classmate
Date
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UNIT - 1

MATHEMATICAL LOGIC

1. Statements and Notations

A statement or proposition is a declarative sentence that is either true or false but not both

Example:

a) The earth is round

b) $2+3=5$

c) $3-x=5$

d) Do you speak English?

e) Take two coins

f) The temperature on the surface of planet Venus is 800°F.

g) The sun will come out tomorrow

Notations:-

Notation	Meaning
$\neg P$	Negation of P
$P \vee Q$	P or Q
$P \wedge Q$	P and Q
$P \rightarrow Q$	P implies Q
$P \leftrightarrow Q$	P if and only if Q
$\Leftrightarrow$	Logical equivalence
$\forall$	For all
$\exists$	there exists
0	False denoted by F
1	True denoted by T

Connectives:-
* Negation ($\neg$ or $\sim$)
* Conjunction ($\wedge$)
* Disjunction ($\vee$)

1. Negation ($\neg$ or $\sim$)

$p = \sim p$

p	~p
T	F
F	T

Example 1:-
If p: I am a boy then

~p: I am not a boy.

Example 2: Give the negation of the following Statements.

(a) p: $2+3>1$

ans:-
~p: 2+3 is not greater than 1

i.e. $\sim p: 2+3<1$

b) Q: It is Cold.

ans:-
~Q: It is not Cold.

2. Conjunction:-
If p & Q are statement the Conjunction of p & Q is the Compound Statement "p and Q", denoted by $p \wedge Q$.

The truth values of $p \wedge Q$ are given in the following truth table.

p	Q	p∧Q
T	T	T
T	F	F
F	T	F
F	F	F

Example:
a) p: It is Snowing, Q: I am Cold.

Ans:-
$p \wedge Q$: It is Snowing and I am Cold.

b) p: $2<3$ Q: $-5>-8$

Ans:-
$p \wedge Q$: $2<3$ and $-5>-8$

3. Disjunction:-

If P & Q are statements the disjunction of P & Q is the compound statement "P or Q" denoted by $P \vee Q$.

Truth values of $P \vee Q$ are given in the following truth table

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Example:

a) P: 2 is a positive integer

Q: $\sqrt{2}$ is a rational number

Ans:-

$P \vee Q$: 2 is a positive integer or $\sqrt{2}$ is a rational number

Since P is true, the disjunction $P \vee Q$ is true, even though Q is false.

4. Conditional statements.

If P & Q are statements the compound statement if P then Q, denoted by $P \to Q$ is called a conditional stmt or implication. The stmt P is called antecedent or hypothesis and the statement Q is called the consequent or conclusion.

Truth values of $P \to Q$ are given in the following truth table.

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Example:

a) P: I am hungry Q: I will eat

Ans:-

$P \to Q$: If I am hungry then I will eat.

If $P \to Q$ is an implication then the converse of $P \to Q$ is the implication $Q \to P$

and the contrapositive of the implication $p \to q$ is the implication $\sim q \to \sim p$.

Example:- Give the converse & the contrapositive of the implication "If it is raining then I get wet."

Solution:-
p: It is raining and
q: I get wet.

The converse is

$q \to p$: If I get wet then it is raining.

The contrapositive is

$\sim q \to \sim p$: If I do not get wet then it is not raining.

5. Biconditional:-

If p and q are statements the compound statement p if and only if q, denoted by $p \leftrightarrow q$ is called a biconditional. $p \leftrightarrow q$ is true only when both p & q are true or when both p & q are false.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example:- Is the following equivalence a true statement?

$3 > 2$ if and only if $0 < 3 - 2$

Solution:-
Let p be the statement $3 > 2$ and let q be the statement $0 < 3 - 2$. Since both p & q are true, we conclude that $p \leftrightarrow q$ is true.

* Statements formulas and Truth Tables :-

P, Q, & R are statements.
There are altogether 2^3 or 8 possible combinations of truth values for P, Q & R.

Step 1: The first n columns of the table are labeled by the component propositional variables. Additional columns are included for all inter-mediate combinations of the variables.

Step 2: Under each of the first n headings we list 2^n possible n-tuples of truth values for the n component statements.

Step 3: For each row, we compute, in sequence, all remaining truth tables.

Example 1: Construct the truth table of the statement $P \wedge \sim Q$

P	Q	$\sim Q$	$P \wedge \sim Q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

A statement that can be either true or false, depending on the truth values of its propositional variables, is called as contingency. The statement $P \wedge \sim Q$ in above example is contingency.

Example 2: Construct truth table of the statement $(P \vee Q) \vee \sim P$

P	Q	$P \vee Q$	$\sim P$	$(P \vee Q) \vee \sim P$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

A statement that is true for all possible values of its propositional variables is called as tautology. The statement $(P \vee Q) \vee \sim P$ in above example is a tautology.

Example 3: Construct truth table of the statement $P \wedge \sim P$

P	$\sim P$	$P \wedge \sim P$
T	F	F
F	T	F

The statement ~~$P \wedge P$~~ $P \wedge \sim P$ in above example is a contradiction

Example 4:- Construct the truth table of the statement
$(P \to Q) \leftrightarrow (\sim Q \to \sim P)$

P	Q	$P \to Q$	$\sim Q$	$\sim P$	$\sim Q \to \sim P$	$(P \to Q) \leftrightarrow (\sim Q \to \sim P)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

$(P \to Q) \leftrightarrow (\sim Q \to \sim P)$ is a tautology

Example 4: Construct the truth table of the statement $(P \to Q) \wedge (P \vee Q)$

P	Q	$P \to Q$	$P \vee Q$	$(P \to Q) \wedge (P \vee Q)$
T	T	T	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	F	F

$(P \to Q) \wedge (P \vee Q)$ is a contingency.

Example 5: Construct the truth table of the statement $(P \to Q) \wedge (Q \to P)$

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \wedge (Q \to P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Example 6: Construct the truth table of the statement
$\sim (P \wedge Q) \leftrightarrow (\sim P \vee \sim Q)$

P	Q	1 $P \wedge Q$	2 $\sim (P \wedge Q)$	3 $\sim P$	4 $\sim Q$	5 $\sim P \vee \sim Q$	6 $2 \leftrightarrow 5$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

Above example is tautology.

