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Attention!!: You will have to upload the final Python notebook to Canvas in the midterm exam assignment to get credit in addition to attaching a print out pdf to the exam you submit! We grade the pdf; the notebook is for posterity (in case there is an academic integrity issue that arises). You must upload a legible pdf to get credit. This is easy to do. Simply use the `print` option under `file` menu, and choose save to pdf. If on JupyterHub, then under the `file` menu, chose to export to a pdf.

Problem 1 (Geometry of SVD) [3 pts]. Consider the 2×2 matrix

$$A = \frac{1}{\sqrt{10}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} + \frac{2}{\sqrt{10}} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix}$$

a. **[1pt]** What is an SVD of A? Express it as $A = USV^\top$, with S the diagonal matrix of singular values ordered in a decreasing fashion. Make sure to check all the properties required for U, S, V.

Solution.

$$A = \frac{1}{\sqrt{10}} \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} + \frac{2}{\sqrt{10}} \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$$

$$A = \frac{1}{\sqrt{10}} \begin{bmatrix} 2-2 & -2-2 \\ 1+4 & -1+4 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 0 & -4 \\ 5 & 3 \end{bmatrix}$$

real nums & $\alpha_2 > \alpha_1$

$AA^T = \frac{1}{5}\begin{bmatrix} 8 & -6 \\ -6 & 17 \end{bmatrix}$, $\lambda_{AA^T} = 4, 1$ $\sqrt{\lambda_{AA^T}} = 2, 1$ $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $U = \begin{bmatrix} -\sqrt{5}/5 & 2\sqrt{5}/5 \\ 2\sqrt{5}/5 & \sqrt{5}/5 \end{bmatrix}$

$\Theta_{AA^T} = [-1\ 2]^T, [2\ 1]^T$

$A^TA = \frac{1}{2}\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$, $\Theta_{A^TA} = [1\ 1]^T, [1\ {-1}]^T$

$$V = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}$$

$UU^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ $V^TV = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$$A = U\Sigma V^T = \begin{bmatrix} -\sqrt{5}/5 & 2\sqrt{5}/5 \\ 2\sqrt{5}/5 & \sqrt{5}/5 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}$$

b. **[1pt]** Find the semi-axis lengths and principal axes of the ellipse

$$\mathcal{E}(S) = \{Sx \mid x \in \mathbb{R}^2, \quad \|x\|_2 \leq 1\},$$

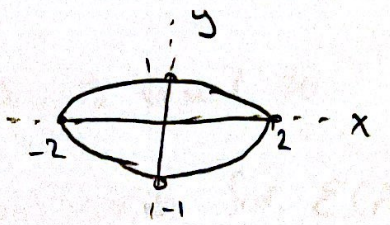
where S is the matrix of singular values above. Note: the principal axes and semi-axis lengths correspond to the largest and smallest distance from the center to the boundary of the ellipse.

Solution.

$$S = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

For every $x \in \mathbb{R}^2$ where $\|x\|_2 \leq 1$, we know x represended as $[x_1, x_2]^T$ yields $\sqrt{x_1^2 + x_2^2} \leq 1$.

$Sx = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ x_2 \end{bmatrix}$. Graphing the range of these values gives us:



, where the semi-axes are equal to 2 and 1 for the x and y axes respectively.

The principal axes are equal to the x and y axes in $\mathbb{R}^2$.

c. **[1pt]** Find the semi-axis lengths and principal axes of the ellipse

$$\mathcal{E}(A) = \{Ax \mid x \in \mathbb{R}^2, \quad \|x\|_2 \le 1\}.$$

Hint: Show that $\mathcal{E}(A) = \{U\bar{y} : \bar{y} \in \mathcal{E}(S)\}$ to reduce to the previous subpart.
Solution.

$$Ax = \frac{1}{\sqrt{10}} \begin{bmatrix} 0 & -4 \\ 5 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = USV^T x = \begin{bmatrix} -\sqrt{5}/5 & 2\sqrt{5}/5 \\ 2\sqrt{5}/5 & \sqrt{5}/5 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}$$

$V^T x = \frac{1}{\sqrt{2}} \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} = \bar{z}$, which we know is on $\mathcal{E}(S)$ because

it $\in \mathbb{R}^2$ and $\frac{1}{\sqrt{2}} \sqrt{(x_1+x_2)^2 + (x_1-x_2)^2} = \frac{\sqrt{2x_1^2 + 2x_2^2}}{\sqrt{2}}$, which has a maximum

value of 1.

