Problems: 8.8, 8.16, 10.13, 10.16, 11.12, 11.24, 7 and 8 from homework file.

1 Problem 1

Since $f(t_i) = \frac{c_1+c_2t_i+c_3t_i^2}{1+d_1t_i+d_2t_i^2} = y_i$. This simplifies to $c_1 + c_2t_i + c_3t_i^2 = y_i + d_1t_iy_i + d_2t_i^2y_i$, or $c_1 + c_2t_i + c_3t_i^2 - d_1t_iy_i - d_2t_i^2y_i = y_i$. We can say that $\theta = [c_1, c_2, c_3, d_1, d_2]$, leaving the terms to be $1 + t_i + t_i^2 - t_iy_i - t_i^2y_i = y_i$ This means we can put these into a matrix A like the following:

$1 \quad t_1 \quad t_1^2 \quad t_1y_1 \quad t_1^2y_1$
$1 \quad t_2 \quad t_2^2 \quad t_2y_2 \quad t_2^2y_2$
...
$1 \quad t_K \quad t_K^2 \quad t_Ky_K \quad t_K^2y_K$

This means $A\theta = [y_1, y_2...y_K]$.

2 Problem 2

Let $\theta = [\theta_1, \theta_2, \theta_3, \theta_4]$ and $b = [F_{11}, F_{12}, F_{21}, F_{22}]$. Then in order for $A\theta = b$ to be true, we need a matrix A that equals the following:

$1 \quad x_1 \quad y_1 \quad x_1y_1$
$1 \quad x_1 \quad y_2 \quad x_1y_2$
$1 \quad x_2 \quad y_1 \quad x_2y_1$
$1 \quad x_2 \quad y_2 \quad x_2y_2$

3 Problem 3

3.1 a

$D(v) = v^TLv = v^T(AA^T)v = (v^TA)(A^Tv) = (A^Tv)^T(A^Tv) = ||A^Tv||^2$. Dirichlet energy is defined as $||A^Tv||^2$, so therefore $D(v) = v^TLv$.

3.2 b

The incidence matrix A contains either 0 or 1 for every index A_{ij}, where 1 represents edge j pointing to node i,, -1 represents edge j leaving from node i, and a 0 represents no edge between the two points. We can add the total number of 1s in A_i to see the total number of edges coming into node i, and likewise for the -1s to find the total number of edges leaving node i. When we calculate $L = AA^T$, we get the resulting matrix L that for any index L_{ii} indicates the sum of each row's -1s and 1s, or the degree of each node. Conversely, for any index L_{ij} where $i \neq j$, we see that the result will be either -1 or 0. It will be -1 if the same edge k connects i and j together, and 0 otherwise. This is because for two entries, A_{ik} and A_{jk}, one will be 1 and the other will be -1, and -1 times 1 is -1. If either is 0, then there is no edge and the product is 0.

4 Problem 4

4.1 a

$\mu_i = avg(a_i) = a_i/n$, so the vector μ is equal to $[\mathbf{1}^T a_1/n, \mathbf{1}^T a_2/n ... \mathbf{1}^T a_n/n] = 1/n[\mathbf{1}^T a_1, \mathbf{1}^T a_2 ... \mathbf{1}^T a_n] = A^T \mathbf{1}/n$.

4.2 b

Demeaning, for any vector a_i, is equal to $\tilde{a_i} = a_i - avg(a_i)\mathbf{1}$, so $\tilde{A} = [a_1 - avg(a_1)\mathbf{1}, a_2 - avg(a_2)\mathbf{1} ... a_n - avg(a_n)\mathbf{1}] = [a_1 - \mu_1\mathbf{1}, a_2 - \mu_2\mathbf{1} ... a_n - \mu_n\mathbf{1}]$. This is equal to the equation $\tilde{A} = A - \mathbf{1}\mu^T$.

4.3 c

The $std(a_i) = \frac{||\tilde{a_i}||}{\sqrt{N}}$, so for $\Sigma = \frac{\tilde{A}^T \tilde{A}}{N}$, $\Sigma_{ii} = \frac{\tilde{a}_i^T \tilde{a}_i}{N} = \frac{||\tilde{a}_i||^2}{N} = std(a_i)$. Similarly, $\Sigma_{ij} = \frac{\tilde{a}_i^T \tilde{a}_j}{N}$. The coefficient correlation is equal to $\rho = \frac{\tilde{a}_i^T \tilde{a}_j}{std(a_i) std(a_j) N}$. Therefore, $\Sigma_{ij} = \frac{\tilde{a}_i^T \tilde{a}_j}{N} = \rho_{ij} std(a_i) std(a_j)$.