* Well Formed Formulas (Wff)

A statement formula is an expression which is a string consisting of variables, parenthesis and connective symbols. Not every string of these symbols is a formula. We shall now give a recursive definition of a statement formula, often called a well formed formula.

A well formed formula can be generated by the following rules:

1. A statement variable standing alone is a well formed formula.
2. If A is a well formed formula then $\sim A$ is a well formed formula.

3. If A & B are well formed formula then $(A \wedge B)$, $(A \vee B)$, $(A \to B)$ & $(A \leftrightarrow B)$ are well formed formulas.
4. A string of symbols containing the statement variables, connectives and parenthesis is a well formed formula, iff it can be obtained by finitely many applications of the rules 1, 2 & 3.

Example:-
i) $\sim(P \wedge Q)$
ii) $\sim(P \vee Q)$
iii) $P \to (P \vee Q)$
iv) $P \to (P \to R)$
v) $(((P \to Q) \wedge (Q \to R)) \leftrightarrow (P \to R))$

The following are not well formed formulas

i) $\sim P \wedge Q$. P & Q are well formed formula. A wff would be either $(\sim P \wedge Q)$ or $\sim(P \wedge Q)$

ii) $(P \to Q) \to (\wedge Q)$. This is not a wff because $\wedge Q$ is not.

iii) $(P \to Q$.
iv) $(P \wedge Q) \to Q)$.

Q. Equivalence of Formulas.

Assume A & B are two statement formulas then formula A is equivalent to formula B if and only if the truth values of formula A is same to the truth values of formula B for all possible interpretations. Equivalence of formula A & formula B is denoted as $A \Leftrightarrow B$.

Example:-
i) $\sim\sim P$ is equivalent to P.
ii) $P \vee P$ is equivalent to P.
iii) $(P \vee \sim P) \vee Q$ is equivalent to Q.
iv) $P \vee \sim P$ is equivalent $Q \vee \sim Q$

Example:- Prove $(P \to Q) \Leftrightarrow (\sim P \vee Q)$

P	Q	$P \to Q$	$\sim P$	$\sim P \vee Q$	$(P \to Q) \Leftrightarrow (\sim P \vee Q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

* Negation Laws

$\sim(\sim P) \Leftrightarrow P$

* Idempotent Laws

$P \vee P \Leftrightarrow P$

$P \wedge P \Leftrightarrow P$

* Commutative Laws

$P \vee Q \Leftrightarrow Q \vee P$

$P \wedge Q \Leftrightarrow Q \wedge P$

* Associative Laws

$P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$

$P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$

* Distributive Laws

$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

* Absorption Laws

- $P \wedge T \Leftrightarrow P$
- $P \wedge F \Leftrightarrow F$
- $P \wedge \sim P \Leftrightarrow F$
- $P \wedge (P \vee Q) \Leftrightarrow P$

* De Morgan's Laws

$\sim(P \vee Q) \Leftrightarrow (\sim P) \wedge (\sim Q)$

$\sim(P \wedge Q) \Leftrightarrow (\sim P) \vee (\sim Q)$

Example 1. Show that

$P \to (Q \to R) \Leftrightarrow P \to (\sim Q \vee R) \Leftrightarrow (P \wedge Q) \to R$

without using truth table.

Solution:-

$P \to (Q \to R)$

$\Leftrightarrow P \to (\sim Q \vee R)$ $\quad (\because Q \to R \Leftrightarrow \sim Q \vee R)$

$\Leftrightarrow \sim P \vee (\sim Q \vee R)$ $\quad (\because P \to Q \Leftrightarrow \sim P \vee Q)$

$\Leftrightarrow (\sim P \vee \sim Q) \vee R$ $\quad (\because P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R)$

$\Leftrightarrow \sim(P \wedge Q) \vee R$ $\quad (\because (\sim P) \vee (\sim Q) \Leftrightarrow \sim(P \wedge Q))$

$\Leftrightarrow (P \wedge Q) \to R$ $\quad (\because \sim P \vee Q \Leftrightarrow P \to Q)$

Example 2: Show that

$((P \vee Q) \wedge \sim(\sim P \wedge (\sim Q \vee \sim R))) \vee (\sim P \wedge \sim Q) \vee (\sim P \wedge \sim R)$ is a tautology.

Solution:-

Using De Morgan's laws, we obtain

$(\sim P) \wedge (\sim Q) \Leftrightarrow \sim(P \vee Q)$ --- (1)

$(\sim P) \wedge (\sim R) \Leftrightarrow \sim(P \vee R)$ ---- (2)

$(\sim P \wedge \sim Q) \vee (\sim P \wedge \sim R) \Leftrightarrow$

$\sim(P \vee Q) \vee \sim(P \vee R)$ by (1) & (2)

$\Leftrightarrow \sim((P \vee Q) \wedge (P \vee R))$

Also

$\sim(\sim P \wedge (\sim Q \vee \sim R))$

$\Leftrightarrow \sim(\sim P \wedge \sim(Q \wedge R))$

$\Leftrightarrow \sim\sim(P \vee (Q \wedge R))$

$\Leftrightarrow (P \vee (Q \wedge R))$

$\Leftrightarrow (P \vee Q) \wedge (P \vee R)$

$(P \vee Q) \wedge ((P \vee Q) \wedge (P \vee R))$

$\Leftrightarrow ((P \vee Q) \wedge (P \vee Q)) \wedge (P \vee R)$

$\Leftrightarrow (P \vee Q) \wedge (P \vee R)$

The given formula is equivalent to

$((P \vee Q) \wedge (P \vee R)) \vee \sim((P \vee Q) \wedge (P \vee R))$

which is a substitution instance of $P \vee \sim P$ whose table is T. Hence it is tautology.

Example 2. Show that $(\sim P \wedge (\sim Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$ without using truth table.

