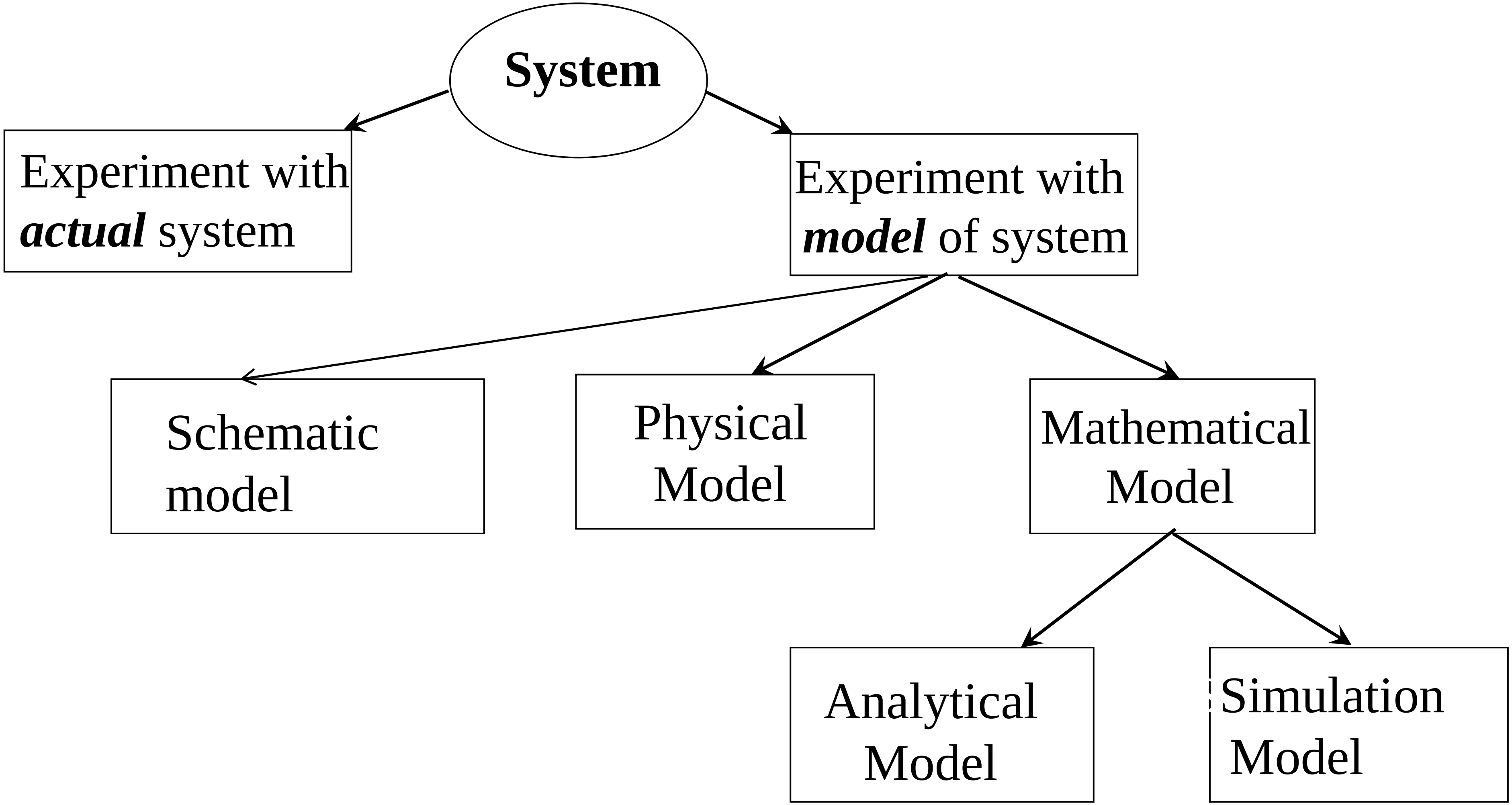
An Overview of Optimization Techniques

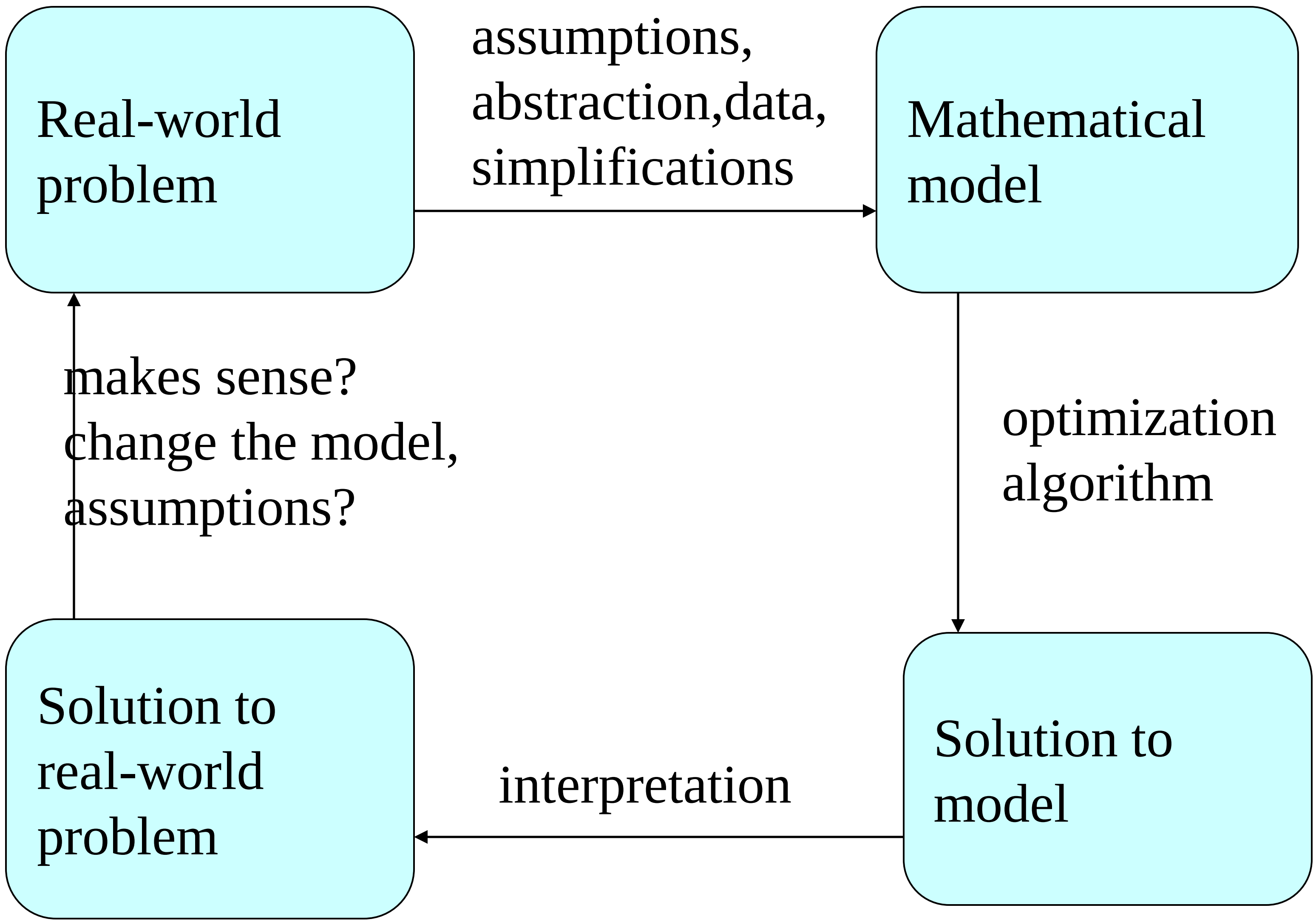
Ways to Study a System



Why optimization?

- In some sense, all engineering design is optimization: choosing design parameters to improve some objective.
- Much of data analysis is also optimization: extracting some model parameters from data while minimizing some error measure (e.g. fitting)
- Most business decisions involve optimization: varying some decision parameters to maximize profit (e.g. investment portfolios, supply chains, etc.)

A schematic view of modeling/optimization process



Mathematical models in Optimization

