06.10.23
Friday

4 - Concepts of Stability & Root Locus

Stability :

* The term stability refers to the stable working condition of control system. for bounded input signal, if the output has constant amplitude, oscillation, then the system may be stable or unstable under some limited constraits, such a system is called Limitedly stable

* If a system output is stable for all variations of its parameters, then the system is called absolutely stable system

* If the system output is stable for limited amount of variation of its parameter is called conditionally stable system

* Routh - Hurwitz Criteria :-

* It is an analytical procedure for determining whether all the roots of polynomial have negative real part or not.

* Stability of the system depending on the location of roots of characteristic equation

i) * If all the roots of the characteristics equation has negative real parts, then the system is stable

ii) If any root of the characteristic equation has a positive real part or if there is a repeated root on the imaginary axis, then the system is unstable

iii) If the condition ① is satisfied except for the presence of one or more non-repeated roots on the imaginary axis, then the system is limitedly or marginally stable

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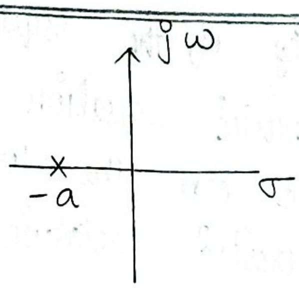
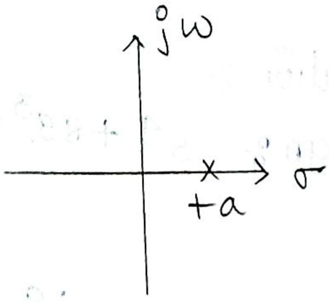
① $M(s) = \dfrac{A}{s+a}$

$\therefore$ root $= -a$



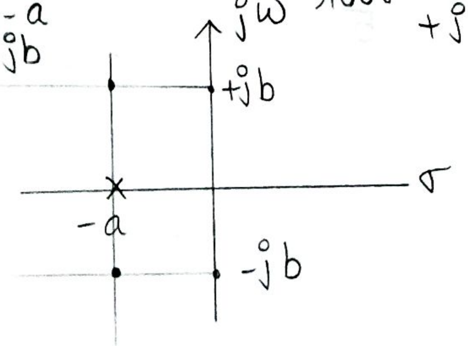
② $M(s) = \dfrac{A}{s-a}$

$\therefore$ root $= +a$



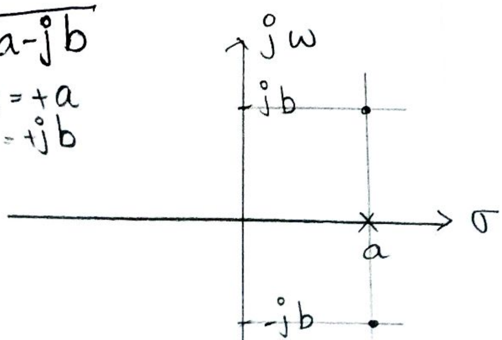
③ $M(s) = \dfrac{A}{s+a+jb} + \dfrac{A^*}{s+a-jb}$

root $= -a - jb$ root $= -a + jb$



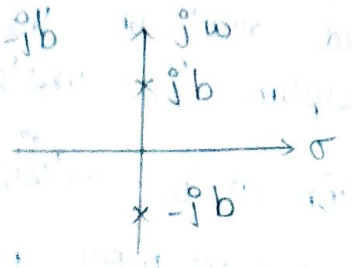
④ $M(s) = \dfrac{A}{s-a+jb} + \dfrac{A^*}{s-a-jb}$

root $= +a - jb$ root $= +a + jb$

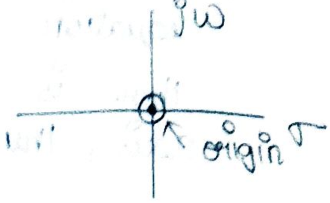


50 $\frac{A}{s+jb} + \frac{A^*}{s-jb} = M(s)$ root $= +jb$

root $= -jb$



(b) $M(s) = \frac{A}{s}$



Problems :-

① Using Routh-Hurwitz criteria, determine the stability of the system represented by the characteristic equation $s^4 + 8s^3 + 18s^2 + 16s + 5 = 0$. Comment on the location of the roots of characteristic equation.

Solution :-

Gn :- $s^4 + 8s^3 + 18s^2 + 16s + 5 = 0$

	a	b	
s^4	1	18	5
s^3	$\overset{c}{8}$ 1	16 d 2	0
s^2	16	5	0
s^1	1.68	0	
s^0	5		

$\frac{(c\times b)-(a\times d)}{c}$

$= \frac{18-2}{1}$

$= \frac{5-0}{1}$

$= 5$

$\frac{32-5}{16} = \frac{27}{16}$

$= 1.68$

$\frac{1.68\times 5 - 0}{1.68}$

$\frac{8.4}{1.68}$

2.) $9s^5 - 20s^4 + 10s^3 - s^2 - 9s - 10 = 0$

s^5	9	10	-9
s^4	-20	-1	-10
s^3	9.55	-13.5	0
s^2	-29.4	-10	
s^1	-16.8	0	
s^0	-10		

unstable
The 1st colum values are -ve, so it lies on positive side of the +ve real axis

① $\dfrac{(-20\times10)-(9\times-1)}{-20}$; $\dfrac{(-20\times-9)-(-10\times9)}{-20}$

$\dfrac{-200+9}{-20} = 9.5$; -13.5

② $\dfrac{(9.5\times-1)-(-13.5\times-20)}{9.5}$; $\dfrac{(9.5\times-10)-0}{9.5}$

-29.4 ; -10

③ $\dfrac{(-29.4\times-13.5)-(-10\times9.55)}{-29.4}$; 0

④ $\dfrac{(-16.8\times-10)-0}{-16.8} = -10$

3. $s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$

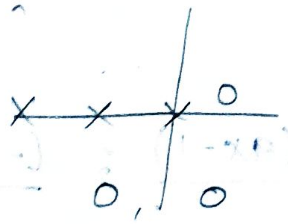
s^6	1	8	20	16
s^5	2	12	16	0
s^4	1 $\cancel{2}$	$\cancel{12}$ 6	$\cancel{16}$ 8	
s^3	1 $\cancel{0}$	$\cancel{0}$ 3	0	
s^2	3	8		
s^1	33	0		
s^0	+8			

→ To overcome this, we go with auxillary equation

$\frac{dA}{ds} = 4s^3 + 12s$

$\cancel{4}$ $1\cancel{2}$

1 3



$-1, -2, \boxed{-1, -3,} -0.33, -3$

marginally stable

14/10/23

4) $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$

s^5	1	2	5
s^4	1	2	5
s^3	0	-2	
s^2	$\frac{2\epsilon+2}{\epsilon}$	5	
s^1	$\frac{-5\epsilon^2 - 4\epsilon - 4}{2\epsilon+2}$ (-2)		
s	5		

6 -

∴ It is unstable

5. Determine the value of 'k' for stability of unity feedback system whose open loop transfer function where $G(s) = \frac{k}{s(s+1)(s+2)}$. comment the system stability using Routh Hurwitz criterion

Solution

$$G(s) = \frac{k}{s(s+1)(s+2)} \quad ; \quad H(s) = 1$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{k/s(s+1)(s+2)}{1+\frac{k}{s(s+1)(s+2)}(1)}$$

$$= \frac{k/\cancel{s(s+1)(s+2)}}{\frac{s(s+1)(s+2)+k}{\cancel{s(s+1)(s+2)}}}$$

$$= k/s(s+1)(s+2)+k$$

$$= \frac{k}{(s^2+s)(s+2)+k}$$

$$= \frac{k}{s^3+2s^2+s^2+2s+k}$$

$$= \frac{k}{s^3+3s^2+2s+k}$$

here, s^3+3s^2+2s+k is characteristic equation

s^3	1	2
s^2	3	k
s^1	$\frac{6-k}{3}$	
s^0	k	

$\frac{6-k}{3} > 0 = 2$ +ve

$\frac{6-5}{3} = 1/3$ +ve

$\frac{6-6}{3} = 0 \rightarrow$ in imaginary axis

∴ To make the system stable, the k value should lies between $0 < k < 6$

19/10/23 Construction of root locus :-

* For adjusting the location of closed loop poles to achieve the desired system performance by varying one or more system performance

1. location of Poles and zeros

$n \to$ no. of poles $m \to$ no. of zeros

2. Root locus on Real axis. In order to determine the part of root locus on real axis, take a test point on real axis. If the total number of poles and zeros on the real axis to the right of this test point is odd numbers, then the test point lies on root locus. If it is even, then the test point does not lies on root locus.

