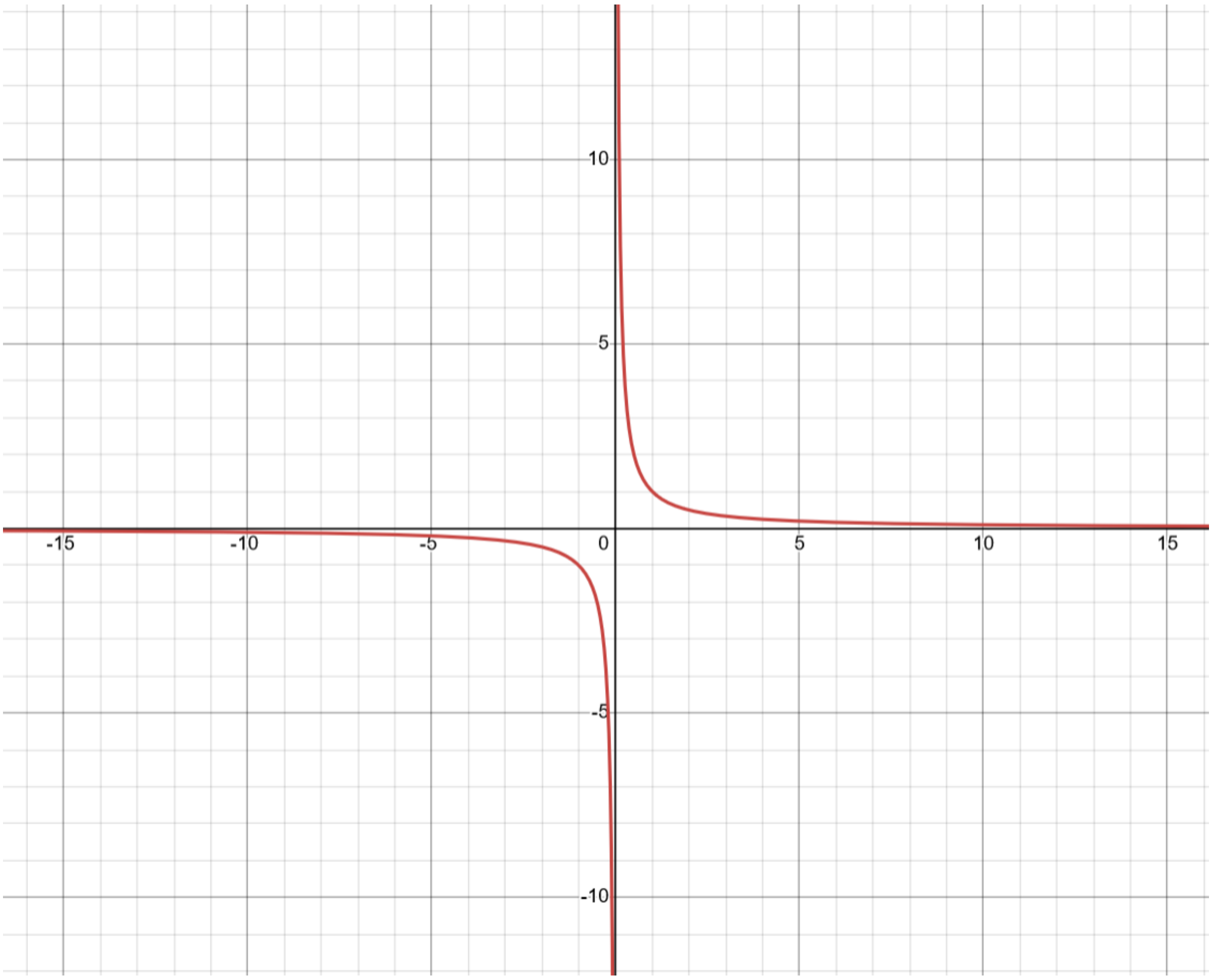
Vertical asymptotes

An asymptote is a line that a curve approaches but does not touch.

Consider the graph $y = \frac{1}{x}$, when $x = 0$, $y = \pm\infty$ and when $y = 0$, $x = \pm\infty$.