Therefore, this is equal to $U\bar{y}$, where $\bar{y} = S\bar{z}$, meaning

the principal axis for $\mathcal{E}(A)$ is the y axis and the lengths

of the semi-axes are $\frac{2}{\sqrt{5}}$ and $\frac{4}{\sqrt{5}}$ respectively for x and y.

Problem 2 (Convex sets)[6 pts]. For each of the sets described below, either show that the set is convex, or provide a counter-example (for example, give two points in the set that violate convexity).

To show convexity, you can either (1) verify the definition of convexity by checking the line segments connecting every pair of points in the set, or (2) use convexity of known sets shown in the lectures (hyperplanes, halfspaces, norm balls, ellipsoids, norm cones) together with properties that preserve convexity. Cite any properties or results you use from the lectures, and show how you use them.

$x_t \geq$ average of prev 3 elements in set vector where $t \geq 4 \dots n$

a. **[2pt]**

$$S = \left\{ x \in \mathbb{R}^n \mid x_t \geq \frac{1}{3} \sum_{i=t-3}^{t-1} x_i, \text{ for } t = 4, 5, \dots, n \right\}.$$

minimum

Solution.

The matrix used to represent this can be seen as

$$A = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 1/3 & 1/3 & 1/3 & \cdots & 0 \\ 0 & 1/3 & 1/3 & 1/3 \cdots & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & 1/3\ 1/3\ 1/3 & 0 \end{bmatrix}$$

, where $S = \{ y \in \mathbb{R}^n \mid \|Ax\|_i^2 \leq y_i \}$

or where y_i is $\geq \|Ax\|_i^2$.

This can be seen as the convex hull where $(1/3 + 1/3 + 1/3) = 1$, $1/3 > 0$ meaning that S is convex.

b. **[2pt]** $S = \{y \in \mathbf{R}^n \mid \|y - a\|_2 \leq r \text{ for all } a \in C\}$, where $C \subset \mathbf{R}^n$. convex There are no other assumptions on C.

Solution.

This is the norm ball repeated for every center a for $a \in C$.

We know that the norm-ball is convex, so we need to show S is obtained from it by an operation that preserves convexity. Affine functions include shifting the location of the norm-ball, such as moving the center. This means this set is convex as it is an affine function applied to a convex function.

c. **[2pt]** $S = \{(x, y) \mid \|x\|_2 \geq \|y\|_2,\ x, y \in \mathbb{R}^n\}$.

Solution.

Let $\|x\|_2 = \sqrt{\Sigma\, x_1^2 + x_2^2 \ldots x_n^2}$ and $\|y\|_2 = \sqrt{\Sigma\, y_1^2 + y_2^2 \ldots y_n^2}$. If this set is convex, then that means for $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in S$, then

$\theta \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + (1-\theta)\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in S$. If $\|x\|_2 \geq \|y\|_2$, then $\|x\|_2 - \|y\|_2 \geq 0$.

By the reverse triangle inequality, we know $\|x-y\|_2 \geq \|x\|_2 - \|y\|_2 \geq 0$.

This means, $\theta\|x_1 - y_1\|_2 \geq \theta\|x_1\|_2 - \theta\|y_1\|_2$, so $\theta\|x_1 - y_1\|_2 + (1-\theta)\|x_2 - y_2\|_2 \geq \theta\|x_1\|_2 - \theta\|y_1\|_2 + (1-\theta)\|x_2\|_2 - (1-\theta)\|y_2\|_2$, showing that this set is not convex.

Problem 3 (Matrix Least Squares Variant) [10 pts]. Consider a data matrix $A \in \mathbb{R}^{m \times n}$ and observation matrix $Y \in \mathbb{R}^{m \times d}$. Recall that the (ordinary) matrix least squares problem $AX \approx Y$ can be seen as projecting the (columns of) Y onto the range of A—i.e., $AX = AA^{\dagger}Y$ where $AA^{\dagger}$ is a projection matrix. In this setting, our model $AX \approx Y$ is such that we are modeling errors in our measurements—that is,

$$Y + \varepsilon_Y = AX.$$

In many applications, however, it is more reasonable to model the errors in both the observations Y and the data A—i.e.,