3. Angle of asymptotes $= \frac{\pm 180 (2q+1)}{n-m}$

$$q = 0, 1, 2, \ldots (n-m)$$

$$\therefore \text{Centroid} = \frac{\text{sum of poles} - \text{sum of zeros}}{n-m}$$

4. Breakaway and Breakin Point :-

* The Breakaway or breakin point either lie on the real axis or exists as complex conjugate pairs. If there is a root locus on real axis from between two ploes, then there exists a breakaway point. If there is a root locus on the real axis between two zeros, then there exist a breakin point. If there is a root locus on the

real axis between poles and zero, then there maybe or maynot be breakaway (or) breakin point

another hint :-

$$\frac{C(s)}{R(s)}$$

↓

find characteristic equation

↓

find 'k' ⇒ $\frac{dk}{ds} = 0$

↓

find roots

↓

substitute roots in 'k' ⇒ +ve Real ⇒ Breakaway / Breakin Point

⑤ · Angle of Depature :-

* Angle of departure from complex Pole,

A = 180 - [sum of angles of vector to the complex pole A from other poles] + [sum of angles of vector to the complex pole A from zeros]

∴ A = 180 - [P] + [Z]

⑥ Angle of Arrival :-

* Angle of arrival at complex pole,

A = 180 - [sum of angles of vector to the complex zero A from all other zeros] + [sum of angles of vector to the complex zero A from poles]

∴ A = 180 - [Z] + [P]

7) Point of intersection of root locus with imaginary axis

characteristic equation

↓

substitute $s \to j\omega$

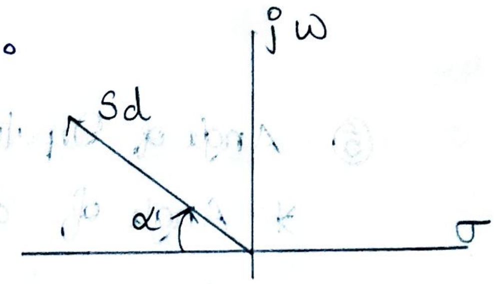
↓

seperate real and imaginary poles

↓

will find intersection / crossing point

(8) Determination of open loop gain for specified damping of dominant roots.



$$\cos\alpha = \frac{\xi\omega_n}{\omega_n}$$

$$\alpha = \cos^{-1}\xi$$

(9) value of k_{sd} (dominant pole)

$$\therefore K_{sd} = \frac{\text{product length of vectors from open loop poles to dominant poles}}{\text{product length of vectors from open loop zeros to dominant poles}}$$

A unity feedback control system has a open-loop transfer function $G(s) = \frac{k}{s(s^2+4s+13)}$. sketch the root locus.

Solution :-

Step 1 :- Location of poles & zeros

m = zeros (o)
n = poles (x)

i) poles :- $s(s^2+4s+13)$

$s = 0$

$s^2+4s+13$

$$= \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{4^2 - 4(1)(13)}}{2}$$

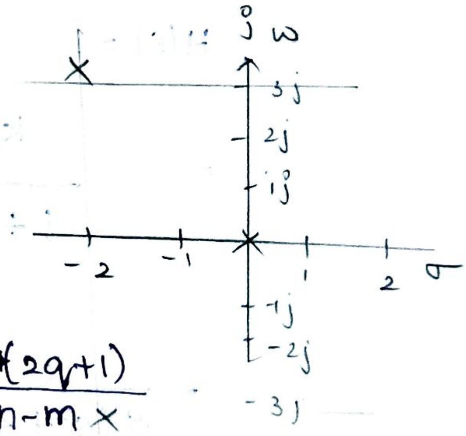
$$= \frac{-4 \pm \sqrt{16-52}}{2}$$

$$= \frac{-4 \pm \sqrt{-36}}{2}$$

$$= \frac{-4 \pm 6j}{2}$$

$$= -2 \pm 3j \qquad \therefore m = 0 ; n = 3$$

2. Root locus on real axis



3. Angle of Asymtotes :-

$$\text{Angle of Asymtotes} = \frac{\pm 180(2q+1)}{n-m}$$

$q = 0, 1, 2 \ldots (n-m)$

$(n-m) = 3-0 = 3$

$q = 0 \Rightarrow \pm \frac{180 \cdot (2(0)+1)}{3} = \pm 60^\circ$

$q = 1 \Rightarrow \pm \frac{180 \cdot (2(1)+1)}{3} = \pm 180^\circ$

$q = 2 \Rightarrow \pm \frac{180 (2(2)+1)}{3} = \pm 300^\circ$

$= \pm 300^\circ - 360^\circ$

$= \pm 60^\circ$

$q = 3 \Rightarrow \pm \frac{180 (2(3)+1)}{3} = \pm 420 = 420^\circ - 360^\circ$

$= \pm 60^\circ$

When we have angles more than 180° then subtract it from 360° because our real axis is in 180°

$$\text{Centroid} = \frac{0 - 2 + \cancel{3j} - 2 - \cancel{3j} - 0}{3} = -\frac{4}{3}$$

$\therefore C = -1.33$

(4) Breakaway or Breakin

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$H(s) = 1$

$$= \frac{k / s(s^2+4s+13)}{1 + \frac{k}{s(s^2+4s+13)}}$$

$$= \frac{k}{s(s^2+4s+13)+k}$$

$$= \frac{k}{\underbrace{s^3+4s^2+13s+k}_{\text{characteristic equation}}}$$

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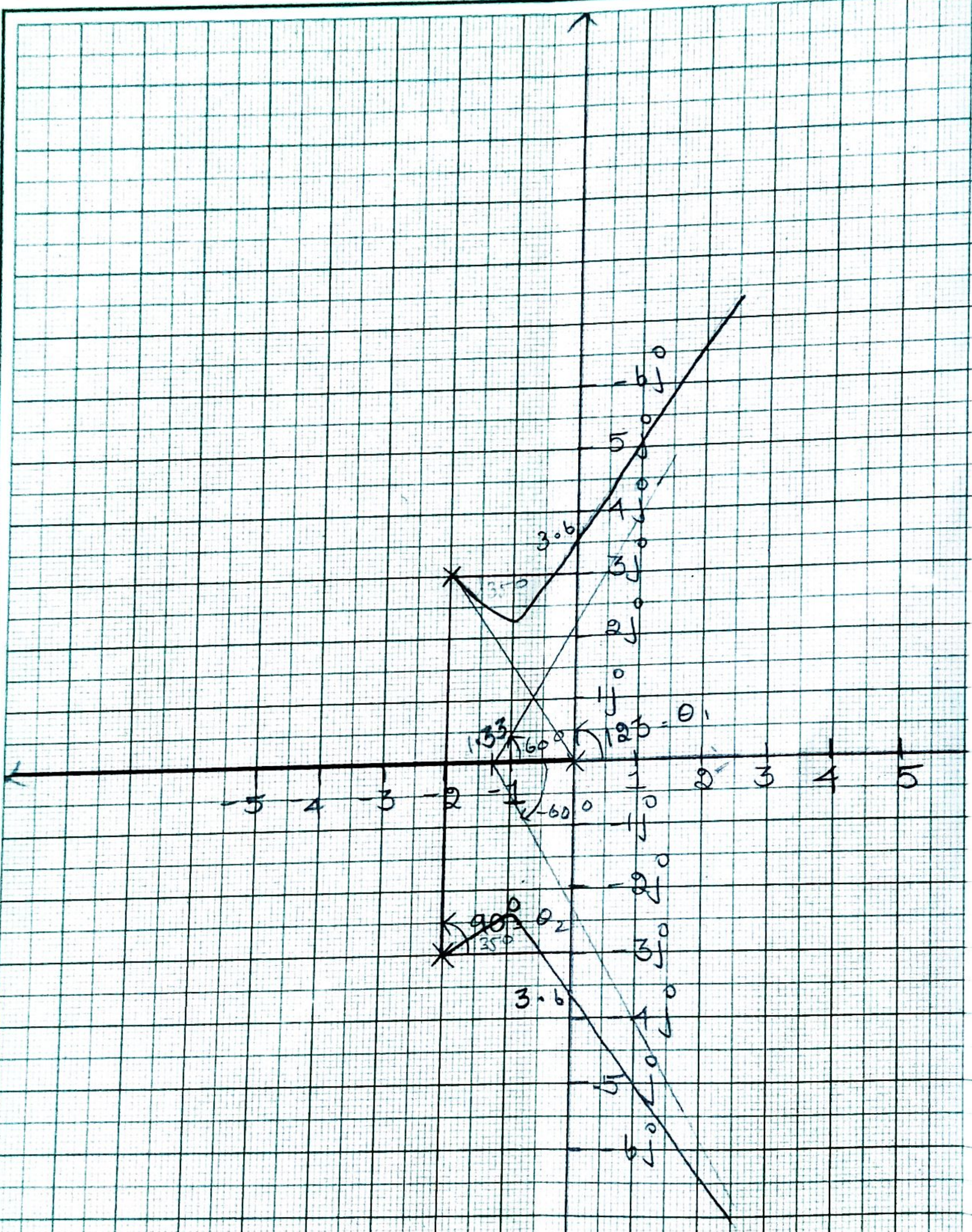

