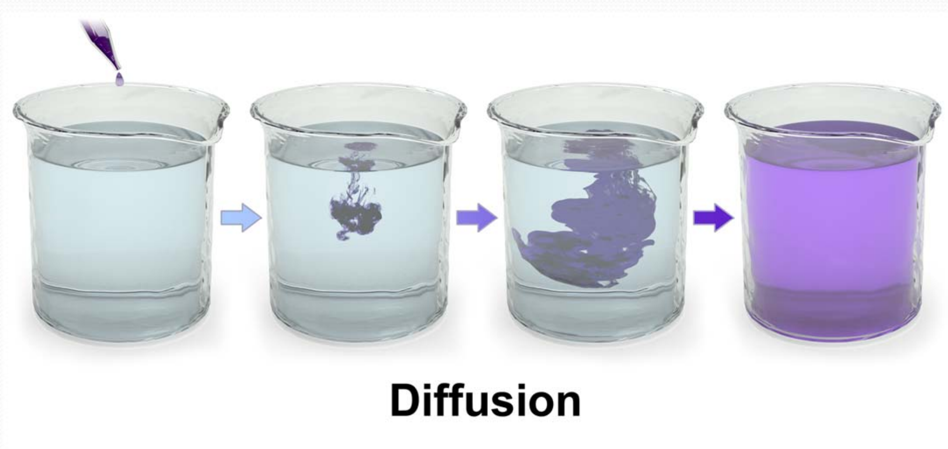
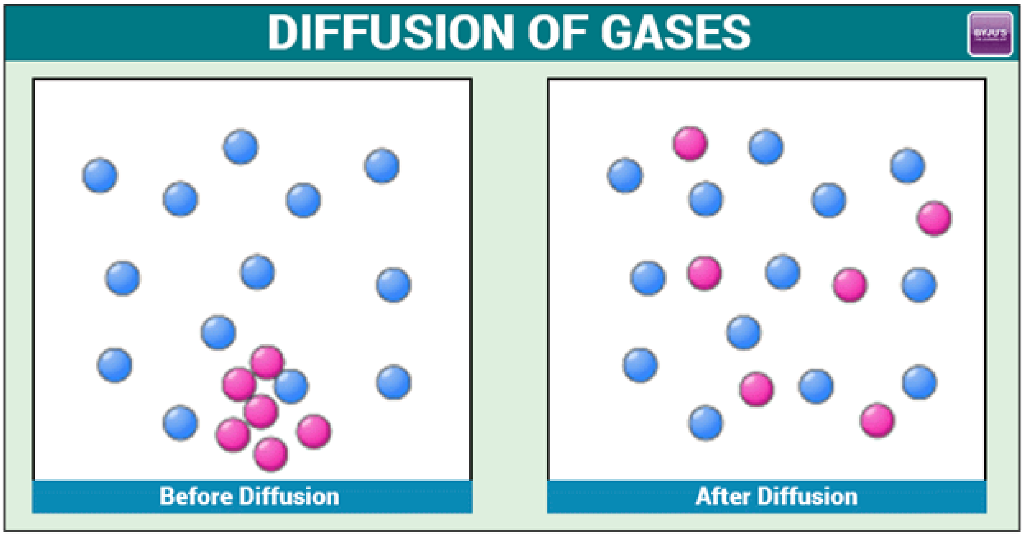
Diffusion & Viscosity:

Navier-Stokes Equation





Diffusion Equation

Imagine a quantity C(x,t) representing a local property in a fluid, eg.

- thermal energy density
- concentration of a pollutant
- density of photons propagating diffusively through a scattering medium

For a fluid at rest, V=0, the diffusive transport of the quantity C in the fluid is described by the Diffusion Equation,

$$\frac{\partial C}{\partial t} = \vec{\nabla} \cdot D \vec{\nabla} C$$

In this expression, D is the diffusion coefficient,

$$D = \frac{v_\sigma \lambda}{3}$$

with v_σ the velocity of the diffusing particles, and λ the mean free path.