Solution:

$(\sim P \wedge (\sim Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$

by distributive law

$\Leftrightarrow (\sim P \wedge (\sim Q \wedge R)) \vee ((Q \vee P) \wedge R)$

by Associative law

$\Leftrightarrow ((\sim P \wedge \sim Q) \wedge R) \vee ((Q \vee P) \wedge R)$

by distributive law

$\Leftrightarrow (((\sim P \wedge \sim Q) \vee (Q \vee P)) \wedge R)$

by Demorgan's Law & Commutative

$\Leftrightarrow ((\sim(P \vee Q) \vee (P \vee Q)) \wedge R)$

by $P \vee \sim P \Leftrightarrow T$

$\Leftrightarrow (T \wedge R)$

by $P \wedge T \Leftrightarrow P$

$\Leftrightarrow R$

b. Duality Law

Two formulas, A and A^* are said to be duals of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. The connectives $\wedge$ & $\vee$ are also called duals of each other. If the formula A contains the special variables T or F then A^* its dual is obtained by replacing T by F and F by T in addition to the above mentioned interchanges.

Example: write the duals of

(a) $(P \vee Q) \wedge R \quad = (P \wedge Q) \vee R$

(b) $(P \wedge Q) \vee T \quad = (P \vee Q) \wedge F$

(c) $\sim(P \vee Q) \wedge (P \vee \sim(Q \wedge \sim S)) =$

$\sim(P \wedge Q) \vee (P \wedge \sim(Q \vee \sim S))$

Theorem 1: Let A and A^* dual formulas and let $P_1, P_2, \ldots, P_n$ be all the atomic variables that occur in A and A^*. That is to say, we may write A as $A(P_1, P_2, \ldots, P_n)$ and A^* as $A^*(P_1, P_2, \ldots, P_n)$. Then though the use of De Morgan's Laws

$P \wedge Q \Leftrightarrow \neg(\neg P \vee \neg Q)$

$P \vee Q \Leftrightarrow \neg(\neg P \wedge \neg Q)$

we can show

$$\neg A(P_1, P_2, \ldots, P_n) \Leftrightarrow A^*(P_1, P_2, \ldots, P_n) \quad \ldots (1)$$

Thus the negation of a formula is equivalent to its dual in which every variable is replaced by its negation. As a consequence of this fact, we also have

$$A(\neg P_1, \neg P_2, \ldots, \neg P_n) \Leftrightarrow A^*(\neg P_1, \neg P_2, \ldots, \neg P_n) \quad \ldots (2)$$

Theorem 2: Let $P_1, P_2, \ldots, P_n$ be all the atomic variables appearing in the formulas A & B. Given that $A \Leftrightarrow B$ means "$A \leftrightarrow B$" is a tautology then the following are also tautologies.

$A(P_1, P_2, \ldots, P_n) \leftrightarrow B(P_1, P_2, \ldots, P_n)$

$A(\neg P_1, \neg P_2, \ldots, \neg P_n) \leftrightarrow B(\neg P_1, \neg P_2, \ldots \neg P_n)$

Using (2), we get

$\neg A^*(P_1, P_2, \ldots, P_n) \Leftrightarrow \neg B^*(P_1, P_2, \ldots P_n)$

Hence $A^* \Leftrightarrow B^*$

Example 1: Show that $\neg(P \wedge Q) \to (\neg P \vee (\neg P \vee Q))$
$\Leftrightarrow (\neg P \vee Q)$

Solution:-

by $P \to Q \Leftrightarrow \neg P \vee Q$

$\Leftrightarrow \neg\neg(P \wedge Q) \vee (\neg P \vee (\neg P \vee Q))$

$\Leftrightarrow (P \wedge Q) \vee (\neg P \vee (\neg P \vee Q))$

by Commutative law

$\Leftrightarrow (P \wedge Q) \vee ((\neg P \vee \neg P \vee Q)$

$\Leftrightarrow (P \wedge Q) \vee (\neg P \vee Q)$

$\Leftrightarrow (P \wedge Q) \vee \neg P \vee Q$

$\Leftrightarrow ((P \vee \neg P) \wedge (Q \vee \neg P)) \vee Q$

$\Leftrightarrow (T \wedge (Q \vee \neg P)) \vee Q \quad \because T \wedge P \Leftrightarrow P$

$\Leftrightarrow (Q \vee \neg P) \vee Q$ (Absorption law)

$\Leftrightarrow (Q \vee Q) \vee (Q \vee \sim P)$

$\Leftrightarrow Q \vee (Q \vee \sim P)$

$\Leftrightarrow Q \vee \sim P$ by Commutative law

$\Leftrightarrow \sim P \vee Q$

By Commutative law it follows that

$(P \wedge Q) \vee (\sim P \vee (\sim P \vee Q)) \Leftrightarrow \sim P \vee Q$

Writing the duals, we obtain by applying theorem 2, we get

$(P \vee Q) \wedge (\sim P \wedge (\sim P \wedge Q)) \Leftrightarrow \sim P \wedge Q$

Example 2: Show that $P \to Q \Rightarrow P \to (P \wedge Q)$

Solution:-

$P \to (P \wedge Q)$ $\quad \because P \to Q \Leftrightarrow \sim P \vee Q$

$\Leftrightarrow \sim P \vee (P \wedge Q)$ by distributive law

$\Leftrightarrow (\sim P \vee P) \wedge (\sim P \vee Q)$ $\quad \because P \vee \sim P \Leftrightarrow T$

$\Leftrightarrow T \wedge (\sim P \vee Q)$

$\Leftrightarrow T \wedge (P \to Q)$ $\quad \because T \wedge P \Leftrightarrow P$

$\Leftrightarrow P \to Q$ Absorption law

$\Leftrightarrow$ L.H.S

Example 3: Show that $(P \to Q) \to Q \Rightarrow P \vee Q$

Solution: $(P \to Q) \to Q$ $\quad (\because P \to Q \Leftrightarrow \sim P \vee Q)$

$\Leftrightarrow (\sim P \vee Q) \to Q$ $\quad (\because P \to Q \Leftrightarrow \sim P \vee Q)$

$\Leftrightarrow \sim(\sim P \vee Q) \vee Q$

$\Leftrightarrow (\sim\sim P \wedge \sim Q) \vee Q$

$\Leftrightarrow (P \wedge \sim Q) \vee Q$ $\quad \because$ (By distributive law)

$\Leftrightarrow (Q \vee P) \wedge (Q \vee \sim Q)$

$\Leftrightarrow (Q \vee P) \wedge T$ $\quad \because Q \vee \sim Q \Leftrightarrow T$

$P \wedge T \Leftrightarrow P$ (Absorption law)

