MODELING CHANNEL FLOW WITH NAVIER-STOKES EQUATIONS

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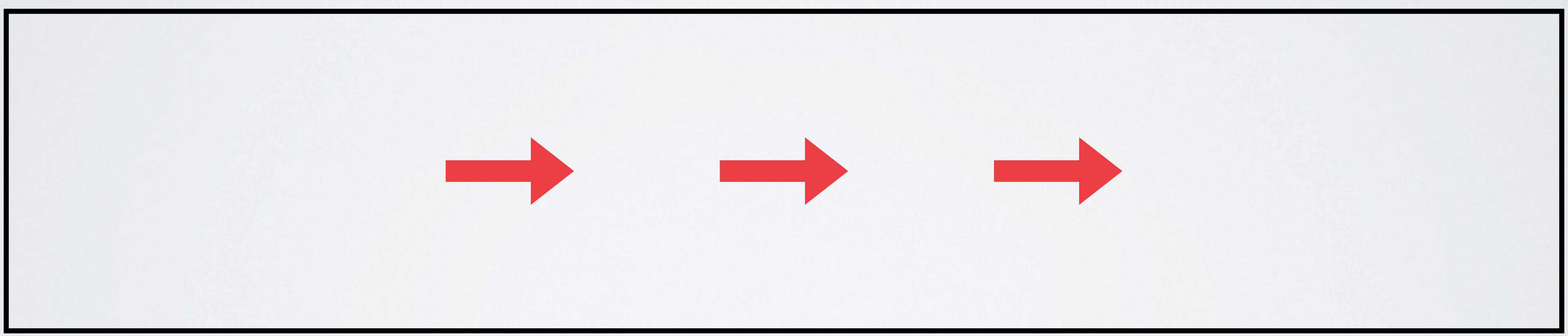
Physics 566: Computational Physics, Spring 2019

INTRODUCTION

- Difficulty of solving for fluids
  - Analytical
  - Leads us to computational methods
- Can be done for specific systems
  - Channel flow

PROBLEM

- We want to model the velocity of a fluid at various points in a channel



$$\frac{\partial \mathbf{u}}{\partial t} + \left(\mathbf{u} \cdot \nabla\right)\mathbf{u} = \nu \Delta \mathbf{u} - \frac{1}{\rho} \nabla P$$

THEORY

- System of 2nd Order PDEs

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\left(\frac{\partial p_0}{\partial x} + \frac{\partial p_1}{\partial x}\right) + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

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$$-\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) = \left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial y}\right)^2$$

