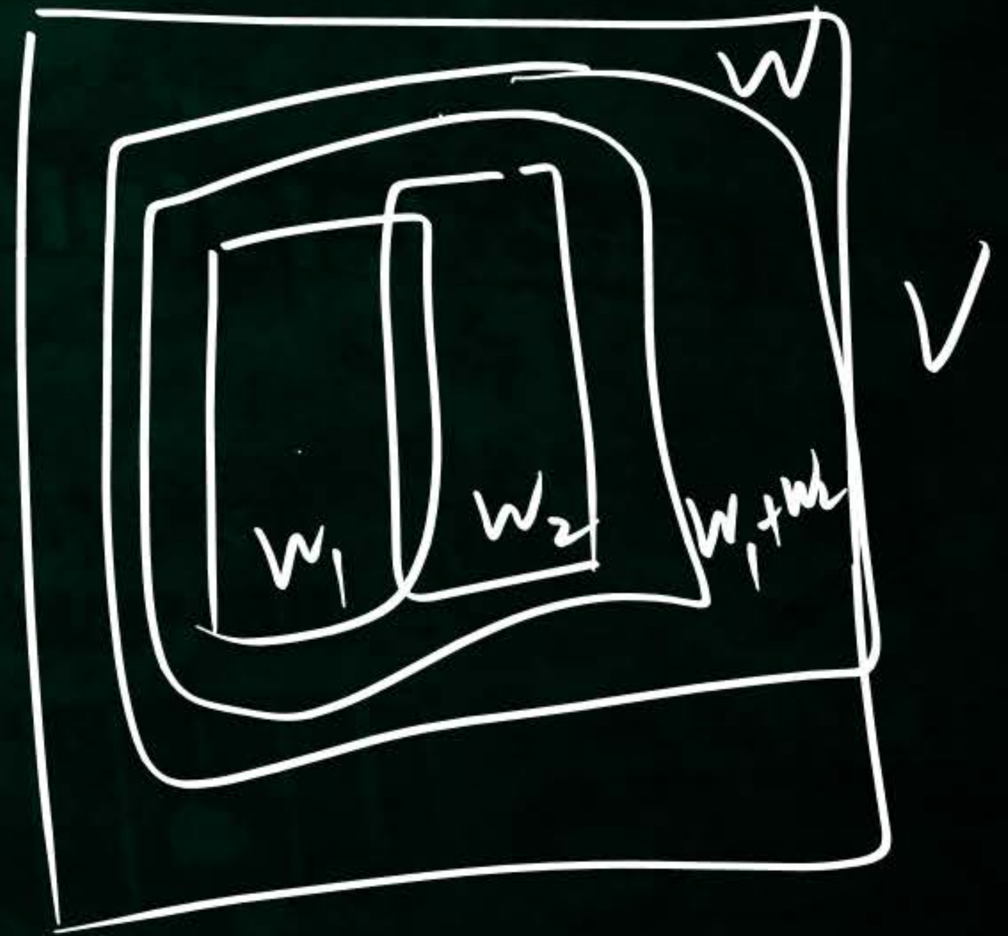
let $z \in W_1 + W_2$

$\Rightarrow z = x + y;\ x \in W_1,\ y \in W_2$

now $x \in W_1 \subseteq W$
$y \in W_2 \subseteq W$ $\Rightarrow x + y \in W$

$\Rightarrow z \in W$

$\therefore W_1 + W_2 \subseteq W$



Ex $V = \mathbb{R}^{2\times 2}$

$$W_1 = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \middle| x \in \mathbb{R} \right\}$$

$$W_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} \middle| y \in \mathbb{R} \right\}$$

$$W_1 + W_2 = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}$$

Direct sum of subspaces

Let $V(F)$ be any vector space, and let W_1, W_2 be two subspaces of V.

Then we say $\boxed{V \text{ is a direct sum of } W_1 \text{ \& } W_2}$ and we write $\boxed{V = W_1 \oplus W_2}$

if

(i) $V = W_1 + W_2$ (i.e. $\forall x \in V \Rightarrow x = u + v;\ u \in W_1,\ v \in W_2$)

(ii) $W_1 \cap W_2 = \{0\}$

Examples of Direct sum

Definition →: let $P_n(F)$ denote the space of all polynomials of degree at most n over F.

i.e. $$\boxed{P_n(F) = \{a_0 + a_1 x + \dots + a_n x^n \mid a_0, a_1, \dots a_n \in F\}}$$

Ex

$P_1(\mathbb{R}) = \{a_0 + a_1 x \mid a_0, a_1 \in \mathbb{R}\}$

$P_2(\mathbb{R}) = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$

Ex

$$P_2(\mathbb{R}) = \{ a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R} \}.$$

$$W_1 = \{ c + d x^2 \mid c, d \in \mathbb{R} \}$$

$$W_2 = \{ e x \mid e \in \mathbb{R} \}$$

Verify ① W_1, W_2 are subspace (H.W)