$\Leftrightarrow Q \vee P$

$\Leftrightarrow P \vee Q$ by Commutative law

$\Leftrightarrow$ R.H.S

~~Soln~~ **Example 4**: Show that

$(Q \to (P \wedge \sim P)) \to (R \to (P \wedge \sim P)) \Rightarrow (R \to Q)$

Solution:- $(Q \to (P \wedge \sim P)) \to (R \to (P \wedge \sim P))$

$\Leftrightarrow (Q \to F) \to (R \to F)$ $\quad \because P \wedge \sim P = F$

$\because Q \to F \Leftrightarrow \sim Q \vee F$

$\Leftrightarrow (\sim Q \vee F) \to (\sim R \vee F)$

$\Leftrightarrow (\sim\sim Q \wedge T) \to (\sim\sim R \wedge T)$

$\Leftrightarrow (Q \wedge T) \to (R \wedge T)$ Absorption law $\quad \because P \wedge T \Leftrightarrow P$

$\Leftrightarrow Q \to R$

$\Leftrightarrow (R \to Q)$

Examples: Show that
$((P \vee \sim P) \to Q) \to ((P \vee \sim P) \to R) \Leftrightarrow (Q$

Solution: - $((P \vee \sim P) \to Q) \to ((P \vee \sim P) \to$
$\Leftrightarrow (T \to Q) \to (T \to R)$
$\Leftrightarrow (F \vee Q) \to (F \vee Q)$
$\Leftrightarrow Q \to R$
$\Leftrightarrow$ R.H.S.

c. Tautological Implications

The Connectives $\vee$, $\wedge$ & $\leftrightarrow$ are Symmetric in the Sense that $P \wedge Q \Leftrightarrow Q \wedge P$, $P \vee Q \Leftrightarrow Q \vee P$ and $P \leftrightarrow Q \Leftrightarrow Q \leftrightarrow P$. In Contras $P \to Q$ is not equivalent to $Q \to P$.

For any statement formula $P \to Q$ the statement formula $Q \to P$ is called its converse $\sim P \to \sim Q$ is called its inverse and $\sim Q \to \sim P$ is called its Contra positive.

A statement 'A' is said to tautologically imply a Statement B if and only if $A \to B$ is a tautology. We shal denote this idea by $A \Rightarrow B$ which is read as "A implies B".

d. Functionally Complete Sets of Connectives.

Any set of Connectives in which every formula can be expressed in terms of an equivalent formula containing the Connectives from this set is called a functionally complete set of Connectives.

$P \leftrightarrow Q \Leftrightarrow (P \to Q) \wedge (Q \to P)$

Example 1: - Write an equivalent formula for $P \wedge (Q \leftrightarrow R)$ which does not contain the bidirectional nor the Conditional.

Solution: -
$P \wedge (Q \leftrightarrow R) \Leftrightarrow P \wedge ((Q \to R) \wedge (R \to Q))$
$\Leftrightarrow P \wedge ((\sim Q \vee R) \wedge (\sim R \vee Q))$

Thus the equivalent formula is $(P \wedge (\sim Q \vee R) \wedge (\sim R \vee Q))$

Example 2: - Write an equivalent formula for $P \wedge (Q \leftrightarrow R) \vee (R \leftrightarrow P)$ which does not contain the bidirectional.

Solution: - $P \wedge (Q \leftrightarrow R) \vee (R \leftrightarrow P)$

Example 2. Obtain the principal disjunctive normal form of
$P \to ((P \to Q) \land \sim(\sim Q \lor \sim P))$

solution:- Using $P \to Q \Leftrightarrow \sim P \lor Q$ and De Morgan's law, we obtain

$\Leftrightarrow \sim P \lor ((\sim P \lor Q) \land (Q \land P))$

$\Leftrightarrow \sim P \lor (\sim P \land (Q \land P)) \lor (Q \land (Q \land P))$

$\Leftrightarrow$

Ex. $(P \to Q) \to Q \Leftrightarrow P \lor Q$

$\Rightarrow$ $\sim(P \to Q) \lor Q$ by Implification

$\sim(\sim P \lor Q) \lor Q$ by Implification

$(P \land \sim Q) \lor Q$ Demorgan's Double negation

$\cancel{P \lor (\sim Q \lor Q)}$ Distributive law

$(P \lor Q) \land (\sim Q \lor Q)$

$(P \lor Q) \land T$ Negation laws

$(P \lor Q)$ Identity

Ex. $\sim(\sim((P \lor Q) \land R) \lor \sim Q) \Leftrightarrow Q \land R$

$\sim(\sim(((P \lor Q) \land R) \land Q)$ Demorgan's Double negation

$((P \lor Q) \land R) \land Q$ by Associative

$((P \lor Q) \land Q) \land R$ Absorption law

$Q \land R$

Ex. $[(P \lor \sim P) \to Q] \to [(P \lor \sim P) \to R] \Leftrightarrow Q \to R$

Negation law

$(T \to Q) \to (T \to R)$ Implification

$(\sim T \lor Q) \to (\sim T \lor R)$

$(F \lor Q) \to (F \lor R)$ Identity law

$Q \to R$

Ex. $(Q \to (P \land \sim P)) \to (R \to (P \land \sim P)) \Leftrightarrow R \to Q$

Negation law

$(Q \to F) \to (R \to F)$ Implification law

$(\sim Q \lor F) \to (\sim R \lor F)$ Identity

$(\sim Q) \to (\sim R)$ Implification

$\sim(\sim Q) \lor (\sim R)$ Negation

$Q \lor \sim R$ Cummutative

$\sim R \lor Q$ $R \to Q$ Implification

Ex. $((Q \wedge A) \to C) \wedge (A \to (P \vee C)) \Leftrightarrow$
$(A \wedge (P \to Q)) \to C$