INITIAL CONDITIONS AND PHYSICAL PARAMETERS

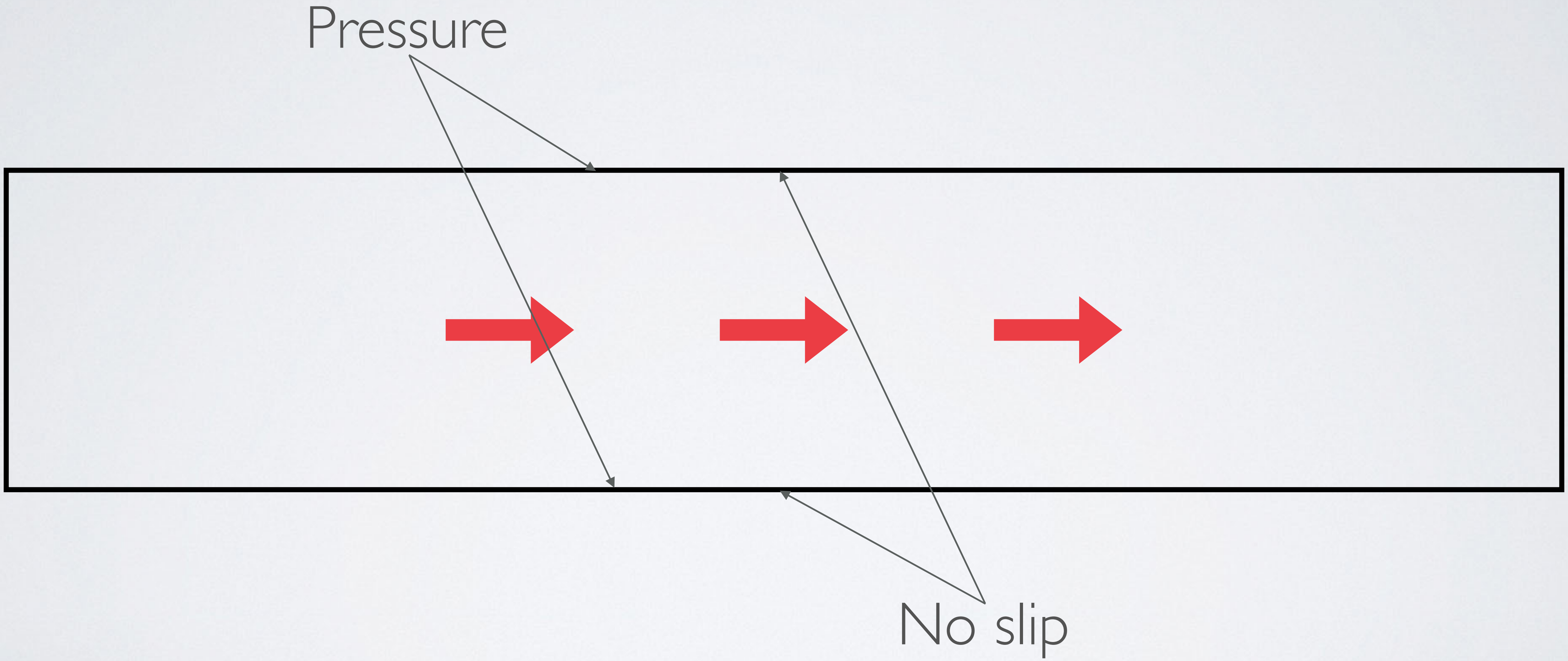
$$u = v = 0$$

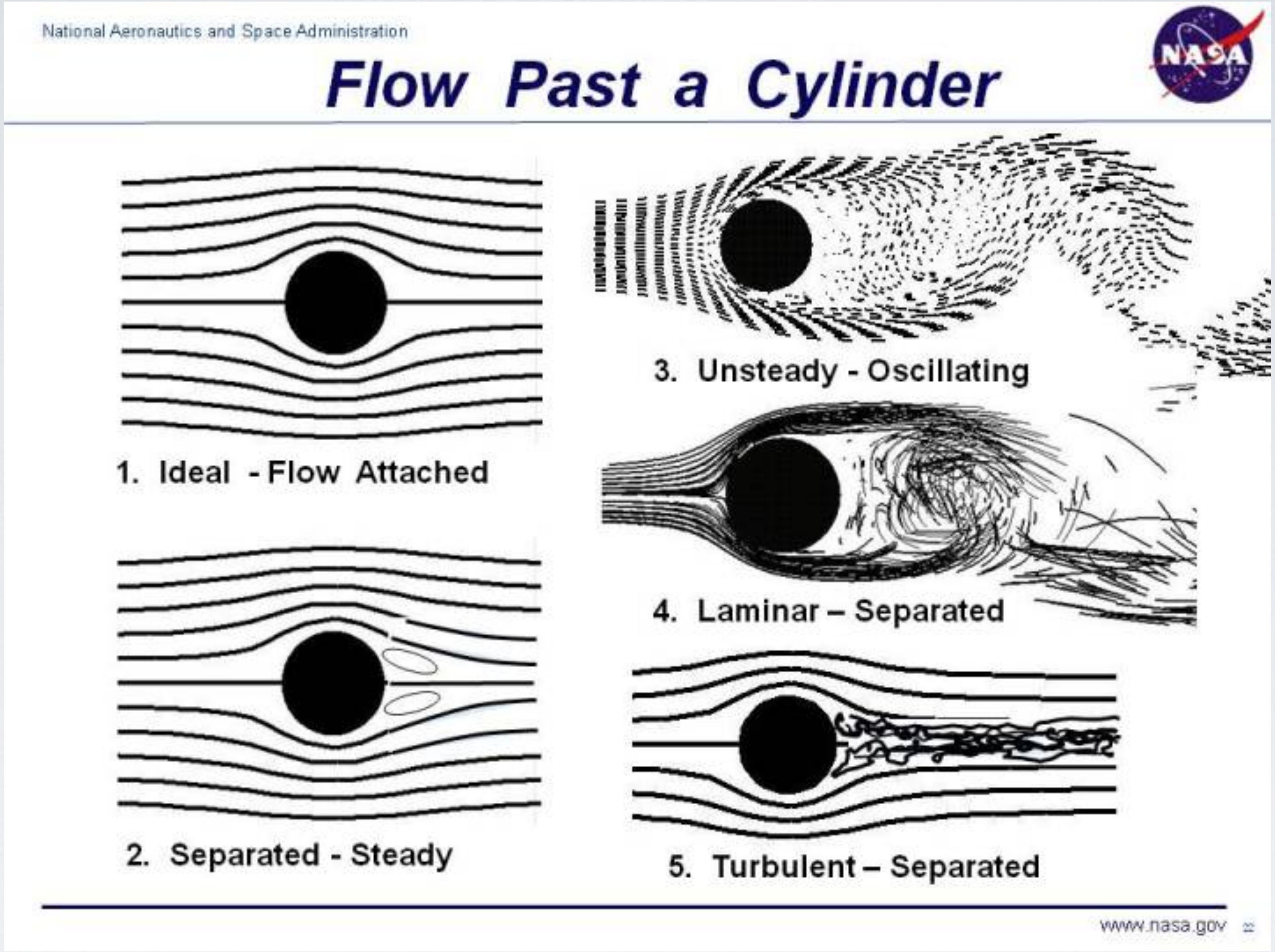
$$P = C$$

$$\rho = 1$$

$$\nu = 0.5$$

BOUNDARY CONDITIONS





TURBULENT FLOWS

- Higher Reynolds number leads to greater turbulence
  - can predict departure from laminar flow

$$\mathrm{Re} = \frac{\rho u L}{\mu}$$

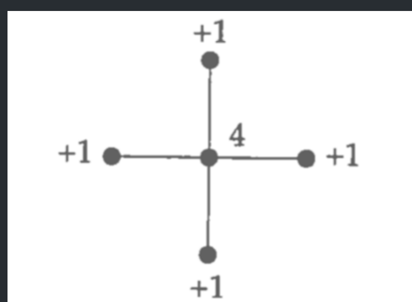
METHODS

- Forward-time centered-space (FTCS) method
  - Extended the 1-d versions done in homework to 2-d
  - Used for more involved mathematics

LOGIC

- while loop
    - set boundary conditions for $y = 0, L/2$
    - check & set obstruction boundary conditions
    - pressure over the region
    - velocity on the interior
    - velocity boundary conditions for $x = 0, L$
    - exchange arrays
- return velocities, pressure

```python
"""Velocity"""
up[1:-1, 1:-1] = (u[1:-1, 1:-1]
                  - u[1:-1, 1:-1] * dt / a
                  * (u[1:-1, 1:-1] - u[1:-1, 0:-2])
                  - v[1:-1, 1:-1] * dt / a
                  * (u[1:-1, 1:-1] - u[0:-2, 1:-1])
                  - dt / (2 * rho * a)
                  * (p[1:-1, 2:] - p[1:-1, 0:-2])
                  + nu * dt / a**2
                  * (u[1:-1, 2:] - 2 * u[1:-1, 1:-1] + u[1:-1, 0:-2]
                  + u[2:, 1:-1] - 2 * u[1:-1, 1:-1] + u[0:-2, 1:-1])
                  + P * dt)
```



x-velocity on the interior of the region

$$u(x,y) \leftarrow u(x,y) - \frac{u(x,y)\mathrm{d}t}{a}\Big[u(x,y) - u(x-a,y)\Big] - \frac{v(x,y)\mathrm{d}t}{a}\Big[u(x,y) - u(x,y-a)\Big]$$

$$-\frac{\mathrm{d}t}{2\rho a}\Big[p(x+a,y) - p(x-a,y)\Big] + P\mathrm{d}t$$

$$+\nu\left[\frac{\mathrm{d}t}{a^2}\Big(\big(u(x+a,y) - 2u(x,y) + u(x-a,y)\big) + \big(u(x,y+a) - 2u(x,y) + u(x,y-a)\big)\Big)\right]$$

