Divide-and-Conquer: Quick Sort

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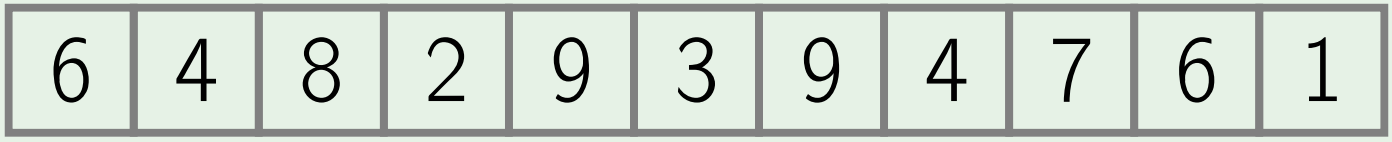
Algorithmic Toolbox
Data Structures and Algorithms

Outline

Quick Sort

- comparison based algorithm
- running time: $O(n \log n)$ (on average)
- efficient in practice

Example: quick sort



Example: quick sort

6	4	8	2	9	3	9	4	7	6	1

partition with respect to $x = A[1]$

in particular, x is in its final position

1	4	2	3	4	6	6	9	7	8	9

≤ 6 $\qquad\qquad$ > 6

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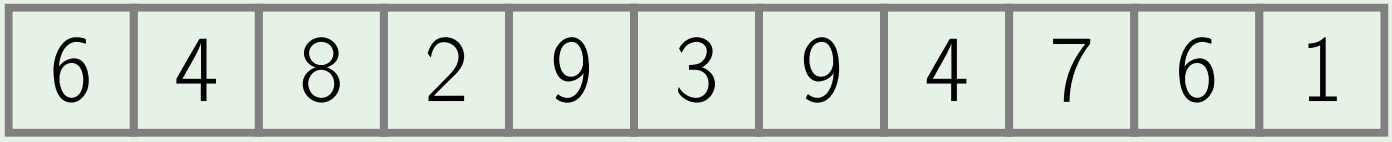

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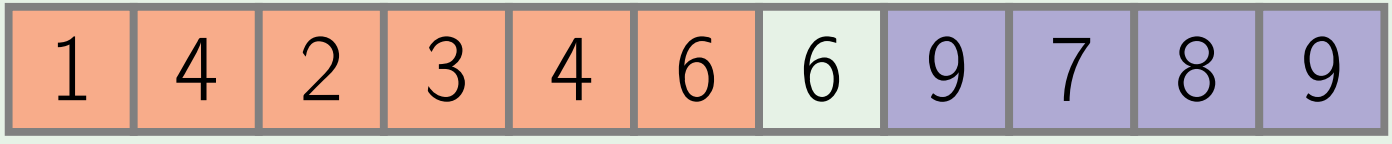
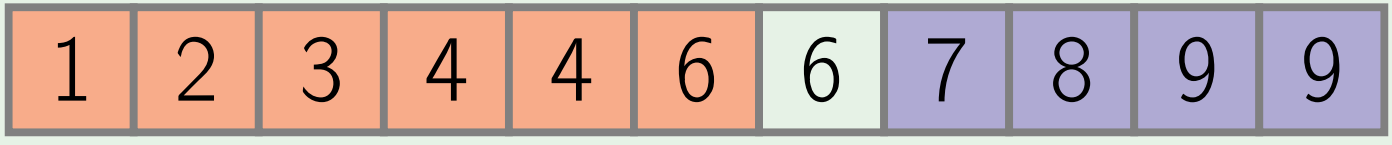