find k:

$$s^3+4s^2+13s+k=0$$

$$k=-[s^3+4s^2+13s] \quad \text{--- ①}$$

$$\frac{dk}{ds}=0$$

$$-[3s^2+8s+13]=0$$

$$3s^2+8s+13=0$$

roots $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$\frac{-8\pm\sqrt{8^2-4(3)(13)}}{2(3)}$$

$$=\frac{-8\pm\sqrt{64-156}}{6}$$

$$=\frac{-8\pm 9.5j}{6}$$

$$s=-1.33\pm 1.58j \quad \text{--- ②}$$

sub ② in ①

$$k=-\left[(-1.33\pm1.58j)^3+4(-1.33\pm1.58j)^2+13(-1.33\pm1.58j)\right]$$

$$=-\left[(-1.33+1.58j)^3+4(-1.33+1.58j)^2+13(-1.33+1.58j)\right]$$

$$,-\left[(-1.33-1.58j)^3+4(-1.33-1.58j)^2+13(-1.33-1.58j)\right]$$

$$k=-12.59+8.169j$$

$$k=-33.13-12.37j$$

∴ Hence, we don't have breakin or breakaway

Angle of departure :-

Angle of departure = 180 − [P] + [Z]

book ans :

$= 180^\circ - [123.7^\circ + 90^\circ]$

$= 180^\circ - 213.7^\circ$

$= 33.7^\circ$

$= 180^\circ - [123^\circ + 90^\circ]$

$= 180^\circ - 213^\circ$

$= 33^\circ$

∴ Angle of departure $= 33^\circ$

step 6 :- To find the crossing point on imaginary axis

The characteristic equation,

$$s^3 + 4s^2 + 13s + k = 0$$

put $s = j\omega$

$$(j\omega)^3 + 4(j\omega)^2 + 13j\omega + k = 0$$

$$-j\omega^3 - 4\omega^2 + 13j\omega + k = 0$$

on equating imaginary part to zero, we get

$$-\omega^3 + 13\omega = 0$$

$$-\omega^2 = -13\omega$$

$$\omega^2 = 13$$

$$\omega = \pm\sqrt{13}$$

$$\therefore \omega = \pm 3.6$$

on equating real part to zero, we get

$$-4\omega^2 + k = 0$$

$$k = 4\omega^2$$

$$= 4 \times 13$$

$$k = 52$$

∴ The crossing point of root locus is $\pm j3.6$. The value of K at this crossing point is $K = 52$ (this is the limiting value of k for the stability of the system).

20 Sketch the root locus of the system whose open loop transfer function $G(s) = \frac{k}{s(s+2)(s+4)}$. Find the value of K so that the damping ratio of closed loop system is 0.5

Solution :-

$$G(s) = \frac{K}{s(s+2)(s+4)}$$

① Location of poles & zeros

zeros $\rightarrow m = 0$

poles $\rightarrow n - 3 \Rightarrow 0, -2, -4$

② Root locus on real axis

③ Angle of Asymptotes :- $= \frac{\pm 180(2q+1)}{n-m}$

$q = 0, 1 \ldots, (n-m)$

$$q = 0, \quad \frac{\pm 180(2(0)+1)}{3} = \pm 60^\circ$$

$$q = 1, \Rightarrow \frac{\pm 180(2(1)+1)}{3} = \pm 180^\circ$$

$$q = 2, \Rightarrow \frac{\pm 180(2(2)+1)}{3} = \pm 300 - 360^\circ$$
$$= \pm 60^\circ$$

$$q = 3 \Rightarrow \frac{\pm 180(2(3)+1)}{3} = \pm 420^\circ - 360$$
$$= \pm 60^\circ$$

Centroid = $\dfrac{\text{sum of poles} - \text{sum of zeros}}{n-m}$

$$= \frac{0-2-4-0}{3} = \frac{-6}{3} = -2$$

④ Breakaway or breakin point

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{\dfrac{k}{s(s+2)(s+4)}}{1+\dfrac{k}{s(s+2)(s+4)}} \cdot (1)$$

$$= \frac{k}{s(s+2)(s+4)+k}$$

$$= \frac{k}{(s^2+2s)(s+4)+k}$$

$$= \frac{k}{s^3+6s^2+8s+k} \longrightarrow \text{characteristic equation}$$

$$s^3+6s^2+8s+k=0$$

$$k = -\left[s^3+6s^2+8s\right] \quad \text{—— ①}$$

$$\frac{dk}{ds} = 0$$

$$0 = -\left[3s^2+12s+8\right]$$

$$3s^2+12s+8 = 0$$

$$= \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

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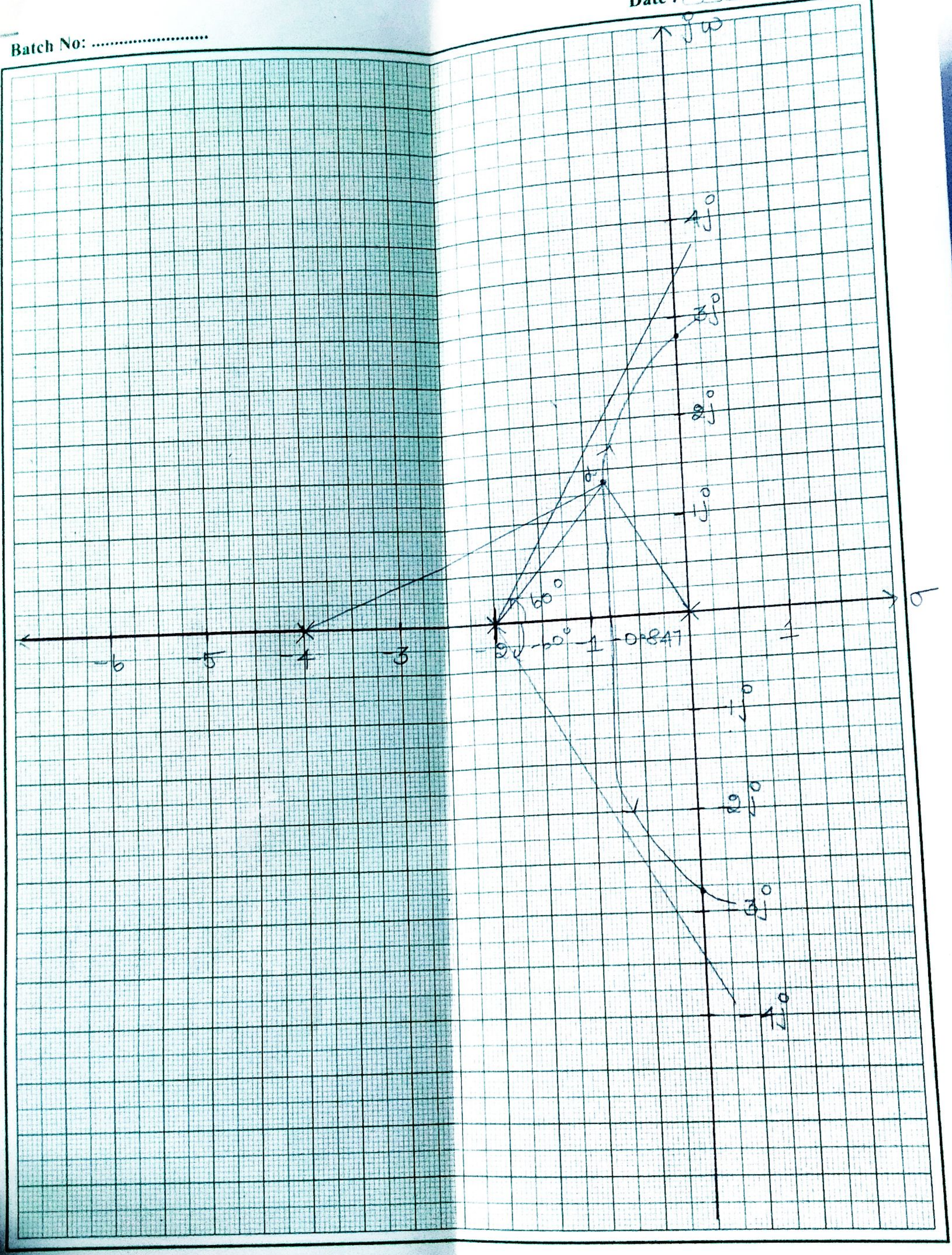