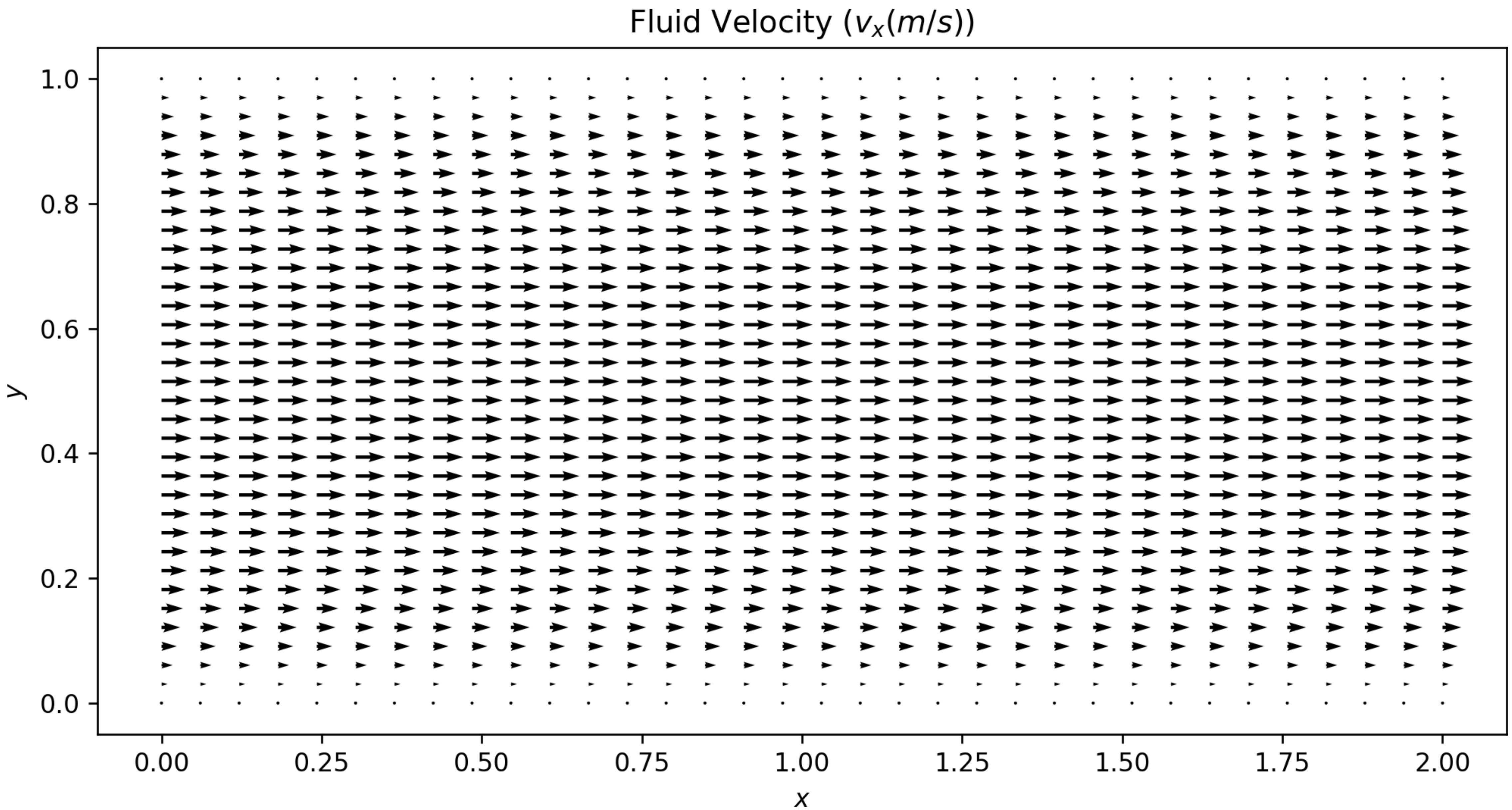
```python
if obstacle == 'ball':
    x0,y0 = N/2,N/2      #origin
    r = 0.1*N      #radius
    xgrid,ygrid = np.ogrid[-x0:N-x0+1,-y0:N-y0+1]
    mask = xgrid**2 + ygrid**2 <= r**2
```