sort the two parts recursively

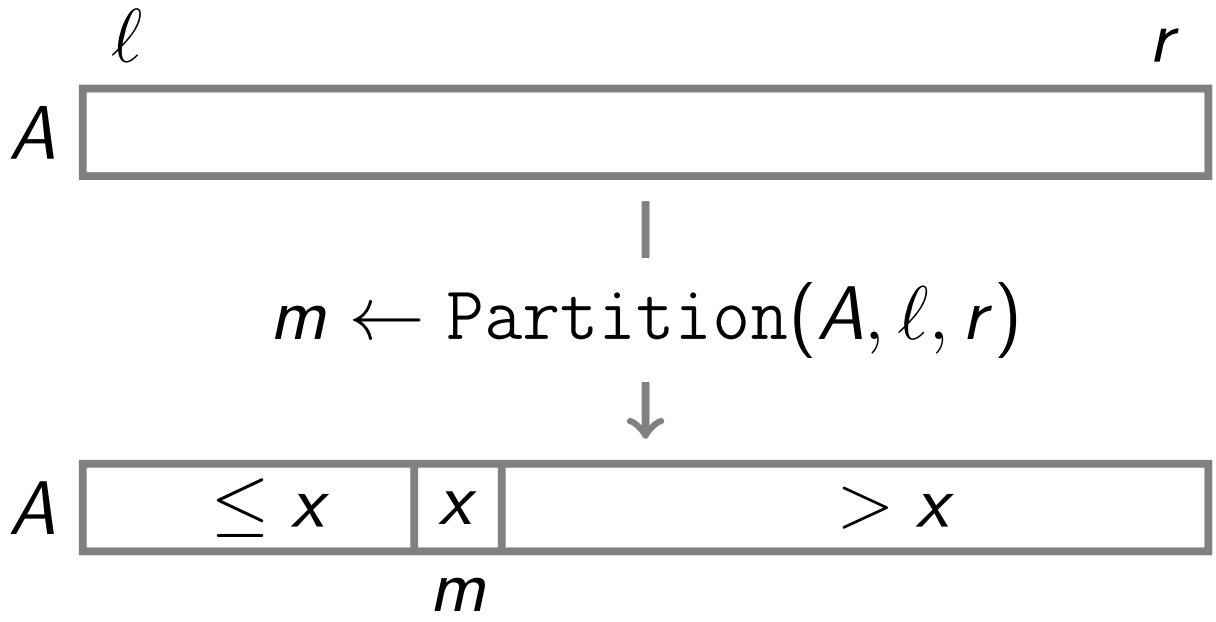


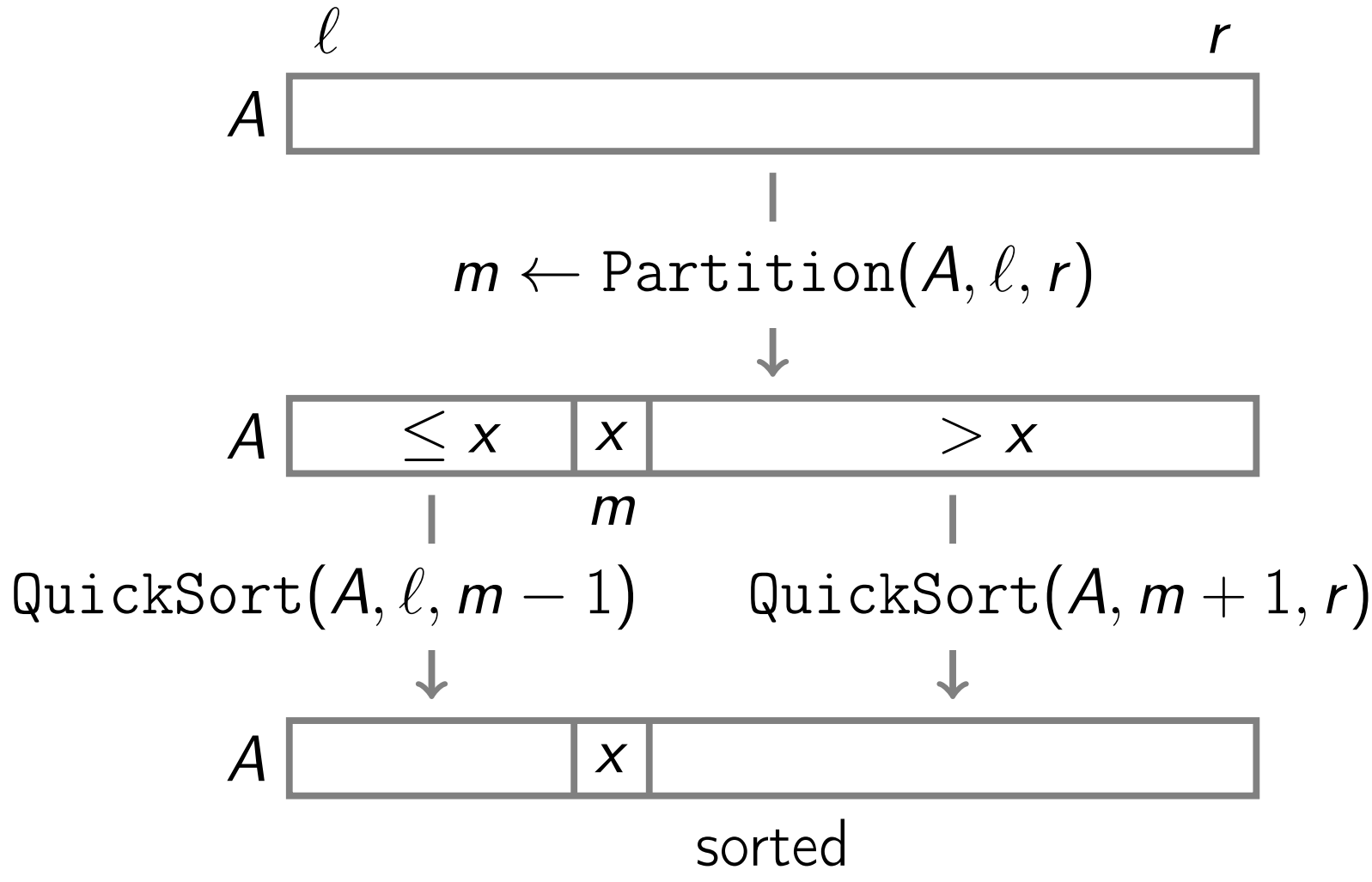
Outline

QuickSort(A, ℓ, r)

```
if ℓ ≥ r:
   return
m ← Partition(A, ℓ, r)
{A[m] is in the final position}
QuickSort(A, ℓ, m − 1)
QuickSort(A, m + 1, r)
```


ℓ
r
A


ℓ
r
A
m ← Partition(A, ℓ, r)
A
≤ x
x
> x
m


ℓ
r
A
$m \leftarrow \texttt{Partition}(A, \ell, r)$
A
$\leq x$
x
$> x$
m
$\texttt{QuickSort}(A, \ell, m-1)$
$\texttt{QuickSort}(A, m+1, r)$
A
x
sorted

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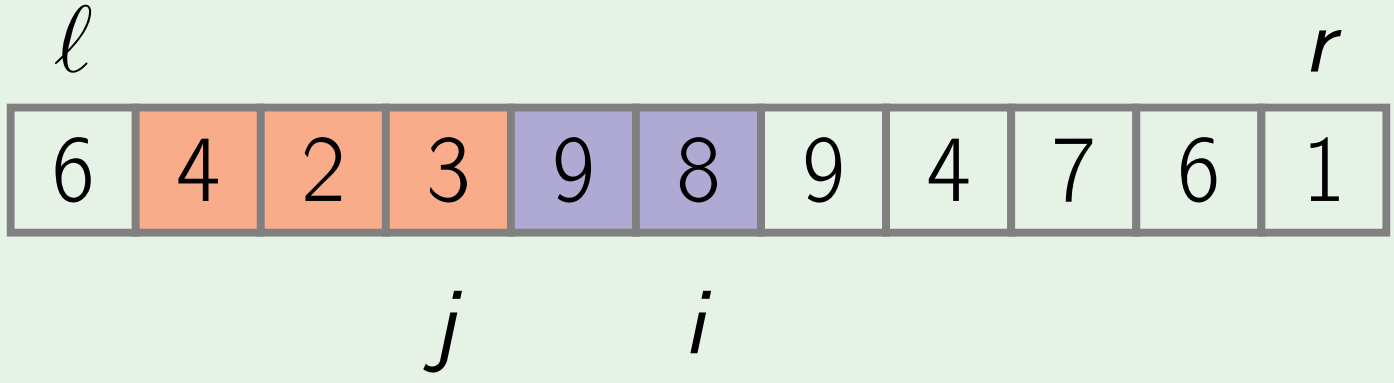

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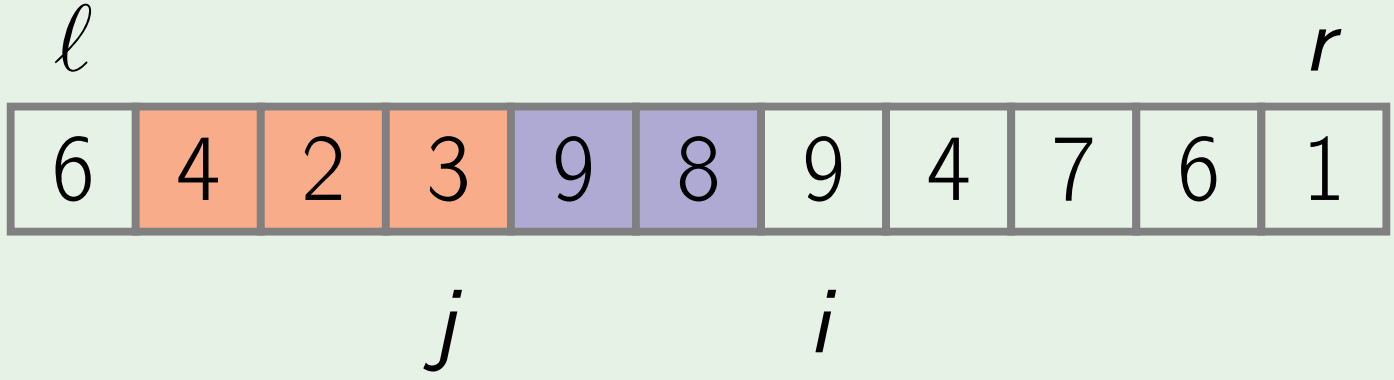

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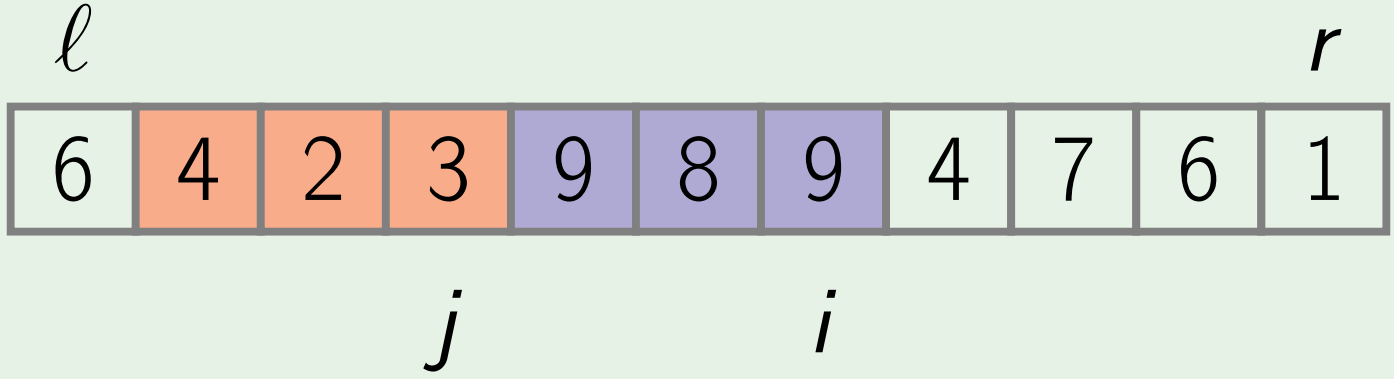

