SAAKAR

FOR IIT JAM 2025

Lecture- 05 | **Linear Algebra**

Questions Practice

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Recap

of previous lecture

1. Examples of vector spaces
2. Properties of vector spaces
3. Subspace

Topics *to be covered*

1. Examples of vector spaces
2. Subspace
3. Properties of Subspace

Properties of Subspaces

#1 let $V(F)$ be a vector space then $W = \{0\}$ is a subspace of V.

$$\begin{cases} \text{(i)} & 0 \in W \\ \text{(ii)} & \text{let } x, y \in W \Rightarrow x = y = 0 \\ & \Rightarrow x - y = 0 - 0 = 0 \in W \\ \text{(iii)} & \text{let } \alpha \in F,\ x \in W \end{cases}$$

$$\Rightarrow \alpha \in F,\ x = 0 \Rightarrow \alpha x = \alpha \cdot 0 = 0 \in W$$

#2, If $V(F)$ is a vector space, then V is a subspace of V.

Note →:

$\{0\}$ is called $\boxed{\text{trivial subspace}}$ of V.

any subspace other than $\{0\}$, is called non-trivial subspace.

Note:

If W is a subspace of V and $W \neq V$ then W is called a proper subspace of V.

Ex $V = \mathbb{R}^2 = \{(a,b) \mid a, b \in \mathbb{R}\}$

$W = \{(x,0) \mid x \in \mathbb{R}\} \neq \{(0,0)\}$

then W is a subspace of V.

also $W \neq V \Rightarrow W$ is a proper subspace of V

\#3 If $W_1, W_2, \ldots, W_n$ are subspaces of $V(F)$ then $W_1 \cap W_2 \cap \cdots \cap W_n$ is also a subspace of V.

Proof: let $\boxed{x, y \in W_1 \cap W_2 \cap \cdots \cap W_n}$ and $\beta, \alpha \in F$

$\Rightarrow \boxed{x, y \in W_i \;\forall\, i = 1, 2, \ldots, n}$

$\Rightarrow \alpha x + \beta y \in W_i \;\forall\, i = 1, 2, \ldots, n$ ($\because W_i$ is subspace)

$\Rightarrow \alpha x + \beta y \in W_1 \cap W_2 \cap \cdots \cap W_n$

#4. Arbitrary intersection of subspaces is a subspace.

#5 Union of two subspaces need not be a subspace.

Ex: $V = \mathbb{R}^2$, $F = \mathbb{R}$

$W_1 = \{(x,0) \mid x \in \mathbb{R}\}$

$W_2 = \{(0,y) \mid y \in \mathbb{R}\}$

then W_1 & W_2 both are subspaces of V.

But $W_1 \cup W_2$ is not a subspace of V

$$\left.\begin{array}{l} x = (1,0) \in W_1 \subseteq W_1 \cup W_2 \\ y = (0,1) \in W_2 \subseteq W_1 \cup W_2 \end{array}\right\}$$

now $x - y = (1,-1) \notin W_1$, $\notin W_2$

$\Rightarrow \boxed{x - y \notin W_1 \cup W_2}$

#6 Union of two subspace is a subspace if and only if one of them is contained in other.

Proof :- let $V(F)$ is V.S

let W_1, W_2 be subspaces of V.

To show :- $W_1 \cup W_2$ is a subspace of V

$\Leftrightarrow$: Either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

$(\longrightarrow)$

let $W_1 \cup W_2$ be a subspace

To show

$W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

let if possible, $\boxed{\text{Neither } W_1 \subseteq W_2 \text{ nor } W_2 \subseteq W_1}$

$\Rightarrow \exists\, x \in W_1 : x \notin W_2$

$\Rightarrow y \in W_2 : y \notin W_1$

now $x \in W_1 \subseteq W_1 \cup W_2$

$y \in W_2 \subseteq W_1 \cup W_2$

$x, y \in W_1 \cup W_2$

$\Rightarrow x + y \in W_1 \cup W_2$ ($\because W_1 \cup W_2$ is subspace)

$\Rightarrow x + y \in W_1$ or $x + y \in W_2$

$\Rightarrow (\cancel{x} + y) - \cancel{x} \in W_1$ or $(x + \cancel{y}) - y \in W_2$