$$Y + \varepsilon_Y = [A + \varepsilon_A]X \tag{1}$$

Hence, we need to formulate an optimization problem that captures trying to find the matrix $X \in \mathbb{R}^{n \times d}$ that is the best approximate solution to the above model. That is, the problem is to find the smallest perturbations $\begin{bmatrix}\varepsilon_A & \varepsilon_Y\end{bmatrix}$ to the measured independent and dependent variables that satisfy (1), where $\begin{bmatrix}\varepsilon_A & \varepsilon_Y\end{bmatrix}$ is the column-wise concatenation of the matrix ε_A with the matrix ε_Y. Indeed, this can be done with the Frobenius norm as follows:

$$\begin{aligned} \min_{X, \varepsilon_Y, \varepsilon_A} \quad & \| \begin{bmatrix}\varepsilon_A & \varepsilon_Y\end{bmatrix} \|_F^2 \\ \text{subject to} \quad & Y + \varepsilon_Y = [A + \varepsilon_A]\, X \end{aligned} \tag{2}$$

Upon examining the constraint we can rewrite it as

$$\begin{bmatrix}A + \varepsilon_A & Y + \varepsilon_Y\end{bmatrix} \begin{bmatrix} X \\ -I_d \end{bmatrix} = 0 \in \mathbb{R}^m, \tag{3}$$

where I_d is the $d \times d$ identity matrix.

Below, we are going to use several of the tools from **Modules 2–4** to solve this problem and implement it in Python. **Note that parts f–h are in the Python Notebook.**

a. **[1pt]** Let the matrix $A \in \mathbb{R}^{m \times n}$ and $Y \in \mathbb{R}^{m \times d}$. Note that this implies $\varepsilon_A \in \mathbb{R}^{m \times n}$ and $\varepsilon_Y \in \mathbb{R}^{m \times d}$. Assuming that $m > n$ and $\operatorname{rank}(A + \varepsilon_A) = n$, using the constraint in the optimization problem (2) explain why $\operatorname{rank}(\begin{bmatrix}A + \varepsilon_A & Y + \varepsilon_Y\end{bmatrix}) = n$.

Solution:

Since we are trying to minimize $\|[\mathcal{E}_A \ \ \mathcal{E}_Y]\|^2$, this means that if $Y + \mathcal{E}_y = [A + \mathcal{E}_A]X$, then $\operatorname{rank}(Y + \mathcal{E}_y) \leq \min(\operatorname{rank}(A + \mathcal{E}_A), \operatorname{rank}(X))$. This is because we know $\operatorname{rank}(AB) \leq \min(\operatorname{rank}(A), \operatorname{rank}(B))$. Therefore $\operatorname{rank}([A + \mathcal{E}_A \ \ Y + \mathcal{E}_Y]) = \operatorname{rank}(A + \mathcal{E}_A)$, as we know $\operatorname{rank}(Y + \mathcal{E}_y)$ $= \operatorname{rank}(A + \mathcal{E}_A)$ at most, $\hat{=} n$

b. **[1pt]** The conclusion that $\operatorname{rank}(\begin{bmatrix} A+\varepsilon_A & Y+\varepsilon_Y \end{bmatrix}) = n$ from part a. tells us that the goal of solving the optimization problem (2) is to find the smallest matrix $\begin{bmatrix} \varepsilon_A & \varepsilon_Y \end{bmatrix}$ that drops the rank of $\begin{bmatrix} A & Y \end{bmatrix}$ to n when added to it.

Recalling the Eckart-Young Theorem from **Module 3**, formulate the optimization problem (2) as a low rank approximation optimization problem in terms of the matrix $C := \begin{bmatrix} A & Y \end{bmatrix}$.

Solution:

Eckart-Young says that $A_k = \sum_{i=1}^{k} \sigma_i \cdot u_i v_i^T$ is the closest matrix of rank k to A. This means the closest matrix to $[A\ Y]$ of rank n is equal to $[A\ Y]_n = \sum_{i=1}^{n} \sigma_i \cdot u_i v_i^T$ where ~~thats~~ is the SVD of $[A\ Y]$. USV^T