Navier-Stokes Equation

Viscous Force

- In general, the viscous force f^{visc} includes 2 different aspects, that of

  - shear viscosity η
  - bulk viscosity ζ

  entailing the following full viscous force

$$\vec{f}^{\,visc} = \eta \nabla^2 \vec{v} + (\varsigma + \frac{1}{3}\eta)\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right)$$

  which for incompressible flow, $\nabla \cdot \vec{v} = 0$, is restricted to

$$\vec{f}^{\,visc} = \eta \nabla^2 \vec{v}$$

Navier-Stokes Equation

- For a fluid with (shear) viscosity η, the equation of motion is called the Navier-Stokes equation. In its most basic form, ie.for incompressible media

$$\boxed{\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \vec{\nabla} \vec{v} = -\vec{\nabla} p + \eta \nabla^2 \vec{v}}$$

- Without any discussion, this is THE most important equation of hydrodynamics.
- While the Euler equation did still allow the description of many analytically tractable problems, the nonlinear viscosity term in the Navier-Stokes equation makes the solving of the NS equation very complicated.
- There are only a few situations that allow analytical solutions for the NS equation, the remainder needs to be solved numerically/computationally.

Navier-Stokes Equation

- The general and full Navier –Stokes equation, for a fluid with
  - shear viscosity η
  - bulk viscosity ζ

  is given by

$$\boxed{\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \vec{\nabla} \vec{v} = -\vec{\nabla} p + \eta \nabla^2 \vec{v} + (\varsigma + \frac{1}{3}\eta) \vec{\nabla}(\vec{\nabla} \cdot \vec{v})}$$

Reynolds Number

- The Reynolds number is the measure of the importance of viscous effects of a flow – hereby assumming the bulk viscosity ζ=0 – and is defined as

  the ratio of the magnitude of the inertial force -
  magnitude of the viscous force

$$\boxed{\mathrm{Re} = \frac{magnitude\ inertial\ \ force}{magnitude\ \ viscous\ \ force} \equiv \frac{|\rho(\vec{v}\cdot\vec{\nabla})\vec{v}|}{|\eta\nabla^2\vec{v}|}}$$

- For large Reynolds number, the flow gets unstable, and finally becomes turbulent.

Reynolds Number

- The Reynolds number is the ratio of the magnitude of the inertial force to the magnitude of the viscous force

$$\boxed{\mathrm{Re} = \frac{magnitude\ inertial\ \ force}{magnitude\ \ viscous\ \ force} \equiv \frac{|\rho(\vec{v}\cdot\vec{\nabla})\vec{v}|}{|\eta\nabla^2\vec{v}|}}$$

- We can find an order of magnitude rough estimate for the Reynolds number. With U the characteristic magnitude of the velocity in a system of characteristic size L, we have

$$\left.\begin{aligned} |(\vec{v}\cdot\vec{\nabla})\vec{v}| &\sim \frac{U^2}{L} \\ |\eta\ \nabla^2\vec{v}| &\sim \frac{\rho\nu U}{L^2} \end{aligned}\right\} \qquad \boxed{\mathrm{Re} \sim \frac{UL}{\nu}}$$

Navier-Stokes Equation: analytical soln's

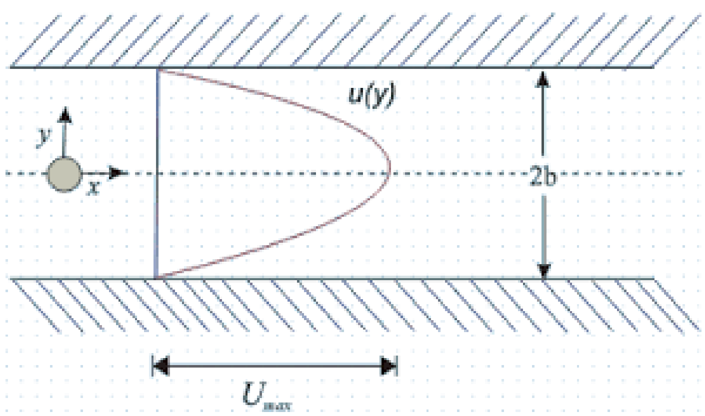
- Due to the high level of nonlinearity and complexity of the full compressible Navier-Stokes equations , there are hardly any analytical solutions known of the Navier-Stokes equation.

$$\boxed{\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \vec{\nabla} \vec{v} = -\vec{\nabla} p + \eta \nabla^2 \vec{v}}$$

- One may try to find some specific configurations that would allow an analytical treatment. This involves simplifying the equations by making the following assumptions:
  - about the fluid
  - about the flow
  - geometry of the problem
- Typical assumptions are:
  - laminar flow
  - steady flow
  - incompressible flow
  - 2-D configuration
  - flow between plates
- Examples are:
  - parallel flow in a channel
  - Couette flow
  - Hagen-Poiseuille flow, ie. flow in a cylindrical pipe.