$\Rightarrow \boxed{y \in W_1 \text{ or } x \in W_2}$ %

$\therefore$ Either $\boxed{\begin{matrix} W_1 \subseteq W_2 \\ \text{or} \quad W_2 \subseteq W_1. \end{matrix}}$

$(\Leftarrow)$ let $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

<u>To Show</u> $W_1 \cup W_2$ is a subspace

$$\left.\begin{matrix} W_1 \cup W_2 = W_2 \\ \text{or} \quad W_1 \cup W_2 = W_1 \end{matrix}\right\} \text{subspace of } V$$

QUESTION- 01

#Q. Let $V = \mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) : a_1, a_2, \ldots, a_n \in \mathbb{R}\}$ considered as a vector space over $\mathbb{R}$.
Then which of the following is (are) subspace(s) of V?

(A) $W_1 = \{(a_1, a_2, \ldots, a_n) \in V : a_1 = 2a_2 + 3a_3\}$

$\Rightarrow a_1 - 2a_2 - 3a_3 = 0$

(B) $W_2 = \{(a_1, a_2, \ldots, a_n) \in V : |a_1| + |a_2| + \cdots + |a_n| = 0\}$ $\Rightarrow |a_1| = 0 = |a_2| \cdots = |a_n|$ $\Rightarrow \boxed{a_1 = 0 = a_2 = \cdots = a_n}$

$= \{(0, 0, \ldots, 0)\}$ $\quad \geq 0 \quad \geq 0 \quad \geq 0$

(C) $W_3 = \{(a_1, a_2, \ldots, a_n) \in V : a_1^2 + a_2^2 + \cdots + a_n^2 = 0\}$ $\Rightarrow a_1^2 = a_2^2 = \cdots = a_n^2 = 0$ $\Rightarrow a_1 = a_2 = \cdots = a_n = 0$

$= \{(0, 0, 0, \ldots, 0)\}$ $\quad \geq 0 \quad \geq 0 \quad \geq$

(D) $0 \notin$ $W_4 = \{(a_1, a_2, \ldots, a_n) \in V : a_1^2 + a_2^2 + \cdots + a_n^2 = 1\}$

$0^2 + 0^2 + \cdots + 0^2 = 0 \neq 1$

QUESTION- 02

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

#Q. Let $V = \mathbb{R}^{n\times n} = \left\{A = \left(a_{ij}\right)_{n\times n} : a_{ij} \in \mathbb{R}, ;\ i, j = 1, 2, \dots, n\right\}$ considered as a vector space over $\mathbb{R}$.

Then which of the following is (are) subspace(s) of V?

(A) $W_1 = \{A = \left(a_{ij}\right)_{n\times n} \in V : \det A = 0\}$

(B) $W_2 = \{A = \left(a_{ij}\right)_{n\times n} \in V : trace(A) = 0\}$

(C) $W_3 = \{A = \left(a_{ij}\right)_{n\times n} \in V : \boxed{A = A^T}\}$ $\Rightarrow$ space of all symmetric matrices

(D) $W_4 = \{A = \left(a_{ij}\right)_{n\times n} \in V : \boxed{-A = A^T}\}$

$\hookrightarrow$ Space of all skew symmetric matrix

$A, B \in W_1 \Rightarrow \det(A) = 0 = \det(B)$

$\det(A - B) = 0$?!

$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \in W_1$

$A - B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \det(A - B) = 1 \neq 0$

$\boxed{A - B \notin W_1}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = (a_{ij})_{n\times n}$$

