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Assignment

12th Week

HW $f(k) = \frac{g}{(2\pi)^3} \frac{1}{\left(\exp\frac{(E-\mu)}{K_BT} \pm 1\right)}$

$E = \sqrt{k^2+m^2}$

$E\,dE = k\,dk$

$$\rho = \int E f(k)\, d^3k = \frac{g}{2\pi^2}\int \frac{k^2 E\, dk}{\left(e^{(E-\mu)/K_BT} \pm 1\right)}$$

$$n = \int f(k) d^3k = \frac{g}{2\pi^2}\int \frac{k^2 dk}{e^{E-\mu/K_BT} \pm 1}$$

$$P = \int \frac{k^2}{3E} f(k) d^3k = \frac{g}{6\pi^2}\int \frac{k^4\, dk}{E\left(e^{(E-\mu/K_BT)} \pm 1\right)}$$

$$\frac{dP}{dT} = \frac{g}{6\pi^2}\int \frac{k^4 dk}{E\left(e^{E-\mu/K_BT} \pm 1\right)^2} \times e^{(E-\mu)/K_BT} \times \frac{(E-\mu)}{K_BT^2} = \frac{g}{6\pi^2 T}\int \frac{k^4 dk}{E\left(e^{E-\mu/K_BT} \pm 1\right)^2} \times \frac{e^{E-\mu/K_BT}(E-\mu)}{K_BT}$$

$$\frac{d}{d\cancel{k}E} \frac{1}{\left(e^{E-\mu/K_BT} \pm 1\right)} = -\frac{e^{(E-\mu)/K_BT}}{\left(e^{E-\mu/K_BT} \pm 1\right)^2} \frac{1}{K_BT} \cancel{dk} \frac{k\,dk}{E}.$$

Let $\frac{1}{\left(e^{E-\mu/K_BT} \pm 1\right)} = A$.

Then

$$\frac{dP}{dT} = -\frac{g}{6\pi^2 T}\int k^3\, dA\, (E-\mu)$$

$$= -\frac{g}{6\pi^2 T}\left[\int E k^3 dA - \mu k^3 dA\right] = \frac{g}{6\pi^2 T}\left[\int k^3 E\, dA - \mu \int k^3 dA\right]$$

$E\,dA = dEA - A\,dE$

$$= -\frac{g}{6\pi^2 T}\left[\int k^3 d(EA) - A\,dE - \mu \int k^3 dA\right]$$

$$= -\frac{g}{6\pi^2 T}\left[\int k^3 dEA - \int k^3 A\, dE - \mu \int k^3 dA\right]$$

$$= \frac{-g}{6\pi T}\left[\cancel{\left| k^3 E A \right|_0^{\infty}} - \int 3k^2 E A\, dk - \mu \cancel{\left| k^3 A \right|_0^{\infty}} + \mu \int 3k^2 A\, dk - \int k^4 dk\, A/E \right.$$

$$= \frac{g}{6\pi T}\left[\int \frac{3k^2 E}{\left(\exp^{E-\mu/K_BT} \pm 1\right)} dk + \int \frac{k^4 dk}{E\left(\exp^{E-\mu/K_BT} \pm 1\right)} - \mu \int \frac{3k^2}{\left(\exp^{E-\mu/K_BT} \pm 1\right)} dk \right]$$

$$= \frac{1}{T}\left[\frac{g}{2\pi} \int \frac{k^2 E\, dk}{\left(\exp E-\mu/K_BT \pm 1\right)} + \frac{g}{6\pi} \int \frac{k^4 dk}{E\left(\exp\left(\frac{E-\mu}{K_BT}\right) \pm 1\right)} - \mu \frac{g}{2\pi} \int \frac{k^2 dk}{\left(\exp \frac{E-\mu}{K_BT} \pm 1\right)} \right]$$

$$= \frac{1}{T}\left(\rho + P - \mu n\right)$$

So $$\frac{dP}{dT} = \frac{(\rho + P - \mu n)}{T}$$

$$S = \frac{dP}{dT} = \left(\frac{\rho + P - \mu n}{T}\right)$$

1. Given cross section times velocity of the equation

$\langle \sigma v \rangle \sim 7.4 \times 10^{-20} cm^3 s^{-1}$

The rate of interaction is $\Gamma = n \langle \sigma v \rangle$

The rate per neutron of $np \longrightarrow {}^2H\gamma$ is:

$\Gamma = n_p \langle \sigma v \rangle = \eta n_\gamma \langle \sigma v \rangle$ [we can take $\eta \sim \frac{n_p}{n_\gamma}$]

$= \eta n_\gamma \times 7 \times 10^{-26} m^3 sec^{-1}$.