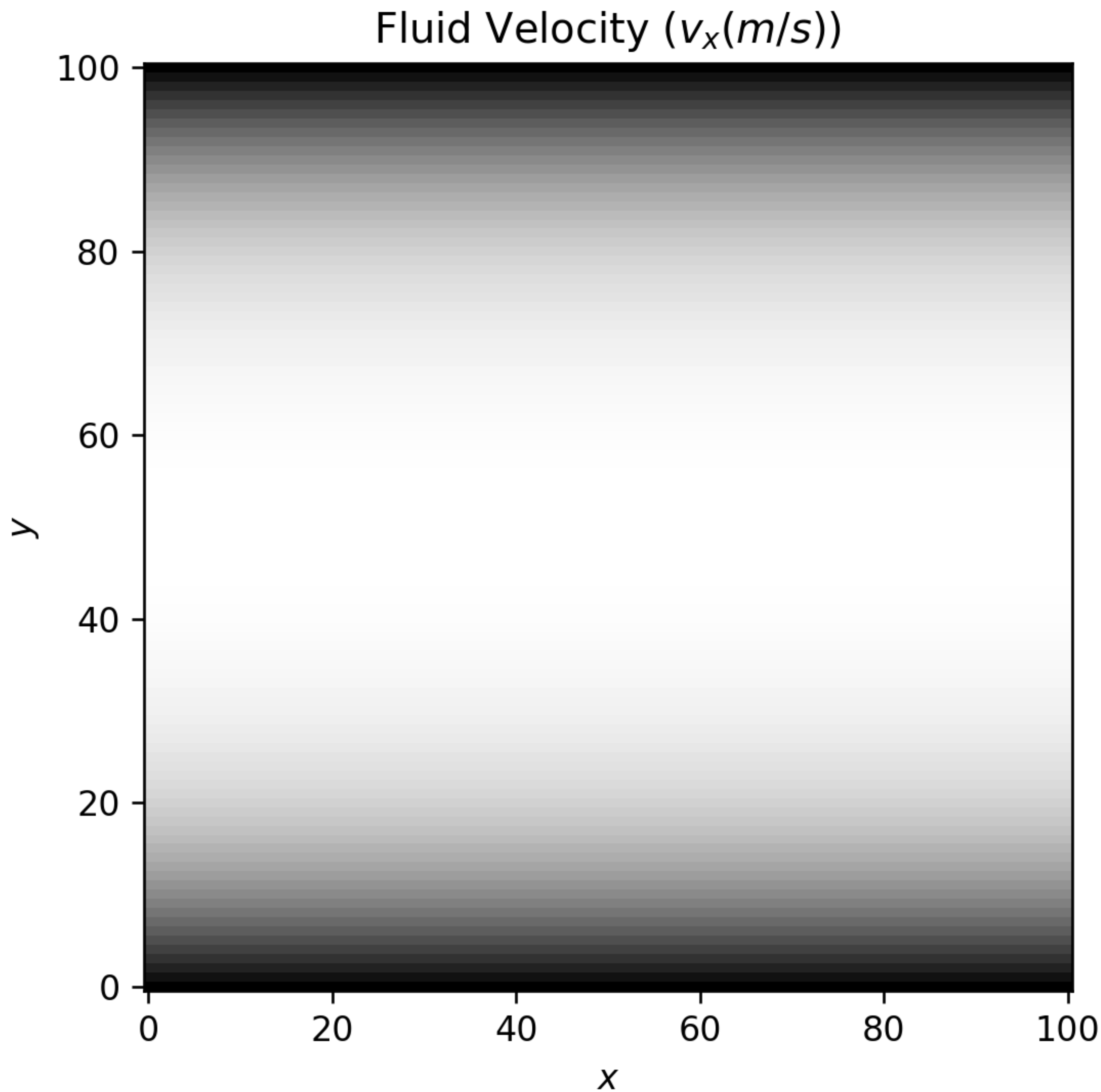
```python
if obstacle != None:
    #velocities in object's interior = 0
    u[mask] = 0.0
    v[mask] = 0.0
```

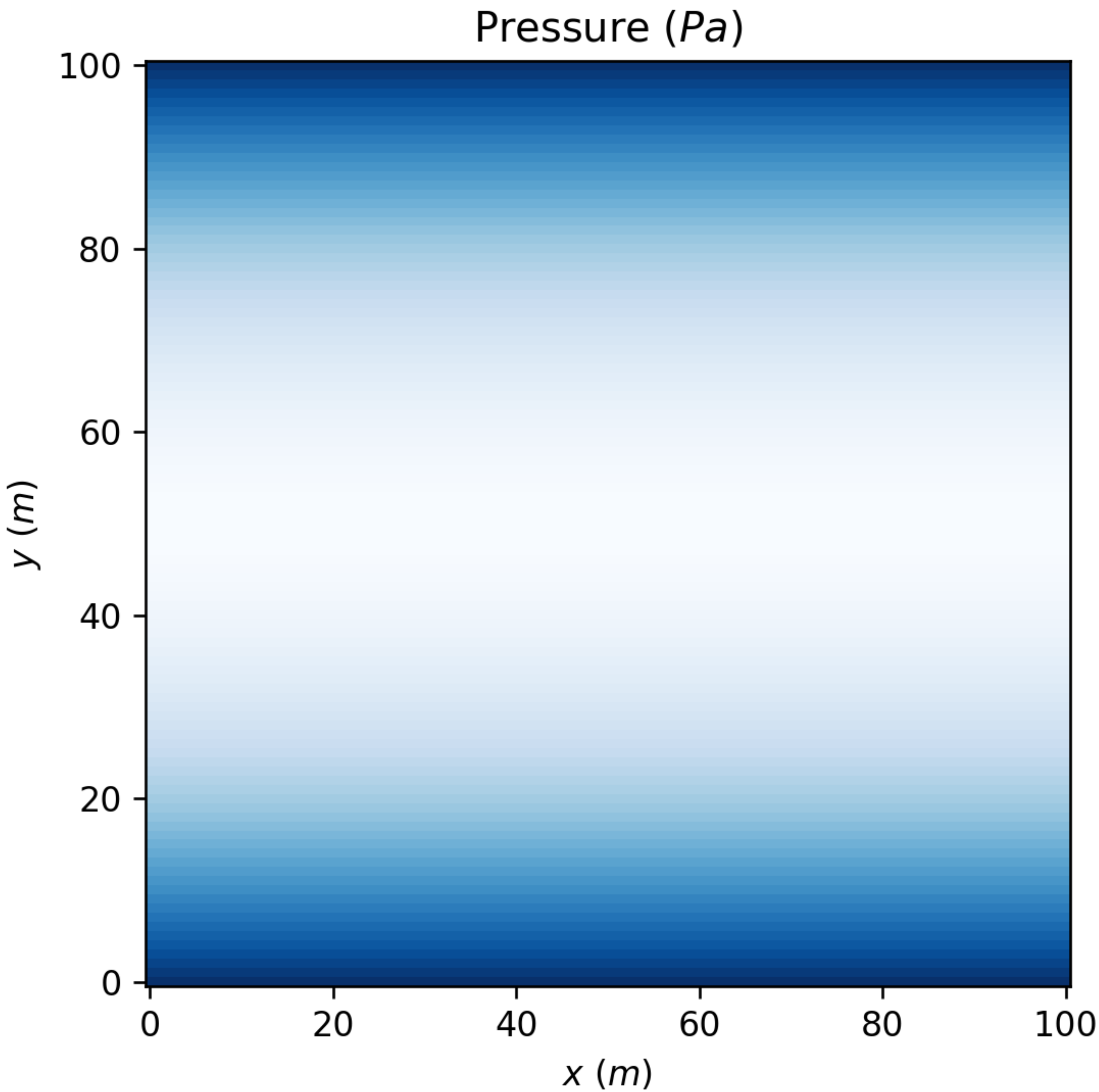
```python
if obstacle == 'wing':
    #derivative of pressure at object boundary = 0
    #along bottom edge
    p[int(N/2),int(N/6):int(5*N/6)] =\
     p[int(N/2)-1,int(N/6):int(5*N/6)]

    #along top edges
    p[int(N/2)+2,int(N/6):int(N/3)] =\
     p[int(N/2)+3,int(N/6):int(N/3)]
    p[int(N/2)+3,int(N/3):int(N/2)] =\
     p[int(N/2)+4,int(N/3):int(N/2)]
    p[int(N/2)+4,int(N/2):int(2*N/3)] =\
     p[int(N/2)+5,int(N/2):int(2*N/3)]
    p[int(N/2)+3,int(2*N/3):int(5*N/6)] =\
     p[int(N/2)+4,int(2*N/3):int(5*N/6)]
```

