14.11.22

UNIT 1: MATRICES

Rank of Matrix:

The order non zero determinant highest order that may be found from the element of matrix, by selecting by orbitarely by equal number of rows and column. from it.

Problem 1:

Find the rank of A : $\begin{bmatrix} 1 & -2 & 5 \\ -2 & 4 & 1 \\ 3 & -6 & -1 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & -2 & 5 \\ -2 & 4 & 1 \\ 3 & -6 & -1 \end{bmatrix}$ $1(-4+6)$

$|A| = 0$

$|A| = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} = 4-4 = 0$

$|A| = \begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix} = 12-12 = 0$

$|A| = \begin{bmatrix} 4 & 1 \\ -6 & -1 \end{bmatrix} = -4+6 = 2 \neq 0$

Rank of A is 2.

2) Find the rank of $A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 4 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & -6 & 2 & -11 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 5R_1$

$= \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & -7 \end{bmatrix}$ $R_3 \to R_3 - 2R_2$

$\therefore$ Number of non zero rows = 3.

$\Rightarrow$ The rank of A is 2.

Consistency system of simultaneous equation:

Consider a system of m linear equations

$$\left.\begin{matrix} a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n = k_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n = k_2 \\ a_{31}x_1 + a_{32}x_2 + \dots\dots + a_{3n}x_n = k_3 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n = k_m \end{matrix}\right] \to (1)$$

where $n = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \to (2)$

$$B = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The matrix $[A, B] = \left[\begin{array}{ccccc} a_{11} & a_{12} & a_{13} \ldots & a_{1m} & k_1 \\ a_{21} & a_{22} & a_{23} \ldots & a_{2m} & k_2 \\ a_{31} & a_{32} & a_{33} \ldots & a_{3m} & k_3 \end{array}\right]$

is called augment matrix

Rouche's theorom:

The system of equations (1) is consistent if and only if the coefficient matrix A and the augmented matric $[A, B]$ are of the same rank Otherwise the system is inconsistent.

Let Rank of A be r and $[A, B]$ be r'

* If $R = R' = n$ the equations are consistent and there is unique solution.
* If $R = R' < n$, the equations are consistent and there are infinite number of solution.
* If $R \neq R'$, the equation are inconsistent, there is no solution.

The system is always inconsistent. If the coefficient matrix A is non-singular, the system has unique solution namely $x_1 = x_2 = \ldots\ldots x_n = 0$.

The unique solution is called trivial solution.

If coefficient matrix A is singular, the system has an infinetly no. of non zero solution. this solution is called non trivial solution.

1) Test wether the equation

$2x + y + z = 5$

$x + y + z = 4$

$x - y + 2z = 1$ are

consistent, if consistent and solve it.

Solution

The system of equations can be written in the form as $AX = B$

Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$

$AX = B$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$$

The Augment matrix is

$$[A, B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 1 & 1 & 1 & 4 \\ 1 & -1 & 2 & 1 \end{array}\right]$$

To prove:

$f(A) = f(A, B)$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 1 & 1 & 1 & 4 \\ 2 & 1 & 1 & 5 \end{array}\right] R_1 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 3 & -3 & 3 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & +3 & -3 \end{array}\right] \quad R_3 \to \overset{2}{\cancel{3}}R_3 - 3R_2$$

$2(3)-3(2)$
$6-6=0$

$+3(2)-3(-1)$
$6+3$

$2(2)-3(3)$
$6-9$

$r(A) = 3 \quad r(A,B) = 3$

$f(A) = f(A,B) = 3 = n$

$\Rightarrow$ The system is consistent.

To find x, y, z

By Back subtution method

$x \cancel{+} y \quad x - y + \overset{2z}{\cancel{z}} = 1 \to ①$

$2y - z = 3 \to ②$

$-3z = -3 \to ③$

$\boxed{z = 1}$

$z = 1$ Sub in ②

$2y - 1 = 3$

$2y = 4$

$\boxed{y = 2}$

$y = 2,\ z = 1$ Sub in ①

~~$x - 2 + 2 = 1$~~
~~$x - 0 = 1$~~
~~$x = 2$~~

$x - 2 + 2(1) = 1$

$x - 2 + 2 = 1$

$x + 0 = 1$

$\boxed{x = 1}$

$x = 1,\ y = 2$ and $z = 1$.

2. Test the consistency of system of equation

$$x+2y+6z=5$$
$$-x+y-2z=3$$
$$x-4y-2z=1.$$

Solution.

$$AX=B$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 6 & 5 \\ -1 & 1 & -2 & 3 \\ 1 & -4 & -2 & 1 \end{array}\right]$$

$$AX=B$$

$$\begin{bmatrix} 1 & 2 & 6 \\ -1 & 1 & -2 \\ 1 & -4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

The Augment Matrix is

$$[A,B] = \left[\begin{array}{ccc|c} 1 & 2 & 6 & 5 \\ -1 & 1 & -2 & 3 \\ 1 & -4 & -2 & 1 \end{array}\right]$$

To prove

$$f(A) = f[A,B]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 6 & 5 \\ 0 & 3 & 4 & 8 \\ 0 & -6 & -8 & -4 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2+R_1 \\ R_3 \to R_3-R_1 \end{array}$$

$-4+16$
-8

$$\left[\begin{array}{ccc|c} 1 & 2 & 6 & 5 \\ 0 & 3 & 4 & 8 \\ 0 & 0 & 0 & 12 \end{array}\right] R_3 \to R_3+2R_2$$

Rank (A) = 2 Rank(A, B) = 3

3. Discuss the consistency of the following equation and if consistent find the solution.

$x+2y+2z = 1$

$2x+y+z = 2$

$3x+2y+2z = 3$

$y+z = 0$

Solution

$AX = B$

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

The Augment Matrix is

$$[A, B] = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 3 & 2 & 2 & 3 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

To prove

$f(A) = f(A, B)$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 0 & -4 & -4 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] R_3 - 3R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 0 & -4 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_4 \to R_3 + 4R_4$$

$\gamma < \gamma' < 3 < 4$.