Substituting for $n_\gamma = 2.85 \times 10^{37} \left(\frac{T}{1MeV}\right)^3$ we get.

$\Gamma = \eta \times 2 \times 10^{12} sec^{-1} \left(\frac{T}{1MeV}\right)^3$

We know that the expansion rate is given by

$H \sim 0.6\, sec^{-1} \left(\frac{T}{1MeV}\right)^2$

Substituting for $T = 60 keV$

$\Gamma = \eta \times 2 \times 10^{12} sec^{-1} \left(\frac{60keV}{1MeV}\right)^3$

$H = 0.6\, sec^{-1} \left(\frac{60keV}{1MeV}\right)^2$

To find the criteria for η where nucleosynthesis is Possible, we need $\Gamma > H$ i.e.

we get $\eta > 4 \times 10^{-12}$

(b) Given $\eta > 4\times10^{-12}$ and $\mu_2 = \mu_p + \mu_n$.

Given the Saha equation:

$$\frac{n_2}{n_p n_n} = \left(\frac{2\pi}{m_n T}\,\frac{2\pi}{m_p T}\,\frac{m_2 T}{2\pi}\right)^{3/2} e^{B/T}$$

$$\frac{n_2}{n_n} = \left(\frac{T}{m_p}\right)^{3/2} e^{B/T} \cdot \left(\frac{4\pi^2}{m_n T^2}\right)^{3/2} n_p$$

Substituting for $n_p = g\left(\frac{1.2}{\pi^2}\right)T^3 \times \frac{3}{4}$ & using the formula for chemical potential we get.

$$\frac{n_2}{n_n} = \left(\frac{T}{m_p}\right)^{3/2} e^{B/T}$$

2. For a collisionless motion, particle freely propogate in phase space according to the Liouville equation, i.e., if f is the distribution function, then,

$$\frac{df}{d\lambda} = 0.$$

Let f be a function of x and p ie $f = f(x^\mu, p^\mu)$.

Then $\frac{df}{d\lambda} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\lambda} + \frac{\partial f}{\partial p^\mu}\frac{dp^\mu}{d\lambda}$, where λ is the affine parameter.

Now ~~&~~ let us consider the geodesic equation on momentum, i.e.,

$$\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta = 0$$

$$\Rightarrow \frac{dp^\mu}{d\lambda} = -\Gamma^\mu_{\alpha\beta} p^\alpha p^\beta$$

$$\Rightarrow \frac{df}{d\lambda} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\lambda} + \frac{\partial f}{\partial p^\mu}\left(-\Gamma^\mu_{\alpha\beta} p^\alpha p^\beta\right)$$

~~[illegible]~~ $\frac{dx^\mu}{d\lambda} = p^\mu$, then,

$$\therefore \frac{df}{d\lambda} = 0 = \frac{\partial f}{\partial x^\mu} p^\mu - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial f}{\partial p^\mu} = 0$$

$$\Rightarrow \frac{\partial f}{\partial x^\mu} p^\mu - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \frac{\partial f}{\partial p^\mu} = 0 \quad \text{—(a)}$$

We know that $g_{\mu\nu} p^\mu p^\nu = -m^2$ for massive particle and ~~Let us find out~~ $g_{\mu\nu} p^\mu p^\nu = 0$ for massless particle.