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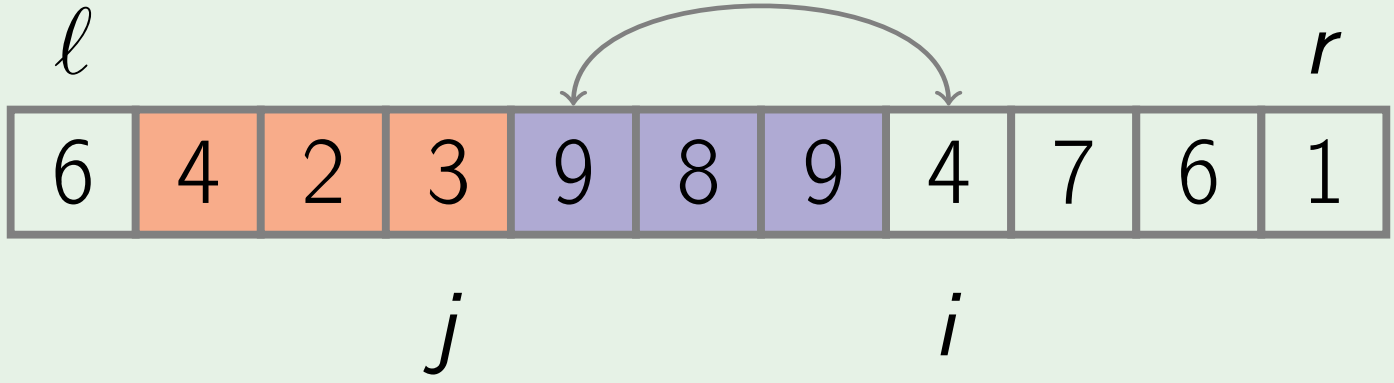

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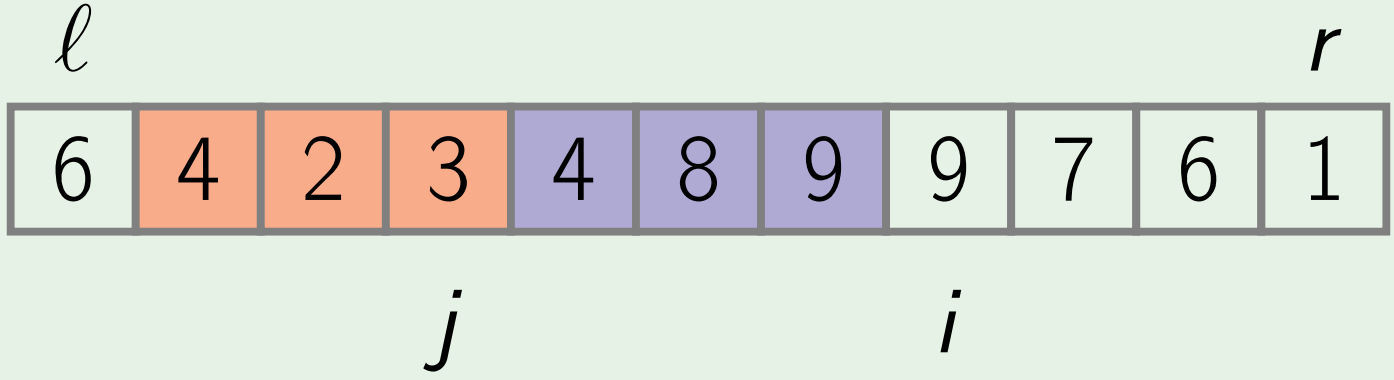

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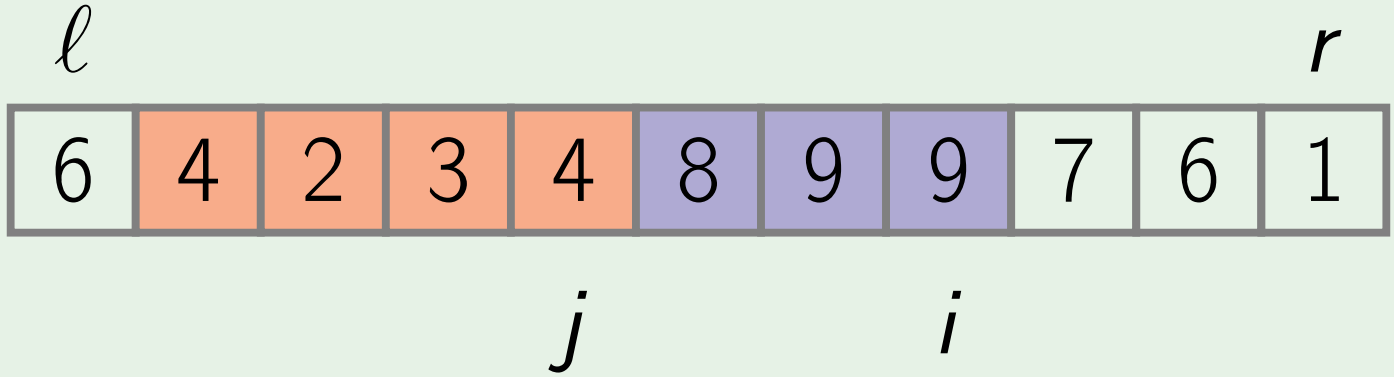

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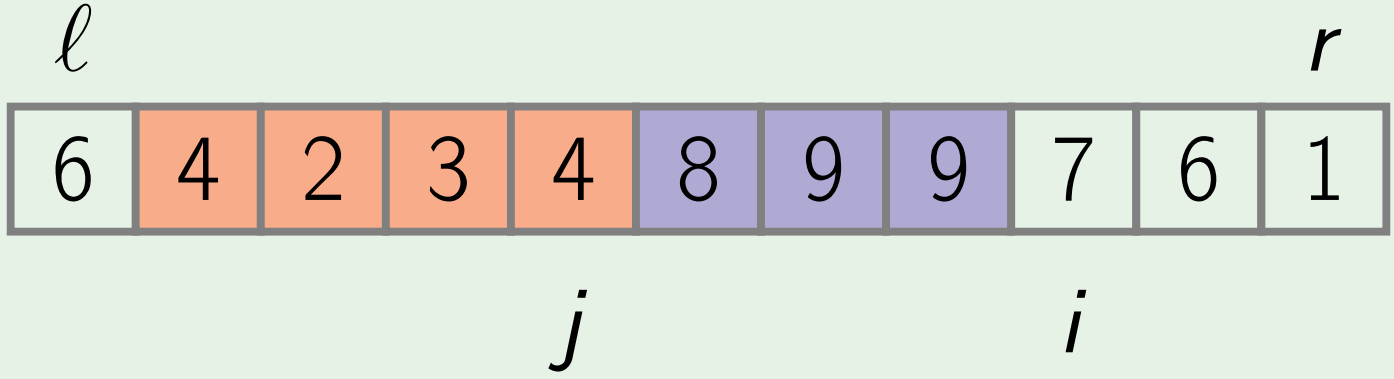

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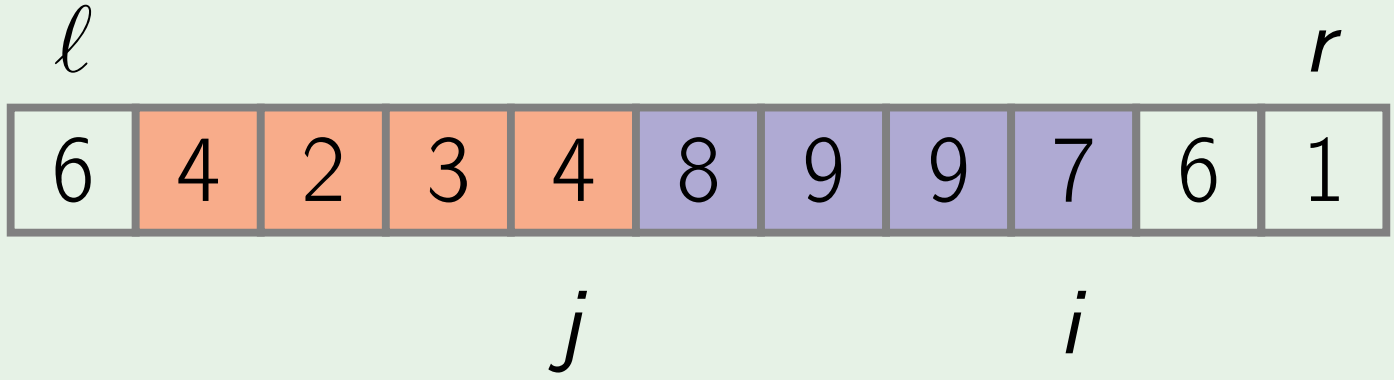