4.4 d

The standardized version of a vector a_i is equal to $z = \frac{a_i - avg(a_i)\mathbf{1}}{std(a_i)} = \frac{\tilde{a_i}}{std(a_i)}$. Therefore, Z is made up of the vectors $[\frac{\tilde{a_1}}{std(a_1)}, \frac{\tilde{a_2}}{std(a_2)} ... \frac{\tilde{a_n}}{std(a_n)}]$. Therefore, $Z = (A - 1\mu^T)S$ where S is a diagonal matrix of dimension kxk where $S_{ii} = \frac{1}{std(a_i)}$ from a_1 to a_k.

5 Problem 5

5.1 a

If A and B are both invertible, then both have a null space of dimension 0 (containing only the 0-vector). Therefore, A+B is not necessarily invertible because we cannot determine its eigenspace for eigenvalue 0- it is possible that it has a nullity ¿ 0, meaning it is not invertible.

5.2 b

The determinant of this matrix is equal AB, which is non-zero because $det(AB) = det(A)det(B)$ is non-zero. Therefore, this matrix is invertible.

5.3 c

The determinant of this matrix is equal to AB-(A+B) = AB-0(A-B), which has determinant det(AB) = det(A)det(B), which is non-zero, meaning this matrix is invertible.

5.4 d

The determinant of an invertible matrix is nonzero and if determinant is nonzero, then the matrix is invertible. Therefore, if A and B are both invertible, they have determinants that are non-zero. Therefore, the matrix ABA has $det(ABA) = det(A)det(B)det(A)$, which is non-zero, meaning ABA is therefore invertible.

6 Problem 6

A matrix C is the left inverse of A if it satisfies $CA = I$ and the left inverse of B if it satisfies $CB = I$. This means that $C[AB] = [II] = (C[AB])^T = ([II])^T = [A^T B^T]^T C^T = [I^T I^T]^T = [II]^T$.

$[A^T B^T]^T$ is equal the following matrix:

1	3	2	2
2	1	1	2
3	1	2	1
2	0	1	3

This is an invertible matrix, so to find C^T we can calculate $([A^T B^T]^T)([A^T B^T]^T)^{-1} C^T = ([A^T B^T]^T)^{-1}[II]^T$, which gives us that C is the following matrix:

$-3/11$	$-2/11$	$12/11$	$-2/11$
$6/11$	$4/11$	$-13/11$	$4/11$

7 Problem 7

The given matrix, A, has dimension nxn, and if we calculate A^{10}, we will get a new matrix. If we count the cells within this matrix, the number of cells with a 10 in the index A_{ij} will be the number of unique paths of length 10. We can tell the number of paths of length 10 by iterating through each cell in A and if it is equal to 10, increment the counter by 1.

8 Problem 8

We can say $z_{t+1} = A_1 z_t + ...A_K z_{t-k}$ for $t = K, K+1....$ We can repeat this for $z_1...z_T$. We can further expand this to mean that $\beta = [A_0, A_1 z_1, A_1 z_2 + A_2 z_2 ... A_1 z_1 + ...A_K z_{t-k}]^T$. The resulting matrix y must reside in the column space of A, so let us say for A with columns $[a_1, a_2 ... a_k]$, y must be within this range. Therefore, we can multiply $A\beta$ to find the end result of $[A_0 a_{11} + A_1 z_1 a_{12} ... + (A_1 z_1 + ...A_K z_{t-k}) a_{1k}, A_0 a_{21} + A_1 z_1 a_{22} ... + (A_1 z_1 + ...A_K z_{t-k}) a_{2k} ... A_0 a_{k1} + A_1 z_1 a_{k2} ... + (A_1 z_1 + ...A_K z_{t-k}) a_{kk}$. This yields the result that y must be some column within a itself of the form $y = [a_{i1}, a_{i1} + a_{i2} ... a_{i1} + a_{i2} + a_{ik}]$, each multiplied by some scalar that is in β. This means that the columns of A are arranged in the order of a Toeplitz matrix, where each column moves down one index from the previous column, hence allowing y to always be in the span of the linearly independent columns of A.