We want $[A+\varepsilon_A \quad Y+\varepsilon_Y] = [A\ Y] + \sum_{i=1}^{n} \sigma_i \cdot u_i v_i^T$

c. **[2pt]** Suppose we express the SVD of $\begin{bmatrix} A & Y \end{bmatrix}$ in terms of submatrices and vectors as follows:

$$C = \begin{bmatrix} A & Y \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{array}{c} \begin{array}{cc} n\times n & n\times d \end{array} \\ \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}^T \\ \begin{array}{cc} d\times n & d\times d \end{array} \end{array} \tag{4}$$

where $\Sigma_1 \in \mathbb{R}^{n\times n}$ contains the top n singular values of C, and $\Sigma_2 \in \mathbb{R}^{d\times d}$ contains the next d singular values. The submatrices of V satisfy the following:

$$V_{11} \in \mathbb{R}^{n\times n}, \quad V_{12} \in \mathbb{R}^{n\times d}, \quad V_{21} \in \mathbb{R}^{d\times n}, \quad V_{22} \in \mathbb{R}^{d\times d}$$

Derive an expression for the solution $\begin{bmatrix} \varepsilon_A & \varepsilon_y \end{bmatrix}$ to the problem in part b. in terms of the SVD of $C = \begin{bmatrix} A \; y \end{bmatrix}$ as given in (4). In addition, derive an expression for the matrix $\begin{bmatrix} A + \varepsilon_A & Y + \varepsilon_Y \end{bmatrix}$ in terms of the SVD and the solution $\begin{bmatrix} \varepsilon_A & \varepsilon_y \end{bmatrix}$. Show your work to get credit.

Solution.

We know rank n approximation tells us $[A\ Y]_n = \sum_{i=1}^{n} \sigma_i \cdot u_i v_i^T$,

which is equal to $[A, Y]_n = \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix} \Sigma_1 \begin{bmatrix} V_{11} & V_{12} \end{bmatrix}$.

This means if $[\varepsilon_A\ \ \varepsilon_y]$ is equal to the difference between full-rank and rank n, then:

$$[\varepsilon_A\ \ \varepsilon_y] = \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix} \Sigma_1 \begin{bmatrix} V_{11} & V_{12} \end{bmatrix}$$

Then $[A+\varepsilon_A\ \ Y+\varepsilon_Y] = [A\ \ Y] + [\varepsilon_A\ \ \varepsilon_Y] = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}^T + \begin{bmatrix} U_{11} & 0 \\ U_{21} & 0 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ 0 & 0 \end{bmatrix}^T$

d. **[2pt]** Given (1) and your solution from the previous parts, derive an expression for X in terms of the SVD of C given in (4). Show your work to get credit.
Hint: Recall the constraint of (2), and the fact that the goal is to find X that satisfies that constraint.
Solution.

$[Y+\mathcal{E}_Y] = [A+\mathcal{E}_A]X$ with $\min \|[\mathcal{E}_A \ \mathcal{E}_Y]\|_F^2$ means we want to minimize $\sqrt{\mathcal{E}_{A_{00}}^2 + \mathcal{E}_{A_{01}}^2 \dots \mathcal{E}_{A_{nd}}^2 + \mathcal{E}_{Y_{00}}^2 + \mathcal{E}_{Y_{01}}^2 \dots \mathcal{E}_{Y_{nd}}^2}$, or in other words, make each element of $\mathcal{E}_A$ and $\mathcal{E}_Y$ as close to 0 as possible.

We want to make $[A+\mathcal{E}_A \ \ Y+\mathcal{E}_Y]\begin{bmatrix} X \\ -I_d \end{bmatrix} = 0$, which is shown by

$$\left(\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}^T + \begin{bmatrix} U_{11} & 0 \\ U_{21} & 0 \end{bmatrix}\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_{11} & V_{12} \\ 0 & 0 \end{bmatrix}^T\right)\begin{bmatrix} X \\ -I_d \end{bmatrix} = 0,$$

$$\begin{bmatrix} 2U_{11} & U_{12} \\ 2U_{21} & U_{22} \end{bmatrix}\begin{bmatrix} 2\Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}\begin{bmatrix} 2V_{11} & 2V_{12} \\ V_{21} & V_{22} \end{bmatrix}^T \begin{bmatrix} X \\ -I_d \end{bmatrix} = 0$$

$$(8\Sigma_1 U_{11} V_{11} + U_{12} V_{21} \Sigma_2) X - (8\Sigma_1 U_{12} V_{12} + U_{12} V_{22} \Sigma_2) I_d = 0$$

$$8\Sigma_1 (U_{11} V_{11} X - U_{12} V_{12} I_d) + U_{12}(V_{21}\Sigma_2 X - V_{22}\Sigma_2 I_d)$$