- The general form of an *optimization model*:

$$\begin{array}{lll} \textit{min} \text{ or } \textit{max} & f(x_1,\ldots,x_n) & \text{(objective function)} \\ \textit{subject to} & g_i(x_1,\ldots,x_n) \geq 0 & \text{(functional constraints)} \\ & x_1,\ldots,x_n \in S & \end{array}$$

- $x_1, \ldots, x_n$ are called decision variables
- In words, the goal is to find $x_1, \ldots, x_n$ that
  - satisfy the constraints
  - achieve min (max) objective function value.

- **Objective Function**

Max (Min) some function of *decision variables*

Subject to (s.t.)

equality (=) *constraints*

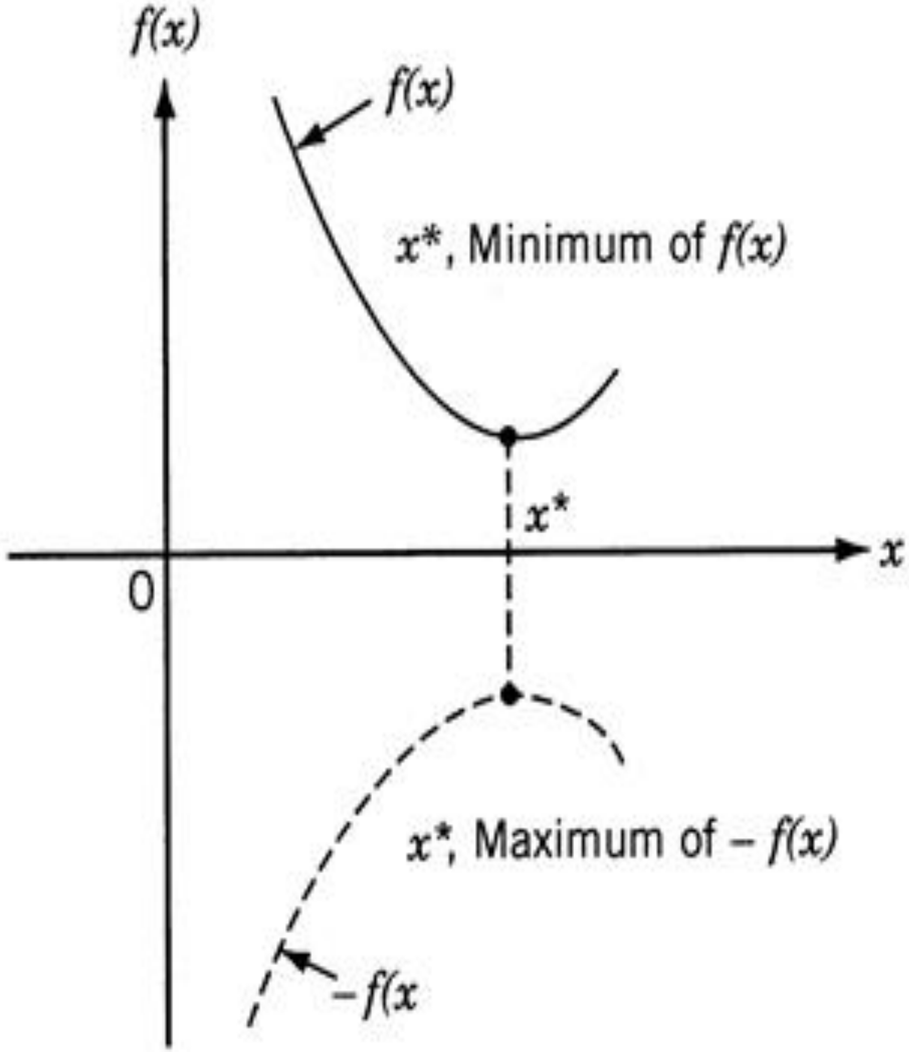
inequality ($<, >, \leq, \geq$) *constraints*