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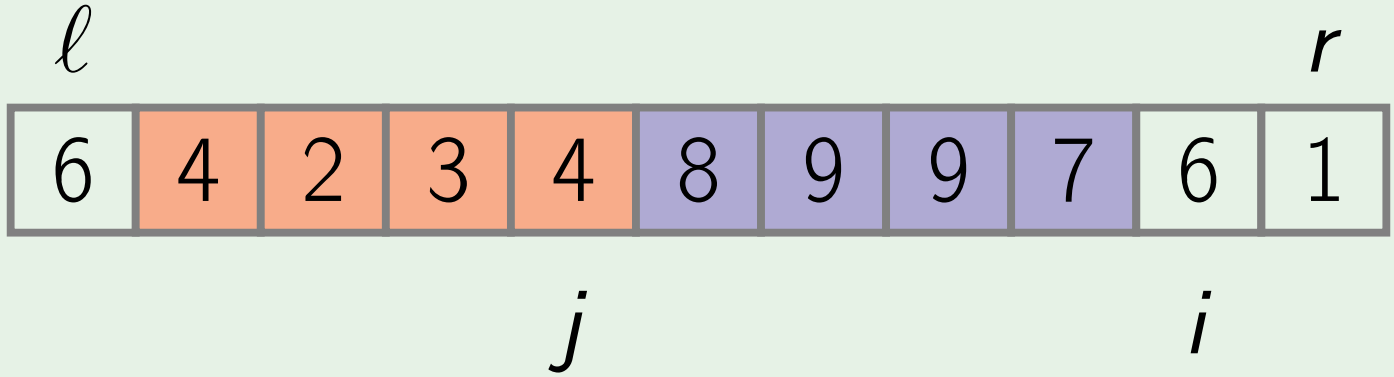

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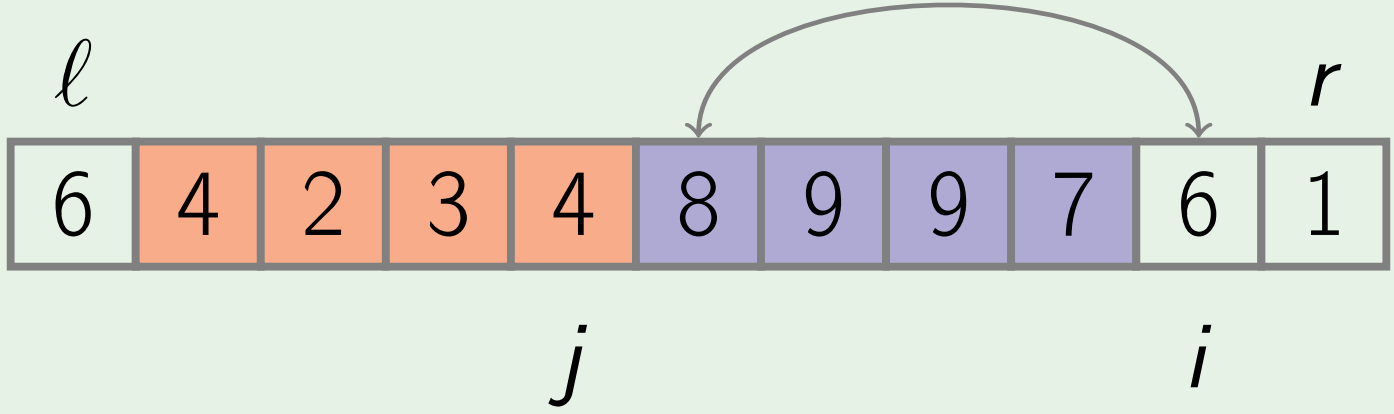

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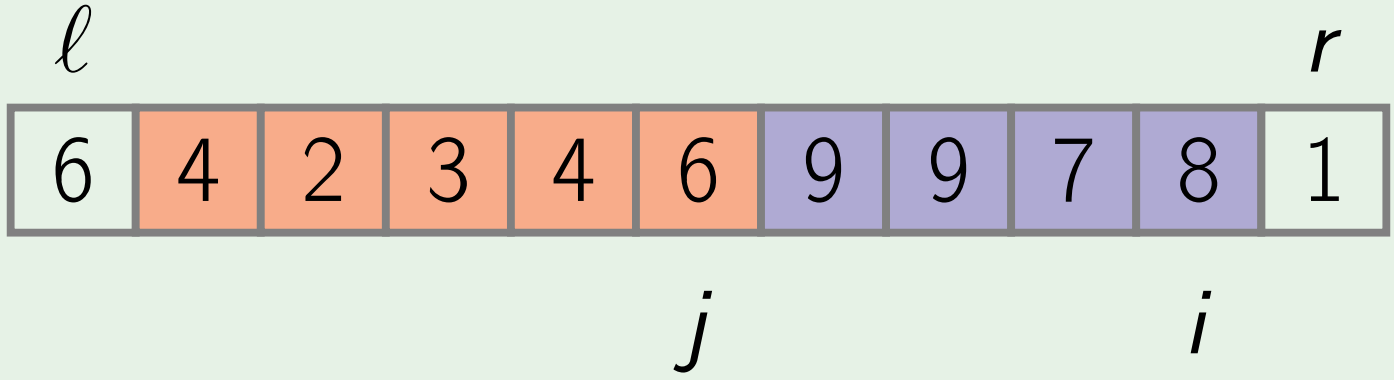

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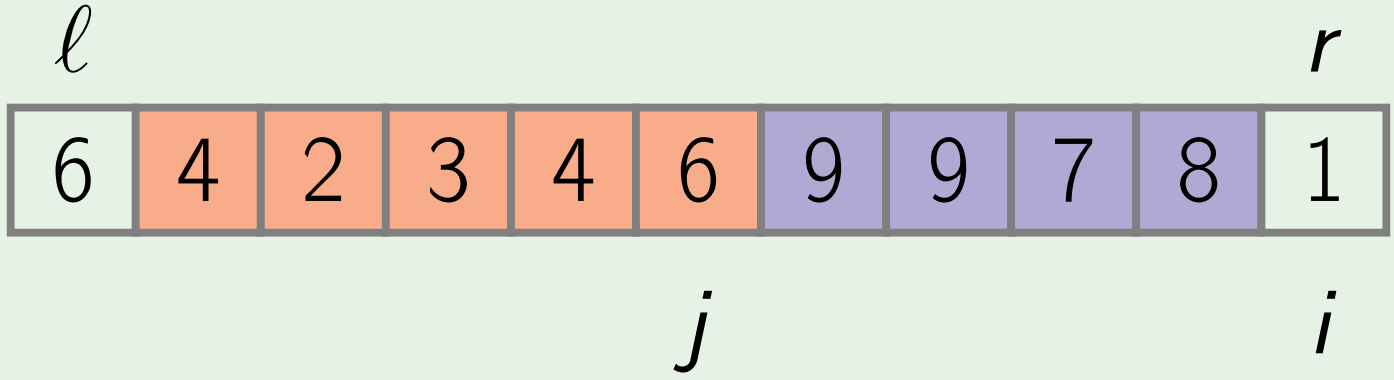

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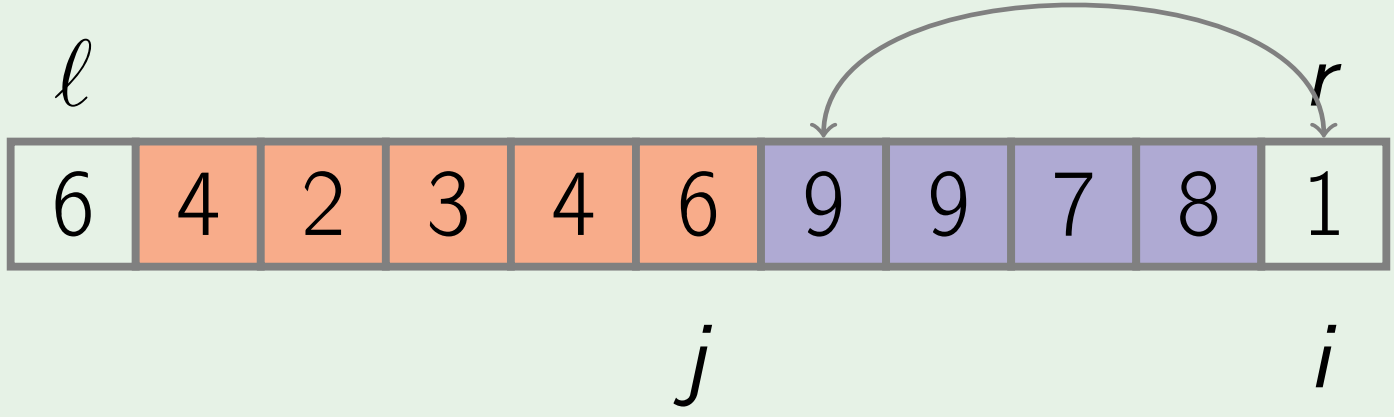