$\rightarrow$ ① $U_{11}V_{11}X = U_{12}V_{12}$ ② $V_{21}X = V_{22}$

$$(8\Sigma_1 U_{21} V_{11} + U_{12} V_{21} \Sigma_2) X - (8\Sigma_1 U_{21} V_{12} + U_{22} V_{22} \Sigma_2) I_d = 0$$

$8\Sigma_1 U_{21} (V_{11} X - V_{12}) = 0$ ④ $U_{12} V_{21} \Sigma_2 X - U_{22} V_{22} \Sigma_2 = 0$

$\rightarrow$ ③ $V_{11} X = V_{12}$

X can be found from taking V_{11}^{-1} since V_{11} is square and setting $X = V_{11}^{-1} V_{12}$ provides a solution

e. **[1pt]** Consider the case where $d = 1$. In this case Σ_2 is simply a scalar—in particular, it is σ_{n+1}. From the previous part, you can see that in this case $\begin{bmatrix} x^\top & -1 \end{bmatrix}^\top$ is a right singular vectors of $\begin{bmatrix} A & y \end{bmatrix}$ where $y \in \mathbb{R}^{m\times 1}$ is a vector since $d = 1$. Use this fact to show that the solution from the previous part solves the equation

$$(A^\top A - \sigma_{n+1}^2 I)x = A^\top y$$

Show your work to get credit.

Solution.

we said $X = V_{11}^{-1} V_{12}$, so we can say:

$$A^T A X - \sigma_{n+1}^2 X = A^T Y$$

$$A^T A V_{11}^{-1} V_{12} - \sigma_{n+1}^2 V_{11}^{-1} V_{12} = A^T Y$$

$$(A^T A - \sigma_{n+1}^2) V_{11}^{-1} V_{12} = A^T Y$$

With the fact that $[x^T -1]^T$ is the right singular vector of $[A\ y]$, then since $\sigma_{n+1}^2 V_{11}^{-1} V_{12}$ = the right singular vector, then this shows that the value we got for X solves this.

Problem 4 (Gradient Descent for Linear Regression) [9 pts]. Consider the linear regression problem:

$$\min_x L(x) = \min_x \|Ax - b\|^2, \tag{5}$$

where $A \in \mathbb{R}^{n\times m}, b \in \mathbb{R}^n$. When the problem is overdetermined i.e $n > m$ and A is full column-rank, we saw in Module 2 that $A^\top A$ is invertible and the solution to problem (5) is $\hat{x}_{\text{OD}} = (A^\top A)^{-1}A^\top b$.

In the midterm, we considered the underdetermined case i.e $n < m$; in this case, the problem (5) has infinitely many solutions with objective 0. If A is full row-rank, then $AA^\top$ is invertible and the minimum norm solution admits the closed form expression $\hat{x}_{\text{UD}} = A^\top(AA^\top)^{-1}b$.

In this problem, we will explore the solution found by gradient descent for problem (5) for both cases. Refer to the initital point of gradient descent as x_0 and the iterates as $\{x_t\}_{t=1}^T$. We say that the algorithm converges if there exists a t' such that $x_t = x_{t'}$ for all $t > t'$, and we refer to $\hat{x}_{\text{GD}} = x_{t'}$ as the solution obtained by gradient descent. This question has parts a - g, and parts f,g can be found on the python notebook.

a. **[1pt]** Consider first the overdetermined case, and assume full column-rank. Write down the solution obtained by gradient descent, assuming that training converges.

Solution.

while $(\|\nabla f(x_t)\| \geq \epsilon$:

$sol = x_k - \alpha \cdot \nabla f(x_k)$

$k = k+1$

If overdetermined, $\text{rank}(A) = m$ & $n > m$ for $A \in \mathbb{R}^{n\times m}$, meaning $x \in \mathbb{R}^m$. In this case, we have many equations to get through, so the solution obtained by gradient descent is equal to

$x^{k+1} = x^k - \alpha_k \cdot A^T \cdot (Ax_k - b)$, giving us $\hat{x} = (A^TA)^{-1}A^Tb$

b. **[1pt]** Now consider the underdetermined case. For parts (b) - (d), we consider a simple special case to develop intuition: let $n = 1, \overset{m}{\cancel{d}} = 2$, and choose $A = [2, 1]$ and $b = [2]$.