RESULTS

- I ran the code for a few different physical systems
  - A simple channel
    - Compared to the analytical solution
  - Obstructed flows
    - Ball
    - Wing


Fluid Velocity ($v_x(m/s)$)
y
1.0
0.8
0.6
0.4
0.2
0.0
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
x


Fluid Velocity ($v_x(m/s)$)
100
80
60
40
20
0
y
0
20
40
60
80
100
x

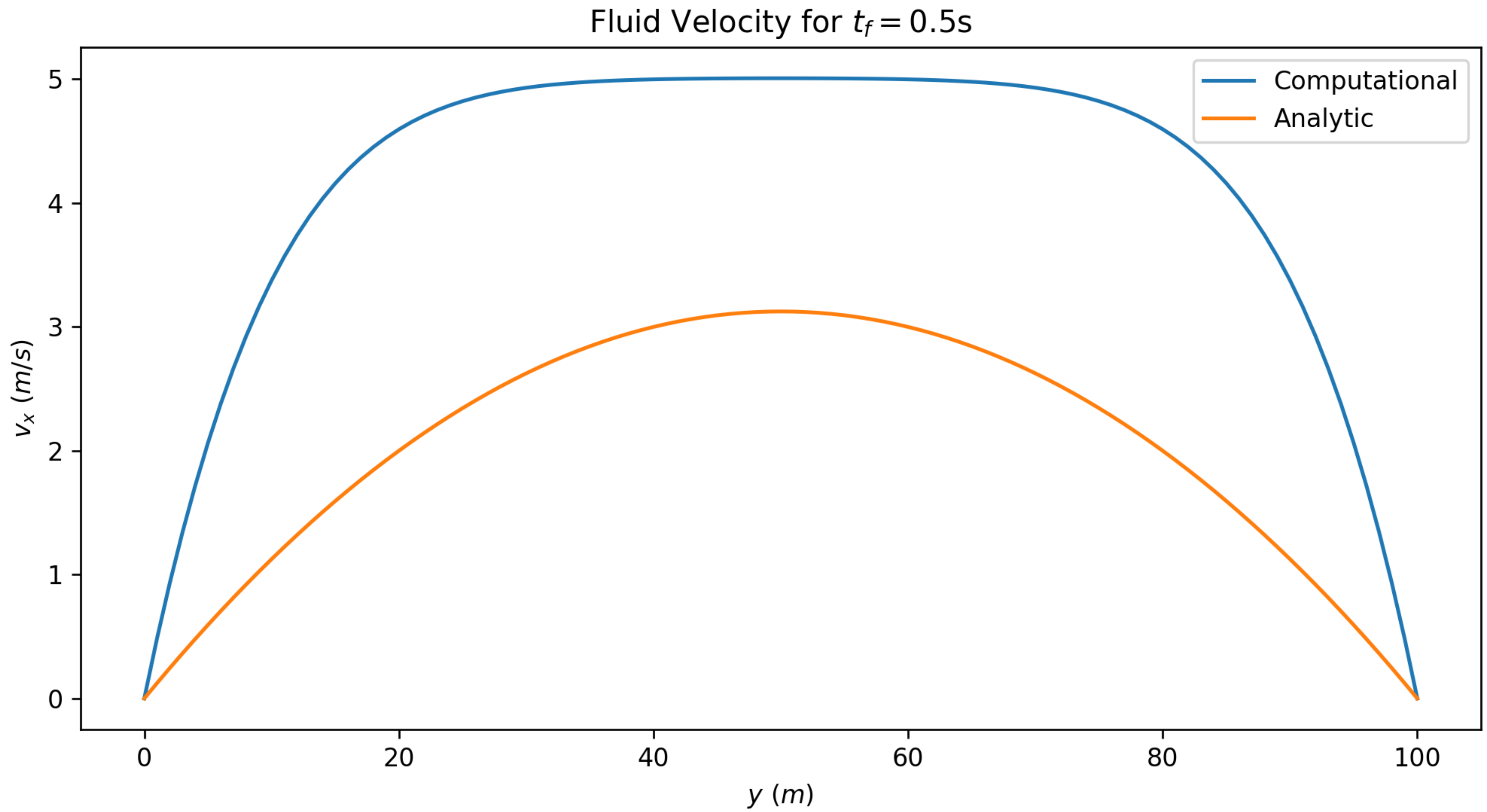

Pressure (Pa)
100
80
60
40
20
0
y (m)
0
20
40
60
80
100
x (m)