$$\text{trace}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

$$\text{trace}(A) = \sum_{i=1}^{n} a_{ii}$$

Properties

(i) $\text{Trace}(A+B) = \text{Trace}(A) + \text{Trace}(B)$

(ii) $\text{Trace}(\alpha A) = \alpha\,\text{Trace}(A)$

(i) $\text{Trace}(0) = 0$

$\Rightarrow 0 \in W_2$

(ii) let $A, B \in W_2$

$\text{Trace}(A) = \text{Trace}(B) = 0$

$\therefore \text{Trace}(A-B) = \text{Trace}(A) - \text{Trace}(B)$

$= 0 - 0$

$= 0$

(iii) $\text{Trace}(\alpha A) = \alpha\,\text{Trace}(A)$

$= \alpha \cdot 0 = 0$

$\alpha A \in W_2$

QUESTION- 03

#Q. Let $V = \mathbb{C}^{n\times n} = \left\{A = (a_{ij})_{n\times n} : a_{ij} \in \mathbb{C}, ; i,j = 1,2,\ldots,n\right\}$ considered as a vector space over $\mathbb{C}$. Then which of the following is (are) subspace(s) of V? Assuming $A^\theta = \overline{(A^T)}$

(A) $W_1 = \{A = (a_{ij})_{n\times n} \in V : \det A = 0\}$

(B) $W_2 = \{A = (a_{ij})_{n\times n} \in V : trace(A) = 0\}$

(C) $W_3 = \{A = (a_{ij})_{n\times n} \in V : A = A^\theta\}$

(D) $W_4 = \{A = (a_{ij})_{n\times n} \in V : -A = A^\theta\}$

Ex $A = \begin{pmatrix} 2+i & 3i \\ 4-i & 1+i \end{pmatrix}$

$A^T = \begin{pmatrix} 2+i & 4-i \\ 3i & 1+i \end{pmatrix}$

$A^\theta = \overline{(A^T)} = \begin{pmatrix} \overline{2+i} & \overline{4-i} \\ \overline{3i} & \overline{1+i} \end{pmatrix}$

$= \begin{pmatrix} 2-i & 4+i \\ -3i & 1-i \end{pmatrix}$

$$O = \begin{pmatrix} 0 & 0 & - & - & 0 \\ 0 & 0 & - & - & 0 \\ \hline 0 & - & - & & 0 \end{pmatrix}_{n \times n}$$

(i) $O^T = O$

$$O^\theta = \overline{(O^T)} = \overline{O} = O$$

$$O \in W_3$$

(ii) let $A, B \in W_3$

$$\Rightarrow A^\theta = A$$

$$B^\theta = B$$

$$(A-B)^\theta = \overline{((A-B)^T)}$$

$$= \overline{(A^T - B^T)}$$

$$= \overline{(A^T)} - \overline{(B^T)}$$

$$= A^\theta - B^\theta$$

$$(A-B)^\theta = (A-B) \Rightarrow \boxed{A-B \in W_3}$$

(iii) let $A \in W_3$, $\boxed{A^\theta = A}$

& $\alpha \in \mathbb{C}$

$$(\alpha A)^\theta = \overline{((\alpha A)^T)} = \overline{(\alpha A^T)}$$

$$= \overline{\alpha}\, \overline{(A^T)} = \overline{\alpha} A^\theta = \boxed{\overline{\alpha} A}$$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

$$A^\theta = \overline{(A^T)} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = A$$

$A \in W_3$

let $\alpha = i$

$$(\alpha A)^\theta = \bar{\alpha} A$$

$$= -iA \neq iA = \alpha A$$

$\alpha A \notin W_3$

QUESTION- 04

#Q. Let $V = \mathbb{C}^{n\times n} = \left\{A = \left(a_{ij}\right)_{n\times n} : a_{ij} \in \mathbb{C}, ; i, j = 1, 2, \dots, n\right\}$ considered as a vector space over $\mathbb{R}$. Then which of the following is (are) subspace(s) of V? Assuming $A^{\theta} = \overline{(A^T)}$

(A) $W_1 = \{A = \left(a_{ij}\right)_{n\times n} \in V : \det A = 0\}$

(B) $W_2 = \{A = \left(a_{ij}\right)_{n\times n} \in V : trace(A) = 0\}$

(C) $W_3 = \{A = \left(a_{ij}\right)_{n\times n} \in V : A = A^{\theta}\}$

(D) $W_4 = \{A = \left(a_{ij}\right)_{n\times n} \in V : -A = A^{\theta}\}$

QUESTION- 05

#Q. Let V be the space of all polynomials with coefficients from the field $\mathbb{R}$, then which of the following is (are) a subspace(s) of V.