Show that there exist infinitely many $\hat{x}$ satisfying $A^\top \hat{x} = b$ on a line ℓ. Write the equation of the line ℓ.

Solution.

$A^T\hat{x} = b$, $\begin{bmatrix}2 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = [2]$, $2x_1 + x_2 = 2$. This gives us an equation, but with 2 variables, meaning x_2 is a free variable. This means there are infinite solutions, any of the values along the line $\ell = -2x_1 + 2 = x_2$.

c. **[1pt]** Starting from $x_0 = 0$, what is the direction of the negative gradient, written as a unit-norm vector? Does the direction change along the trajectory?

Solution.

$x_{k+1} = x_k - \mu A^T A x_k + \mu A^T b$, so $x_1 = 0 - 0 + \mu A^T b = \mu [2\ 1]^T [2] = \mu [4\ 2]^T$

we move on the direction of $[2, 1]$, or $[\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}]$.

The direction does not change with the trajectory because we are moving on line with it till we hit the line with minimized norm dist.

d. **[2pt]** Now, we explore for the simple example which solution gradient descent finds. Assume that gradient descent converges.

(i) Using your findings from the previous part, which solution $\hat{x}_{GD}$ on line ℓ does gradient descent find?

(ii) Verify that for any other solution $\hat{x} \in \ell$, it holds that $(\hat{x} - \hat{x}_{GD}) \perp \hat{x}_{GD}$.

(iii) Through a pictorial sketch, argue that $\hat{x}_{GD}$ has the smallest Euclidean norm across all the solutions on the line ℓ.

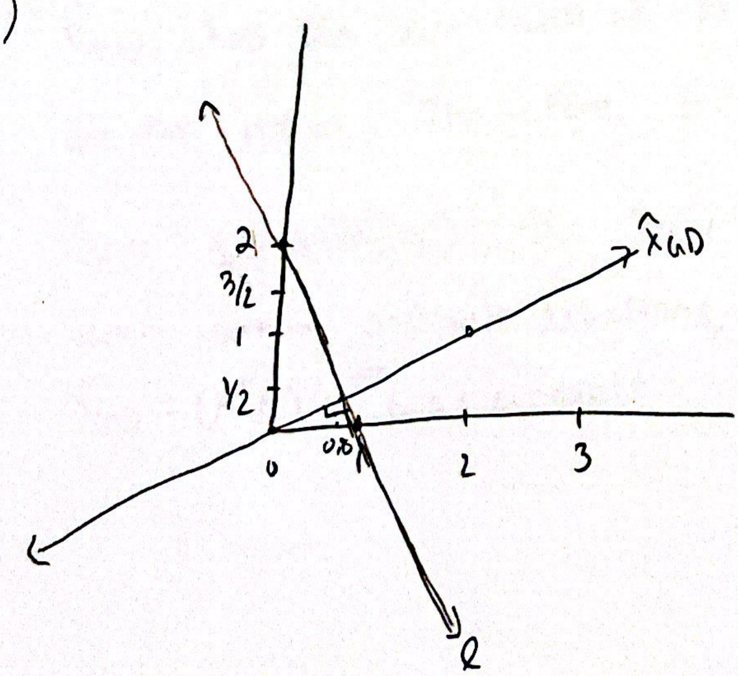
Hint: Pythagoras theorem may be helpful here.

Solution.

i) From $x_0 = 0$, the solution we find is the point $(\frac{4}{5}, \frac{2}{5})$.

ii) For any point $(x_1, -2x_1+2)$, we know this is $\perp$ to $(\frac{4}{5}, \frac{2}{5})$ because $(\frac{4}{5}, \frac{2}{5}) \cdot (x_1, -2x_1) = \frac{4}{5}x_1 - \frac{4}{5}x_1 = 0$.

iii)



We can see the two form a right angle, where the dist is $\sqrt{0.8^2+0.4^2} = \sqrt{0.8} = 0.894$, any other angle will yield a longer distance since the shortest distance between two points is equal to the distance at the point at a $\perp$ angle.

e. **[2pt]** Now, we generalize the argument from the simple example to general underdetermined systems. Assume full row-rank.

(i) Show that for $x_0 = 0$, the gradient vector is always spanned by the rows of A.