COMPARISON

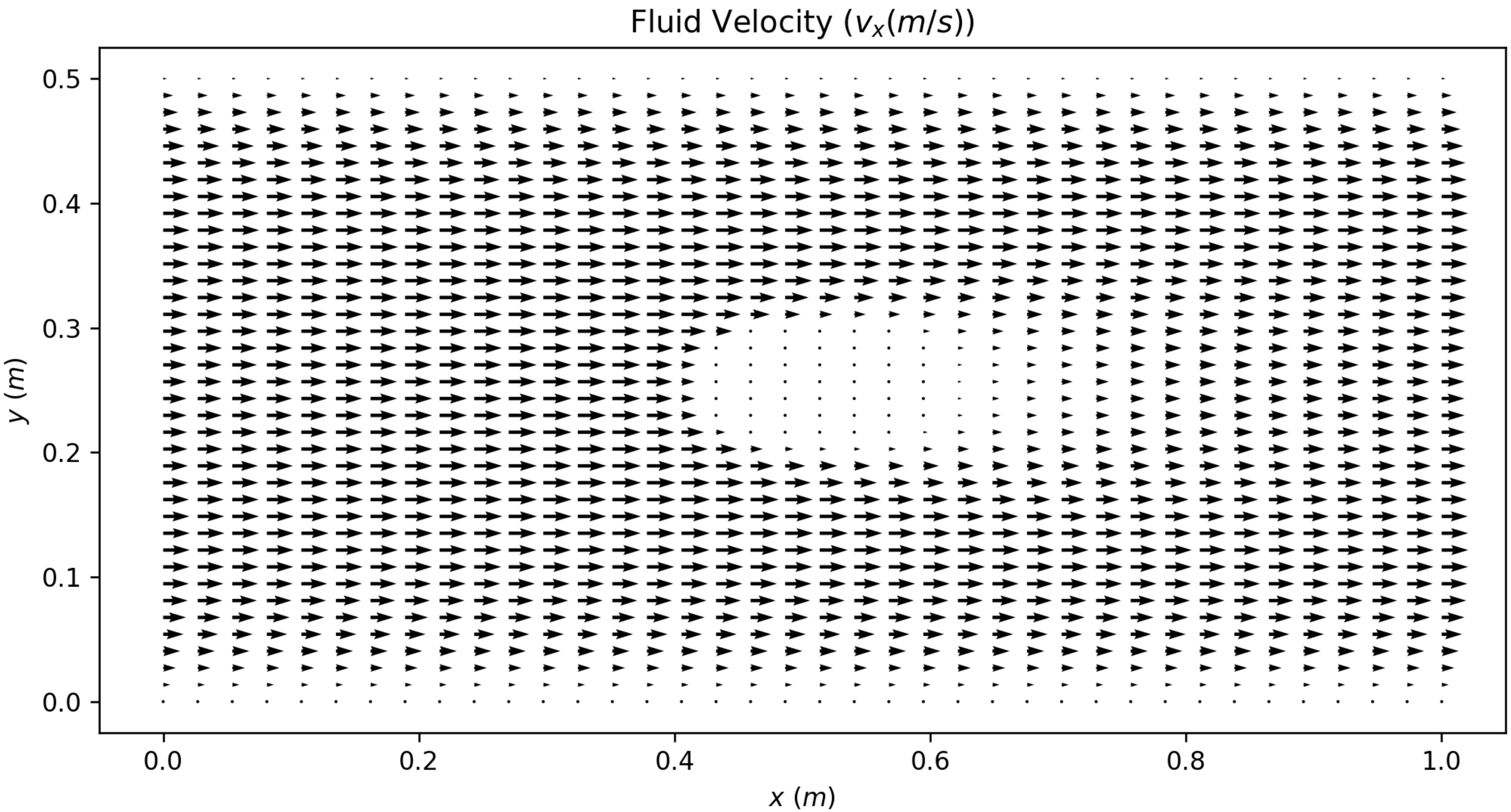
- For the simple channel flow case, an analytical solution exists