Rank (A) = 3

Rank (A,B) = 3

This system is consistent.

$\gamma = \gamma' \alpha = 3 \alpha 4$

By back solution

$-4y - 4z = 0$

$y + z = 0$

$\boxed{y = -z}$

Let $\boxed{z = k}$ assuming

$2x + y + z = 2$

$2x - k + k = 2$

$2x = 2$

$\boxed{x = 1}$

$\therefore$ The system has infinitely many solution.

4. Investigate the value of λ and μ, so that the equation $2x + 3y + 5z = 9$

$$7x + 3y - 2y = 8$$

$$2x + 3y + \lambda z = \mu$$

have first sub division no solution

ii) unique solution iii) infinit number of solution

solution

$AX = B$

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

The Augment matrix is

$$[A, B] = \left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right]$$

$$= \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{bmatrix} R_3 - R_1$$

Rank of $(A) = 3$, Rank of $(A|B) = 3$

$\therefore$ The system is consistent.

To find

Case 1:

If $\lambda = 5$, $\mu = 9$

Augment matrix is

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\gamma(A) = \gamma(A|B) = 2 < 3$

$\therefore$ The system is consistent but infinetly many solution.

Case 2:

If $\lambda = 5$ $\mu \neq 9$

Augment matrix is.

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 0 & 0 & 0 & \mu-9 \end{array}\right]$$

$\gamma(A) = 2$ $\gamma(A|B) = 3$

$\therefore$ The system is inconsistent and no solution.

Case 3:-

If $\lambda \neq 5$, $\mu = 9$

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 0 & 0 & \lambda-5 & 0 \end{array}\right]$$

$\gamma(A) = 3$ $\gamma(A|B) = 3$

consistent, but many solution.

5) Find the value of k to the system of equation has i) unique ii) many solution iii) no solution

$kx + y + z = 1$

$x + ky + z = 1$

$x + y + kz = 1$

solution.

The system of equation can be written in the form as

$$AX = B$$

$$\begin{bmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Augment matrix is

$$[A,B] = \left[\begin{array}{ccc|c} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \end{array}\right]$$

To find k by choose $|A| = 0$.

$$\begin{vmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{vmatrix} = 0$$

$k(k^2-1) - 1(k-1) + 1(1-k)$

$k(k^2-1) - k + 1 + 1 - k = 0$

$k^3 - k - k + 1 + 1 - k = 0$

$k^3 - 3k + 2 = 0$

$$\begin{array}{c|cccc} 1 & 1 & 0 & -3 & 2 \\ & \downarrow & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

$$k^2+k-2=0$$

-2, $+2$ -1

$$(k+2)(k-1)=0$$

$$k=-2 \quad k=+1$$

$$\boxed{k=1,-2}$$

Augmented matrix is

$$\left[\begin{array}{ccc|c} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \end{array}\right]$$

case 1:

$k \neq 1$ $k=2$

$$[A,B] = \left[\begin{array}{ccc|c} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \end{array}\right]$$

$$\sim \begin{bmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 0 & 3 & 3 & 3 \end{bmatrix} R_3 \to R_1 + 2R_3$$

$\gamma(A) = \gamma(A,B) = 3$

consistent and it has unique solution.

case 2:

$k=1$ $k \neq 2$

$$[A,B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$\gamma(A) = \gamma(A, B) = 1 < 3$

$\therefore$ The system is consistent and it has many solution

Case 3:

Put $k \neq 1$, $k \neq -2$

$\therefore$ The system is inconsistent and it has no solution.

Linear dependence and independence of vectors.

The vectors $x_1, x_2 \ldots x_n$ are said to be linearly dependent if there exist a number $\lambda_1, \lambda_2 \ldots \lambda_n$ not all zero such that $\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = 0$.

If such number other than zero exist the vector are said to linearly independent

1. Show that the vector $x_1 = [1 \ 1 \ 2]$ $x_2 = [1 \ 2 \ 5]$ and $x_3 = [5 \ 3 \ 4]$ are linearly dependent also express each vectors as a linear combination other two.

Solution.

The given vector can be written is the form as

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -6 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 5R_1 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + 2R_2$$

$r(A) = 2 < 3$

$\therefore$ the given vectors are linearly dependent.

To find the expression

$R_3 + 2R_2 = 0$

$\cancel{R_3 - 5R_1 + R_2 - R_1 + 0}$

$\cancel{R_3 + R_2 - 6R_1 + 0}$

$R_3 + 2(R_2 - R_1) = 0$

$R_3 - 5R_1 + 2R_2 - 5R_1 = 0.$

$$R_3 + 2R_2 - 7R_1 = 0$$

$$x_3 + 2x_2 - 7x_1 = 0$$

$$\boxed{-7x_1 + 2x_2 + x_3 = 0}$$

$$-7x_1 = -2x_2 - x_3 = 0$$

$$\boxed{x_1 = \frac{2}{7}x_2 + \frac{1}{7}x_3}$$

$$2x_2 = 7x_1 - x_3$$

$$\boxed{x_2 = \frac{7}{2}x_1 - \frac{1}{2}x_3}$$

$$\boxed{x_3 = 7x_1 - 2x_2}$$

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2. Show that the vector $x_1 = [2\ -2\ 1]$ $x_2 = [1\ 4\ -1]$ & $x_3 = [4\ 6\ -3]$ are linearly independent also find the expression for each vectors.