$\Rightarrow ((Q \wedge A) \to C)$ Implification

$= \sim(Q \wedge A) \vee C$ Demorgan's

$= \sim Q \vee \sim A \vee C$ ~~Ass~~

$= \sim A \vee \sim Q \vee C$

$(A \to (P \vee C))$ Implification

$= \sim A \vee (P \vee C)$ Associative

$= \sim A \vee C \vee P$

$= \sim A \vee P \vee C$

$(\sim A \vee C \vee \sim Q) \wedge (\sim A \vee C \vee P)$

$= (\sim A \vee \sim Q \vee C) \wedge (\sim A \vee P \vee C)$

$= ((\sim A \vee \sim Q) \vee C) \wedge ((\sim A \vee P) \vee C)$ Distributive

$= ((\sim A \vee \sim Q) \wedge (\sim A \vee P)) \vee C$ Distributive

$= C \vee (\sim A \vee (\sim Q \wedge P))$ Double nega

$= C \vee (\sim A \vee \sim(\sim(\sim Q \wedge P)))$ Demorgan's

$= C \vee (\sim A \vee \sim(Q \vee \sim P))$ Commutative

$= C \vee (\sim A \vee \sim(\sim P \vee Q))$ Implification

$= C \vee (\sim A \vee \sim(P \to Q))$ Double nega

$= C \vee \sim(\sim(\sim A \vee (P \to Q)))$

$= C \vee \sim(A \wedge \sim(P \to Q))$

$= C \vee \sim(\sim(\sim A \vee \sim(P \to Q)))$ Demorgan's

$= C \vee \sim(\sim(\sim(A \wedge (P \to Q))))$ Double negation

$= C \vee \sim(A \wedge (P \to Q))$

$= \sim(A \wedge (P \to Q)) \vee C$ Implification

$= (A \wedge (P \to Q)) \to C$

Ex. $(P \vee Q) \vee (\sim P \wedge \sim Q \wedge R) \Leftrightarrow P \vee Q \vee R$

$(P \vee Q) \vee (\sim P \wedge \sim Q \wedge R)$ Associative

$(P \vee Q) \vee ((\sim P \wedge \sim Q) \wedge R)$

$(P \vee Q) \vee (\sim(P \vee Q) \wedge R)$ Demorgan's

Distributive

$((P \vee Q) \vee \sim(P \vee Q)) \wedge ((P \vee Q) \vee R)$

$T \wedge (P \vee Q \vee R)$ Negation

Identity

$(P \vee Q \vee R)$

Duality Law

Ex. $P \to Q$

$\sim P \vee Q$

Duality

$\sim P \wedge Q$

Implication

$(P \to Q) \to R$

$\sim(P \to Q) \vee R$

$\sim(\sim P \vee Q) \vee R$

$(P \wedge \sim Q) \vee R = (P \vee \sim Q)$

Ex. Verify the principle of duality for the logical equivalence

$\sim(P \wedge Q) \to (\sim P \vee (\sim P \vee Q)) \Leftrightarrow (\sim P \vee Q)$

$\sim(\sim(P \wedge Q)) \vee (\sim P \vee (\sim P \vee Q))$

$(P \wedge Q) \vee ((\sim P \vee \sim P) \vee Q))$

$(P \wedge Q) \vee (\sim P \vee Q)$

$(P \wedge Q) \vee \sim P \vee Q$

$((P \vee \sim P) \wedge (Q \vee \sim P)) \vee Q$

$(T \wedge (Q \vee \sim P)) \vee Q$

$(Q \vee \sim P) \vee Q$

$(Q \vee Q \vee \sim P)$

$(\sim P \vee Q)$

duality

L.H.S $(\sim P \wedge Q)$

R.H.S $= \sim P \vee Q$

$= \sim P \wedge Q$

L.H.S = R.H.S

∴ duality is equivalent

Tautological Implication

Ex. $\underbrace{P \wedge (P \to Q)}_{A} \Rightarrow \underbrace{Q}_{B}$

$\Rightarrow B$

$P \wedge (P \to Q) \to Q$

P	Q	$P \to Q$	$P \wedge (P \to Q) = A$	$A \Rightarrow B$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Ex. $\underbrace{(P \to Q) \wedge (Q \to R)}_{A} \Rightarrow \underbrace{(P \to R)}_{B}$

P	Q	R	$P \to Q$	$Q \to R$	$(P \to Q) \wedge (Q \to R)$ (A)	$(P \to R)$ (B)	$A \to B$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

$A \Rightarrow B$

rules of Inference

1. Ex Demonstrate that R is a valid inference from the premises $P \to Q$, $Q \to R$ and P, $\sim P \vee Q$

{1} (1) P Rule P
{2} (2) $P \to Q$ Rule P
{1,2} (3) Q Rule T (1),(2) $P, P \to Q \Rightarrow Q$
{4} (4) $Q \to R$ Rule P
{1,2,4} (5) R Rule T (3),(4) $P, P \to Q \Rightarrow Q$

$P \to Q$, $Q \to R$
$P \to R$, $P \Rightarrow R$

2 Ex. show that $R \vee S$ follows logically from the premises
$C \vee D$, $(C \vee D) \to \sim H$, $\sim H \to (A \wedge \sim B)$ and $(A \wedge \sim B) \to R \vee S$

{1} (1) $(C \vee D) \to \sim H$ Rule P
{2} (2) $\sim H \to (A \wedge \sim B)$ Rule P
{1,2} (3) $(C \vee D) \to (A \wedge \sim B)$ Rule T (1),(2) $P \to Q$, $Q \to R \Rightarrow$
{4} (4) $(A \wedge \sim B) \to R \vee S$ Rule P
{1,2,4} (5) $(C \vee D) \to R \vee S$ Rule T $P \to Q, Q \to R \Rightarrow$
{6} (6) $C \vee D$ Rule P
{1,2,4,6} (7) $R \vee S$ Rule T $P, P \to Q$

$\sim(P \to Q)$
$\sim(\sim P) \vee Q$
$P \vee Q$

3. Ex. Show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \to R) \wedge (Q \to S)$

{1} (1) $P \vee Q$ Rule P
{1} (2) $\sim P \to Q$ Rule T $P \to Q \Rightarrow \sim P \vee Q$, $\sim(\sim P) \Leftrightarrow P$
{3} (3) $Q \to S$ Rule P
{1,3} (4) $\sim P \to S$ Rule T $P \to Q$, $Q \to R \Rightarrow$ (2),(3) $P \to R$
{1,3} (5) $\sim S \to P$ Rule T $P \to Q \Leftrightarrow \sim Q \to \sim P$
{6} (6) $P \to R$ Rule P
{1,3,6} (7) $\sim S \to R$ Rule T $P \to Q$, $Q \to R$ (5),(6) $\Rightarrow P \to R$
{1,3,6} (8) $\sim(\sim S) \vee R$ Rule T $P \to Q \Leftrightarrow \sim P \vee Q$
{1,3,6} (9) $S \vee R$ Rule T $\sim(\sim P) = P$

4. Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \to R$, $P \to M$ and $\sim M$

$\Rightarrow$

{1}	(1)	$P \to M$	Rule P
{2}	(2)	$\sim M$	Rule P
{1,2}	(3)	$\sim P$	Rule T (1),(2) $\sim Q, P \to Q \Rightarrow \sim P$
{4}	(4)	$P \vee Q$	Rule P
{1,2,4}	(5)	Q	Rule T (3),(4) $\sim P, P \vee Q \Rightarrow Q$
{6}	(6)	$Q \to R$	Rule P
{1,2,4,6}	(7)	R	Rule T (5),(6) $Q, Q \to Q \Rightarrow Q$
{1,2,4,6}	(8)	$R \wedge (P \vee Q)$	Rule T (4),(7) $P, Q \Rightarrow P \wedge Q$

5. Show $\sim Q, P \to Q \Rightarrow \sim P$

$\Rightarrow$

{1}	(1)	$P \to Q$	Rule P
{1}	(2)	$\sim Q \to \sim P$	Rule T $P \to Q \Leftrightarrow \sim Q \to \sim P$
{3}	(3)	$\sim Q$	Rule P
{1,3}	(4)	$\sim P$	Rule T (2),(3) $P, P \to Q \Rightarrow Q$

6. Show that $R \to S$ can be derived from the premises $P \to (Q \to S)$, $\sim R \vee P$ and Q.

$\Rightarrow$ R is additional premise & show first S.

{1}	(1)	R	Rule P (Additional)
{2}	(2)	$\sim R \vee P$ $(R \to P)$	Rule P
{1,2}	(3)	P	Rule T (1),(2) ~~$\sim P, R \vee Q \Rightarrow Q$~~ $P, P \to Q \Rightarrow Q$
{4}	(4)	$P \to (Q \to S)$	Rule P
{1,2,4}	(5)	$Q \to S$	Rule T (3),(4) $P, P \to Q \Rightarrow Q$
{6}	(6)	Q	Rule P
{1,2,4,6}	(7)	S	Rule T (5),(6) $P, P \to Q \Rightarrow Q$
{1,2,4,6}	(8)	$R \to S$	Rule CP

7. $(P\to Q), (Q\to \sim R), R$ and $(P\vee (T\wedge S))$
$\Rightarrow T\wedge S$

{1}	(1)	$P\to Q$	Rule P
{2}	(2)	$Q\to \sim R$	Rule P
{1,2}	(3)	$P\to \sim R$	Rule T (1), (2) $(P\to Q, Q\to R \Rightarrow P\to R)$
{1,2}	(4)	$R\to \sim P$	Rule T(3) contrapositive $P\to Q \Leftrightarrow \sim Q \to \sim P$
{5}	(5)	R	Rule P
{1,2,5}	(6)	$\sim P$	Rule T (4), (5) $P, P\to Q \Rightarrow Q$
{7}	(7)	$(P\vee (T\wedge S))$	Rule P
{1,2,5,7}	(8)	$(T\wedge S)$	Rule T (6), (7) $\sim P, P\vee Q \Rightarrow Q$

8. ~~Show that~~ Derive $P\to (Q\to S)$ using ~~E~~ Rule CP from
$P\to (Q\to R), Q\to (R\to S)$

{1}	(1)	P	Rule P (Assumed)
{2}	(2)	$P\to (Q\to R)$	Rule P
{1,2}	(3)	$Q\to R$	Rule T (1) (2) $P, P\to Q \Rightarrow Q$
{1,2} ~~{4}~~	(4)	~~$Q\to (R\to S)$~~ $\sim Q\vee R$	~~Rule P~~ Rule T (3), (4) $P\to Q \Leftrightarrow \sim P\vee Q$
{5}	(5)	$Q\to (R\to S)$	Rule P
{5}	(6)	$\sim Q\vee (\sim R\vee S)$	Rule T (5) $P\to Q \Rightarrow \sim P\vee Q$
{1,2,5}	(7)	$(\sim Q\vee R)\wedge (\sim Q\vee (\sim R\vee S))$	Rule T (4) (6) $P, Q \Rightarrow P\wedge Q$

$\sim Q\vee (R\wedge (\sim R\vee S))$
$\sim Q\vee ((R\wedge \sim R)\vee (R\vee S))$ $\quad P\wedge Q = Q$
$\sim Q\vee (F\vee S)$

{1,2,5}	(7)	$\sim Q\vee S$	
{1,2,5}	(8)	$Q\to S$	Rule T (7) $P\to Q \Leftrightarrow \sim P\vee Q$
{1,2,5}	(9)	$P\to (Q\to S)$	Rule CP

Ex 9. If there was a ball game then travelling was difficult, If ~~they~~ arrived on time, then travelling was not difficult. they arrived on time. therefore there was no ball game.

P: There was a ball game
Q: Travelling was difficult
R: they arrived on time,

$P\to Q,\ R\to \sim Q,\ R \ \therefore \sim P$

$\sim Q,$

Rule P

{1}	(1)	R	Rule P
{2}	(2)	$R \to \sim Q$	Rule P
{1, 2}	(3)	$\sim Q$	Rule T (1), (2) $P, P \to Q \Rightarrow Q$
{4}	(4)	$P \to Q$	Rule P
{1,2,4}	(5)	$\sim P$	Rule T (3) (4) $P, P \to Q \Rightarrow$ $P, \sim P \vee Q = Q$

Ex. If A works hard the either B or C enjoy themselves. If B enjoy himself then A will not work hard. If D enjoy himself then C will not enjoy Therefore If A works hard then D will not enjoy himself

A: A works hard
B: B enjoys himself
C: C enjoy him self
D: D enjoy him self

$A \to (B \vee C)$
$B \to \sim A$
$D \to \sim C$
Conclusion
$A \to \sim D$
Apply CP

