B180441CS

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05) By Karush-Kuhn-Tucker (KKT) method,

should

maximise $f(x) = 8x_1^2 + \cancel{\ } 2x_2^2$

subject to $x_1^2 + x_2^2 \leq 9$

$x_1 \leq 2.$

Here constraints are

$g_1 = x_1^2 + x_2^2 \leq 9$

and $g_2 = x_1 \leq 2.$

The KKT necessary conditions are:

(a) $\frac{\partial f}{\partial x_i} + \lambda_1 \frac{\partial g_1}{\partial x_i} + \lambda_2 \frac{\partial g_2}{\partial x_i} = 0 \qquad i = 1, 2.$

(b) $\lambda_j [g_j - b_j] = 0 \qquad j = 1, 2$

(c) $\lambda_j \leq 0 \qquad j = 1, 2.$

$\Rightarrow \frac{\partial f}{\partial x_1} + \lambda_1 \frac{\partial g_1}{\partial x_1} + \lambda_2 \frac{\partial g_2}{\partial x_1} = 0.$

$16x_1 + \lambda_1(2x_1) + \lambda_2(1) = 0.$ —— ①

$\frac{\partial f}{\partial x_2} + \lambda_1 \frac{\partial g_1}{\partial x_2} + \lambda_2 \cdot \frac{\partial g_2}{\partial x_2} = 0$

$4x_2 + \lambda_1(2x_2) + \lambda_2(0) = 0$

$4x_2 + 2x_2\lambda_1 = 0$ —— ②.

From (b)

$\lambda_1 + (x_1^2 + x_2^2 - 9) = 0$ —— ⑤

$\lambda_2 + (x_1 - 2) = 0$ —— ⑥

$\Rightarrow \lambda_1 \leq 0$ —— ③

$\Rightarrow \lambda_2 \leq 0$ —— ④

Four case, arises

case 1: $\lambda_1 = 0$ $\lambda_2 = 0$

case 2: $\lambda_1 = 0$ $\lambda_2 \neq 0$

case 3: $\lambda_1 \neq 0$ $\lambda_2 = 0$

case 4: $\lambda_1 \neq 0$ $\lambda_2 \neq 0$.

case 1

$\lambda_1 = 0, \lambda_2 = 0.$

① $\Rightarrow 16x_1 + 2x_1\lambda_1 + \lambda_2 = 0$

$16x_1 = 0$

$\boxed{x_1 = 0}$

② $\Rightarrow 4x_2 + 2x_2\lambda_1 = 0$

$4x_2 = 0$

$\boxed{x_2 = 0}$

$x_1 = 0$
$x_2 = 0.$

~~case 2~~ ~~$\lambda_1 = 0, \lambda_2 \neq 0.$~~

~~From ①~~

put x_1, x_2, λ_1 in ⑤

$9 = 0$

wrong :

$\therefore$ (x_1, x_2) doesn't satisfy ⑤.

case 2

$\lambda_1 = 0, \lambda_2 \neq 0.$

From ⑤

$x_1^2 + x_2^2 = 9$

$x_1^2 = 9$

$\boxed{x_1 = \pm 3}$

From ①

$16x_1 + \lambda_2 = 0$

From ②

$4x_2 = 0$

$\boxed{x_2 = 0}$

~~From~~

put x_1, λ_2 in ①

$16(-3) + (-1) = 0$

doesn't satisfy.

put $x_1 x_2$ in ⑥

$\lambda_2 + 1 = 0$

$\boxed{\lambda_2 = -1}$

case 3

$\lambda_1 \neq 0$

$\boxed{\lambda_2 = 0.}$

put x_2 in 6

$\boxed{x_1 = 2.}$

put x_1 in 1

$16(2) + 4\lambda_1 + \lambda_2 = 0.$

$32 + 4\lambda_1 + \lambda_2 = 0.$

$32 + 4\lambda_1 = 0.$

$\boxed{\lambda_1 = -8.}$

$x_1, \lambda_1.$

put λ_2 in 1

$\cancel{16x_2 + 2x_2\lambda_1 = 0}.$

$4(x_2) + 2x_2(-8) = 0$

$4x_2 - 16x_2 = 0.$

$-12x_2 = 0.$

$\boxed{x_2 = 0.}$

put x_2, x_2, x_1 in 5

It fails.

case 4

$\lambda_1 \neq 0$

$\lambda_2 \neq 0.$

From 5

$\lambda_2 = 2 - x_1$ — 7

From 1

$16x_1 + 2\lambda_1 x_1 + \lambda_2 = 0.$

$\lambda_2 = -(16x_1 + 2\lambda_1 x_1)$

$\boxed{\lambda_2 = -x_1(16 + 2\lambda_1)}$ — 6'

From 2

$4x_2 + 2x_2\lambda_1 = 0.$

$\lambda_1 = \frac{-4x_2}{2x_2}$

$\lambda_1 = -2.$ (or) $x_2 = 0.$

From 5

$\lambda_1 + (x_1^2 + x_2^2 - 9) = 0.$

case: $\lambda_1 = -2$

$x_1^2 + x_2^2 - 9 = 2$

$\boxed{x_1^2 + x_2^2 = 11}$

From 7

$x_1 = 4.$

From 6 (8)

$x_2^2 < 0.$ Impossible.

$\therefore \underline{x_2 = 0}$ ✓.

$\lambda_1 + x_1^2 = 9.$

$\lambda_2 = 2 - x_1.$

$\Rightarrow \boxed{x_1 = 2 - \lambda_1}$

$\Rightarrow \boxed{x_2 = 0}$

$\lambda_2 = \cancel{29}$
$-16x_1 - 2x_1\lambda_1.$

$\lambda_2 = -16x_1 - 2(2-\lambda_1)\lambda_1$

$x_2 = -16(2-\lambda_1) - 2(2-\lambda_1$

(7)

solution will be $(0, \underline{2-\lambda_1})$.

From (7)

$\lambda_2 = -(2-\lambda_1)(16+2\lambda_1)$

$= -(32 - 4\lambda_1 - 16\lambda_1 + 4\lambda_1^2)$

$\lambda_2 = -32 + 4\lambda_1 + 16\lambda_1 - 4\lambda_1^2$

$\boxed{\lambda_2 = -32 + 20\lambda_1 - 4\lambda_1^2}$

$\Rightarrow 2x_1^3 - 33x_1 + 2 = 0.$

$\Rightarrow \boxed{x_1 = -4}$.

$\therefore$ solution $(x_1, x_2) = (-4, \underline{0})$ ✓.

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