Solution

The given vector can be written in the form

$$A = \begin{bmatrix} 2 & -2 & 1 \\ 1 & 4 & -1 \\ 4 & 6 & 38 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & 1 \\ 0 & 10 & -3 \\ & & -5 \\ 0 & 10 & 6 \end{bmatrix} \begin{matrix} R_1 - 2R_1 \\ R_2 \to 2R_2 - R_1 \\ \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

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$$= \begin{bmatrix} 2 & -2 & 1 \\ 0 & 10 & -3 \\ 0 & 10 & \overset{-2}{\cancel{9}} \end{bmatrix} R_3 \to R_3 - R_2$$

To find expression

$$R_3 - R_2 = 0$$

$$R_3 - 2R_1 - (\overset{R_1 - 2R_2}{\cancel{2R_2 - R_1}}) = 0$$

$$R_3 - 2R_1 + 2R_2 \mp R_1 = 0$$

$$R_3 - 3R_1 + 2R_2 = 0$$

$$R_3 + 2R_2 - 3R_1 = 0$$

$$\boxed{-3x_1 + 2x_2 + x_3 = 0}$$

$$-3x_1 = -2x_2 - x_3$$

$$\boxed{x_1 = \frac{2}{3}x_2 + \frac{1}{3}x_3}$$

$$2x_2 = 3x_1 \mp x_3$$

$$\boxed{x_2 = \frac{3}{2}x_1 - \frac{1}{2}x_3}$$

$$\boxed{x_3 = 3x_1 - 2x_2}$$

Homogeneous system of linear equation.

Basic problem based on trivial, non-trivial, orthogonal matrix.

1. Examine the system has trivial or non trivial solution.

$$x_1 - x_2 + x_3 = 0$$
$$x_1 + 2x_2 - x_3 = 0$$
$$2x_1 + x_2 + 3x_3 = 0$$

trivial (non singular) $\neq 0$.

Solution

The given equation of solution in the matrix form

$$|A| = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ & & \end{bmatrix} \quad R_2 \to R_2 - R_1$$

$$= 1(6+1) + 1(3+2) + 1(1-4)$$
$$= 1(7) + 1(5) + 1(-2)$$
$$= 7 + 5 - 3$$
$$= 9$$

$|A| = 9 \neq 0$.

$\therefore$ It is trivial solution.

2. Prove the following matrix is orthogonal

$$A = \begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$$

Solution

To prove A is orthogonal

That is to prove

$AA^T = I$

$$A = \begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix}$$

$$AA^T = 1/3 \begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix} 1/3 \begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$$

$$= 1/3 \begin{bmatrix} 4+1+4 & -4+2+2 & -2-2+4 \\ -4+2+2 & 4+4+1 & 2-4+2 \\ -2-2+4 & 2-4+2 & 1+4+4 \end{bmatrix}$$

$$= 1/3 \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 9/3 & 0 & 0 \\ 0 & 9/3 & 0 \\ 0 & 0 & 9/3 \end{bmatrix}$$

LHS = 3I

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Properties of Eigen values and eigen vectors.

Property 1:

The sum of Eigen values of matrix is equal to sum of elements of main dignols.

Produd of the aigen values is equal to determinent of matrix.

Property 2 :-

Square matrix A and its A^T have the same Aigen values. (characteristic value)

Property 3:-

The characterisitic roots (Eigan values) of a triangular are just the dignol elements of the matrix

Property 4:-

I λ is an Aigen values of A then $\frac{1}{\lambda}$ is Aigen value of A^{-1}

Property 5:-

If λ is an Aigen value of orthogonal matrix then $1/\lambda$ is also its Aigen values.

Property 6 :-

If $\lambda_1, \lambda_2, \lambda_3, \dots \lambda_n$ are aigen value of matrix A then A^m has aigen value of $\lambda^m, \lambda_2^m \dots \lambda_n^m$

Property 7:

The aigen value of the real symetric matrix are real numbers.

Property 8:

The similar matrices have same aigen value.

Property 9:

The aigen corresponding to distinct aigen value of a real symetric matrix or orthogonal.

Property 10:

The aigen vector X_i of ^two^ matrix A is not unique.

Property 11:

Two aigen vector X_1, X_2 are called orthogonal vectors. If $x_1^T x_2 = 0$.

Problems :-

1. Find the sum and product of Eigan values of matrix

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Solution.

By the property.

Sum of the Eigen values = sum of main diagonal

$= -1-1-1 = -3$

Product of Eigan values = $|A|$

$= -1(1-1) - 1(-1-1) + 1(1+1)$

$= -1(0) - 1(-2) + 1(2)$

$= +2+2$

$= 4.$

2) The product of two Eigen values of the matrix

$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & 1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16. Find the 3^{rd} Eigen value.

Solution.

By the property.

Product of the Eigen values = $|A|$.

Given that $\lambda_1 \lambda_2 = 16$

Product of eigen value $= |A|$

$(\lambda_1 \cdot \lambda_2) \cdot \lambda_3 = |A|$

$16 \cdot \lambda_3 = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

$|A| = 6(9+1) + 2(-6-2) + 2(2-6)$

$= 6(10) + 2(-\cancel{2}8) + 2(-4)$

$= 60 + \cancel{24}-16 - 8$

$= 60 - 32$

$|A| = 32$

$16 \cdot \lambda_3 = 32$

$\lambda_3 = \frac{32}{16}$

$\lambda_3 = 2$

3\. If 3 & 15 are two eigen values of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & 4 \\ -2 & -4 & 3 \end{bmatrix}$ find $|A|$ without expand the determinant.