now $\forall$ $\boxed{v \in P_2(\mathbb{R})}$

$\Rightarrow v = a_0 + a_1 x + a_2 x^2$

$\Rightarrow \boxed{v = \boxed{a_0 + a_2 x^2} + \boxed{a_1 x}}$

$$\boxed{P_2(\mathbb{R}) = W_1 + W_2}$$

(II) $a_0 + a_1 x + a_2 x^2 \in W_1 \cap W_2$

$\Rightarrow a_0 + \boxed{a_1} x + a_2 x^2 \in W_1$

& $a_0 + a_1 x + a_2 x^2 \in W_2$

$\Rightarrow \boxed{a_1 = 0}$ & $\boxed{a_0 = a_2 = 0}$

$a_0 + a_1 x + a_2 x^2 = 0 + 0x + 0x^2$

$\Rightarrow W_1 \cap W_2 = \{0\}$

$\Rightarrow P_2(\mathbb{R}) = W_1 \oplus W_2$

Ex: let $V = \mathbb{R}^2 = \{(x,y) \mid x, y \in \mathbb{R}\}$

$$\boxed{\begin{aligned} W_1 &= \{(a,0) \mid a \in \mathbb{R}\} \\ W_2 &= \{(0,b) \mid b \in \mathbb{R}\} \end{aligned}}$$

Now, $\forall\ (x,y) \in \mathbb{R}^2$

$\Rightarrow (x,y) = \underbrace{(x,0)}_{\in W_1} + \underbrace{(0,y)}_{\in W_2}$

$\Rightarrow \boxed{\mathbb{R}^2 = W_1 + W_2}$

now

$(x,y) \in \boxed{W_1 \cap W_2}$

$\Rightarrow (x,y) \in W_1$ & $(x,y) \in W_2$

$\Rightarrow y = 0$ & $x = 0$

$\Rightarrow (x,y) = (0,0)$

$\Rightarrow W_1 \cap W_2 = \{(0,0)\}$

$\Rightarrow \boxed{\mathbb{R}^2 = W_1 \oplus W_2}$

Ex let $V = \mathbb{C}^{n\times n}$, consider $V(\mathbb{C})$

define $\left.\begin{aligned} W_1 &= \{A \in V \mid A^T = A\} \\ W_2 &= \{A \in V \mid A^T = -A\} \end{aligned}\right\}$ are subspaces of V.

now ① $\forall\ X \in V$

<u>we want</u> $\boxed{X = B + C;\ B \in W_1,\ C \in W_2}$.

$\Rightarrow\ X^T = B^T + C^T$

$\Rightarrow\ X^T = B + (-C)$

$X^T = B - C$

$X = B + C$

$X^T + X = 2B$

$\Rightarrow \boxed{B = \frac{1}{2}(X + X^T) \in W_1}$

$X - X^T = 2C$

$\boxed{C = \frac{1}{2}(X - X^T) \in W_2}$

verify $\boxed{B^T \overset{?}{=} B \quad C^T = -C}$

$X = B + C$

$$\boxed{X = \frac{1}{2}(X + X^T) + \frac{1}{2}(X - X^T)}$$

(ii) let $X \in W_1 \cap W_2$

$\Rightarrow X \in W_1$ & $X \in W_2$

$\Rightarrow X^T = X$ & $X^T = -X$

$\Rightarrow X = -X$

$\Rightarrow \boxed{2X = 0}$

$\boxed{X = 0}$ ✓

$W_1 \cap W_2 = \{0\}$

$\Rightarrow \boxed{\mathbb{C}^{n \times n} = W_1 \oplus W_2}$

Ex:-

let $V = \mathbb{R}^{\mathbb{R}} = \{f \mid f: \mathbb{R} \to \mathbb{R}\}$

let $W_1 = \{f \in V \mid f(-x) = f(x)\}$

$W_2 = \{f \in V \mid f(-x) = -f(x)\}$

Show that

$$\boxed{V = W_1 \oplus W_2}$$

Sol^n

let $f \in V$

We want

$$\boxed{f = g + h, \; g \in W_1, \; h \in W_2}$$

$$\boxed{f(x) = g(x) + h(x)}, \quad g \in W_1, \; h \in W_2$$

$$\Rightarrow f(-x) = g(-x) + h(-x)$$

$$\Rightarrow \boxed{f(-x) = g(x) - h(x)}$$

$$\Rightarrow f(x) + f(-x) = 2g(x)$$

$$\Rightarrow \boxed{g(x) = \frac{1}{2}\left(f(x) + f(-x)\right)}$$

$$f(x) - f(-x) = 2h(x)$$

$$\boxed{h(x) = \frac{1}{2}\left(f(x) - f(-x)\right)}$$

verify; $g(-x) = g(x)$

$h(-x) = -h(x)$

$$f(x) = \underbrace{\tfrac{1}{2}\left(f(x) + f(-x)\right)}_{\in W_1} + \underbrace{\tfrac{1}{2}\left(f(x) - f(-x)\right)}_{\in W_2}$$

$f \in V$

$\Rightarrow V = W_1 + W_2$

let $f \in W_1 \cap W_2$

$\Rightarrow f \in W_1$ & $f \in W_2$

$\Rightarrow f(-x) = f(x)$ & $f(-x) = -f(x)$

$\Rightarrow f(x) = -f(x) \quad \forall x \in \mathbb{R}$

$\Rightarrow 2f(x) = 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow f(x) = 0$

$\Rightarrow W_1 \cap W_2 = \{0\}$

$\Rightarrow V = W_1 \oplus W_2$

2 Mins Summary

1. Direct Sum of Subspaces
2. Examples of Some Direct sums

THANK YOU