- **Search Space**

Range or values of decisions variables that will be searched during optimization. Often a calculable size in combinatorial optimization

Types of Solutions

- A *solution* to an optimization problem specifies the values of the decision variables, and therefore also the value of the objective function.
- A *feasible solution* satisfies all constraints.
- An *optimal solution* is feasible and provides the best objective function value.
- A *near-optimal solution* is feasible and provides a superior objective function value, but not necessarily the best.



Minimum of $f(x)$ is same as maximum of $-f(x)$.

Continuous vs Combinatorial

- Optimization problems can be *continuous* (an infinite number of feasible solutions) or *combinatorial* (a finite number of feasible solutions).

- Continuous problem generally maximize or minimize a function of continuous variables such as min $4x + 5y$ where x and y are *real numbers*

- Combinatorial problems generally maximize or minimize a function of discrete variables such as min $4x + 5y$ where x and y are *countable items* (e.g. integer only).

Definition of Combinatorial Optimization

- Combinatorial optimization is the mathematical study of finding an optimal arrangement, grouping, ordering, or selection of discrete objects usually finite in numbers.

  - *Lawler*, 1976

- In practice, combinatorial problems are often more difficult because there is *no derivative information* and the *surfaces are not smooth*.

Constraints

- Constraints can be *hard* (must be satisfied) or *soft* (is desirable to satisfy).

  Example: In your course schedule, a hard constraint is that no classes overlap. A soft constraint is that no class be before 10 AM.

- Constraints can be *explicit* (stated in the problem) or *implicit* (obvious to the problem).

- **TSP** (Traveling Salesman Problem)

  Given the coordinates of n cities, find the *shortest closed tour* which visits each *once and only once* (i.e. exactly once).

  Example: In the TSP, an implicit constraint is that all cities be visited once and only once.

Aspects of an Optimization Problem

- Continuous or Combinatorial
- **Size** – number of decision variables, range/count of possible values of decision variables, search space size
- **Degree of constraints** – number of constraints, difficulty of satisfying constraints, proportion of feasible solutions to search space
- *Single* or *Multiple* objectives

Aspects of an Optimization Problem

- *Deterministic* (all variables are deterministic) or *Stochastic* (the objective function and/or some decision variables and/or some constraints are random variables)

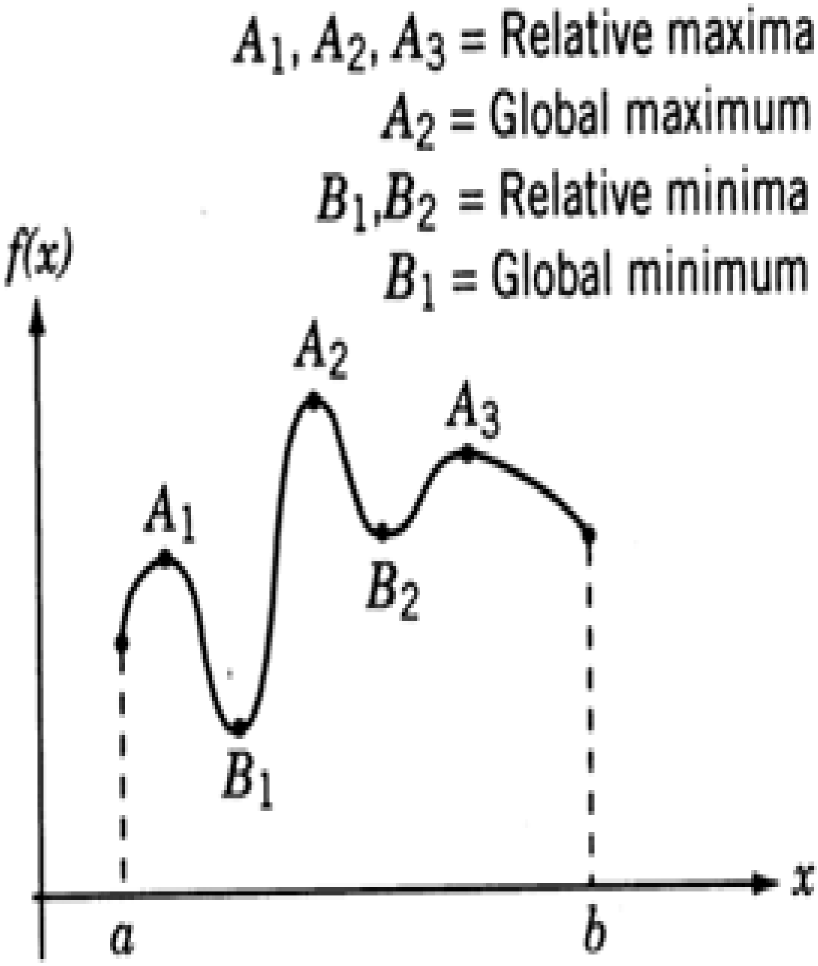
- **Decomposition** – decompose a problem into series problems, and then solve them independently