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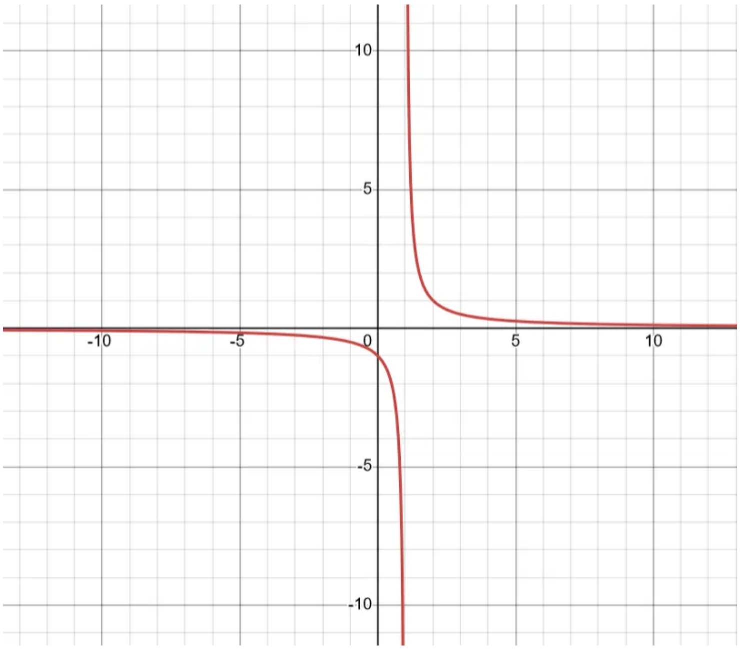
Vertical asymptotes

So the vertical asymptote is basically the x value of which y cannot touch. An easy way to find the asymptote is to make the denominator equal to 0.

For e.g., $y = \frac{1}{x-1}$ has the vertical asymptote of $x = 1$



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Horizontal asymptotes

This is the line which x will approach but not touch. In both examples above, the horizontal asymptote is 0.

Top-heavy fractions

This is a fraction in which the degree of polynomial of the numerator is higher than or equal to the denominator. Examples are $\frac{x}{x-1}$, $\frac{x^2}{x-1}$, $\frac{2x}{3-x}$, etc.