$$= \frac{-12 \pm \sqrt{12^2 - (4)(3)(8)}}{2(3)}$$

$$= \frac{-12 \pm \sqrt{144-96}}{6}$$

$$= \frac{-12 \pm \sqrt{48}}{6}$$

$$= \frac{-12 \pm 6.92}{6}$$

$\therefore S = -2 + 1.153 = -0.847$

$\therefore 6 = -2 - 1.153 = -3.153$

$$K = -\left[(-0.847)^3 + 6(-0.847)^2 + 8(-0.847)\right]$$

$\therefore K = 3.079$

$$K = -\left[(-3.153)^3 + 6(-3.153)^2 + 8(-3.153)\right]$$

$\boxed{\therefore K = -3.079}$

⑤ Angle of departure :-

* we dont have complex poles and complex zero. Therefore there is no angle of departure or arrival

⑥ point of intersection

$$s^3 + 6s^2 + 8s + k = 0$$

$$s \rightarrow j\omega$$

$$(j\omega)^3 + 6(j\omega)^2 + 8(j\omega) + k = 0$$

$$-j\omega^3 - 6\omega^2 + 8j\omega + k = 0$$

$$-\omega^3+8\omega=0$$

$$\cancel{-}\omega^{\cancel{3}}=\cancel{+}8\omega$$

$$\omega^2=8$$

$$\omega=\sqrt{8}$$

$$\therefore \omega=2.82$$

$$-6\omega^2+k=0$$

$$-6(2.82)^2+k=0$$

$$k=47.7$$

By given condition $\xi=0.5$

$$\therefore \alpha=\cos^{-1}\xi$$

$$\alpha=\cos^{-1}0.5$$

$$\therefore \alpha=60^\circ$$

$$\cancel{K_{sd}}=\cancel{\frac{l_1\times l_2\times l_3}{1}}$$

$$=3.2\times3.5\times7$$

$$=\cancel{8/3/6\times6}$$

$$=\cancel{71.8}\ 78.4$$

$$\therefore K_{sd}=\frac{l_1\times l_2\times l_3}{1}$$

$$l_1=3.2\times0.5=1.63$$

$$l_2=3.5\times0.5=1.75$$

$$l_3=7\times0.5=3.5$$

$$K_{sd}=1.3\times1.75\times3.5$$

$$K_{sd}=7.96\simeq8$$

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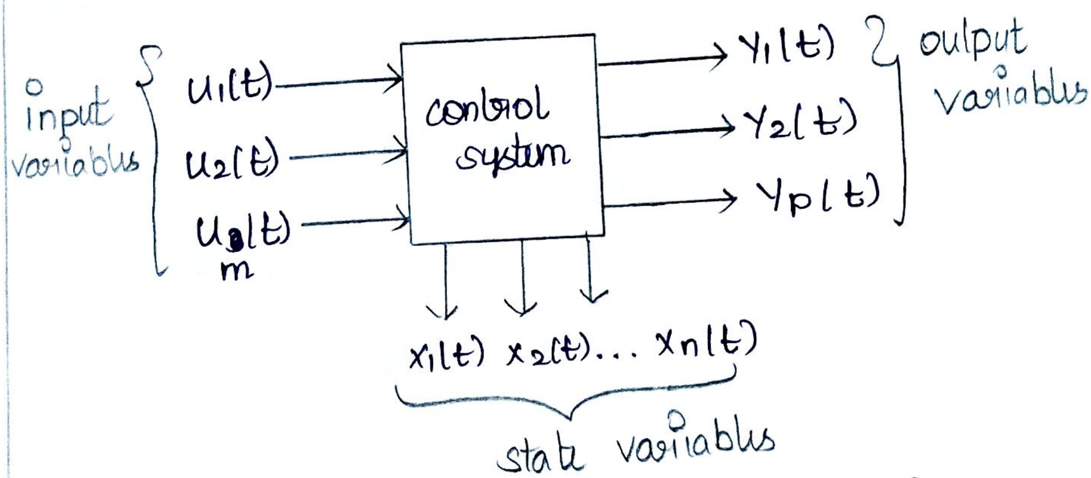
Unit - 5

State Space Analysis

State Space Formulation :-

* The State of dynamic system is a minimal set of variables such that the knowledge of this variables at $t = t_0$ together with the knowledge of the inputs for $t \geq t_0$, completely determines the behaviour of the system $t > t_0$

* A set of variables which describes the system at any time instant are called state variables



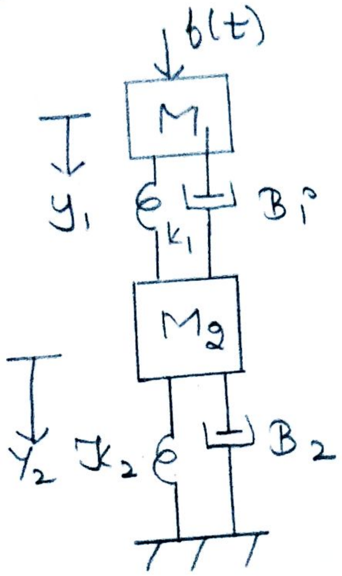
* 2M ·X··X··X·

* state equation ⟶ $\dot{x}(t) = Ax(t) + Bu(t)$ → one time differentiation

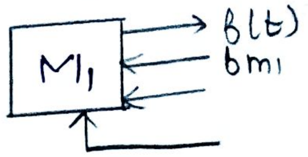
output equation ⟶ $y(t) = Cx(t) + Du(t)$

* State Space Representation Using Physical Variable

1. Construct a state model of a machinical system show in a figure



step 1 :- free body diagram of M_1

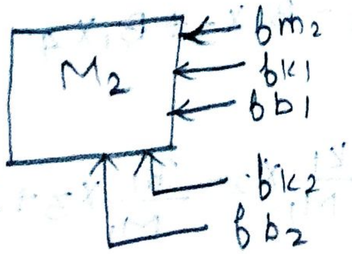


force - balance equation

$$f(t) = f_{m1} + f_{k1} + f_{b1}$$

$$F(t) = M_1 \frac{d^2 y_1}{dt^2} + k_1 (y_1 - Y_2) + B_1 \frac{d}{dt}(Y_1 - Y_2)$$

$$F(t) = M_1 \frac{d^2 Y_1}{dt^2} + k_1 Y_1 - k_1 Y_2 + B_1 \frac{dY_1}{dt} - B_1 \frac{dY_2}{dt} \quad (1)$$



$$0 = bm_2 + bk_1 + bb_1 + bk_2 + bb_2$$

$$0 = M_2 \frac{d^2 Y_2}{dt^2} + K_1 (Y_2 - Y_1) + B_1 \frac{d}{dt}(Y_2 - Y_1) + k_2 Y_2 + B_2 \frac{dY_2}{dt} \quad \text{—②}$$