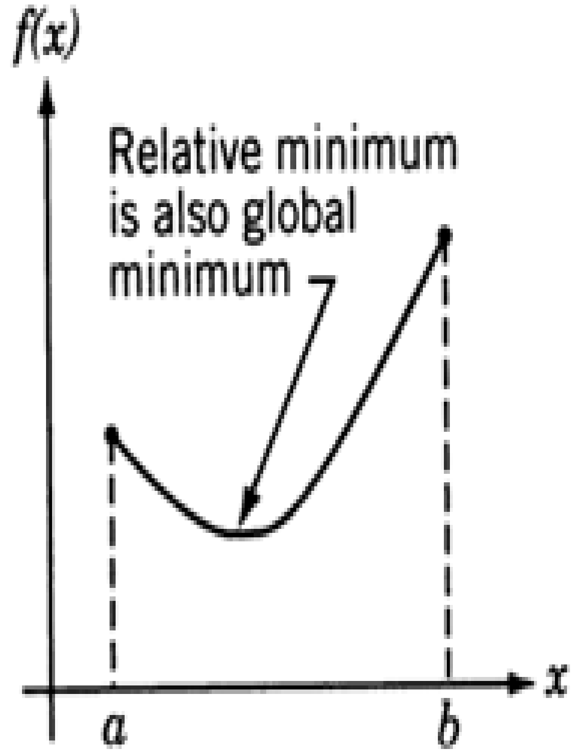
- **Relaxation** – solve problem beyond relaxation, and then can solve it back easier

Simple	Hard
• Few decision variables	• Many decision variables
• Differentiable	• Discontinuous, Combinatorial
• Single modal	• Multi-modal
• Objective easy to calculate	• Objective difficult to calculate
• No or light constraints	• Severely constraints
• Feasibility easy to determine	• Feasibility difficult to determine
• Single objective	• Multiple objective
• Deterministic	• Stochastic

- For Simple problems, enumeration or exact methods such as *differentiation* or *mathematical programming* or *branch and bound* will work best.

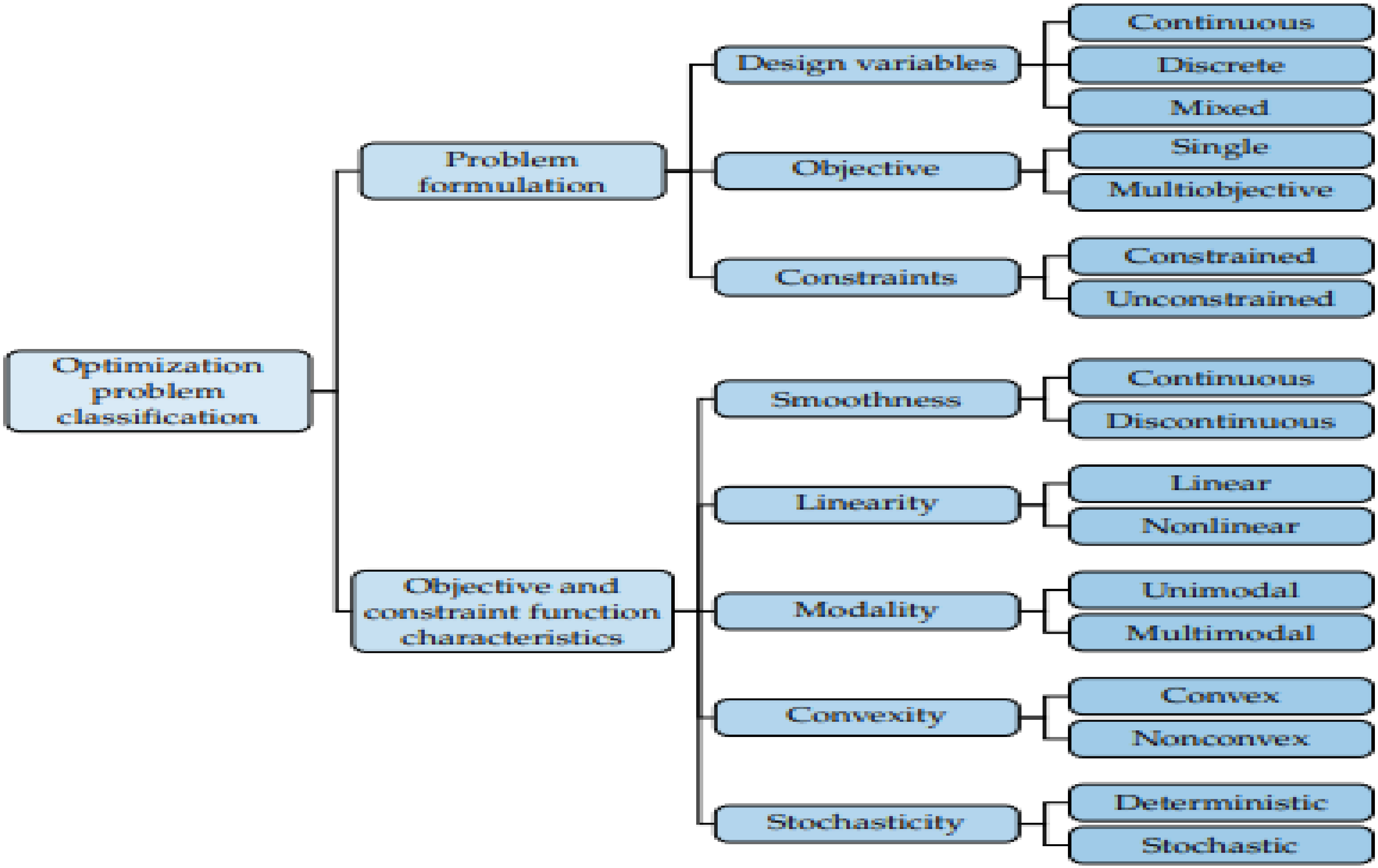
- For Hard problems, differentiation is not possible and enumeration and other exact methods such as math programming are not computationally practical. For these problems, *heuristics* are used.