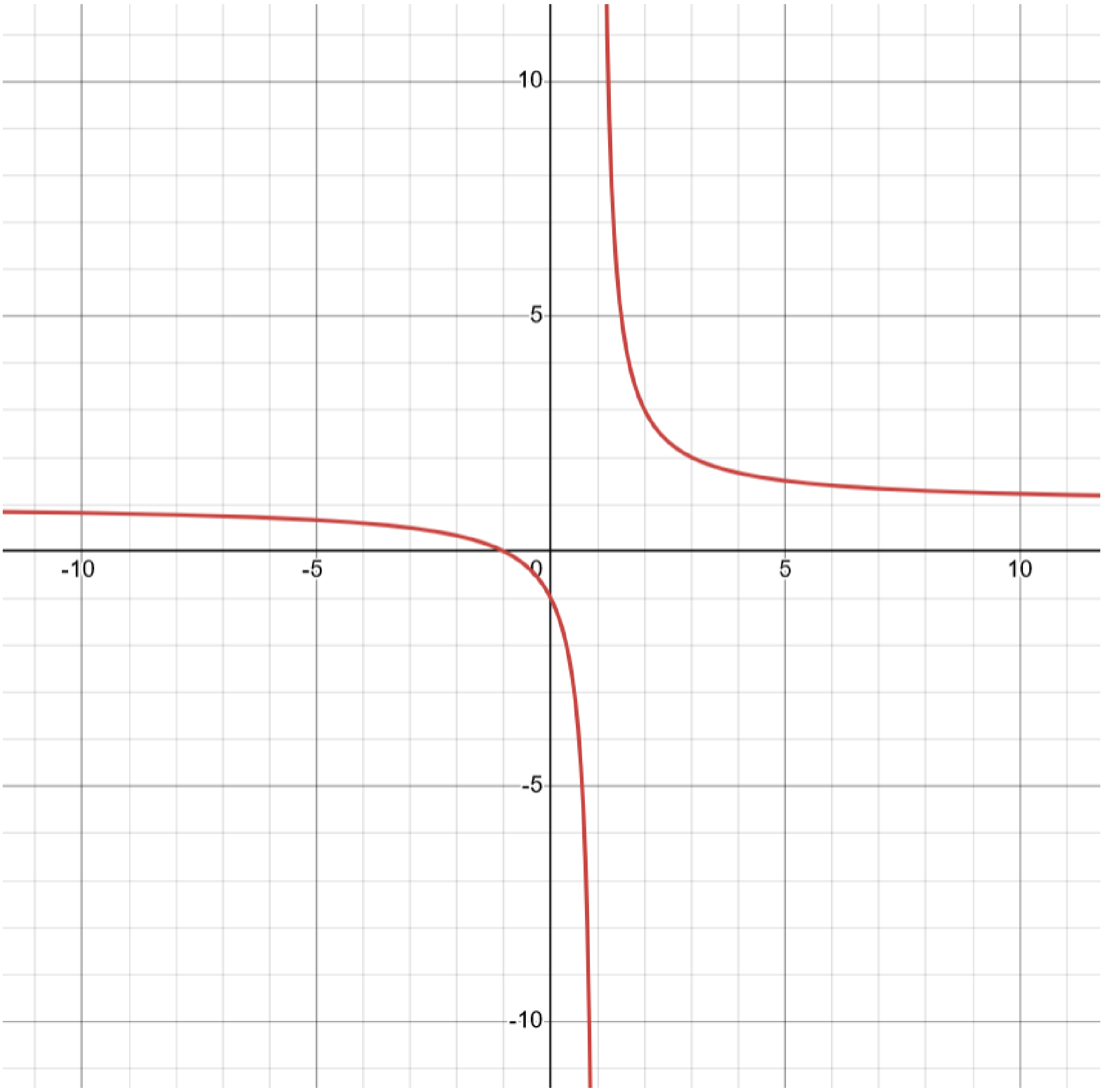
There are 2 methods to find the horizontal asymptote for top-heavy fractions.

Consider $y = \frac{x+1}{x-1}$.

The first way is to consider $|x| = \infty$. As x gets bigger and bigger, $y \approx \frac{x}{x}$, so $y = 1$. This is our horizontal asymptote.

The second way is to split/rewrite the equation into smaller parts. $\frac{x+1}{x-1}$ can be written as $\frac{(x-1)+2}{x-1} \to 1 + \frac{2}{x-1}$.

Now, as the denominator is approaching 0, $y = 1$. So this is our horizontal asymptote.