(ii) Show that $\hat{x}_{\mathrm{GD}} = \hat{x}_{\mathrm{UD}}$

This result is surprising and nontrivial. It states that amongst infinitely many solutions that attain the optimal objective value, gradient descent chooses the one with the minimum norm, which is a nice desirable property. This phenomenon, know as *implicit regularization*, helps explain the success of using gradient-based methods to train overparameterized models, like deep neural networks; this has been a very active area of research over the last 5 years!

Solution.

i) For $x_0 = 0$, $\nabla f(x) = \mu A^T b$, which has rank = rank(A), which on this case with full-row rank = row(A)

ii) $\hat{x}_{GD} = \hat{x}_{UD}$ because as we iterate through possible solutions, $\hat{x}_{GD}$ has the full-rank of A and minimizes the distance for the norm. Therefore, for x_{k+1} that is the solution,

$x_{k+1} = x_k - \mu \nabla f(x_k)$, the $x_{k+1} \approx x_k$ will be the solution, meaning even among infinite solutions, $\hat{x}_{GD}$ is the optimal choice.

$\hat{x}_{GD} = (A^T A)^{-1} A^T b$ as a result

f. **[1pt]** See Notebook

g. **[1pt]** See Notebook

Problem 5 (Convexity and income inequality measures)[4 pts]. Let $x \in \mathbb{R}^n$ represent the vector of incomes of n individuals, where $x_i > 0$, $i = 1, \ldots, n$. Let g_i be the fraction of the total income earned by i individuals who have the lowest income,

g_1 = lowest frac

g_2 = lowest frac of bottom 2

...

g_n = 100% = 1

$$g_i = \frac{1}{\mathbf{1}^\top x} \sum_{j=1}^{i} x_{(j)}, \quad i = 1, \ldots, n,$$

where $x_{(j)}$ denotes the jth *smallest* number among $\{x_1, \ldots, x_n\}$. Note that if income is distributed equally among all individuals we have $g_i = i/n$ for all i. One measure of *income inequality* in a society is given by

$$f(x) = \frac{2}{n} \sum_{i=1}^{n} \left(\frac{i}{n} - g_i \right).$$

This measure ranges between 0 (for equal distribution) and $1 - 1/n$ (when all income is earned by one person). Here we are interested to examine the properties of the set of bounded-inequality incomes: $C = \{x \in \mathbb{R}^n_{++} \mid f(x) \leq \frac{1}{2}\}$ (where $\mathbb{R}_{++}$ denotes positive real numbers).

$x \geq 0$, $f(x) \leq 1/2 \rightarrow$ nums where

Interesting Aside: this leads to interesting economic measures that are beyond our scope here.

Answer the following questions:

a. **[2pt]** Recall that the "sum-k-largest" function $\sum_{i=1}^{k} x_{[i]}$ is a convex function of x, where $x_{[i]}$ denotes the ith largest entry in vector x. Using this, answer the following: consider the function $S_i(x) = \sum_{j=1}^{i} x_{(j)}$, that is, the sum of the i lowest incomes in x. For some fixed i, is the function $S_i(x)$ convex, concave, or neither? Justify your answer (cite any property/definition you use from class and show how it is used).

Solution.

The sum k-largest function is the reverse of $S_i(x)$ because it adds the k largest, not k-smallest. This means $S_i(x)$ is not convex, as the reason sum-k-largest is convex is the pointwise max for its convex - this does not hold for $S_i(x)$. $S_i(x)$ is instead concave because if we take $\sum_{i=1}^{n} x_i + (-S_p(x)) = \sum_{i=1}^{n-p} x_i$, where $0 \leq p \leq n$, then we get that the sum of 2 convex functions is convex, since $-(-)=+$. Therefore, the negation of $S_i(x)$ is convex, so $S_i(x)$ is concave.

b. [2pt] It can be shown that (you don't need to show this) the set C defined above can be equivalently expressed as

$$\left\{ x \in \mathbb{R}^n_{++} \;\middle|\; \frac{\sum_{i=1}^n S_i(x)}{\mathbf{1}^\top x} > \left(\frac{n}{4} + \frac{1}{2}\right) \right\}.$$

Is this set convex or nonconvex? Justify your answer. **Hint:** use the result of part a about $S_i(x)$.

Solution.

This set is not convex because S_i is concave, meaning the set contains elements x that are passed through a concave, non-convex function.