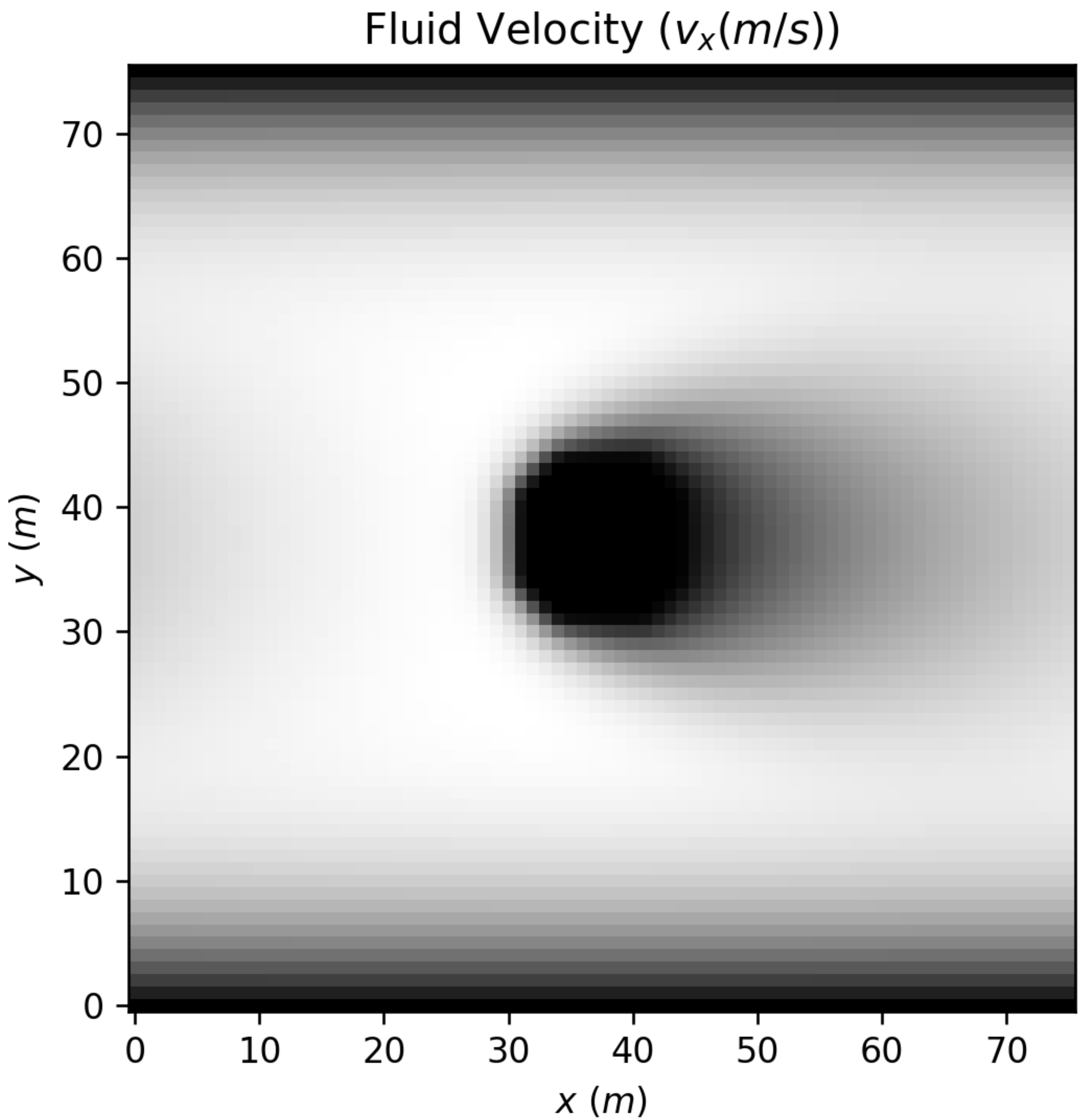
$$u(y) = -\frac{10}{\nu}\left(\frac{y^2}{2} + C_1 y + C_2\right)$$