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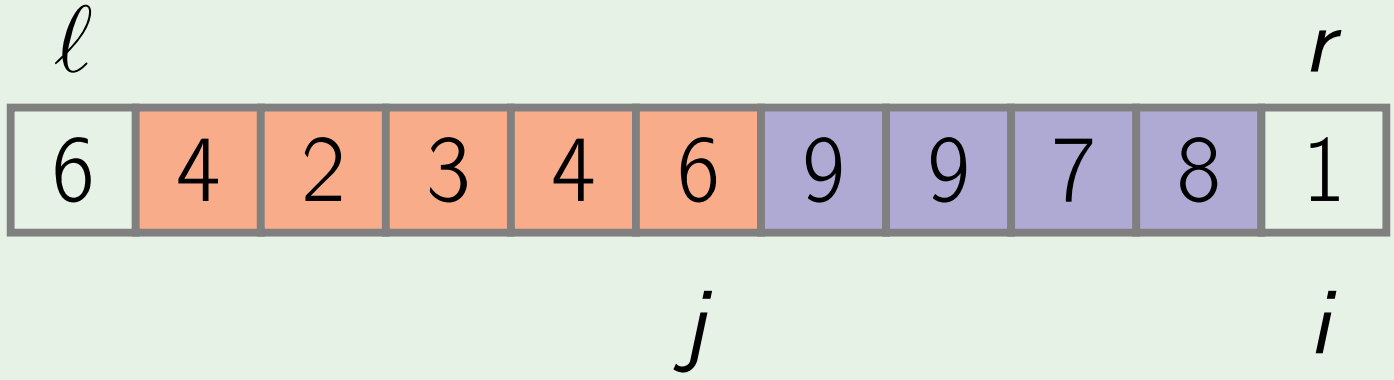

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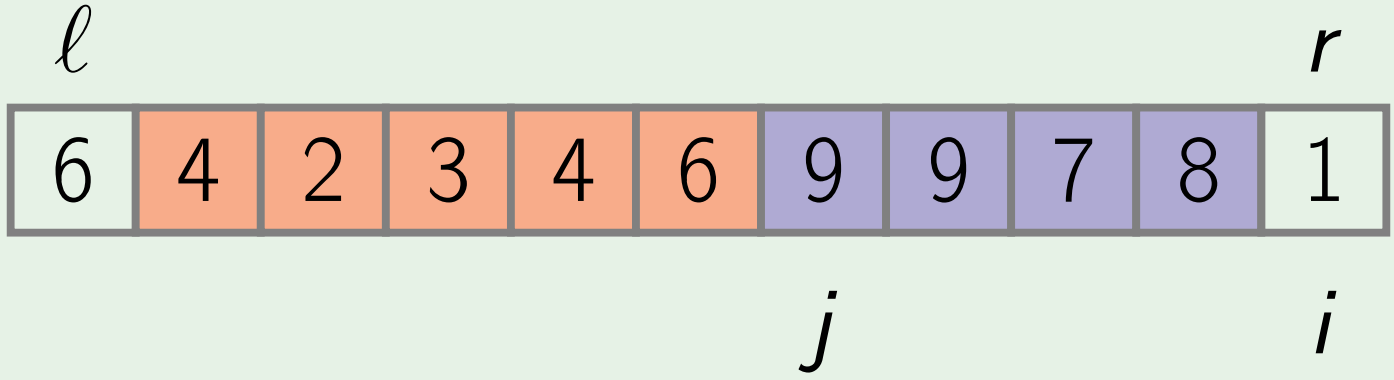

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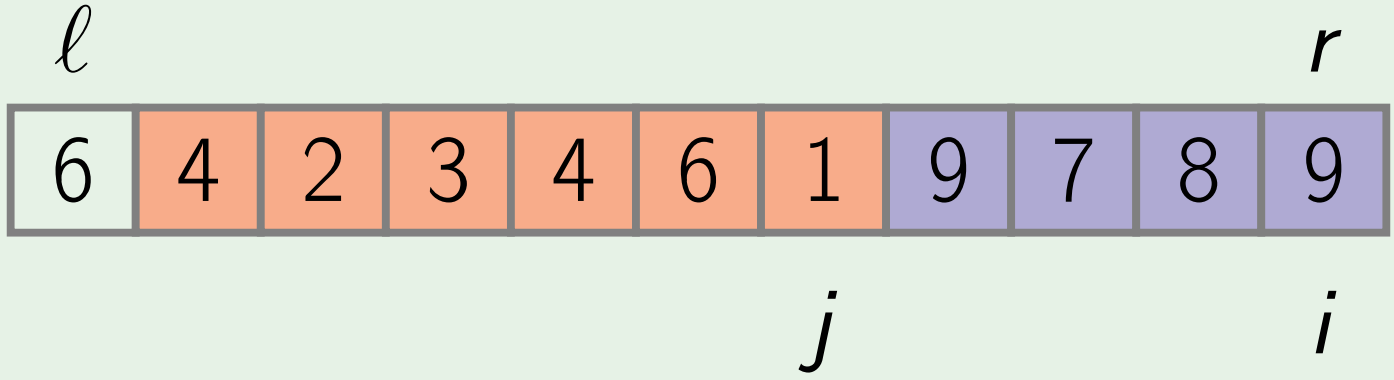

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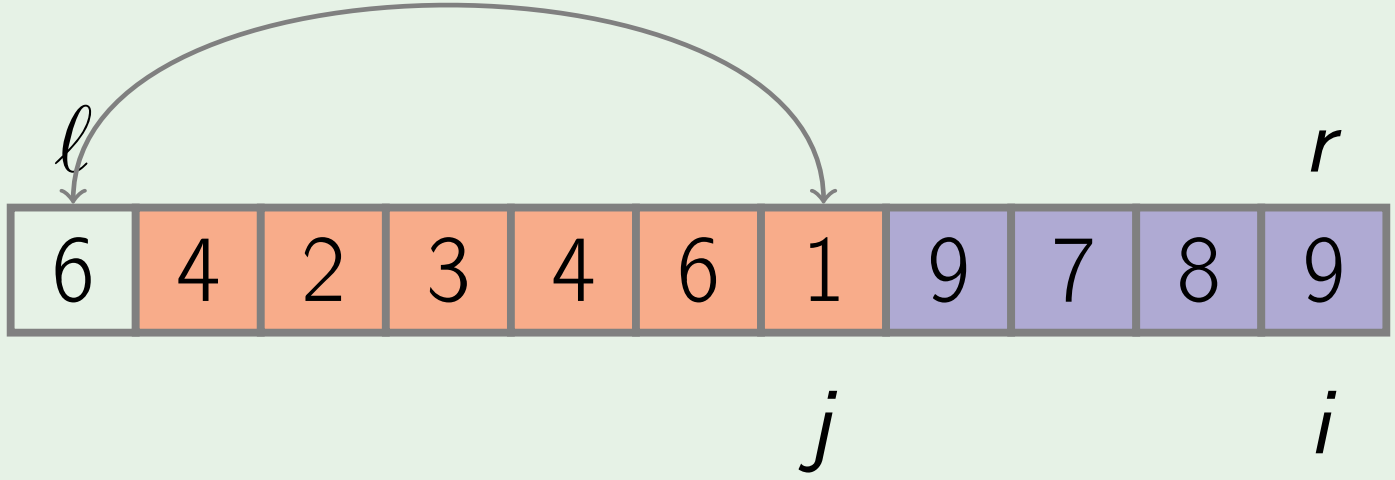

