Substitutions

Let’s say that we are given $x^2 + 3x + 5 = 0$ with roots α and β, and we are asked to find a polynomial equation with roots 2α and 2β.

We can find this using what we have learned and conclude that the new polynomial is $y^2 - (2\alpha + 2\beta)y + 4\alpha\beta = 0$ which if we find using $\alpha + \beta = -\frac{b}{a}$, etc, is $y^2 + 6y + 20 = 0$.

Or, we could just use $y = 2x$ since the roots are doubled.

$x = \frac{y}{2}$, and if we substitute this, we get:

$(\frac{y}{2})^2 + 3(\frac{y}{2}) + 5 = 0$

Multiply by 4 to get a whole equation.

$y^2 + 6y + 20 = 0$

Using this method, we do not need to find each individual root, we only need the relation of the roots.

In the case of reciprocals, it is the same.

For $x^3 + x^2 + 7 = 0$ with roots α, β, γ, we can find the polynomial with roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $\frac{1}{\gamma}$.

$y = \frac{1}{x}$, so $x = \frac{1}{y}$

$(\frac{1}{y})^3 + (\frac{1}{y})^2 + 7 = 0$

Multiply both sides by y^3,

$7y^3 + y + 1 = 0$

In the case of power of roots, we can do like this.

For $2x^3 + 7x^2 - 1 = 0$ with roots α, β, γ, we can find the cubic with roots α^2, β^2, γ^2.

$y = x^2$, so $x = \sqrt{y}$

$2(\sqrt{y})^3 + 7y - 1 = 0$

$7y - 1 = -2(\sqrt{y})^3$

Square both sides,

$(7y - 1) = 4y^3$

$4y^3 = 49y^2 - 14y + 1$

$4y^3 - 49y^2 + 14y - 1 = 0$

Using substitution, we can find $S4$, $S6$, and so on easily.

In $x^4 + x^3 - 5 = 0$ with roots α, β, γ, δ, we can find $S4$ by substituting $y = x^2$ instead.

$x^4 - 5 = -x^3$

Square both sides

$x^8 - 10x^4 + 25 = x^6$

$x^8 - x^6 - 10x^4 + 25 = 0$

Now substitute with y

$y^4 - y^3 - 10y^2 + 25 = 0$

$S4$ of the x polynomial will be $S2$ of the y polynomial.

$S1 = \alpha + \beta + \gamma + \delta = 1$

$S2 = 1^2 - 2\Sigma\alpha\beta$

$S2 = 1 - 2(-10)$

$S2 = 21$

So $S4$ of the x polynomial is 21.