$\dot{x}(t) = Ax(t) + Bu(t) \rightarrow$ state equation

$Y(t) = Cx(t) + DU(t) \rightarrow$ O/P

State variables :-

$Y_1 \rightarrow X_1$

$Y_2 \rightarrow X_2$

$\frac{dY_1}{dt} \rightarrow X_3 \Rightarrow \dot{X}_1$

$\frac{d^2Y_1}{dt^2} \rightarrow \dot{X}_3$

$\frac{dY_2}{dt} \rightarrow X_4 \Rightarrow \dot{X}_2$

$\frac{d^2Y_2}{dt^2} \rightarrow \dot{X}_4$

$$U = M_1 \dot{X}_3 + k_1 X_1 - k_1 X_2 + B_1 X_3 - B_1 X_4$$

$$M_1 \dot{X}_3 = U - k_1 X_1 + k_1 X_2 - B_1 X_3 + B_1 X_4$$

$$\dot{X}_3 = \frac{U}{M_1} - \frac{k_1}{M_1} X_1 + \frac{k_1}{M_1} X_2 - \frac{B_1}{M_1} X_3 + \frac{B_1}{M_1} X_4$$

$$0 = M_2\dot{x}_4 + k_1 x_2 - k_1 x_1 + B_1 x_3 + k_2 x_2 + B_2 x_4$$

$$M_2\dot{x}_4 = -k_1 x_2 + k_1 x_1 - B_1 x_4 + B_1 x_3 - k_2 x_2 - B_2 x_4$$

$$\dot{x}_4 = \frac{-k_1}{M_2} x_2 + \frac{k_1 x_1}{M_2} - \frac{B_1 x_4}{M_2} + \frac{B_1 x_3}{M_2} - \frac{k_2 x_2}{M_2} - \frac{B_2 x_4}{M_2}$$

$$\dot{x}_4 = \frac{-(k_1+k_2) x_2}{M_2} + \frac{k_1 x_1}{M_2} - \frac{(B_1+B_2) x_4}{M_2} + \frac{B_1 x_3}{M_2}$$

state equation:-

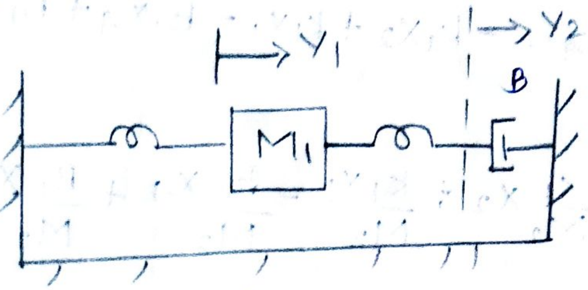
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_1}{M_1} & \frac{k_1}{M_1} & \frac{-B_1}{M_1} & \frac{B_1}{M_1} \\ \frac{k_1}{M_2} & \frac{-(k_1+k_2)}{M_2} & \frac{B_1}{M_2} & \frac{-(B_1+B_2)}{M_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/M_1 \\ 0 \end{bmatrix} \begin{bmatrix} U \end{bmatrix}$$

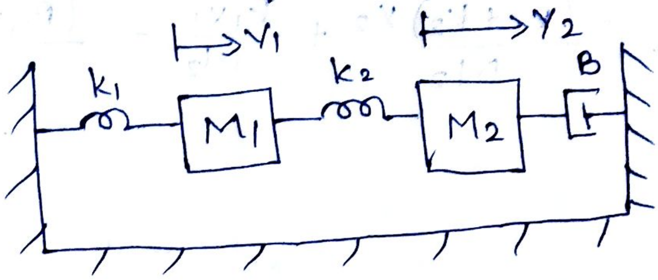
output

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} U$$

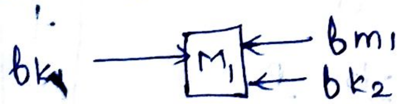
$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Construct a state model of a machinical system shown below





* free-body diagram



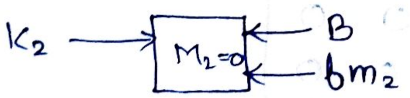
force-balance equation,

$$0 = f_{k_1} + f_{m_1} + f_{k_2}$$

$$0 = M_1 \frac{d^2 y_1}{dt^2} + k_1 y_1 + k_2 (\overset{y_1 - y_2}{\cancel{x_2 - x_1}})$$

$$0 = M_1 \frac{d^2 y_1}{dt^2} + k_1 y_1 + k_2 y_1 - k_2 y_2 \longrightarrow (1)$$

* free-body diagram



force-balance equation

$$0 = k_2 (y_2 - y_1) + \frac{B dy_2}{dt} \quad -(2)$$

$$0 = k_2 y_2 - k_2 y_1 + \frac{B dy_2}{dt}$$

* state variable :-

$Y_1 \rightarrow x_1$

$Y_2 \rightarrow x_2$

$\frac{dY_1}{dt} \rightarrow x_3 = \dot{x}_1$

$\frac{d^2Y_1}{dt^2} \rightarrow \dot{x}_3$

$\frac{dY_2}{dt} \rightarrow x_4 = \dot{x}_2$

$\frac{d^2Y_2}{dt} \rightarrow \dot{x}_4$ we will not use this, so 3 variable used

$$0 = M_1 \dot{x}_3 + k_1 x_1 - k_2 x_2 + k_2 x_1$$

$$M_1 \dot{x}_3 = -k_1 x_1 + k_2 x_2 + k_2 x_1$$

$$\dot{x}_3 = \frac{-k_1}{M_1} x_1 + \frac{k_2}{M_1} x_2 + \frac{k_2}{M_1} x_1$$

$$\therefore \dot{x}_3 = \left[-\frac{k_1 + k_2}{M}\right] x_1 + \frac{k_2}{M_1} x_2$$

$$0 = k_2 x_2 - k_2 x_1 + B\dot{x}_4$$

$$0 = k_2 x_2 - k_2 x_1 + B\dot{x}_4$$

$$0 = k_2 x_2 - k_2 x_1 + B\dot{x}_2$$

$$B\dot{x}_2 = -k_2 x_2 + k_2 x_1$$

$$\therefore \dot{x}_2 = \frac{-k_2}{B} x_2 + \frac{k_2}{B} x_1$$

$\dot{=} x$

$\ddot{=} x$

State equation :-

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ \frac{k_2}{B} & \frac{-k_2}{B} & 0 & 0 \\ \frac{-k_1+k_2}{M_1} & \frac{-k_2}{M_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

my ans

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{k_2}{B} & \frac{-k_2}{B} & 0 \\ \frac{-k_1+k_2}{M_1} & \frac{k_2}{M_1} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

book ans

Output

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

my

$$y(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

book ans

* State Space Representation using Phase variables :-

① Construct a state model for a system characterised by the differential equation $\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y + U = 0$. give block diagram representation of the state model.

Solution :-

$$\text{gn :-}\quad \frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y + U = 0$$

* state variables :-

$$* \; y \rightarrow x_1$$

$$* \; \frac{dy}{dt} \rightarrow x_2 = \dot{x}_1$$

$$* \; \frac{d^2y}{dt^2} \rightarrow x_3 = \dot{x}_2$$

$$* \; \frac{d^3y}{dt^3} \rightarrow \dot{x}_3$$

sub in above eq,

$$\dot{x}_3 + 6x_3 + 11x_2 + 6x_1 + U = 0$$

$$\dot{x}_3 = -\left[6x_3 + 11x_2 + 6x_1 + U\right]$$

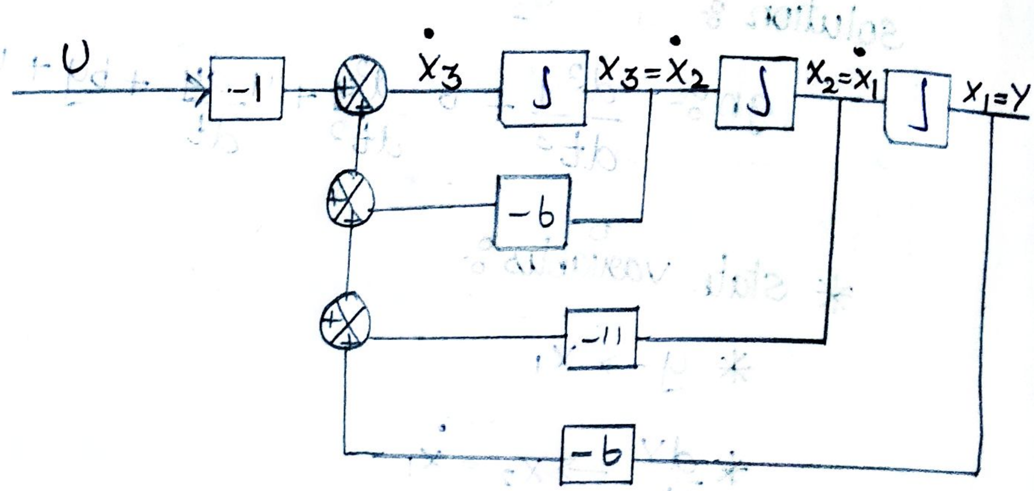
∴ state equation $\dot{x}(t) = Ax(t) + BU(t)$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} U \end{bmatrix}$$