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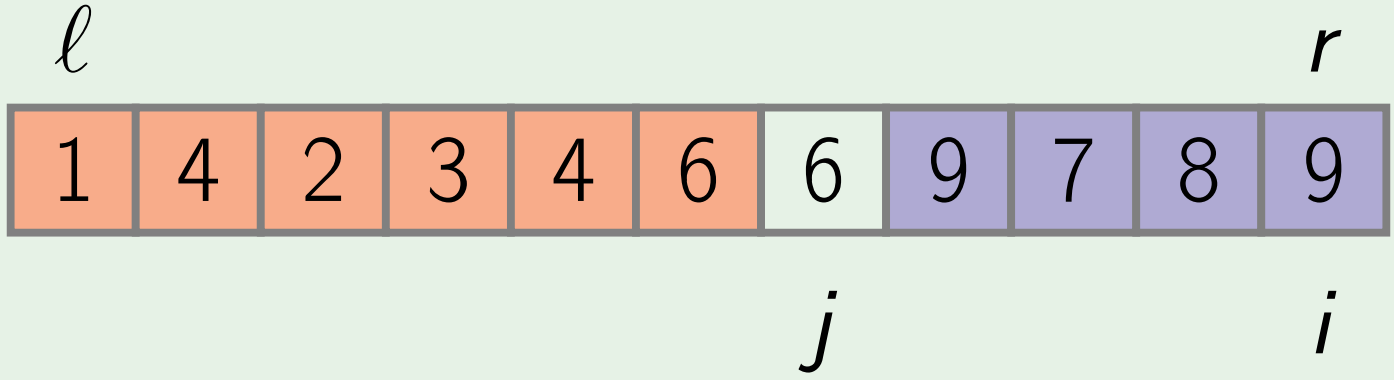

Partition(A, ℓ, r)

$x \leftarrow A[\ell]$ {pivot}
$j \leftarrow \ell$
for i from $\ell + 1$ to r:
  if $A[i] \leq x$:
    $j \leftarrow j + 1$
    swap $A[j]$ and $A[i]$
  $\{A[\ell + 1 \ldots j] \leq x,\ A[j + 1 \ldots i] > x\}$
swap $A[\ell]$ and $A[j]$
return j

Outline

Unbalanced Partitions

- $T(n) = n + T(n-1)$:

$$T(n) = n+(n-1)+(n-2)+\cdots = \Theta(n^2)$$

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- $T(n) = n + T(n-5) + T(4)$:

  $T(n) \geq n+(n-5)+(n-10)+\cdots = \Theta(n^2)$

Balanced Partitions

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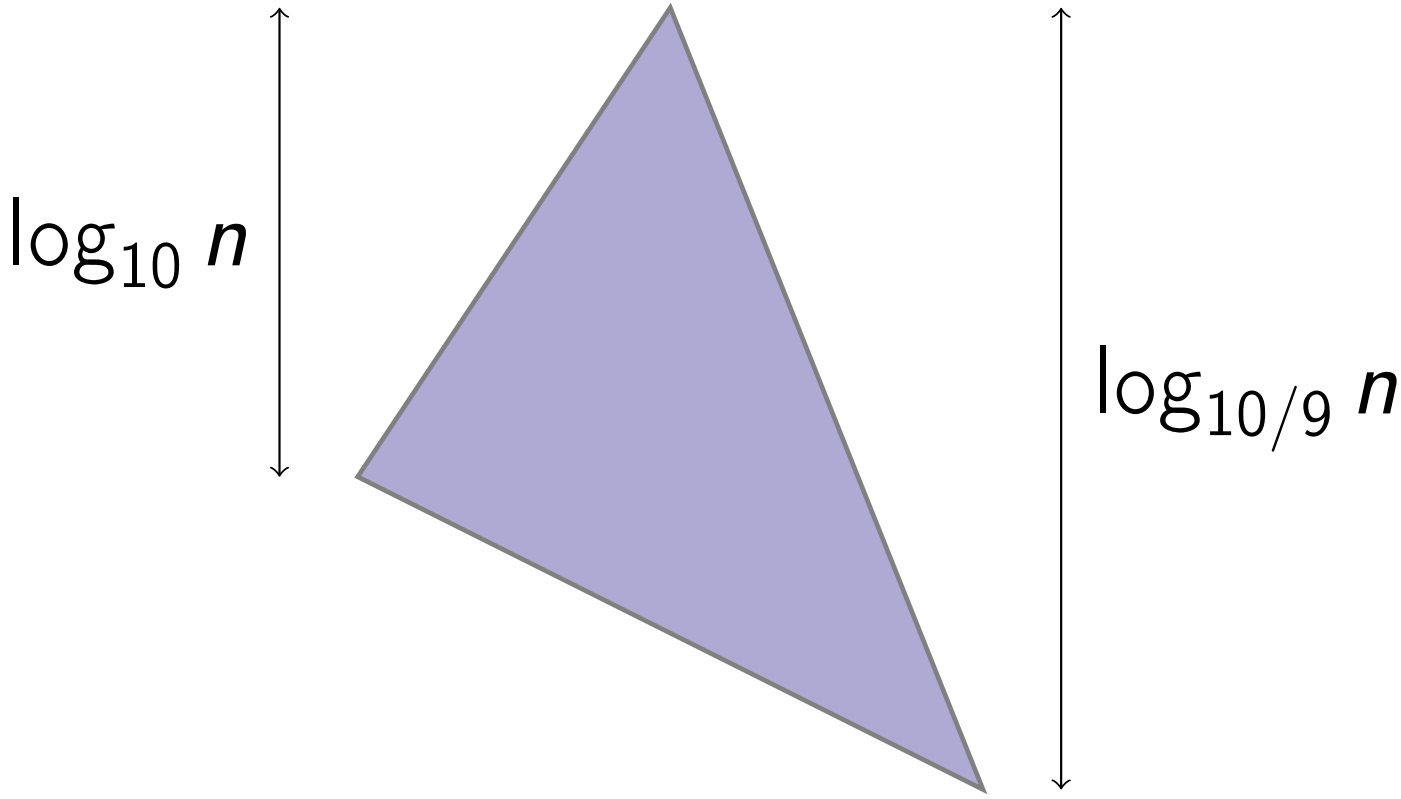

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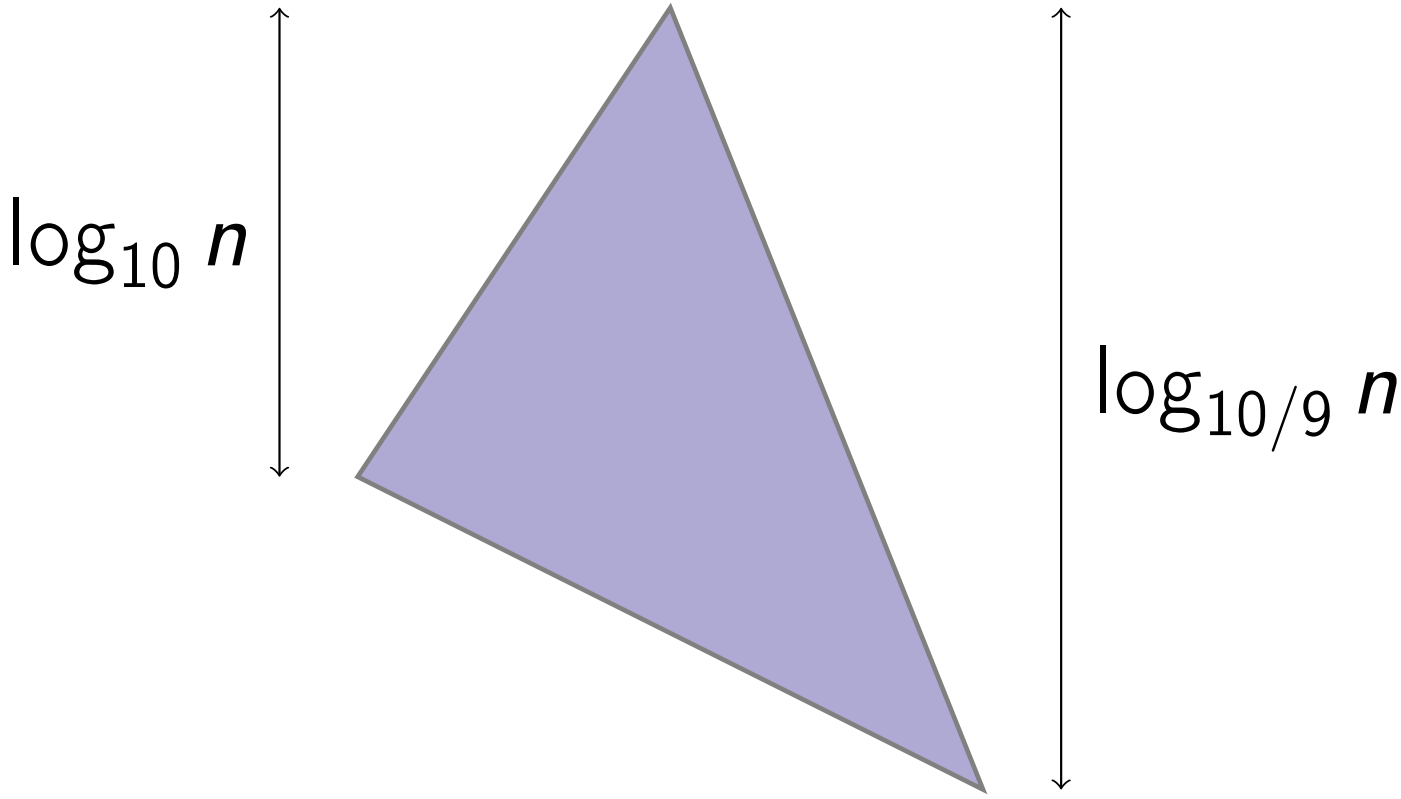