$$u(y) = -\frac{10}{\nu}\left(\frac{y^2}{2} - \frac{L}{4}y\right)$$

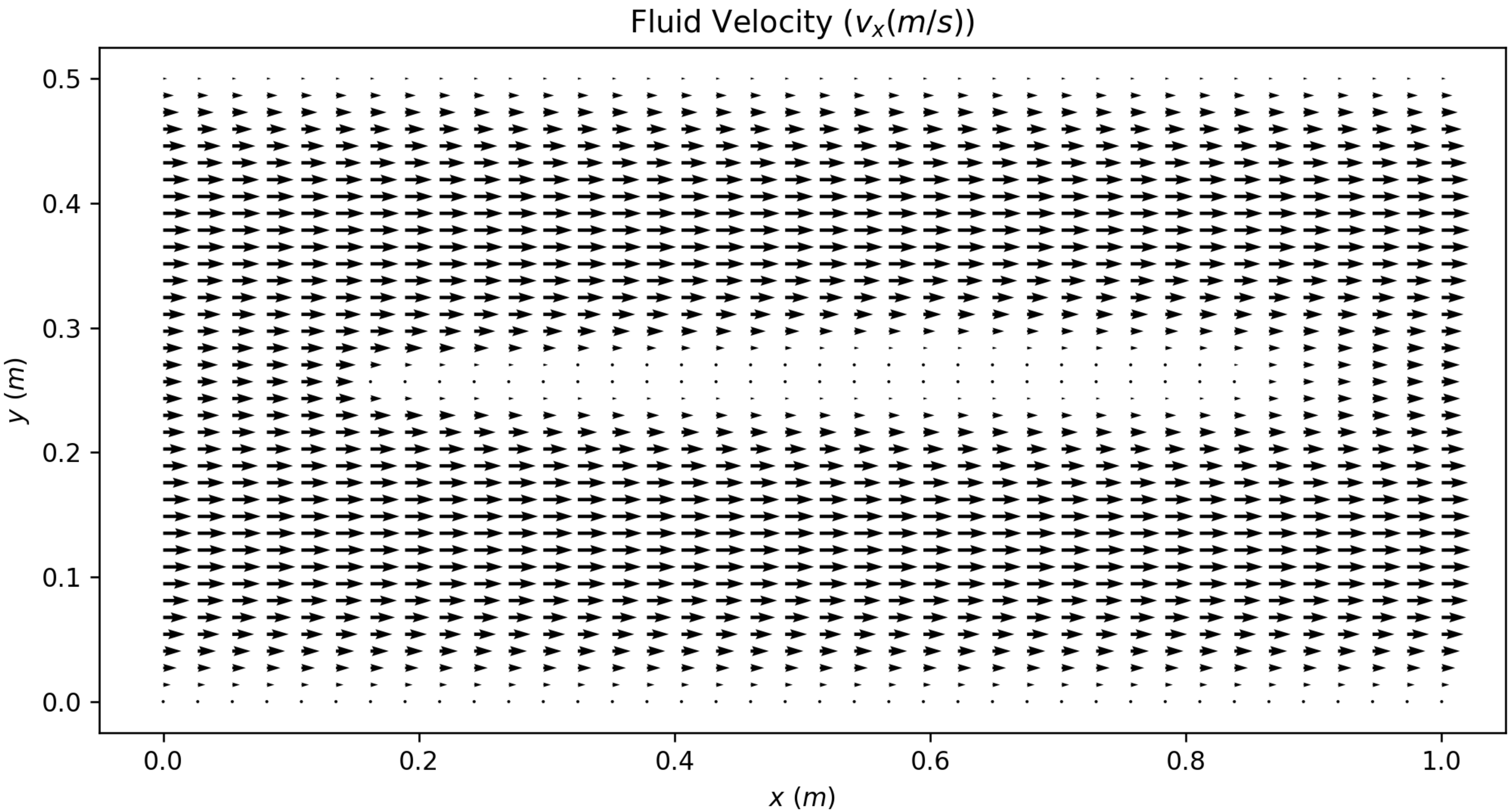


Fluid Velocity for $t_f = 0.5$s
Computational
Analytic
v_x (m/s)
y (m)
0
1
2
3
4
5
0
20
40
60
80
100

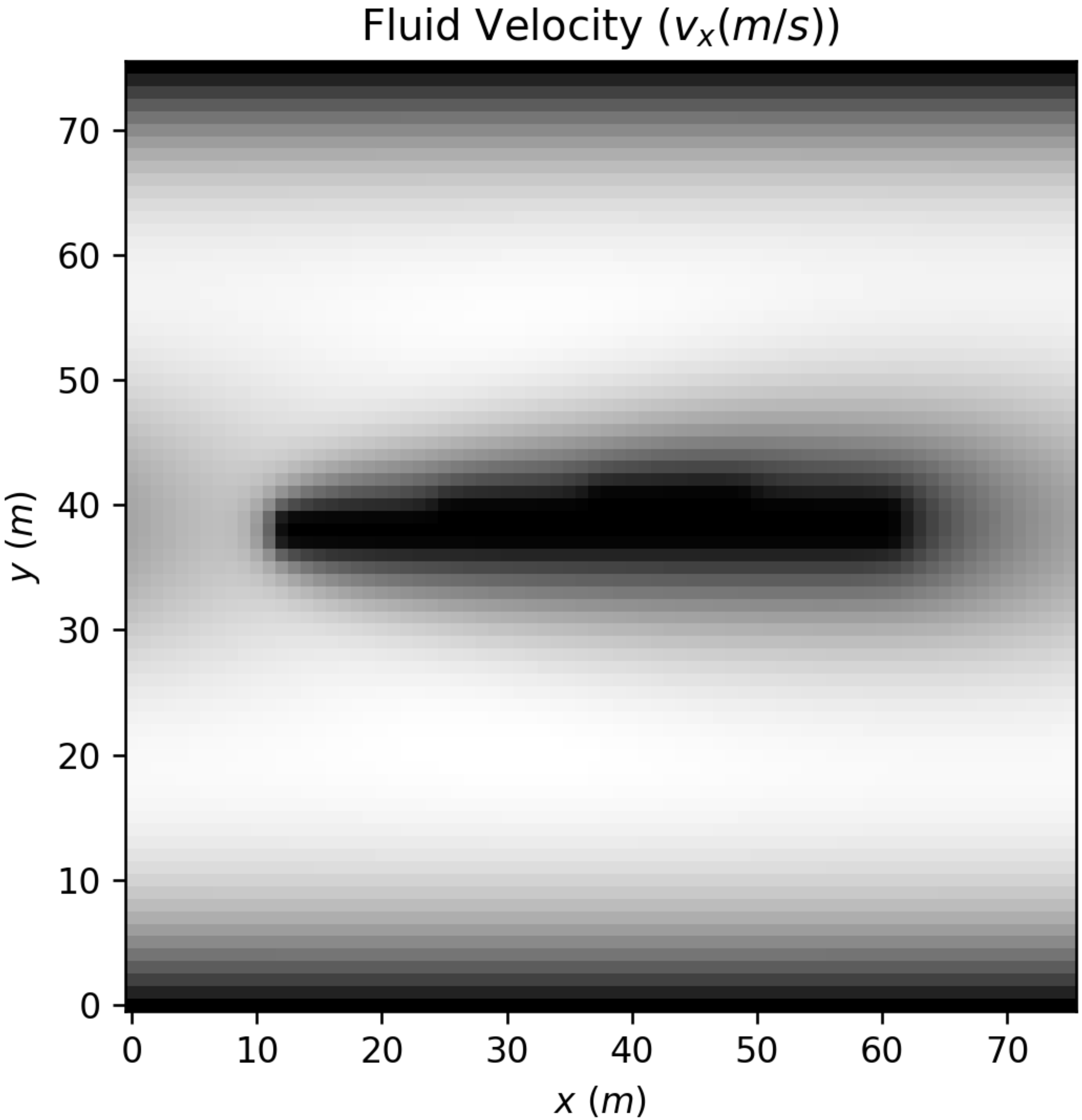

Fluid Velocity ($v_x(m/s)$)
0.5
0.4
0.3
0.2
0.1
0.0
y (m)
0.0
0.2
0.4
0.6
0.8
1.0
x (m)



Fluid Velocity ($v_x(m/s)$)
70
60
50
40
30
20
10
0
0
10
20
30
40
50
60
70
y (m)
x (m)



Fluid Velocity ($v_x(m/s)$)
0.0
0.1
0.2
0.3
0.4
0.5
y (m)
0.0
0.2
0.4
0.6
0.8
1.0
x (m)


Fluid Velocity ($v_x(m/s)$)
y (m)
x (m)
0
10
20
30
40
50
60
70

CONCLUSION

- Computational Fluid Dynamics
  - Used to model complex systems w/o closed form solutions
  - Computationally expensive
  - Simplified systems
- Monitor convergence of solutions
  - Mine did relatively well under the time limits I tested

REFERENCES

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- Nair Presentation.pdf
- Nair Final Project.ipynb
- Nair Final Project.pdf
- channel_flow.py