Here we are considering photons, so $g_{\mu\nu} p^\mu p^\nu = 0$

Let us find what $\Gamma^0_{ij} p^i p^j$ is.

$$\Gamma^0_{ij}\, p^i p^j = -g^{00} g_{ij}\, \delta_{ij} \frac{\dot{a}(t)}{a(t)} p^i p^j$$

$$= -g^{00} \delta_{ij} \frac{\dot{a}}{a} (g_{ij} p^i p^j) = 0 \quad \text{(for photons.)}$$

So, eqn (a) becomes,

$$p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^i_{\alpha\beta}\, p^\alpha p^\beta \frac{\partial f}{\partial p^i} = 0 \quad \text{— (b)}$$

For homogeneous universe, the distribution function is independent of position. , then eqn (b) becomes,

$$p^0 \frac{\partial f}{\partial x^0} - \Gamma^i_{\alpha\beta}\, p^\alpha p^\beta \frac{\partial f}{\partial p^i} = 0 \quad \text{— (c)}$$

For a flat universe in FRW metric, $(k=0)$, non zero Γ are

for $\quad \Gamma^i_{0j} = \tfrac{1}{2} g^{ii}\, \partial_0 g_{ij}\, \delta_{ij}$

$$= \tfrac{1}{2} g^{ii} g_{ij}\, 2 \frac{\dot{a}(t)}{a(t)} \delta_{ij} = \frac{\dot{a}(t)}{a(t)} = H.$$

rest all terms are 0. $\quad [ds^2 = c^2 dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)]$

Then eqn c become,

$$p^0 \frac{\partial f}{\partial x^0} - \cancel{\Gamma^i_{\alpha\beta}}\, \Gamma^i_{0j}\, p^0 p^j \frac{\partial f}{\partial p^i} = 0$$

$$\Rightarrow \frac{\partial f}{\partial t} - \Gamma^i_{0i}\, p^i \frac{\partial f}{\partial p^i} = 0 \quad [i=j,\ \text{then only } \Gamma^i_{0j} \text{ exist}].$$

$$\Rightarrow \frac{\partial f}{\partial t} - \frac{\dot{a}}{a} p^i \frac{\partial f}{\partial p^i} = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial t} = -\frac{\dot{a}}{a}\, p_i \frac{\partial f}{\partial p_i} \quad \text{—(d)}$$

The number density $n_\gamma = \int \frac{d^3p}{(2\pi)^3} f = \frac{1}{2\pi^2}\int p^2 f\, dp$

$$= \frac{1}{2\pi^2}\left[\frac{p^3}{3} f\right]_0^\infty - \frac{1}{2\pi^2}\int \frac{p^3}{3}\frac{\partial f}{\partial p}\, dp$$

$p^3 f \to 0$ as $p \to 0$

$p^3 f \to 0$ as $p \to \infty$ as $f \to 0$

So $\quad n_\gamma = -\frac{1}{3}\frac{1}{2\pi^2}\int p^3 \frac{\partial f}{\partial p}\, dp = -\frac{1}{3}\frac{1}{(2\pi)^3}\int p \frac{\partial f}{\partial p}\, d^3p$

$$\Rightarrow \quad -3n_\gamma = \frac{1}{(2\pi)^3}\int p\frac{\partial f}{\partial p}\, d^3p \quad \text{— (e)}$$

So in equation d, if we integrate by d^3p on both side, we get,

$$\int \frac{1}{(2\pi)^3}\frac{\partial f}{\partial t}\, d^3p_i = -\frac{\dot{a}}{a}\int p_i \frac{\partial f}{\partial p_i}\frac{d^3p_i}{(2\pi)^3} \quad \text{—(f)}$$

$$n_\gamma = \int \frac{d^3p}{(2\pi)^3} f$$

$$\Rightarrow \quad \frac{\partial n_\gamma}{\partial t} = \int \frac{d^3p}{(2\pi)^3}\frac{\partial f}{\partial t} \quad \text{—(g)}$$

Using eqn (g) and (e) on eqn (f), we get

$$\frac{\partial n_\gamma}{\partial t} = -\frac{\dot{a}}{a}\, (-3n_\gamma)$$

$$= -3H n_\gamma$$

$$\Rightarrow \quad \dot{n}_\gamma + 3H n_\gamma = 0.$$

$$\frac{dn_\gamma}{n_\gamma} = -3\frac{da}{a} \quad \Rightarrow \ln n_\gamma = \ln a^{-3} + k.$$

$$\boxed{\Rightarrow \quad n_\gamma = k a^{-3} \qquad n_\gamma \propto a^{-3}}$$