$$T(n) = O(n \log n)$$

Random Pivot

RandomizedQuickSort(A, ℓ, r)

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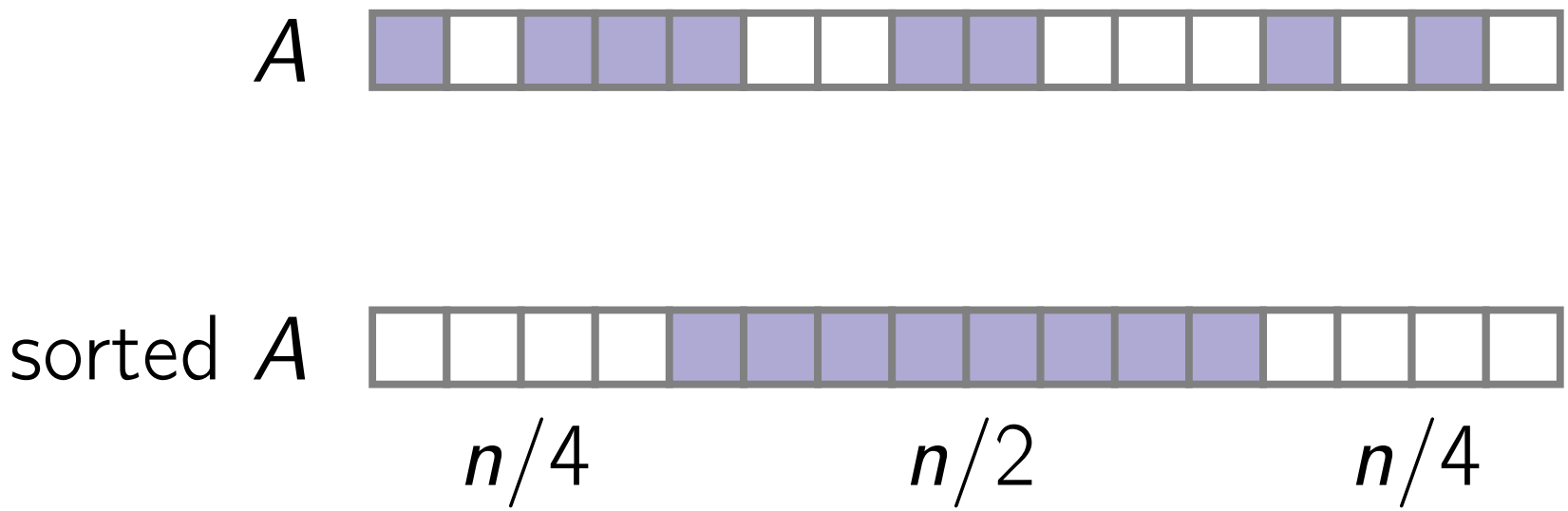
if ℓ ≥ r:
    return
k ← random number between ℓ and r
swap A[ℓ] and A[k]
m ← Partition(A, ℓ, r)
{A[m] is in the final position}
RandomizedQuickSort(A, ℓ, m − 1)
RandomizedQuickSort(A, m + 1, r)
```

Why Random?

half of the elements of A guarantees a balanced partition:



Theorem

Assume that all the elements of $A[1 \ldots n]$ are pairwise different. Then the average running time of `RandomizedQuickSort`(A) is $O(n \log n)$ while the worst case running time is $O(n^2)$.

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Remark

Averaging is over random numbers used by the algorithm, but not over the inputs.

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Proof Ideas: Comparisons

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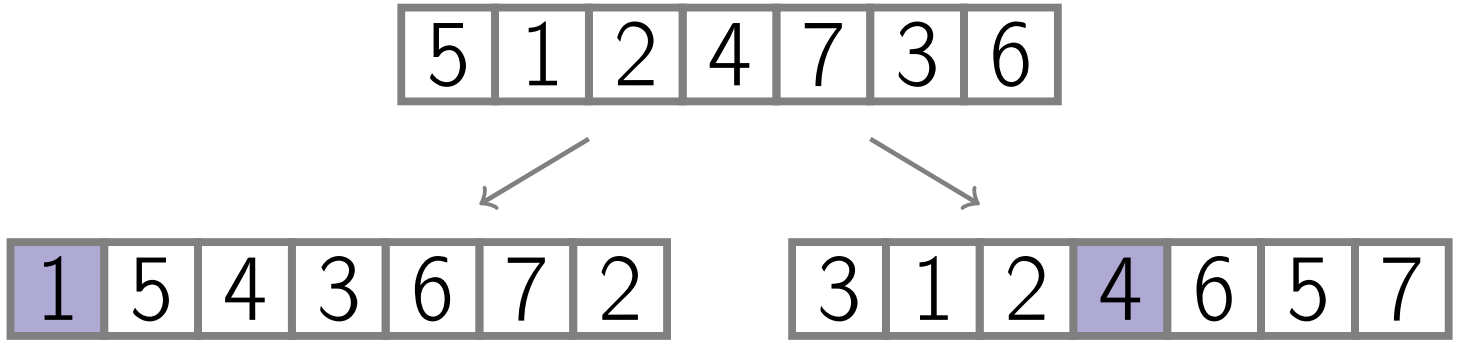
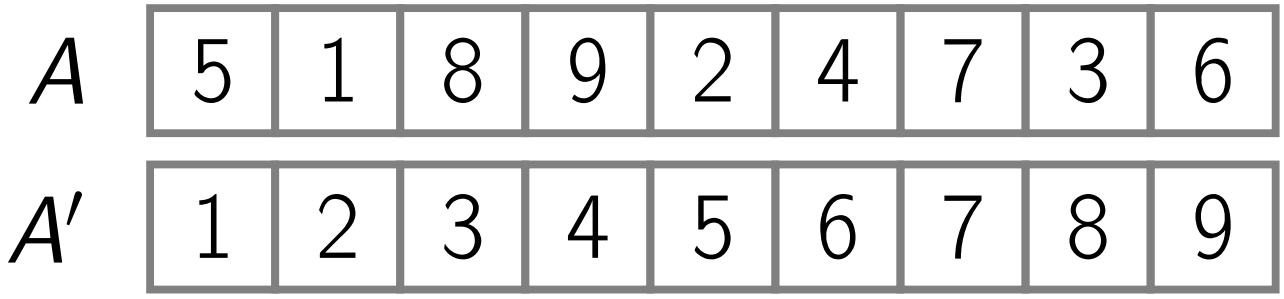
- balanced partition are better since they reduce the number of comparisons needed:



Proof Ideas: Probability



Proof Ideas: Probability

A	5	1	8	9	2	4	7	3	6
A'	1	2	3	4	5	6	7	8	9

$$\text{Prob}\,(1 \text{ and } 9 \text{ are compared}) =$$

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A'	1	2	3	4	5	6	7	8	9

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$$\text{Prob}\,(3 \text{ and } 4 \text{ are compared}) =$$

Proof Ideas: Probability

A	5	1	8	9	2	4	7	3	6
A'	1	2	3	4	5	6	7	8	9

$$\text{Prob}\,(1 \text{ and } 9 \text{ are compared}) = \frac{2}{9}$$

$$\text{Prob}\,(3 \text{ and } 4 \text{ are compared}) = 1$$

Proof

- let, for $i < j$,

$$\chi_{ij} = \begin{cases} 1 & A'[i] \text{ and } A'[j] \text{ are compared} \\ 0 & \text{otherwise} \end{cases}$$

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- for all $i < j$, $A'[i]$ and $A'[j]$ are either compared exactly once or not compared at all (as we compare with a pivot)
- this, in particular, implies that the worst case running time is $O(n^2)$

Proof (continued)

- crucial observation: $\chi_{ij} = 1$ iff the first selected pivot in $A'[i \ldots j]$ is $A'[i]$ or $A'[j]$

Proof (continued)

- crucial observation: $\chi_{ij} = 1$ iff the first selected pivot in $A'[i \ldots j]$ is $A'[i]$ or $A'[j]$
- then $\text{Prob}(\chi_{ij}) = \frac{2}{j-i+1}$ and $\text{E}(\chi_{ij}) = \frac{2}{j-i+1}$