Solution.

By the property

Given that $\lambda_1 = 3$ $\lambda_2 = 15$

~~Product~~ Sum of eigen values = sum of digonals

$\lambda_1 + \lambda_2 + \lambda_3 = 8 + 7 + 3$

$3 + 15 + \lambda_3 = 18$

$\lambda_3 = 18 - 18$

$\lambda_3 = 0$

By property

Product of Eigen value $= |A|$

$\lambda_1 \lambda_2 \lambda_3 = |A|$

$(8)(15)(0) = |A|$

$0 = |A|$

4. If 2, 2, 3 are the eigen value of $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ find the eigen value of A^T

Solution.

By the property

A square matric A and it A^T have a same eigen values.

That is eigen values of A^T are 2, 2, 3

5. Find the eigen value of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ without using characteristic equation.

Solution

By the property,

The Eigen value of triangular matrix are just the digonal element

$\therefore$ Eigen values are 2, 2, 0.

6. Two of the eigen values of $A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 8 \end{bmatrix}$ are 3 & 6, find eigen values of A^{-1}.

Solution.

By the property

sum of the eigen values = sum of main digonals

$\lambda_1 + \lambda_2 + \lambda_3 = 3+5+\cancel{8}3$

$3+6+\lambda_3 = \cancel{16}11$

$9+\lambda_3 = \cancel{16}11$

$\lambda_3 = \cancel{16}/\cancel{9} \cdot 11-9$

$\lambda_3 = 2$

By the property

If λ is an eigen value of A then $\frac{1}{\lambda}$ is an eigen value of A^{-1}

$\Rightarrow$ If 2, 3, 6 are eigen value of A.

then $1/2, 1/3, 1/6$ are eigen value of A^{-1}.

7. Find the eigen value of inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$

Solution

By the property.

the eigen value of triangular matrix are just the digonal elements.

$\therefore$ eigen values A are 2, 3, 4

By the property.

If λ is the eigen value of A then eigen value of A^{-1}.

2, 3, 4 are Eigen value of A,

$1/2, 1/3, 1/4$ are Eigen value of A^{-1}.

8. Two eigen value of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal to 1 each. Find eigen value of A^{-1}.

Solution.

$\lambda_1 = \lambda_2 = 1 \quad \lambda_3 = ?$

By the property.

Sum of eigen values = ~~&~~ sum of main digonal

$\lambda_1 + \lambda_2 + \lambda_3 = 2 + 3 + 2$

$1 + 1 + \lambda_3 = 7$

$2 + \lambda_3 = 7$

$\boxed{\lambda_3 = 5}$

1, 1, 5 are eigen value of A.

By the property.

If λ is eigen value of A then $\frac{1}{A}$ is eigen value of A^{-1}.

1, 1, 5 are eigen value of A.

$1/1, 1/1, 1/5$ are eigen value of A^{-1}

1. Find the eigen values of A^3, $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix}$

Solution

Given that $= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix}$

By the property (3)

The eigen value triangular matrix are the just the digonal elements.

The eigen values of A are 1, 2, 3

By the property (A^3)

If λ is an eigen values of A^3, then λ^m is an eigen values of A^3.

$\Rightarrow$ The eigen values of A^3 are 1, 8, 27.

2. If 1, 1, 5 are the eigen values of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

find the eigen values of 5A.

Solution

Given that, eigen values of A are 1, 1, 5

By the property.

The eigen value of 5A are 5, 5, 25.

3. If 2, −1, −3 are the eigen values of A. Find the eigen values of matrix $A^2 - 2I$.

Solution.

Given that,

Eigen value of A are $2,-1,-3$

eigen value of A^2 are $2^2, 1^2, -3^2 = 4, 1, 9$.

Eigen value of A^2-2I are $= 4-2, 1-2, 9-2 = 2, -1, 7$.

4. What are the eigen values of the matrix A^2+3I. If the eigen values of matrix $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$ are 6 & -1.

Solution.

Eigen values of A are $6, -1$.

Eigen values of A^2 are $36, 1$.

Eigen values of $A^2+3I = 39, 4$.

5. If the eigen values of A are of order order 3×3 are 2, 3 and 1 the find the eigen values of adj A.

Solution.

The eigen value of A are $2, 3, 1$.

We know that eigen value of $A^{-1} \Rightarrow 1/2, 1/3, 1$

To find the eigen of adj A.

$A^{-1} = \frac{1}{|A|}$ adj A.

$A^{-1} \cdot |A| =$ adj $A \rightarrow$ ①.

By the property.

$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = |A|$

$2 \cdot 3 \cdot 1 = |A|$

$6 = |A|$.

Product

Eigen values of adj A $= 6(1/2, 1/3, 1)$

$= 6/2, 6/3, 6$

adj A $= (3, 2, 6)$

6. Two eigen values $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal and third are 1/2 times to third. Find them.

Solution.

$\lambda_1 = \frac{1}{5}\lambda_3$, $\lambda_2 = \frac{1}{5}\lambda_2$, $\lambda_3 = ?$

Sum of eigen values = Sum of all digonals.

$\lambda_1 + \lambda_2 + \lambda_3 = 2+3+2$

$\frac{1}{5}\lambda_3 + \frac{1}{5}\lambda_3 + \lambda_3 = 7.$

$\lambda_3\left(\frac{1}{5} + \frac{1}{5} + 1\right) = 7.$

$\lambda_3\left(\frac{7}{5}\right) = 7.$

$\lambda_3 = \frac{7}{\frac{7}{5}}$

$\lambda_3 = \cancel{7} \times \frac{5}{\cancel{7}}$

$\lambda_3 = 5$

$\lambda_1 = \frac{1}{\cancel{5}}(\cancel{5})$ $\lambda_2 = \frac{1}{\cancel{5}}(\cancel{5})$

$\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 5$.