Navier-Stokes Equation: Channel flow

- Consider the following configuration:
  - flow of a fluid through a channel
  - steady flow
  - incompressible flow
  - axisymmetric geometry (2-D problem)



- the 2-D flow field is represented by a 2-D velocity field, with u the component in the x-direction, v in the y-direction $\vec{v} = \begin{pmatrix} u \\ v \end{pmatrix}$

Navier-Stokes Equation: Channel flow

- the 2-D flow field is represented by a 2-D velocity field, with u the component in the x-direction, v in the y-direction

- the flow of the system is then described by the
  (a) continuity equation
  (b) Navier-Stokes equation

$$\boxed{\rho\frac{\partial\vec{v}}{\partial t}+\rho\vec{v}\cdot\vec{\nabla}\vec{v}=-\vec{\nabla}p+\eta\,\nabla^2\vec{v}}$$

- which for the system at hand simplify to:

continuity equation:
(notice: incompressibility)

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

x-momentum (NS):

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\eta\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

y-momentum (NS):

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\eta\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

Navier-Stokes Equation: Channel flow

- Boundary condition:
  the flow is constrained by flat parallel walls of the channel,

$$v_y=v=0$$

$$\Downarrow$$

$$\frac{\partial v}{\partial y}=\frac{\partial v}{\partial x}=\frac{\partial^2 v}{\partial y^2}=\frac{\partial^2 v}{\partial x^2}=0$$

- Continuity equation:

$$\frac{\partial u}{\partial x}=-\frac{\partial v}{\partial y}=0; \qquad \frac{\partial^2 u}{\partial x^2}=0$$

- Using these relations, we end up with the Navier-Stokes equations:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}+\eta\frac{\partial^2 u}{\partial y^2}=0$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y}=0$$

Navier-Stokes Equation: Channel flow

- Given that

$$\frac{\partial u}{\partial x} = 0$$

we immediately infer that u(x,y) must be independent of x. Hence

$$\eta \frac{\partial^2 u}{\partial y^2}$$

can only be a function of y, i.e u(x,y)=u(y). This implies, via the relation,

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} + \eta \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{that,} \qquad \frac{\partial p}{\partial x} = \frac{dp}{dx} = cst.$$

and that the general solution for u(y) is given by

$$\boxed{u(y) = \frac{1}{2}\frac{1}{\rho\eta}\frac{\partial p}{\partial x} y^2 + Ay + B}$$

Navier-Stokes Equation: Channel flow

- The general solution for u(y) is given by

$$\boxed{u(y) = \frac{1}{2}\frac{1}{\rho\eta}\frac{\partial p}{\partial x} y^2 + Ay + B}$$

- Using the boundary conditions that the velocity u=0 at the border of the channel, ie. u(±R)=0, the constants A and B get fixed

$$A = 0; \qquad B = -\frac{1}{2}\frac{R^2}{\rho\eta}\frac{dp}{dx}$$

which yields the complete solution for the flow velocity u(y) through the channel:

$$\boxed{u(y) = -\frac{1}{2}\frac{R^2}{\rho\eta}\frac{dp}{dx}\left[1 - \left(\frac{y}{R}\right)^2\right]}$$

Navier-Stokes Equation: Channel flow

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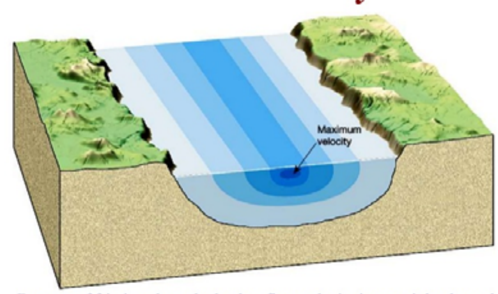
$$u(y) = -\frac{1}{2}\frac{R^2}{\rho\eta}\frac{dp}{dx}\left[1-\left(\frac{y}{R}\right)^2\right]$$

- Flow through a channel thus displays a parabolic velocity distribution, summetric about the central axis. The maximum velocity u_{max} is attained along the central axis,

$$u_{\max} = -\frac{1}{2}\frac{R^2}{\rho\eta}\frac{dp}{dx}$$



Because of friction along the banks, flow velocity in a straight channel is highest near the surface in the middle of the stream.