Relative and global minima.

Classification of Optimization Problems



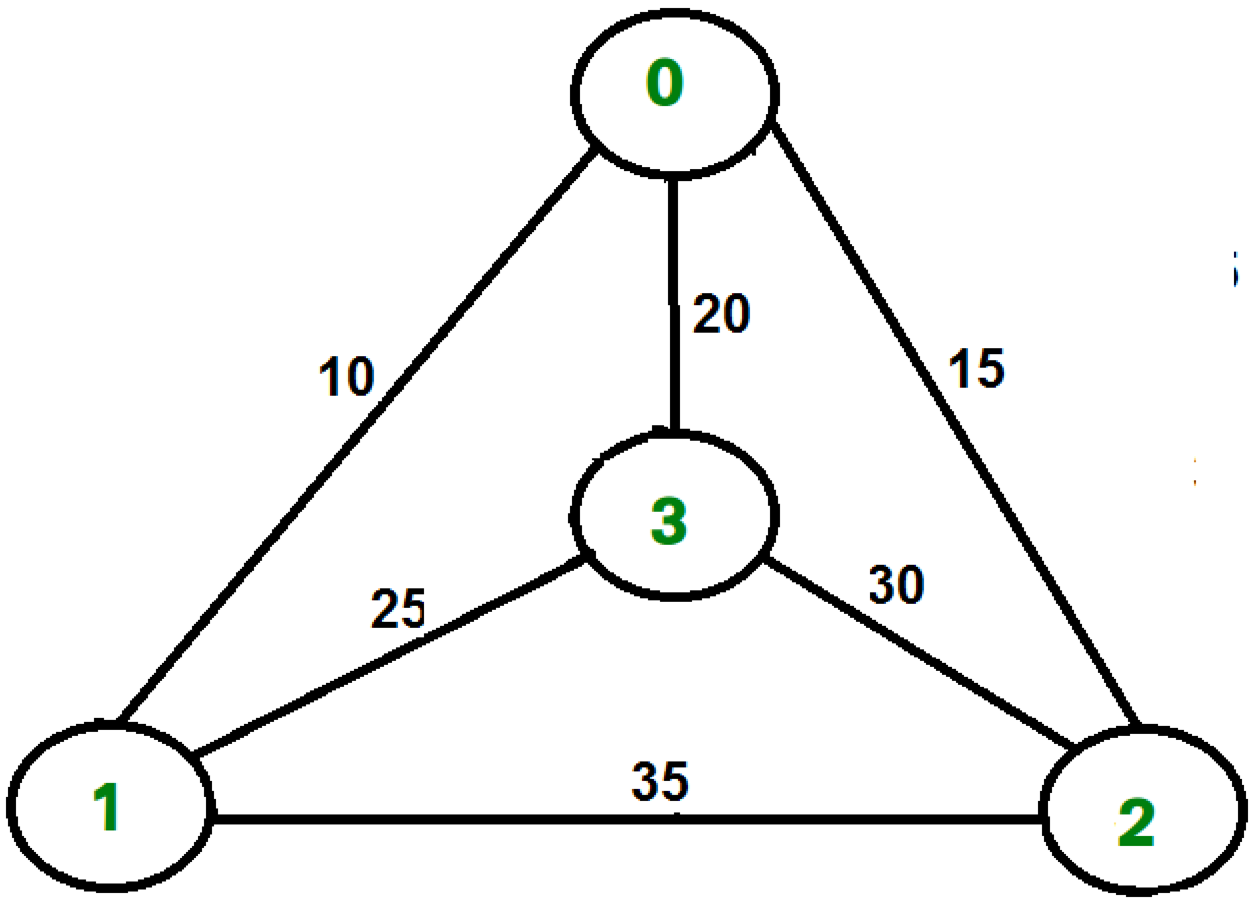
Complexity

- Combinatorial complexity
  - It is related to the number of components in a system or the number of combinations of system components that are possible.