7. Prove that the eigen ~~values~~ vectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ are orthogonal pairs.

Solution

$(AA^T = 0)$

Let

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad x_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$x_1 x_2^T = 0 \quad x_2 x_3^T = 0 \quad x_3 x_1^T = 0.$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} [1 \ -1 \ 1] \qquad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} [1 \cdot 2 \ 1] \qquad \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} [\cancel{0} 1 \ 0 \ -1]$$

$= [1 + 0 - 1]$ $\quad = 1 - 2 + 1$ $\quad = 1 + 0 - 1$

$= 0$ $\quad = 0$ $\quad = 0.$

Eigen values and Eigen vectors.

Problems based on non-symmetric matrix with non-repeated eigen values.

1. Find the eigen values and eigen vectors of

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

Solution.

Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Step 1:

To find the ~~eat~~ characteristic Eigen the general term is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$

$S_1 =$ sum of digonal.

$S_1 = 1+2+3$

$S_1 = 6$

$S_2 =$ sum of determinent of principle minors.

$$S_2 = \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$

$= (6-2) + (3+2) + (2-0)$

$= 4+5+2$

$S_2 = 11$

$S_3 = |A|$

$$= \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix}$$

$= 1(6-2) \mp 0(3-2) - 1(2-4)$

$= 1(4) + 0 - 1(-2)$

$= 4+2$

$S_3 = 6$

$\Rightarrow$ The characteristic of $|A - \lambda I|$ is

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$

Step 2:

To find the characteristic roots (Eigen values).

By synthetic division.

1	1	-6	11	-6
	$\downarrow$	1	-5	6
	1	-5	6	0

$\lambda^2 - 5\lambda + 6 = 0.$

$\lambda^2 - 5\lambda + 6 = 0$

$\lambda^2 - 3\lambda - 2\lambda + 6 = 0.$

$(\lambda-3)\ (\lambda-2) = 0.$

$\boxed{\lambda = 3}$ $\boxed{\lambda = 2}$

6
-3 -2

Step 3:

To find eigen vector. by

$(A - \lambda I)x = 0.$

$$\left[\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (1)$$

case (i) put $\lambda = 1$.

$0x_1 + 0x_2 - x_3 = 0 \rightarrow (1)$

$x_1 + 1x_2 + x_3 = 0 \rightarrow (2)$

$2x_1 + 2x_2 + 2x_3 = 0; \Rightarrow \div 2$

$x_1 + x_2 + x_3 = 0 \rightarrow (3)$

consider (1) and (2) (different).

$$\frac{x_1}{0+1} = \frac{x_2}{-1+0} = \frac{x_3}{0}$$

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

eigen vectors: $x_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

case(2):

Put $\lambda = 2$

$-1x_1 + 0x_2 - 1x_3 = 0 \rightarrow ①$

$x_1 + 0x_2 + 1x_3 = 0 \rightarrow ②$

$2x_1 + 2x_2 + 1x_3 = 0 \rightarrow ③$

$① \rightarrow \times by(-) = x_1 + \cancel{0}x_2 + x_3 = 0 \rightarrow ①$

consider ② & ③

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 2 & 1 & 2 & 2 \end{bmatrix}$$

$$\frac{x_1}{0-2} = \frac{x_2}{2-1} = \frac{x_3}{2-0}$$

$x_2 = \begin{bmatrix} +2 \\ 1 \\ 2 \end{bmatrix}$ first value is always (+) ve.

case 3:- $\lambda = 3$

$-2x_1 + 0x_2 - 1x_3 = 0 \rightarrow ①$

$x_1 - 1x_2 + x_3 = 0 \rightarrow ②$

$2x_1 + 2x_2 + 0x_3 = 0 \rightarrow ③$

consider ① & ③.

$$\begin{bmatrix} 0 & -1 & -2 & 0 \\ 2 & 0 & 2 & 2 \end{bmatrix}$$

$$\frac{x_1}{0+2} = \frac{x_2}{-2+0} = \frac{x_3}{-4+0}$$

$$\frac{x_1}{\cancel{2}\,1} = \frac{x_2}{\cancel{-2}\,-1} = \frac{x_3}{\cancel{-4}\,-2}$$

$\div 2/2$

eigen vector $x_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

The eigen values of A are $\lambda = 1, 2, 3$ with the corresponding eigen ~~values~~ vectors.

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad x_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

Problem based on non symmetric matrices with repeated eigen values ($A \neq A^T$, λ = same).

1. Find the eigen values and eigen vectors of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Solution

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Step 1:

To find the characteristic eigen the general form is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

S_1 = sum of digonal

$= -2+1+0$

$S_1 = -1$

S_2 = sum of determinand of principle minors.

$$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$$

$= (0-12) + (0-3) + (4-2)$

$= -12 - 3 - 6$

$S = -21$

$S_3 = |A|$

$$= \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$S_3 =$
$|A| = 45$

The CE of $|A-\lambda I|$ is

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$.

$$\begin{array}{c|cccc} 3 & 1 & 1 & -21 & -45 \\ & \downarrow & 3 & 12 & -27 \\ \hline & 1 & +4 & -9 & \end{array}$$

$$\begin{array}{c|cccc} -3 & 1 & 1 & -21 & -45 \\ & \downarrow & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array}$$

$$\begin{array}{c|cccc} 4 & 1 & 1 & -21 & -45 \\ & \downarrow & 4 & 20 & 4 \\ \hline & 1 & 5 & -1 & \boxed{-41} \end{array}$$

Let $\lambda = -3$

$\lambda^2 - 2\lambda - 15 = 0$

$$\begin{array}{c} -15 \\ \wedge \\ -5 \quad 3 \end{array}$$

$(\lambda-5)(\lambda+3) = 0$

$\boxed{\lambda = 5}$ $\boxed{\lambda = -3}$

$\therefore$ The eigen values $\lambda = -3, -3, 5$. (λ values are same ~~repeated~~).