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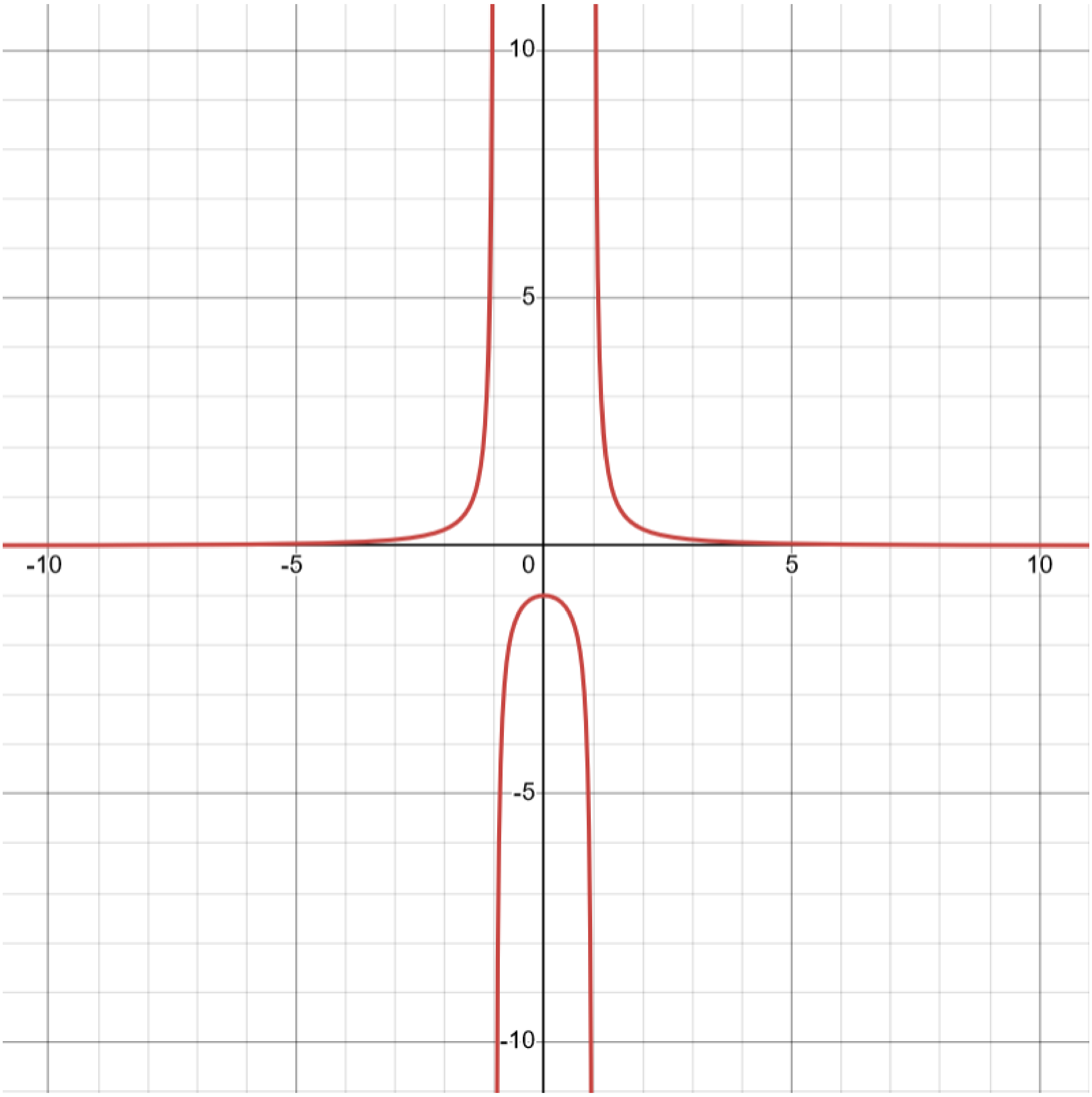
Quadratic denominators

Sometimes, the denominator will have a quadratic expression. In $y = \frac{1}{(x-1)(x+1)}$, how do we find the asymptotes?

Well, it’s quite simple. Just let the quadratic expression equal to zero. In this case, $(x-1)(x+1) = 0$ will give us $x = 1$ and $x = -1$ as our 2 vertical asymptotes. Then for the horizontal asymptote, when $|x|$ approaches infinity, $y = 0$. The graph will look like this.



We can see that there is a turning point in the bottom. To find the turning point, we can use 2 methods.

1. Differentiation
2. $\frac{dy}{dx} = \frac{-2x}{(x^2-1)^2}$

$\frac{dy}{dx} = 0 \rightarrow \frac{-2x}{(x^2-1)^2} = 0 \rightarrow -2x = 0$

$x = 0$

$x = 0 \rightarrow y = -1$

So, the turning point is $(0, -1)$.

2. Using the discriminant
3. $y = \frac{1}{x^2-1}$

$x^2y - y - 1 = 0$

Using the discriminant $b^2 - 4ac < 0$,

$(0)^2 + 4(y)(y-1) < 0$

$4y(y-1) < 0$

$0 < y < 1$

Note that this is to find the **invalid y value**, so we want to take the opposite where $y > 0$ and $y \leq 1$. So we can see that $y = 1$.

Quadratic numerators AND denominators

When $y = \frac{ax^2+bx+c}{x^2+d}$, the horizontal asymptote will be at a. Depending on the numerator, there will be zero, one, two turning points.

However, normal differentiation can be very time consuming. Instead, split the equation and differentiation will be easier.

$y = \frac{(x-1)(x-2)}{(2x-1)(x+1)}$ can be split into $y = \frac{1}{2} + \frac{1}{2(2x-1)} - \frac{2}{x-1}$ which is much easier to differentiate.