Assignment - 1
(Bonus)

(2) (a)

$$i^{(t)} = \sigma(W_{ix}x^{(t)} + W_{ih}h^{(t-1)})$$
$$f^{(t)} = \sigma(W_{fx}x^{(t)} + W_{fh}h^{(t-1)})$$
$$o^{(t)} = \sigma(W_{ox}x^{(t)} + W_{oh}h^{(t-1)})$$
$$g^{(t)} = \tanh(W_{gx}x^{(t)} + W_{gh}h^{(t-1)})$$
$$c^{(t)} = f^{(t)}c^{(t-1)} + i^{(t)}g^{(t)}$$
$$h^{(t)} = o^{(t)}\tanh(c^{(t)})$$

let the gradient pass down by the LSTM cell be $\bar{h}^{(t)}$

where $\bar{h}^{(t)} = \partial L / \partial h_t$

if we are using MSE, then $\bar{h}^{(t)} = y - h(x)$
where $h(x)$ is the predicted value & y is the original value.

$\Rightarrow$ gradient w.r.t output gate

$$\bar{o}^{(t)} = \partial L/\partial o = \partial L/\partial h^{(t)} \times \frac{\partial h^{(t)}}{\partial o^{(t)}}$$
$$= \bar{h}^{(t)} \frac{\partial}{\partial o^{(t)}}\left(o^{(t)}\tanh(c^{(t)})\right)$$
$$= \bar{h}^{(t)}\left(\tanh c^{(t)}\right)$$

$\Rightarrow$ gradient w.r.t $c^{(t)}$

$$\bar{c}^{(t)} = \partial L/\partial c^{(t)} = \partial L/\partial h^{(t)} \times \frac{\partial h^{(t)}}{\partial c^{(t)}}$$
$$= \bar{h}^{(t)} \frac{\partial}{\partial c^{(t)}}\left(o^{(t)}\tanh(c^{(t)})\right)$$
$$= \bar{h}^{(t)} o^{(t)}\left(1 - \tanh^2 c^{(t)}\right)$$

$\Rightarrow$ gradient w.r.t input gate $i^{(t)}$ & $g^{(t)}$

$$\bar{i}^{(t)} = \partial L/\partial i^{(t)} = \partial L/\partial c^{(t)} \times \frac{\partial c^{(t)}}{\partial i^{(t)}}$$
$$= \bar{h}^{(t)} o^{(t)}\left(1 - \tanh^2 c^{(t)}\right) \times \frac{\partial}{\partial i^{(t)}}\left(f^{(t)}c^{(t-1)} + i^{(t)}g^{(t)}\right)$$
$$= \bar{h}^{(t)} o^{(t)}\left(1 - \tanh^2 c^{(t)}\right) g^{(t)}$$

$$\bar{g}^{(t)} = \partial L / \partial g^{(t)} = \partial L / \partial c^{(t)} \times \partial c^{(t)} / \partial g^{(t)}$$

$$= \bar{h}^{(t)} o^{(t)} \left(1 - \tanh^2 c^{(t)}\right) \times \partial / \partial g^{(t)} \left(f^{(t)} c^{(t-1)} + i^{(t)} g^{(t)} \right)$$

$$= \bar{h}^{(t)} o^{(t)} \left(1 - \tanh^2 c^{(t)}\right) i^{(t)}$$

gradient w.r.t forget gate

$$\bar{f}^{(t)} = \partial L / \partial f^{(t)} = \partial L / \partial c^{(t)} \times \partial c^{(t)} / \partial f^{(t)}$$

$$= \bar{h}^{(t)} o^{(t)} \left(1 - \tanh^2 c^{(t)}\right) \times \partial / \partial f^{(t)} \left(f^{(t)} c^{(t-1)} + i^{(t)} g^{(t)} \right)$$

$$= \bar{h}^{(t)} o^{(t)} \left(1 - \tanh^2 c^{(t)}\right) c^{(t-1)}$$

(b) $$\bar{W}_{ix} = \partial L / \partial W_{ix} = \partial L / \partial i^{(t)} \times \partial i^{(t)} / \partial W_{ix}$$

$$= \bar{h}^{(t)} o^{(t)} \left(1 - \tanh^2 c^{(t)}\right) g^{(t)} \times \partial / \partial W_{ix} \left(\sigma\left(W_{ix} x^{(t)} + W_{ih} h^{(t-1)} \right) \right)$$

$$= \bar{h}^{(t)} o^{(t)} \left(1 - \tanh^2 c^{(t)}\right) g^{(t)} \sigma' \left(W_{ix} x^{(t)} + W_{ih} h^{(t-1)} \right) x^{(t)}$$