$\sim B \to C$
$\sim (\sim B) \vee C$
$B \vee C$

{1}	(1)	A	Rule P (Assumed)
{2}	(2)	$A \to (B \vee C)$	Rule P
{1, 2}	(3)	$(B \vee C)$	Rule T (1), (2) $P, P \to Q \Rightarrow Q$
{1, 2}	(4)	$\sim B \to C$	Rule T (3) $P \vee Q \Leftrightarrow \sim P \to Q$
{1, 2}	(5)	$\sim C \to B$	Rule T (4) $P \to Q \Leftrightarrow \sim Q \to \sim P$
{6}	(6)	$B \to \sim A$	Rule P
{1, 2, 6}	(7)	$\sim C \to \sim A$	Rule T (5), (6) $P \to Q, Q \to R \Rightarrow P \to R$
{1, 2, 6}	(8)	$A \to C$	$P \to Q \Leftrightarrow \sim Q \to \sim P$
{8}	(9)	$D \to \sim C$	Rule P
{8}	(10)	$C \to \sim D$	Rule T $P \to Q \Leftrightarrow \sim Q \to \sim P$
{1, 2, 6, 8}	(11)	$A \to \sim D$	Rule T (8), (10) $P \to Q, Q \to R \Rightarrow P \to R$
{1, 2, 6, 8}	(12)	$\sim D$	Rule T (1), (11) $P, P \to Q \Rightarrow Q$
{1, 2, 6, 8}	(13)	$A \to \sim D$	Rule CP

Unit - 2

Predicate Logic or Predicate Calculus

Defn: - Predicate logic is an extension of propositional logic.

Ex.

P: The Ball colour is Red

colour(Ball, Red)

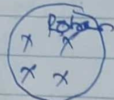
Predicate or relation — Arguments, or object

Ex. Rohan likes Bananas

like(Rohan, Bananas)

Ex. All students are intelligents
Rohan is a student

Rohan is intelligents



object, relation, propositional

Ex. Everybody Loves Somebody

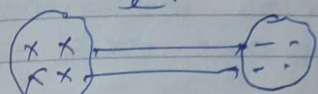
Quantifiers: -

We need quantifiers to express english words including All & Some.

Universal — $\forall$ (for all)

Existential — $\exists$ (there exist)

Everybody (x) love Somebody (y)



$L(x, y)$

$\forall x \exists y\ L(x, y)$

Ex. Ramu is a good boy
Ramu — subject; is a good boy — predicate

Ex. $x > 3$

Ex. x is a student
$S(x)$ → variable
S — Predicate; $S(x)$ → proposition funn or stmt funn

Ex. x is a man
$M(x)$

x is a woman
$W(x)$

(various operation) we can construct compound stmt funn

$M(x) \land W(x)$
$M(x) \lor W(x)$
$\sim M(x)$ $\quad M(x) \to W(x)$, $M(x) \leftrightarrow W(x)$

Quantifiers:-

1. Universal Quantifier ($\forall$)

phrases such as $\rightarrow$ For, all, for every, for each, everything, for any

2. Existential Quantifier ($\exists$)

there exist, for some, there is atleast one, exist some

1. Ex. a. Something is good

$(\exists x)\ G(x)$

b. Everything is good

$(\forall x)\ G(x)$

c. Something is not good

$(\exists x)(\sim G(x))$

d. Nothing is good

$(\forall x)(\sim G(x))$

Sol^n: $\Rightarrow$ $G(x)$: x is good.

Ex. 2 a All mens are motal

b Every Apple is red

c Any integers is either positive or negative.

Sol^n: $\Rightarrow$ $M(x)$: x is a man

$L(x)$: x is mortal

$A(x)$: x is Apple

$R(x)$: x is Red

$I(x)$: x is integer

$P(x)$: x is +ve or -ve

a $\Rightarrow$ $(\forall x)(M(x) \rightarrow L(x))$

b $\Rightarrow$ $(\forall x)(A(x) \rightarrow R(x))$

c $\Rightarrow$ $(\forall x)(I(x) \rightarrow P(x))$

Ex. 3. a. There exist a man

b. Some men are clever

c. Some real numbers are rationals.

Sol^n: - $M(x)$: x is a man

$C(x)$: x is clever

$R(x)$: x is real no.

$L(x)$: x is rational.

a. $(\exists x)(M(x))$

b. $(\exists x)(M(x) \wedge C(x))$

c. $(\exists x)(R(x) \wedge L(x))$

Equivalence Relation :-

If the given relation satisfies all the three properties i.e. reflexive, Symmetric and transitive then it is called as equivalence relation:

Ex. X = {1,2,3,4,5,6}

R = {(1,1), (1,3) (1,5), (3,5), (3,3), (5,5), (2,6), (2,2), (6,6), (6,2), (3,1), (5,1), (5,3), (4,4)}

i) Reflexive

(x,x) ∈ R for any x ∈ X

(1,1) ∈ R (4,4) ∈ R
(2,2) ∈ R (5,5) ∈ R
(3,3) ∈ R (6,6) ∈ R

ii) Symmetric

(x,y) ∈ R & (y,x) ∈ R

(1,3) (3,1) ∈ R
(1,5) (5,1) ∈ R
(3,5) (5,3) ∈ R
(2,6) (6,2) ∈ R

iii) Transitive

(x,y) ∈ R (y,z) ∈ R then (x,z) ∈ R

(1,1), (1,3) → (1,3) ∈ R
(1,1), (1,5) → (1,5) ∈ R

(1,3), (3,5) → (1,5) ∈ R
(1,3), (3,3) → (1,3) ∈ R
(1,3), (3,1) → (1,1) ∈ R

(1,5), (5,5) → (1,5) ∈ R
(1,5), (5,1) → (1,1) ∈ R
(1,5), (5,3) → (1,3) ∈ R

(3,5), (5,5) → (3,5) ∈ R
(3,5), (5,1) → (3,1) ∈ R
(3,5), (5,3) → (3,3) ∈ R

(3,3), (3,1) → (3,1) ∈ R

(5,5), (5,1) → (5,1) ∈ R
(5,5), (5,3) → (5,3) ∈ R
(2,6), (6,6) → (2,6) ∈ R
(2,6), (6,2) → (2,2) ∈ R
(2,2), (2,6) → (2,6) ∈ R
(6,6), (6,2) → (6,2) ∈ R
(6,2), (2,6) → (6,6) ∈ R
(6,2), (2,2) → (6,2) ∈ R
(3,1), (1,1) → (3,1) ∈ R
(3,1), (1,3) → (3,3) ∈ R
(3,1), (1,5) → (3,5) ∈ R
(5,1), (1,1) → (5,1) ∈ R
(5,1), (1,3) → (5,3) ∈ R
(5,1), (1,5) → (5,5) ∈ R
(5,3), (3,3) → (5,3) ∈ R
(5,3), (3,5) → (5,5) ∈ R