Step 3:

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (1)$$

case (1)

Put $\lambda = -3$ in (1).

$$\begin{bmatrix} \cancel{+5}\,1 & 2 & -3 \\ 2 & +4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 - 3x_3 = 0 \rightarrow ①$

$2x_1 + 4x_2 - 6x_3 = 0 \div 2 \Rightarrow x_1 + 2x_2 - 3x_3 \rightarrow ②$

$-x_1 - 2x_2 + 3x_3 = 0.\ (x-)\ x_1 + 2x_2 - 3x_3 \rightarrow ③$

all the three equation are same.

consider $x_1 + 2x_2 - 3x_3 = 0$.

choose $x_1 = 0$.

$0 + 2x_2 - 3x_3 = 0.$

$2x_2 = 3x_3$

$\frac{x_2}{3} = \frac{x_3}{2}$

$\therefore$ Eigen vector of $x_1 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

choose $x_3 = 0$.

$x_1 + 2x_2 + 0 = 0$

$x_1 = -2x_2$

$\frac{x_1}{-2} = \frac{x_2}{1}$

$\therefore$ Eigen vector of $x_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

case ii)

$\lambda = 5$ in ①.

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ 1 & -2 & -5 \end{bmatrix}$$

$-7x_1 + 2x_2 - 3x_3 = 0 \rightarrow ①$

$2x_1 - 4x_2 - 6x_3 = 0 \rightarrow ②$

$1x_1 - 2x_2 - 5x_3 = 0 \rightarrow ③$.

consider ② and ③

$\frac{x_1}{20-12} = \frac{x_2}{-6+10} = \frac{x_3}{-4+4}$

$$\begin{bmatrix} -4 & -6 & 2 & -4 \\ -2 & -5 & 1 & -2 \end{bmatrix}$$

$\frac{x_1}{8\,\cancel{2}} = \frac{x_2}{\cancel{4}\,1} = \frac{x_3}{0}.$

Eigen vector of $x_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Problems based on symmetric matrices with non repeated eigen values.

1. Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$

Solution

Step 1: To find the equation

The characteristic equation of $(A-\lambda I)$ is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$

$S_1 =$ sum of diagonals.

$7+6+5 = 18.$

$S_2 =$ sum of determinant of diagonals.

$$\begin{vmatrix} 6 & -2 \\ -2 & 5 \end{vmatrix} + \begin{vmatrix} 7 & 0 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 7 & -2 \\ -2 & 6 \end{vmatrix}$$

$= (30-4) + (35-0) + (42-4)$

$= 26+35+38$

$= 99$

$S_3 = |A|$

$= 7(30-4) + 2(-10+0) + 0(0)$

$= 7(26) + 2(-10)$

$= 182 - 20$

$S_3 = 162$

The characteristic equation is $\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0.$

Step 2:

To find eigen values.

$89\times 2 = 178$

$$\begin{array}{r|rrrr} 8 & 1 & -18 & +99 & -162 \\ & \downarrow & 8 & -80 & 178 \\ \hline & 1 & -10 & 89 & \end{array}$$

$$\begin{array}{r|rrrr} -8 & 1 & -18 & 99 & -162 \\ & \downarrow & -8 & & \\ \hline & & -26 & & \end{array}$$

8×19

$$\begin{array}{r|rrrr} 3 & 1 & -18 & 99 & -162 \\ & \downarrow & 3 & -45 & 162 \\ \hline & 1 & -15 & 54 & 0 \end{array}$$

3×69

$99 - 45 = 54$

$\lambda^2 - 15\lambda + 54 = 0$

$(\lambda-9)(\lambda-6) = 0$

$\boxed{\lambda = 9}$ $\boxed{\lambda = 6}$

54
$9, 6$

$\lambda = 3, 6, 9.$

Step 3:

To find the eigen values.

By $(A-\lambda I)x = 0$

$$\begin{bmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case i) Put $\lambda = 3$

$$\begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4x_1 - 2x_2 + 0x_3 = 0. \rightarrow (1)$

$-2x_1 + 3x_2 - 2x_3 = 0. \rightarrow (2)$

$0x_1 - 2x_2 + 2x_3 = 0 \rightarrow (3).$

consider (2) and (3)

$$\frac{x_1}{6-4} = \frac{x_2}{0+4} = \frac{x_3}{4-0} \qquad \begin{bmatrix} 3 & -2 & -2 & 3 \\ -2 & 2 & 0 & -2 \end{bmatrix}$$

$$\frac{x_1}{2_1} = \frac{x_2}{4_2} = \frac{x_3}{4_2}$$

Eigen vector $x_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Case ii)

$\lambda = 6$ in ①.

$$\begin{bmatrix} +1 & -2 & 0 \\ -2 & +0 & -2 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 - 2x_2 + 0x_3 = 0 \rightarrow$ ①

