1 Problem 1

1.1 a

Let $\theta = 0.5$, then we need to show that $f(1/2x + 1/2y)$ is not always $<=$ $1/2f(x) + 1/2f(y)$ for $x = (x_1, x_2), y = (y_1, y_2)$. $f(1/2x + 1/2y) = f(1/2(x + y)) = 1/2(x_1 + y_1)(x_2 + y_2)$ and $1/2f(x) + 1/2f(y) = 1/2x_1x_2 + 1/2y_1y_2$.

For $x = (0.5, 1)$ and $y = (1, 0.5)$, we get $f(1/2x + 1/2y) = 1/2 * 1.5 * 1.5 = 1.1125$ and $1/2f(x) + 1/2f(y) = 1/2 * 1 * 0.5 + 1/2 * 1 * 0.5 = 0.5$. Since $1.1125 > 0.5$, this does not hold, meaning it is not convex.

1.2 b

For this to be concave, we need to prove the opposite, that $f(1/2x + 1/2y)$ is not always $>= 1/2f(x) + 1/2f(y)$. For $\theta = 0.1, f(0.1x + 0.9y) = (0.1x_1 + 0.9y_1)(0.1x_2 + 0.9y_2)$ and $0.1f(x) + 0.9f(y) = 0.1x_1x_2 + 0.9y_1y_2$. For $x = (1, 1), y = (10, 10)$, $f(0.1x + 0.9y) = 82.81$ and $0.1f(x) + 0.9f(y) = 90.1$. Since $82.81 < 90.1$, this does not hold, meaning it is not concave.

2 Problem 2

In order to show that this element is concave, we just need to show that the overall set is concave, as partial minimization shows that the minimum element of a concave set is concave.

For two vectors $x, y \in R^m$, let f(z) be the weighted least squares problem, then $f(x) = \sum_{i=1}^{n} w_i(a_i^T x - b_i)^2$ and $f(y) = \sum_{i=1}^{n} w_i(a_i^T y - b_i)^2$. For $\lambda = 1/2$, we need to show that $f(1/2x + 1/2y) >= 1/2f(x) + 1/2f(y)$ always.

$f(1/2x + 1/2y) = \sum_{i=1}^{n} w_i(a_i^T(1/2x + 1/2y) - b_i)^2 = \sum_{i=1}^{n} w_i(1/2a_i^T x + 1/2a_i^T y - b_i)^2 = \sum_{i=1}^{n} 1/4w_i(a_i^T x + a_i^T y - b_i)^2$.

$1/2f(x) + 1/2f(y) = \sum_{i=1}^{n} 1/2w_i(a_i^T x - b_i)^2 + \sum_{i=1}^{n} 1/2w_i(a_i^T y - b_i)^2 = \sum_{i=1}^{n} 1/2w_i(a_i^T x - b_i)^2 + 1/2w_i(a_i^T y - b_i)^2 = \sum_{i=1}^{n} 1/2w_i((a_i^T x - b_i)^2 + (a_i^T y - b_i)^2)$.

Comparing these, we get $\sum_{i=1}^{n} 1/4w_i(a_i^T x + a_i^T y - b_i)^2 >= \sum_{i=1}^{n} 1/2w_i((a_i^T x - b_i)^2 + (a_i^T y - b_i)^2)$, or $\sum_{i=1}^{n}(a_i^T x + a_i^T y - b_i)^2 >= \sum_{i=1}^{n} 2((a_i^T x - b_i)^2 + (a_i^T y - b_i)^2)$. We know this is true, because as we try to minimize the vector in the least squares problem, this will be as close as possible to 0 for $(a_i^T x - b_i)^2$ and $(a_i^T y - b_i)^2$. Therefore, we can say that this is approximately equal to $\sum_{i=1}^{n}(a_i^T x)^2 >= 0$, meaning this is concave.

3 Problem 3

3.1 a

For λ between 0 and 1, we need to show that $f(\lambda x + (1 - \lambda)y) <= \lambda f(x) + (1 - \lambda)f(y)$ for two vectors x and y in R^n. If we let $\lambda = 1/2$, then we need to show that $f(1/2x + 1/2y) <= 1/2f(x) + 1/2f(y)$.

If we let $x = (x_1, x_2...x_n)$ and $y = (y_1, y_2...y_n)$, then $f(1/2x + 1/2y) = max(x_a, y_b) - min(x_i - y_j)$ where x_a, y_b are the maximum values and x_i, y_j are the minimum values in x and y respectively.

$1/2f(x)+1/2f(y) = 1/2x_a+1/2y_b-1/2x_i-1/2y_j = 1/2(x_a+y_b-x_i-y_j)$. This means the function is convex if and only if $max(x_a, y_b) - min(x_i - y_j) <= 1/2x_a + 1/2y_b - 1/2x_i - 1/2y_j$. Let us say $x_a = 1, x_i = 0, y_b = 0, y_j = 0$, then this yields $1 <= 1/2$, which is false. Therefore, this function is not convex.

3.2 b

$f(x) = (1/n \sum_{i=1}^{n} x_i^2 - (1/n \sum_{i=1}^{n} x_i)^2)^{1/2}$. This can be rewritten as $f(x) = (1/n(x - 1/nx^T 1)^T (x - 1/n1^T x1))^{1/2} = (1/n||(I - 1/n11^T)||^2)^{1/2} = 1/\sqrt{n}||(I - P)x||$. We know that vector norms are convex, so this function is also convex as it is a scaled-by-n function of a norm.

3.3 c

If we take the the space the function covers, the entirety of R^n is under this function, since every vector in the space is a possible mapping. Therefore, this function is convex.

4 Problem 4

We know that $g(x) = \sum_{i=1}^{n} x_{[i]}$ is convex, so we know that $f(x) = \sum_{i=1}^{r} \alpha_i x_{[i]}$ is also convex since it is a nonnegative multiple. Since all the $\alpha_1...\alpha_r >= 0$ and f(x) is convex, $\alpha f(x)$ is also convex for all the alpha values.