Example:

- The travelling salesman problem
  - A salesperson has to make a series of visits to potential customers during a day. The aim is to find the shortest route around those customers.
  - As the number of cities increases, the number of combinations grows at an increasing rate.
  - The problem is subject to combinatorial complexity.

Given a set of cities and distance between every pair of cities, the problem is to find the shortest possible tour that visits every city exactly once and returns to the starting point.



A TSP tour in the graph is 0-1-3-2-0. The cost of the tour is 10+25+30+15 which is 80.

In general, for an n city TSP, there are $(n-1)!$ possible routes the salesman can take. The following Table shows the value of $(n-1)!$ for several n.

Alternative Solutions in TSP

n	$(n-1)!$
3	2
5	24
9	40,320
13	479,001,600
17	20,922,789,888,000
20	121,645,100,408,832,000

Sarker, R.A. and Newton, C.S., (2008), Optimization Modelling: A Practical Approach, CRC Press.

	Job 1	Job 2	Job 3	Job 4
A	9	2	7	8
B	6	4	3	7
C	5	8	1	8
D	7	6	9	4

Worker A takes 8 units of time to finish job 4.

An example job assignment problem. Green values show optimal job assignment that is A-Job4, B-Job1 C-Job3 and D-Job4

The number of feasible combinations of matching can be calculated as follows:
Theoretically, there are n! = 3,628,800 different combinations (where n = 10).

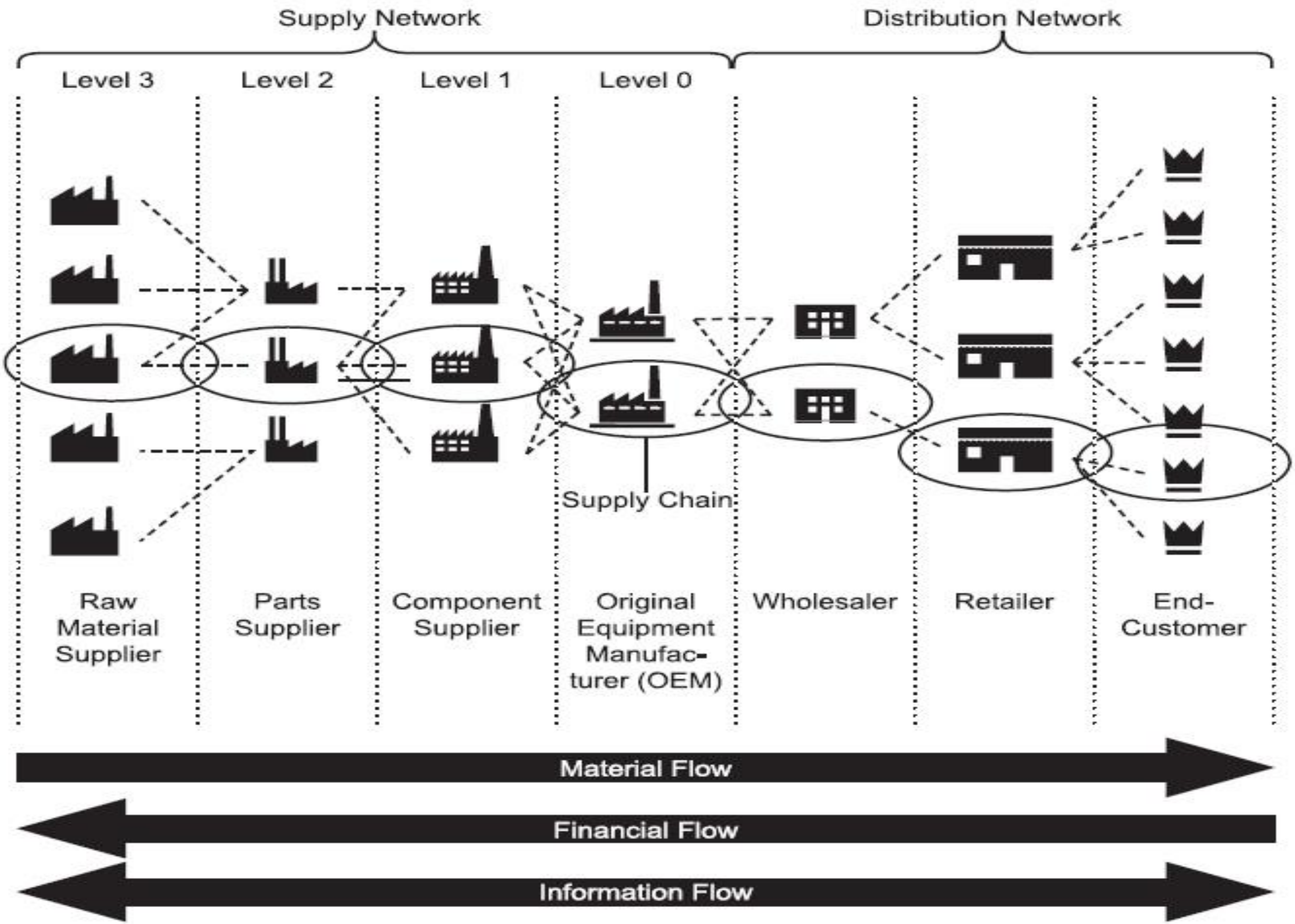
If n = 11 (a 10% increase in the number of candidates and jobs), the number of alternative matchings increases by 1,100%, i.e., to 39,916,800.