$-2x_1 + 0x_2 - 2x_3 = 0 \rightarrow$ ②

$0x_1 - 2x_2 - 1x_3 = 0 \rightarrow$ ③

Consider ① and ③

$$\frac{x_1}{2+0} = \frac{x_2}{0+1} = \frac{x_3}{-2+0} \qquad \begin{bmatrix} -2 & 0 & 1 & -2 \\ -2 & -1 & 0 & -2 \end{bmatrix}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

Eigen vector $x_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

Case iii)

Put $\lambda = 9$ in equ ①.

$$\begin{bmatrix}-2&-2&0\\-2&-3&-2\\0&-2&-4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$-2x_1 - 2x_2 + 0x_3 = 0 \rightarrow ①$

$-2x_1 - 3x_2 - 2x_3 = 0 \rightarrow ②$

$0x_1 - 2x_2 - 4x_3 = 0 \rightarrow ③$

consider ① and ②.

$$\frac{x_1}{4+0} = \frac{x_2}{0-4} = \frac{x_3}{6-4} \qquad \begin{bmatrix}-2 & 0 & -2 & -2\\ -3 & -2 & -2 & -3\end{bmatrix}$$

$$\frac{x_1}{4_2} = \frac{x}{-4_{-2}} = \frac{x_3}{2_1}$$

Eigen vector of $x_3 = \begin{bmatrix}2\\-2\\1\end{bmatrix}$

The eigen ~~vector~~ value ~~of~~ A are 3, 6, 9 with corresponding vectors are $x_1 = \begin{bmatrix}1\\2\\2\end{bmatrix}$ $x_2 = \begin{bmatrix}2\\1\\2\end{bmatrix}$ $x_3 = \begin{bmatrix}1\\-1\\-2\end{bmatrix}$

Problems based on Symmetric matrices with repeated eigen values:

1. Find the eigen values and vector of the matrix $\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$

Solution.

Step 1:

To find characteristic equation $|A-\lambda I|$ is

The general equation is ~~|A|−λ~~ $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3$

S_1 = Sum of digonal.

$S_1 = 0+0+0$

$S_1 = 0$

S_2 = Sum of determinant of digonal.

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= (0-1) + (0-1) + (0-1)$$

$S_2 = -3$

$$S_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= 0(0) - 1(0-1) + 1(1-0)$$

$$= 1+1$$

$S_3 = 2$

$\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0$

Step 2:-

To find the eigen values.

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -3 & -2 \\ & \downarrow & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

$\lambda^2 - \lambda - 2 = 0$ -2 $\wedge$ $+1\ -2$

$(\lambda+1)(\lambda-2) = 0$

$\boxed{\lambda = -1}$ $\boxed{\lambda = 2}$

$\lambda = -1, -1, 2.$

Step 3:

To find the eigen vector By using $(A-\lambda I)x=0$.

$$\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

case (i) $\lambda = 2$ in ①

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & 1-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_1+x_2+x_3=0 \rightarrow$ ①

$x_1-2x_2+x_3=0 \rightarrow$ ②

$x_1+x_2-2x_3=0 \rightarrow$ ③.

consider ① and ②.

$$\frac{x_1}{1+2}=\frac{x_2}{1+2}=\frac{x_3}{1+2}$$

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ -2 & 1 & 1 & -2 \end{bmatrix}$$

$$\frac{x_1}{3}=\frac{x_2}{3}=\frac{x_3}{3}$$

* Eigen vector $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

case ii)

$\lambda=-1$ sub in ①.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1+x_2+x_3=0 \rightarrow$ ①

$x_1+x_2+x_3=0 \rightarrow$ ②

$x_1+x_2+x_3=0 \rightarrow$ ③

All the above equations are equal.

choose $x_3 = 0$.

$x_1 + x_2 = 0$

$x_1 = -x_2$

$\frac{x_1}{-1} = +\frac{x_2}{1}$

Eigen vector of $x_2 = \begin{bmatrix} +1 \\ -1 \\ 0 \end{bmatrix}$

case iii)

choose $x_2 = 0$

$x_1 + x_3 = 0$

$x_1 = -x_3$

$\frac{x_1}{-1} = \frac{x_3}{1}$

Eigen vector $x_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

The eigen

Orthogonal $= x_1^T x_2$

case iii)

choose $x_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$ is orthogonal to x_1^T, x_2^T vectors

$\Rightarrow x_1^T x_3 = 0$

$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$

$l + m + n = 0 \rightarrow ①$.

$\Rightarrow x_2^T x_3$

$$\begin{bmatrix}1 & -1 & 0\end{bmatrix}\begin{bmatrix}l\\ m\\ n\end{bmatrix} = 0$$

$$l - m + 0n = 0 \rightarrow ②$$

$$\frac{l}{0+1} = \frac{m}{+1-0} = \frac{n}{-1-1}$$

$$\frac{l}{1} = \frac{m}{1} = \frac{n}{-2}$$

Eigen vectors $x_3 = \begin{bmatrix}1\\ 1\\ -2\end{bmatrix}$

The eigen values of $A = -1, -1, 2$ with corresponding vectors are $= x_1 = \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$ $x_2 = \begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$ $x_3 = \begin{bmatrix}1\\ 1\\ -2\end{bmatrix}$

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Unit - II

Diagonalisation of Matrices.

Cayley Hamilton theorem - Reduction of Quadratic form to canonical form through orthogonal Transformation.

Cayley

Statement: Every square satisfies its own characteristic equation.

Uses of Cayley Hamilton theorem:

To calculate inverse of non singular square matrix A ($|A| \neq 0$).

To calculate the positive integral powers of A. (highest power of A).

Problem:

1. Verify Cayley Hamilton theorem, find A^4 and inverse of the matrix A (A^{-1}). when $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Solution.