* output

$$Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Diagram :



2. Obtain the state model of the system whose transfer function is given as $\frac{Y(s)}{U(s)} = \frac{10}{s^3+4s^2+2s+1}$

Solution :-

gn :- $\frac{Y(s)}{U(s)} = \frac{10}{s^3+4s^2+2s+1}$

$\therefore Y(s)\left[s^3+4s^2+2s+1\right] = 10\,U(s)$

$$s^3 Y(s) + 4s^2 Y(s) + 2s Y(s) + Y(s) = 10U(s)$$

$L^{-1} \Rightarrow \quad x \rightarrow X(s)$

$$\frac{dx}{dt} \rightarrow s \cdot X(s) \qquad \frac{d^3x}{dt^3} \rightarrow s^3 \cdot X(s)$$

$$\frac{d^2x}{dt^2} \rightarrow s^2 \cdot X(s)$$

$$\frac{d^3y}{dt^3} + 4 \cdot \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 10U$$

* state variables :

* $y \rightarrow x_1$

* $\frac{dy}{dt} \rightarrow x_2 = \dot{x}_1$

* $\frac{d^2y}{dt^2} \rightarrow x_3 = \dot{x}_2$

* $\frac{d^3y}{dt^3} \rightarrow \dot{x}_3$

$$\Rightarrow \dot{x}_3 + 4x_3 + 2x_2 + x_1 = 10U$$

$$\therefore \dot{x}_3 = 10U - 4x_3 - 2x_2 - x_1$$

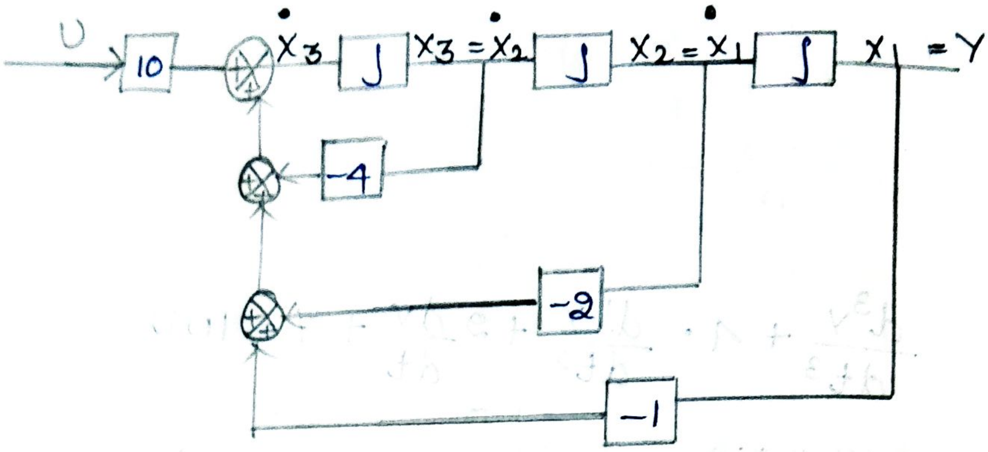
state equation :-

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} \begin{bmatrix} U \end{bmatrix}$$

output :-

$$\therefore y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Diagram :-



* State Space Representation using Canonical variables :-

$$* \quad \frac{Y(s)}{U(s)} = b_0 + \frac{C_1}{s+\lambda_1} + \frac{C_2}{s+\lambda_2}, \dots, \frac{C_n}{s+\lambda_n}$$

where C_1, C_2, C_3, C_n = residues

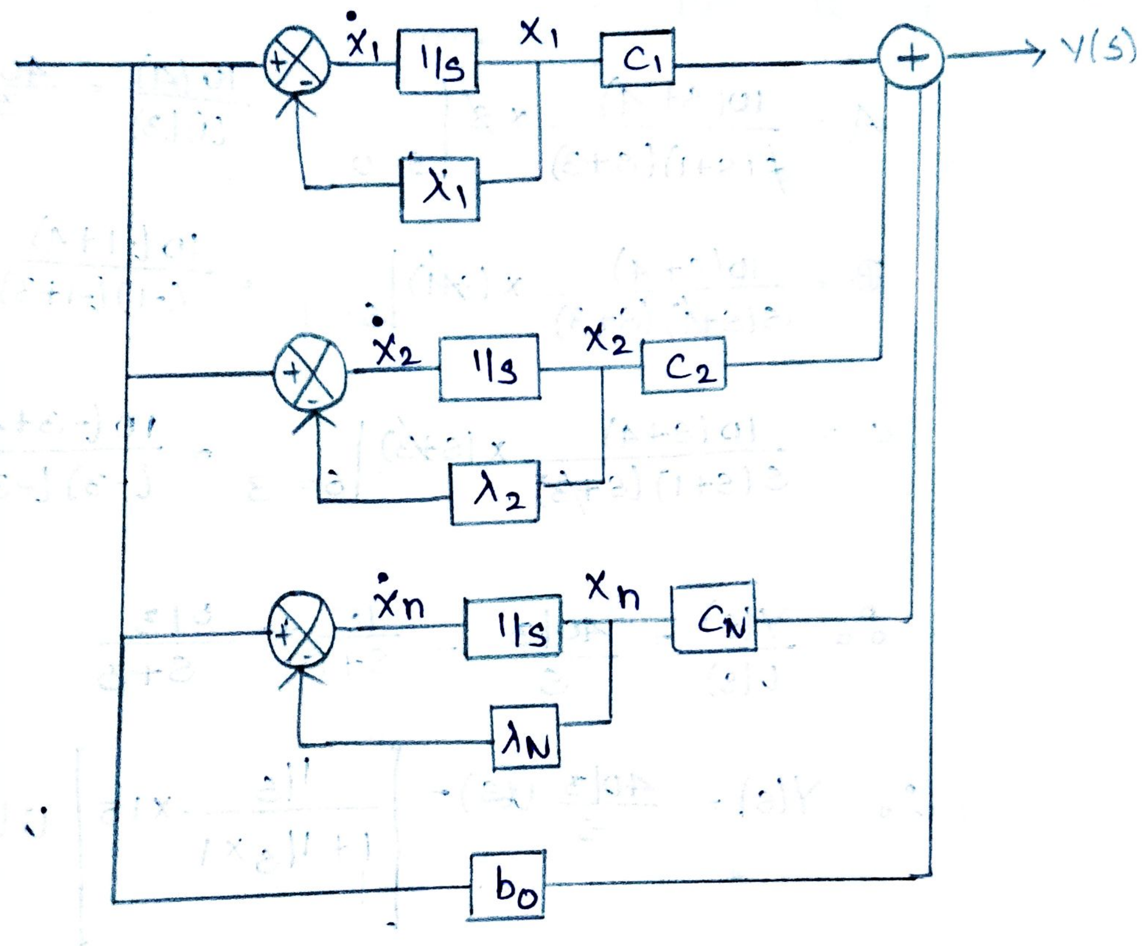
$\lambda_1, \lambda_2, \lambda_n$ = $\underbrace{\text{Dinr}}_{\text{denominator}}$ polynomials / Poles

$$\therefore \frac{Y(s)}{U(s)} = b_0 + \frac{C_1}{s\left(1+\frac{\lambda_1}{s}\right)} + \frac{C_2}{s\left(1+\frac{\lambda_2}{s}\right)} + \dots, \frac{C_n}{s\left(1+\frac{\lambda_n}{s}\right)}$$

$$= b_0 + \frac{C_1/s}{\left(1+\frac{\lambda_1}{s}\right)} + \frac{C_2/s}{\left(1+\frac{\lambda_2}{s}\right)} + \dots, \frac{C_n/s}{\left(1+\frac{\lambda_n}{s}\right)}$$

$$Y(s) = b_0 U(s) + \left[\frac{1/s}{\left(1+\frac{1}{s}\times\lambda_1\right)} \times C_1\right] U(s) + \left[\frac{1/s}{\left(1+\frac{1}{s}\times\lambda_2\right)} \times C_2\right] U(s)$$

$$+ \left[\frac{1/s}{\left(1+\frac{1}{s}\times\lambda_n\right)} \times C_n\right] U(s)$$