If n = 12 (a 20% increase), the number of alternative matchings increases by 13,200%, i.e., to 480,000,000

- Dynamic complexity
  - It is not necessarily related to size.
  - It arises from the interaction of components in a system over time.
  - This can occur in systems that are small, as well as large.
  - Systems that are highly interconnected are likely to display dynamic complexity.

Example:

- A supply chain consisting of a retailer, wholesaler and factory.
- The retailer orders an item from the wholesaler, who in turn orders from the factory.
- There is a delay between placing an order and receiving the item.
- A small perturbation in the number of items sold by the retailer can cause large shifts in the quantity stored and produced by the wholesaler and factory respectively.
- The system is subject to dynamic complexity.



Supply Network
Distribution Network
Level 3
Level 2
Level 1
Level 0
Supply Chain
Raw Material Supplier
Parts Supplier
Component Supplier
Original Equipment Manufac-turer (OEM)
Wholesaler
Retailer
End-Customer
Material Flow
Financial Flow
Information Flow

Linear and Non-linear systems

- A system is linear if it satisfies the super position principle.
- A system satisfies the superposition principle if the following conditions are satisfied:
  - Multiplying the input by any constant, multiplies the output by the same constant. (homogeneity)
  - The response to several inputs applied simultaneously is the sum of individual response to each input applied separately. (additivity)

Linear Equation

- **A mathematical equation is said to be linear if the following properties hold.**
  - **homogeneity (**i.e. the output of a linear system is always directly proportional to the input)
  - **additivity (**i.e. the output corresponding to the sum of any two inputs is the sum of the two outputs)

Examples:

y = 4x

y = 4x + 2

(Test for x =1; x = 2)

Examples

- **$y = 4x$**

Let $x = x_1$, then $y_1 = 4x_1$

Let $x = x_2$, then $y_2 = 4x_2$

Then $y = y_1 + y_2 = 4x_1 + 4x_2$ (1)

Also, we note,

$y = f(x_1 + x_2) = 4(x_1 + x_2) = 4x_1 + 4x_2$ (2)

Since Equations (1) and (2) are identical, the additivity property holds.

Examples

Given, $y = 4x + 2$.

Let $x = x_1$, then $y_1 = 4x_1 + 2$

Let $x = x_2$, then $y_2 = 4x_2 + 2$

Then $y = y_1 + y_2 = 4x_1+2 + 4x_2+2 = 4(x_1+x_2) + 4$ (3)

Also,

$y = f(x_1 + x_2) = 4(x_1 + x_2) + 2$ (4)

Since Equations (3) and (4) are not identical, the additivity property does not hold.

The Fixed-Charge Problem

- In a typical production planning problem involving n products, the production cost for product j may consist of a variable per-unit cost c_j , and a fixed cost (charge) K_j (>0), which occurs only if product j is produced.
- Thus, if x_j = the production level of product j, then its production cost $C_j\ (x_j\ )$ is

$$C_j\ (x_j) = \begin{cases} K_j + c_j x_j, & x_j > 0 \\ 0, & x_j = 0 \end{cases}$$

(Ref: Ravindran, A.R. *Operations Research Methodologies*, CRC Press, 2009).

This cost function is depicted in Figure 3.1. The objective would be to minimize $Z = \sum_j C_j(x_j)$. This objective function is nonlinear in variables x_j due to the discontinuity at the origin, and can be converted into a linear function by introducing additional

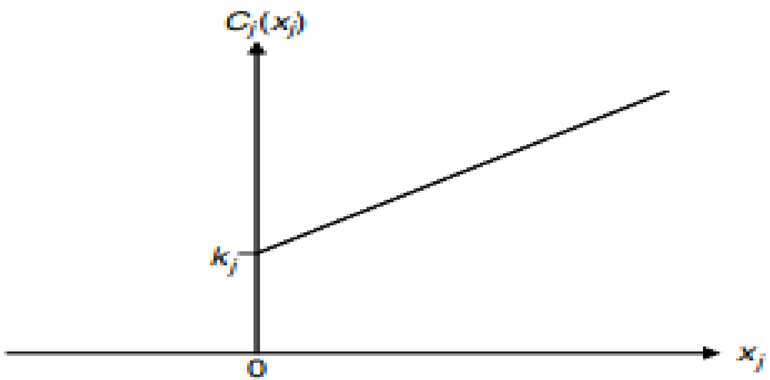


FIGURE 3.1 Fixed-charge cost function.

binary variables as follows. Define

$$y_j = \begin{cases} 1, & x_j > 0 \\ 0, & x_j = 0 \end{cases}$$

This condition can then be expressed as a single linear constraint as

$$x_j \le M y_j$$

where $M\,(>0)$ is sufficiently large to guarantee that constraint $x_j \le M$ is redundant whenever $y_j = 1$. On the other hand, the minimization of the linear objective function $Z = \sum_j (c_j x_j + K_j y_j)$ assures that $y_j = 0$, whenever $x_j = 0$.