Proof (continued)

Then (the expected value of) the running time is

$$\begin{aligned}
\mathrm{E}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\chi_{ij} &= \sum_{i=1}^{n}\sum_{j=i+1}^{n}\mathrm{E}(\chi_{ij}) \\
&= \sum_{i<j}\frac{2}{j-i+1} \\
&\leq 2n\cdot\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right) \\
&= \Theta(n\log n)
\end{aligned}$$

Outline

Equal Elements

- what if all the elements of the given array are equal to each other?

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- the array is always split into two parts of size 0 and $n - 1$
- $T(n) = n + T(n-1) + T(0)$ and hence $T(n) = \Theta(n^2)$!

To handle equal elements, we replace the line

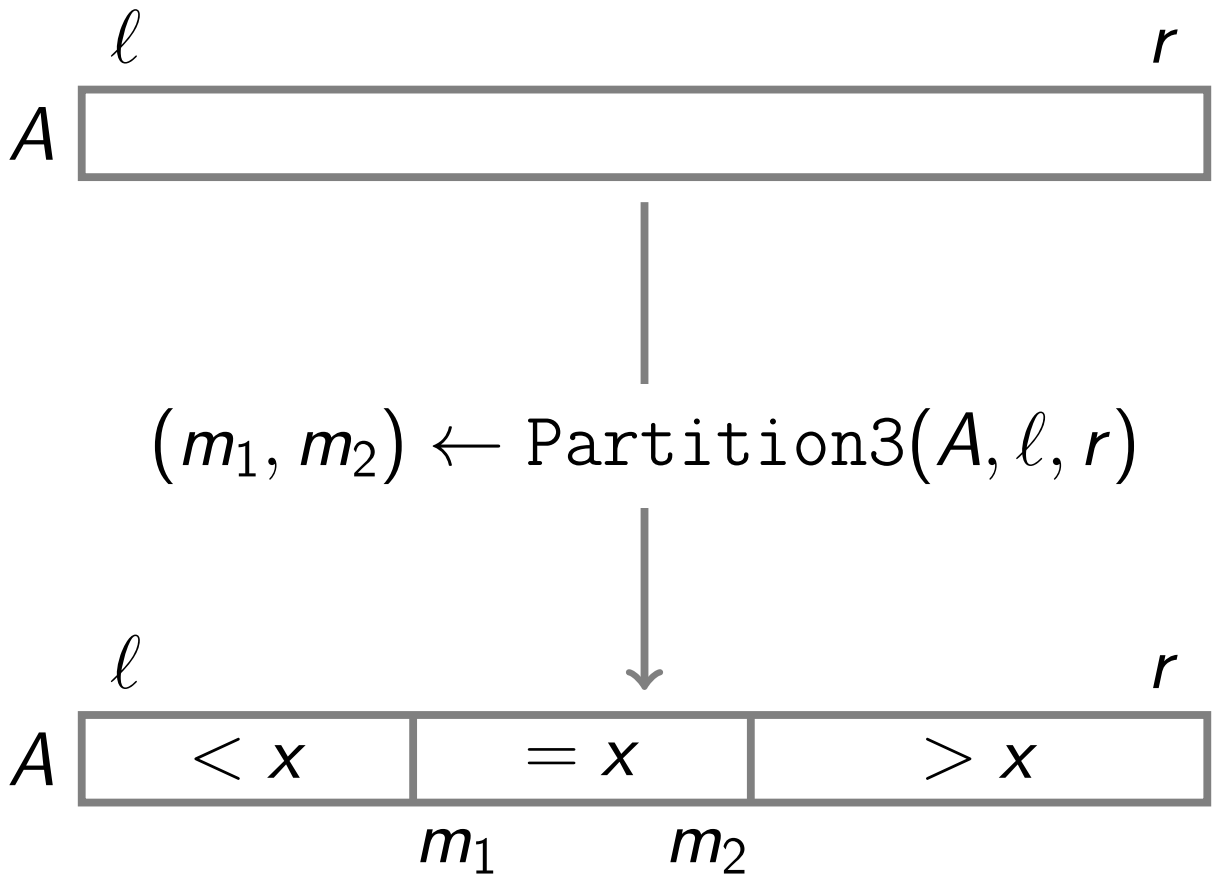
$$m \leftarrow \texttt{Partition}(A, \ell, r)$$

with the line

$$(m_1, m_2) \leftarrow \texttt{Partition3}(A, \ell, r)$$

such that

- for all $\ell \leq k \leq m_1 - 1$, $A[k] < x$
- for all $m_1 \leq k \leq m_2$, $A[k] = x$
- for all $m_2 + 1 \leq k \leq r$, $A[k] > x$


ℓ
r
A
$(m_1, m_2) \leftarrow \texttt{Partition3}(A, \ell, r)$
ℓ
r
A
$< x$
$= x$
$> x$
m_1
m_2

RandomizedQuickSort(A, ℓ, r)

if $\ell \geq r$:

  return

$k \leftarrow$ random number between ℓ and r

swap $A[\ell]$ and $A[k]$

$(m_1, m_2) \leftarrow$ Partition3(A, ℓ, r)

{$A[m_1 \ldots m_2]$ is in final position}

RandomizedQuickSort$(A, \ell, m_1 - 1)$

RandomizedQuickSort$(A, m_2 + 1, r)$

Outline

Tail Recursion Elimination

QuickSort(A, ℓ, r)

```
while ℓ < r:
    m ← Partition(A, ℓ, r)
    QuickSort(A, ℓ, m − 1)
    ℓ ← m + 1
```

QuickSort(A, ℓ, r)

```
while ℓ < r:
    m ← Partition(A, ℓ, r)
    if (m − ℓ) < (r − m):
        QuickSort(A, ℓ, m − 1)
        ℓ ← m + 1
    else:
        QuickSort(A, m + 1, r)
        r ← m − 1
```

QuickSort(A, ℓ, r)

```
while ℓ < r:
    m ← Partition(A, ℓ, r)
    if (m − ℓ) < (r − m):
        QuickSort(A, ℓ, m − 1)
        ℓ ← m + 1
    else:
        QuickSort(A, m + 1, r)
        r ← m − 1
```

Worst-case space requirement: $O(\log n)$

Intro Sort

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- if the recursion depth exceeds a certain threshold $c \log n$ the algorithm switches to heap sort
- the running time is $O(n \log n)$ in the worst case

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- Quick sort is a comparison based algorithm
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- Efficient in practice