Step 1:

To find the characteristic equation ($A - \lambda I$)

general equation $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

$S_1 =$ sum of diagonal.

$= 2+2+2$

$S_1 = 6.$

S_2 = sum of determinant of digonal.

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix}$$

$= (4-1) + (4-2) + (-2+1)$

$= 3+2-1$

$S_2 = \cancel{4}8$

$S_3 = |A|.$

$$= \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$= 2(4-1) + 1(-2+1) + 2(1-2)$

$= 6 \cancel{+} 1 - 2$

$|A| = 3$

$\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0.$

Step 2:

By cayley Hamilton theorem,

"Every square matrix A satisfies it own C.E."

The characteristic equation is

$A^3 - 6A^2 + 8A - 3I = 0.$

verification

$A^2 = A \times A.$

$$\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+2 & -2-2-2 & 4+1+4 \\ -2-2-1 & 1+4+1 & -2-2-2 \\ 2+1+2 & -1-2-2 & 2+1+4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = A^2 \times A.$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

$$= \begin{bmatrix} 14+6+9 & -7-12-9 & 14+6+18 \\ -10-6-6 & 5+12+6 & -10-6-12 \\ 10+5+7 & -5-10-7 & 10+5+14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}.$$

$$A^3 - 6A^2 + 8A - 3I = 0$$

$$\begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - 6 \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} + 8 \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$-3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix}$$

$$- \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

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$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

hence verified.

Step 3:

To find A^{-1}

The characteristic equation is $A^3 - 6A^2 + 8A - 3I = 0$

$\div A \Rightarrow A^2 - 6A + 8I - \frac{3I}{A} = 0.$

$A^2 - 6A + 8I - 3A^{-1} = 0.$

$3A^{-1} = A^2 - 6A + 8I$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 6 \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 12 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$3A^{-1} = \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3/3 & 0/3 & -3/3 \\ 1/3 & 2/3 & 0/3 \\ -1/3 & 1/3 & 3/3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 1/3 & 2/3 & 0 \\ -1/3 & 1/3 & 1 \end{bmatrix}$$

Step 4:

To find A^4

The characteristic equation is $A^3 - 6A^2 + 8A - 3I = 0$.

$(\times A) \Rightarrow A^4 - 6A^3 + 8A^2 - 3A = 0$.

$A^4 = 6A^3 - 8A^2 + 3A$.

$$= 6\begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & 22 & 29 \end{bmatrix} + 8\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} + 3\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 174 & -168 & 228 \\ -132 & 138 & -168 \\ 132 & 132 & 174 \end{bmatrix} - \begin{bmatrix} 56 & -48 & 72 \\ -40 & 48 & -48 \\ 40 & -40 & 56 \end{bmatrix} + \begin{bmatrix} 6 & -3 & 6 \\ -3 & 6 & -3 \\ 3 & -3 & 6 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & \overset{-95}{\cancel{1677}} & 124 \end{bmatrix}$$

---

2) verify cayley Hamilton therom and find its invers of $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

solution.

Step 1:

To find the characteristic equation $(A-\lambda I)$ is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

S_1 = sum of digonal.

$= 7 - 1 - 1$

$S_1 = 5$

S_2 = sum of determinant of all digonal.

$$\begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 7 & -2 \\ 6 & -1 \end{vmatrix} + \begin{vmatrix} 7 & 2 \\ -6 & -1 \end{vmatrix}$$

$$= (1-4) + (-7+12) + (-7+12)$$

$$= -3+5+5.$$

$$S_2 = 7.$$

$S_3 = |A|$

$$|A| = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$$

$$= 7(1-4) - 2(6-12) - 2(-12+6)$$

$$= 7(-3) - 2(-6) - 2(-6)$$

$$= -21+12+12$$

$$S_3 = 3.$$

Step 2:

By cayley Hamilton theorem.

The characteristic equation is

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0.$$

$$A^3 - 5A^2 + 7A - 3I = 0.$$

verification

$$A^2 = A \times A$$

$$A^2 = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 49-12-12 & 14-2-4 & -14+4+2 \\ -42+6+12 & -12+1+4 & 12-2-2 \\ 42-12-6 & 12-2-2 & -12+4+1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}$$

$$A^3 = A^2 \times A$$

$$= \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} \times \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 175-48-48 & 50-8-16 & -50+16+8 \\ -168+42+48 & -48+7+16 & +48-14-8 \\ 168-48-42 & 48-8-14 & -48+16+7 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$$

$$A^3 - 5A^2 + 7A - 3I = 0.$$

$$= \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -15 \end{bmatrix} - 5\begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} + 7\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} -$$

$$3\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence verified.

Step 3:

To find the A^{-1}.

The characteristic equation is $A^3 - 5A^2 + 7A - 3I = 0$

$\div A^{-1} \Rightarrow A^2 - 5A + 7I - 3A^{-1} = 0$

$3A^{-1} = A^2 - 5A + 7I.$

$$= \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} - 5 \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} - \begin{bmatrix} 35 & 10 & -10 \\ -30 & -5 & 10 \\ 30 & 10 & -5 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$3A^{-1} = \begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & +5 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -3/3 & -2/3 & 2/3 \\ 6/3 & 5/3 & -2/3 \\ -6/3 & -2/3 & 5/3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1 & -2/3 & 2/3 \\ 2 & 5/3 & -2/3 \\ -2 & -2/3 & 5/3 \end{bmatrix}$$

Step 4:

To find A^4.

The characteristic equation $A^3 - 5A^2 + 7A - 3I = 0$.

$(X) A \Rightarrow A^4 - 5A$