1. $\dfrac{Y(s)}{U(s)} = \dfrac{10(s+4)}{s(s+1)(s+3)}$

solution:-

$$\text{Gn:-}\quad \frac{Y(s)}{U(s)} = \frac{10(s+4)}{s(s+1)(s+3)}$$

$$\frac{Y(s)}{U(s)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

Partial fraction:

$$A = \frac{10(s+4)}{\cancel{s}(s+1)(s+3)} \times \cancel{s}\Big|_{s=0} = \frac{10(4)}{(1)(3)} = \frac{40}{3}$$

$$B = \frac{10(s+4)}{s\cancel{(s+1)}(s+3)} \times \cancel{(s+1)}\Big|_{s=-1} = \frac{10(-1+4)}{(-1)(-1+3)} = \frac{-30}{2} = -15$$

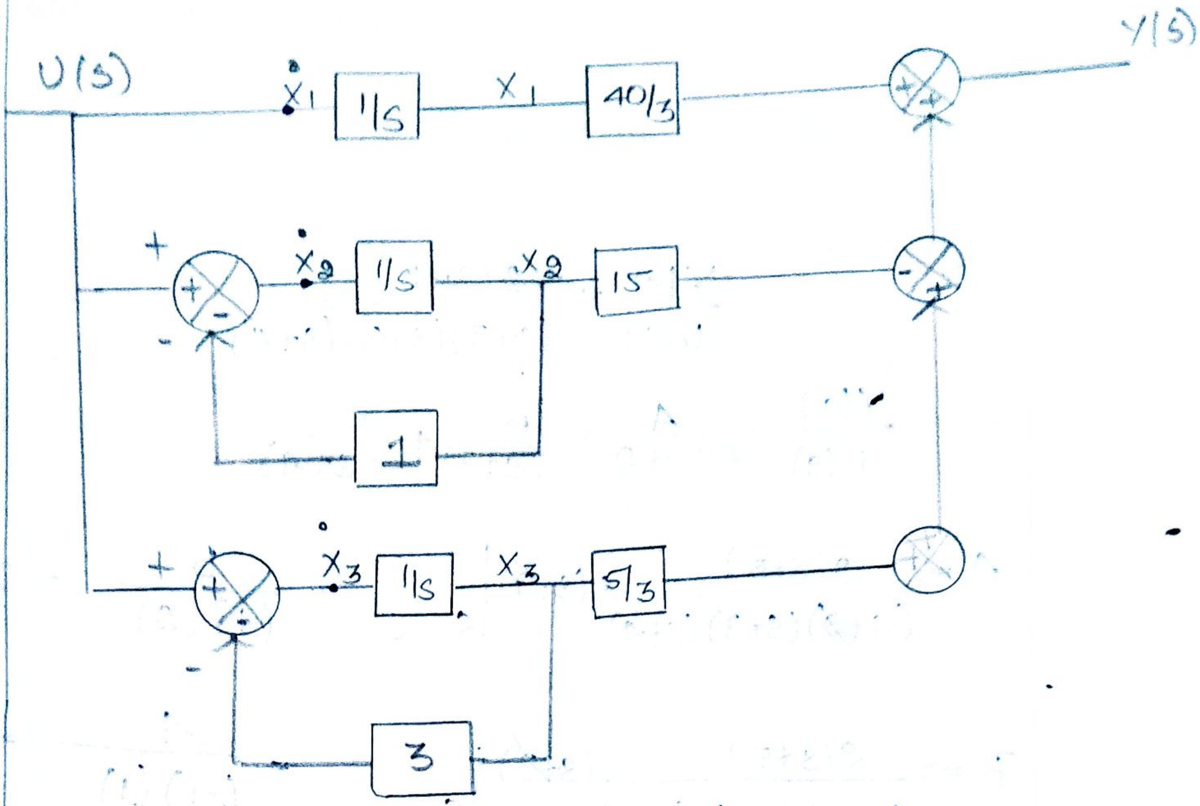
$$C = \frac{10(s+4)}{s(s+1)\cancel{(s+3)}} \times \cancel{(s+3)}\Big|_{s=-3} = \frac{10(-3+4)}{(-3)(-3+1)} = \frac{10}{6}$$

$$= 5/3$$

$$\therefore \frac{Y(s)}{U(s)} = \frac{40/3}{s} - \frac{15}{s+1} + \frac{5/3}{s+3}$$

$$\therefore Y(s) = \frac{40/3}{s}U(s) - \left[\frac{1/s}{1+1/s \times 1} \times 15\right] U(s)$$

$$+ \left[\frac{1/s}{1+1/s \times 3} \times 5/3\right] U(s)$$



$$\dot{x}_1 = u$$

$$\dot{x}_2 = u - x_2$$

$$\dot{x}_3 = u - 3x_3$$

$\therefore \; x(t) = Ax(t) + Bu(t)$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} u \end{bmatrix}$$

$y(t) = Cx(t) + Bu(t)$

$$y = \frac{40}{3} x_1 - 15 x_2 + \frac{5}{3} x_3$$

$$y = \begin{bmatrix} \frac{40}{3} \cancel{x_1} & -15 & \frac{5}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

2. $\frac{Y(s)}{U(s)} = \frac{2(s+5)}{(s+2)(s+3)(s+4)}$ solve the equation using canonical method

solution

$$\frac{Y(s)}{U(s)} = \frac{2(s+5)}{(s+2)(s+3)(s+4)}$$

$$\frac{Y(s)}{U(s)} = \frac{A}{s+2} + \frac{B}{(s+3)} + \frac{C}{(s+4)}$$

$$A \Rightarrow \frac{2(s+5)}{(s+\cancel{2})(s+3)(s+4)} \times (s+\cancel{2}) \Big|_{s=-2} = \frac{6}{(1)(2)} = 3$$

$$B \Rightarrow \frac{2(s+5)}{(s+2)(s+\cancel{3})(s+4)} \times (s+\cancel{3}) \Big|_{s=-3} = \frac{4}{(-1)(1)} = -4$$

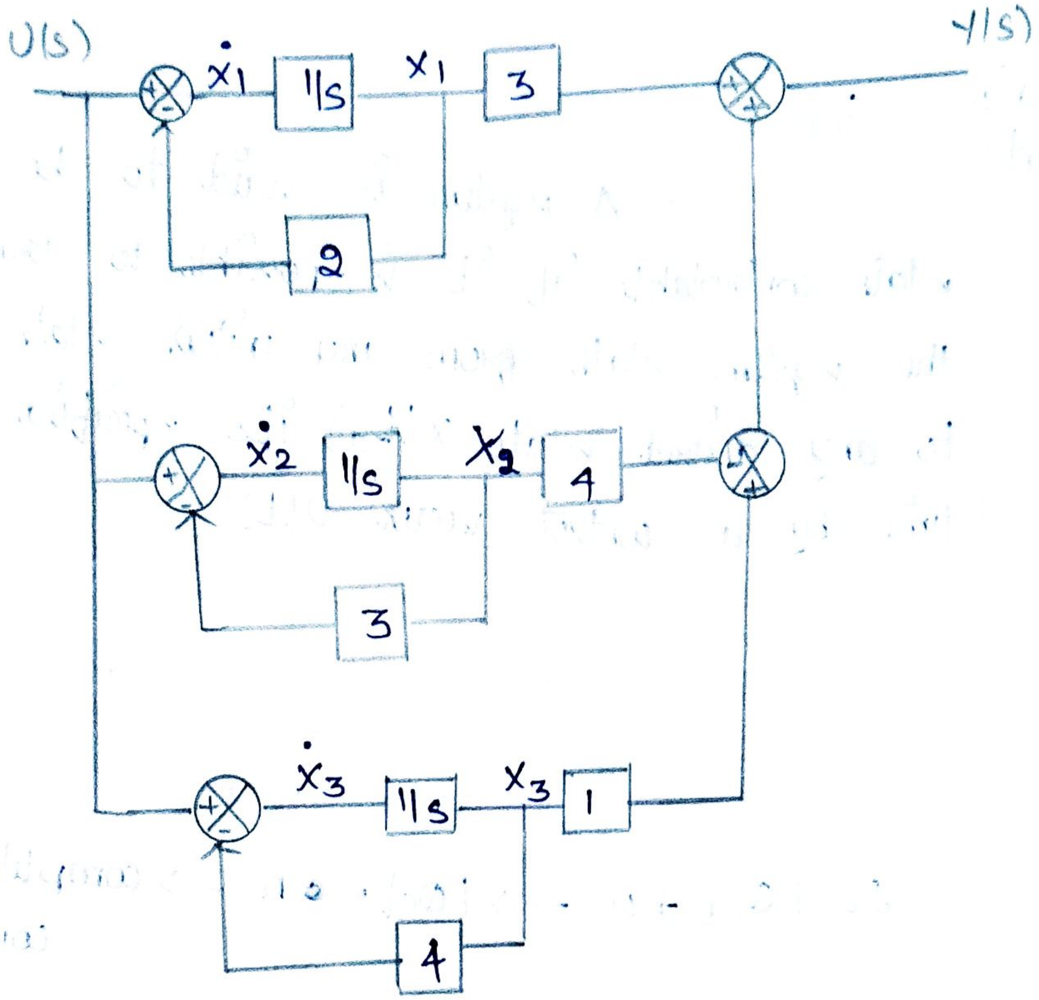
$$C \Rightarrow \frac{2(s+5)}{(s+2)(s+3)(s+\cancel{4})} \times (s+\cancel{4}) \Big|_{s=-4} = \frac{2}{-2(-1)} = 1$$

