

Intro: Greatest Common Divisors II

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Data Structures and Algorithms
Algorithmic Toolbox

Learning Objectives

- Implement the Euclidean Algorithm.
- Approximate the runtime.

GCDs

Definition

For integers, a and b , their **greatest common divisor** or $\gcd(a, b)$ is the largest integer d so that d divides both a and b .

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Compute GCD

Input: Integers $a, b \geq 0$.

Output: $\gcd(a, b)$.

Key Lemma

Lemma

Let a' be the remainder when a is divided by b , then

$$\gcd(a, b) = \gcd(a', b) = \gcd(b, a').$$

Proof

Proof (sketch)

- $a = a' + bq$ for some q
- d divides a and b if and only if it divides a' and b

Euclidean Algorithm

Function EuclidGCD(a, b)

if $b = 0$:

 return a

$a' \leftarrow$ the remainder when a is
 divided by b

return EuclidGCD(b, a')

Euclidean Algorithm

Function EuclidGCD(a, b)

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if  $b = 0$ :  
    return  $a$   
 $a' \leftarrow$  the remainder when  $a$  is  
    divided by  $b$   
return EuclidGCD( $b, a'$ )
```

Produces correct result by Lemma.

Example

$\text{gcd}(3918848, 1653264)$

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Example

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Runtime

- Each step reduces the size of numbers by about a factor of 2.
- Takes about $\log(ab)$ steps.

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- Each step reduces the size of numbers by about a factor of 2.
- Takes about $\log(ab)$ steps.
- GCDs of 100 digit numbers takes about 600 steps.
- Each step a single division.

Summary

- Naive algorithm is too slow.
- The correct algorithm is much better.
- Finding the correct algorithm requires knowing something interesting about the problem.