Search Basics

- **Search** is the term used for constructing/improving solutions to obtain the optimum or near-optimum.

Solution	Encoding (representing the solution)
Neighborhood	Nearby solutions (in the encoding or solution space)
Move	Transforming current solution to another (usually neighboring) solution
Evaluation	The solutions' feasibility and objective function value

- **Constructive** search techniques
  - work by constructing a solution step by step, evaluating that solution for (a) feasibility and (b) objective function.

- **Improvement** search techniques
  - work by constructing a solution, moving to a neighboring solution, evaluating that solution for (a) feasibility and (b) objective function.

- Search techniques may be *deterministic* (always arrive at the same final solution through the same sequence of solutions, although they may depend on the initial solution).
  - Examples are LP (simplex method), tabu search, simple heuristics like FIFO, LIFO, and greedy heuristics.

- Search techniques may be *stochastic* where the solutions considered and their order are different depending on random variables.
  - Examples are simulated annealing, ant colony optimization and genetic algorithms.

- Search techniques may be *local*, that is, they find the nearest optimum which may not be the real optimum. Example: greedy heuristic (local optimizers).

- Search techniques may be *global*, that is, they find the true optimum even if it involves moving to worst solutions during search (non-greedy).

Heuristics

- Heuristics are rules to search to find optimal or near-optimal solutions.
  - Examples are FIFO, LIFO, earliest due date first, largest processing time first, shortest distance first, etc.

- Heuristics can be *constructive* (build a solution piece by piece) or *improvement* (take a solution and alter it to find a better solution).

- Many constructive heuristics are *greedy* or *myopic*, that is, they take the best thing next without regard for the rest of the solution.

  Example: A constructive heuristic for TSP is to take the nearest city next. An improvement heuristic for TSP is to take a tour and swap the order of two cities.

Meta-Heuristics

An iterative generation process which guides a subordinate heuristic

by combining intelligently different concepts derived from

classical heuristics, artificial intelligence, biological evolution, natural and physical sciences

for exploring and exploiting the search spaces using learning strategies

to structure information in order to find efficiently near-optimal solutions.

- *Osman and Kelly*, 1996

- Meta-heuristics are not tied to any special problem type and are general methods that can be altered to fit the specific problem.
- The inspiration from nature is:

  Simulated Annealing (SA) – molecule/crystal arrangement during cool down

  Evolutionary Computation (EC) – biological evolution; genetic algorithm

  Tabu Search (TS) – long and short term memory

  Ant Colony and **Swarms** - individual and group behavior using communication between agents

Advantages of Meta-Heuristics

- Very flexible
- Often global optimizers
- Often robust to problem size, problem instance and random variables
- May be only practical alternative

Disadvantages of Meta-Heuristics

- Often need problem specific information / techniques
- Optimality (convergence) may not be guaranteed
- Lack of theoretic basis
- Different searches may yield different solutions to the same problem (stochastic)
- Stopping criteria
- Multiple search parameters

Historical Development of Optimization Techniques

- Isaac Newton (1642-1727)
  (The development of differential calculus
  methods of optimization)

- Joseph-Louis Lagrange (1736-1813)
  (Calculus of variations, minimization of functionals,
  method of optimization for constrained problems)

- Augustin-Louis Cauchy (1789-1857)
  (Solution by direct substitution, steepest
  descent method for unconstrained optimization)

• Leonhard Euler (1707-1783)
(Calculus of variations, minimization of functionals)

• Gottfried Leibnitz (1646-1716)
(Differential calculus methods of optimization)

İsim: Gottfired Wilhelm von Leibniz

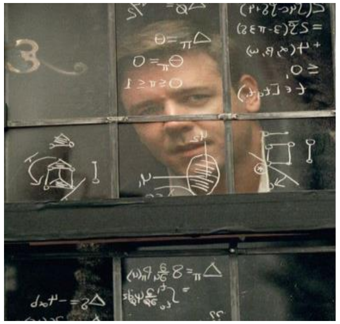
• George Bernard Dantzig (1914-2005)
(Linear programming and Simplex method (1947))

• Richard Bellman (1920-1984)

(Principle of optimality in dynamic programming problems)

• Harold William Kuhn (1925- 2014)

(Necessary and sufficient conditions for the optimal solution of programming problems)



• Albert William Tucker (1905-1995)

(Necessary and sufficient conditions for the optimal solution of programming problems, nonlinear programming, game theory)

- Von Neumann (1903-1957)
  (Game theory)

- Narendra Karmarkar (1957-   )
  (a polynomial algorithm for
  linear programming
  also known as
  the interior point method)

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- D. Nagesh Kumar, Optimization Methods. [online] Available: http://nptel.ac.in/course.php

Optimization Literature - Journals

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- Engineering Optimization
- ASME Journal of Mechnical Design
- ASCE Journal of Structural Engineering
- International Journal for Numerical Methods in Engineering
- Journal of Optimization Theory and Applications
- Computers and Operations Research
- Operations Research
- Management Science

Optimization Literature - Journals

- European Journal of Operational Research
- Journal of Operational Research Society
- Journal of Heuristics
- Applied Mathematical Modelling
- Applied Soft Computing
- International Journal of Mathematical Modelling and Numerical Optimization

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