$$\therefore \frac{Y(s)}{U(s)} = \frac{3}{s+2} - \frac{4}{s+3} + \frac{1}{s+4}$$

$$\frac{Y(s)}{U(s)} = \left[\frac{1/s}{1+1/s \cdot 2} \times 3\right] - \left[\frac{1/s}{1+1/s \cdot 3} \times 4\right] + \left[\frac{1/s}{1+\frac{1}{s} \times 4} \times 1\right]$$

$$Y(s) = \left[\frac{1/s}{1+1/s \cdot 2} \times 3\right] U(s) - \left[\frac{1/s}{1+1/s \cdot 3} \times 4\right] U(s)$$

$$+ \left[\frac{1/s}{1+1/s \times 4} \times 1\right] U(s)$$



$$\dot{x}_1 = U - 2x_1$$

$$\dot{x}_2 = U - 3x_2$$

$$\dot{x}_3 = U - 4x_3$$

$$\dot{x}(t) = Ax(t) + BU(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} U \end{bmatrix}$$

$$Y(t) = Cx(t) + DU(t)$$

$$Y = 3x_1 - 4x_2 + x_3$$

$$\therefore Y = \begin{bmatrix} 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

* Controlability & Observability :- 16 M topic

* Colven's method

Controlability :

* A system is said to be completly state controlable if it is possible to transfer the system state from any initial state $x(t_0)$ to any desired state $x(t_d)$ in specified finite time by a control vector $u(t)$

$$Q_c = \begin{bmatrix} B & AB & A^2B & \dots, & A^{n-1}B \end{bmatrix}$$

$n \rightarrow$ order of the system

$\therefore |Q_c| \neq 0 \longrightarrow |Q_c| = n \longrightarrow$ completly controllable

* Observability :-

* A system is said to be completly observable if every state $x(t)$ can be completly indentified by measurments of the output $y(t)$ over a finite time interval

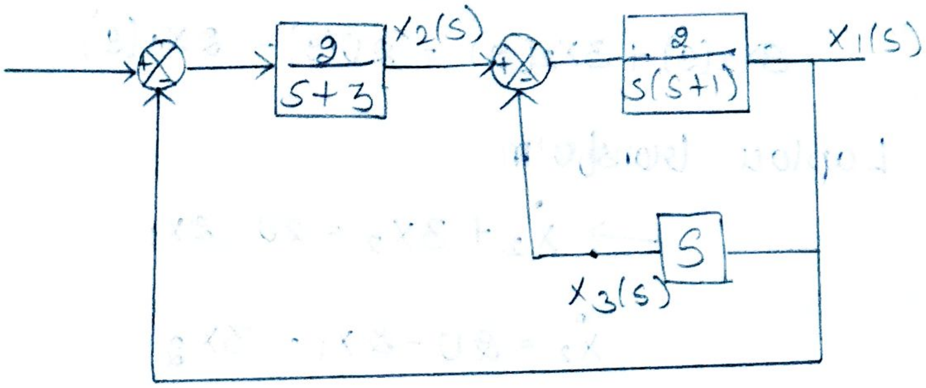
$$Q_o = \begin{bmatrix} C^T & A^TC^T & (A^T)^2C^T \dots & (A^T)^{n-1}C^T \end{bmatrix}$$

$|Q_o| \neq 0 \rightarrow |Q_o| = n \rightarrow$ completly observable

6-11-23
monday

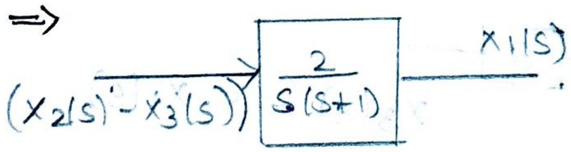
Controllability & Observability :-

16M1. Write the state equation for the system shown in a figure in which x_1, x_2, x_3 consitute the state vector. Determine whether the system is completely controllable & observable.



solution :-

$\therefore X_1(s) \Rightarrow$



$$\therefore X_1(S) = [X_2(S) - X_3(S)] \left[\frac{2}{S(S+1)}\right]$$

$$X_1(S)[S(S+1)] = 2X_2(S) - 2X_3(S)$$

$$S^2 X_1(S) + S X_1(S) = 2X_2(S) - 2X_3(S)$$

Laplace transform,

$$\rightarrow \ddot{x}_1 + \dot{x}_1 = 2x_2 - 2x_3$$

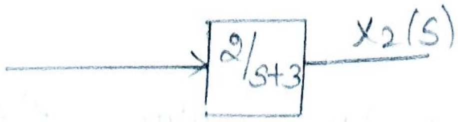
$$\dot{x}_3 + x_3 = 2x_2 - 2x_3$$

$$\therefore \dot{x}_3 = 2x_2 - 3x_3$$

L.T

* $x \rightarrow x(s)$
* $\frac{dx}{dt} = Sx(s) = \dot{x}$
* $\frac{d^2x}{dt^2} = S^2 x(s) = \ddot{x}$

* $X_2(s)$:-



$$\therefore X_2(s) = U(s) - X_1(s)\left[\frac{2}{s+3}\right]$$

$$X_2(s)[s+3] = 2U(s) - 2X_1(s)$$

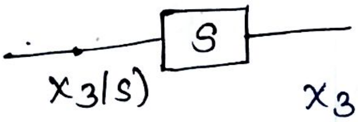
$$sX_2(s) + 3X_2(s) = 2U(s) - 2X_1(s)$$

Laplace transform

$$\rightarrow \dot{X}_2 + 3X_2 = 2U - 2X_1$$

$$\dot{X}_2 = 2U - 2X_1 - 3X_2$$

$X_3(s)$



$$X_3(s) = s \cdot X_1(s)$$

Laplace transform,

$$X_3 = \dot{X}_1$$

state equation

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}\begin{bmatrix} U \end{bmatrix}$$

output equation,

$$\therefore Y = X_1$$

$$Y = \overset{C}{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

Controlability :-

$* \ Q_C = [B \quad AB \quad A^2B \ \dots \ A^{n-1}B]$

$n = 3$

$\therefore n = n-1$

$n = 3-1$

$n = 2$

$Q_C = [B \quad AB \quad A^2B]$

$A^2 = \begin{bmatrix} 0 & 2 & -3 \\ 6 & 9 & -2 \\ -4 & -12 & 9 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 4 \\ 2 & -6 & 18 \\ 0 & 4 & -24 \end{bmatrix}$

$\therefore |Q_C| = 32$

$\therefore$ Hence the system is controlable

Observability :-

$Q_O = [C^T \quad A^TC^T \quad (A^T)^2C^T \ \dots \ (A^T)^{n-1}C^T]$

$n = 3-1$

$n = 2$

$Q_O = [C^T \quad A^TC^T \quad (A^T)^2C^T]$

$C = [1 \ 0 \ 0]$

$C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & 2 \\ 1 & 0 & -3 \end{bmatrix}$

$A^TC^T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\therefore \quad Q_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$

$$|Q_0| = -2 \qquad |Q_0| \neq 0$$

$\therefore$ Hence the system is observable