(A) $W_1 = \{p(x) \in V : p(x) = p(-x)\}$

(B) $W_2 = \{p(x) \in V : p(x) = -p(-x)\}$

(C) $W_3 = \{p(x) \in V : p(x) \text{ has degree } 3\}$ $\quad 0 \notin W_3$

(D) $W_4 = \{p(x) \in V : p(x) \text{ has degree at most } 3\}$

$0(x) = 0 + 0x + 0x^2 + \cdots = 0$

$0(-x) = 0 + 0(-x) + 0(-x)^2 + \cdots = 0$

$0(x) = 0(-x) \Rightarrow 0 \in W_1$

let $p(x), q(x) \in W_1$, $\alpha, \beta \in \mathbb{R}$

$\Rightarrow p(-x) = p(x)$

$q(-x) = q(x)$

let

$r(x) = \alpha p(x) + \beta q(x)$

<u>To show</u> $r(x) \in W_1$

$r(-x) = \alpha p(-x) + \beta q(-x)$

$= \alpha p(x) + \beta q(x)$

$r(-x) = r(x)$

$\Rightarrow r(x) \in W_1$

$\Rightarrow \alpha p(x) + \beta q(x) \in W_1 \Rightarrow W_1$ is a subspace

$W_4 = \{ p(x) \in V \mid \deg p(x) \le 3 \}$

$= \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R} \}$

(i) $0 \in W_4$

(ii) $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$

$\alpha p(x) + \beta q(x)$

$= \underbrace{(\alpha a_0 + \beta b_0)} + \underbrace{(\alpha a_1 + \beta b_1)} x + (\alpha a_2 + \beta b_2) x^2 + (\alpha a_3 + \beta b_3) x^3$

$\in W_4$

Note →:

Although degree of zero polynomial is not defined but from now onwards in this course we'll assume

degree of zero polynomial to be -1.

$V = P(\mathbb{R}),\ F = \mathbb{R}$

$W = \left\{ p(x) \in V \mid \begin{array}{l} \text{degree } p(x) = 3 \\ \text{or } \quad p(x) = 0 \end{array} \right\}$

$\longrightarrow$

$p(x) = x - x^3 \in W$

$q(x) = -x - x^3 \in W$

$p(x) - q(x)$

$= \boxed{2x} \notin W$

is W a subspace of V?

(A) Yes

(B) NO

QUESTION- 06

$F^S = \{f \mid f : S \to F\}$

#Q. Let V be the space of all functions from $\mathbb{R}$ to $\mathbb{R}$ over the field $\mathbb{R}$, then which of the following is (are) a subspace(s) of V.

$V = \mathbb{R}^{\mathbb{R}} = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$

(A) $W_1 = \{f(x) \in V : f(x) = f(-x)\}$ ✓

(B) $W_2 = \{f(x) \in V : f(x) = -f(-x)\}$ ✓

(C) $W_3 = \{f(x) \in V : f(x) \text{ is increasing}\}$ ✗

(D) $W_4 = \{f(x) \in V : f(x) \text{ is decreasing}\}$ ✗

$f(x) = x$ is $\uparrow$

$(-1) f(x) = -x$ is $\downarrow$

$\langle a_n \rangle, \langle b_n \rangle \in S_4$

$$\left.\begin{aligned} \lim a_n &= 0 \\ \lim b_n &= 0 \end{aligned}\right\}$$

$\forall\ \alpha, \beta \in \mathbb{R}$

$\lim(\underline{\alpha a_n} + \underline{\beta b_n})$

$= \alpha \lim a_n + \beta \lim b_n$

$= \alpha \cdot 0 + \beta \cdot 0 = 0$

$a_n = 0 \ \forall n \in \mathbb{N}$

$\lim a_n = 0$

$S_1 = \{\langle a_n \rangle \mid \lim a_n = 1\}$ ✗

$S_2 = \{\langle a_n \rangle \mid \lim a_n = 2\}$ ✗

$S_3 = \{\langle a_n \rangle \mid \lim a_n = 3\}$ ✗

$S_4 = \{\langle a_n \rangle \mid \lim a_n = 0\}$ ✓

2 Mins Summary

1. Examples of vector spaces
2. Subspace
3. Properties of Subspace

THANK YOU