Ex2. Given $A = \{1,2,3,4\}$ and relation on set A
$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

→ Reflexive
$(x,x) \in R$ for any $x \in A$
$(1,1) \in R$
$(2,2) \in R$
$(3,3) \in R$
$(4,4) \in R$
R is reflexive

Symmetric
(x,y) & $(y,x) \in R$
$(1,2)$ but $(2,1) \notin R$
$(1,3)$ but $(3,1) \notin R$
$(1,4)$ but $(4,1) \notin R$
$(2,3)$ but $(3,2) \notin R$
$(2,4)$ but $(4,2) \notin R$
$(3,4)$ but $(4,3) \notin R$
R is not symmetric

Transitive
$(1,1), (1,2) \rightarrow (1,2) \in R$
$(1,1), (1,3) \rightarrow (1,3) \in R$
$(1,1), (1,4) \rightarrow (1,4) \in R$
$(1,2), (2,2) \rightarrow (1,2) \in R$
$(1,2), (2,3) \rightarrow (1,3) \in R$
$(2,4) \rightarrow (1,4) \in R$

Partial order Relation or POSET

1) Which of the following relation on set $\{0,1,2,3\}$ are partial ordering?

a) $R = \{(0,0), (2,2), (3,3)\}$
b) $R = \{(0,0), (1,1), (2,2), (2,3), (3,3)\}$
c) $R = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (3,3)\}$
d) $R = \{(0,0), (1,1), (1,2), (1,3), (2,0), (2,2), (2,3), (3,0), (3,3)\}$

2). $A = \{1,2,3,4,5,6,7,8\}$
$R = \{(x,y) \mid y \text{ divides } x\}$
$\underline{y \mid x}$
$R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (2,2), (2,4), (2,6), (2,8), (3,3), (3,6), (4,4), (4,8), (5,5), (6,6), (7,7), (8,8)\}$

Ex2. Given $A = \{1,2,3,4\}$
and relation on set A
$R = \{(1,1), (1,2), (1,3), (1,4),$
$(2,3), (2,4), (3,3), (3,4), (4,$

→ Reflexive
$(x,x) \in R$ for any $x \in A$
$(1,1) \in R$
$(2,2) \in R$
$(3,3) \in R$
$(4,4) \in R$
R is reflexive
Symmetric
(x,y) & $(y,x) \in R$
$(1,2)$ but $(2,1) \notin R$
$(1,3)$ but $(3,1) \notin R$
$(1,4)$ but $(4,1) \notin R$
$(2,3)$ but $(3,2) \notin R$
$(2,4)$ but $(4,2) \notin R$
$(3,4)$ but $(4,3) \notin R$
R is not Symmetric
Transitive
$(1,1), (1,2) \to (1,2) \in R$
$(1,1), (1,3) \to (1,3) \in R$
$(1,1), (1,4) \to (1,4) \in R$
$(1,2), (2,2) \to (1,2) \in R$
$(1,2), (2,3) \to (1,3) \in R$
$(2,4) \to (1,4) \in R$

partial order Relation
or POSET

1) which of the following relation on set $\{0,1,2,3\}$ are partial ordering?

a) $R = \{(0,0), (2,2), (3,3)\}$
b) $R = \{(0,0), (1,1), (2,2), (2,3), (3,3)\}$
c) $R = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (3,3)\}$
d) $R = \{(0,0), (1,1), (1,2), (1,3), (2,0), (2,2), (2,3), (3,0), (3,3)\}$

2). $A = \{1,2,3,4,5,6,7,8\}$
$R = \{(x,y) \mid y \text{ divides } x\}$
$\underline{y \mid x}$
$R = \{(1,1), (1,2), (1,3), (1,4), (1,5)$
$(1,6), (1,7), (1,8), (2,2), (2,4)$
$(2,6), (2,8), (3,3), (3,6), (4,4)$
$(4,8), (5,5), (6,6), (7,7), (8,8)\}$

Ex.2. Given $A = \{1,2,3,4\}$ and relation on set A
$R = \{(1,1), (1,2), (1,3), (1,4), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

→ Reflexive
$(x,x) \in R$ for any $x \in A$
$(1,1) \in R$
$(2,2) \in R$
$(3,3) \in R$
$(4,4) \in R$
R is reflexive

Symmetric
(x,y) & $(y,x) \in R$
$(1,2)$ but $(2,1) \notin R$
$(1,3)$ but $(3,1) \notin R$
$(1,4)$ but $(4,1) \notin R$
$(2,3)$ but $(3,2) \notin R$
$(2,4)$ but $(4,2) \notin R$
$(3,4)$ but $(4,3) \notin R$
R is not symmetric

Transitive
$(1,1), (1,2) \to (1,2) \in R$
$(1,1), (1,3) \to (1,3) \in R$
$(1,1), (1,4) \to (1,4) \in R$
$(1,2), (2,2) \to (1,2) \in R$
$(1,2), (2,3) \to (1,3) \in R$
$(2,4) \to (1,4) \in R$

Partial order Relation or POSET

1) Which of the following relation on set $\{0,1,2,3\}$ are partial ordering?

a) $R = \{(0,0), (2,2), (3,3)\}$
b) $R = \{(0,0), (1,1), (2,2), (2,3), (3,3)\}$
c) $R = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (3,3)\}$
d) $R = \{(0,0), (1,1), (1,2), (1,3), (2,0), (2,2), (2,3), (3,0), (3,3)\}$

2). $A = \{1,2,3,4,5,6,7,8\}$
$R = \{(x,y) \mid y \text{ divides } x\}$
$y \mid x$
$R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (2,2), (2,4), (2,6), (2,8), (3,3), (3,6), (4,4), (4,8), (5,5), (6,6), (7,7), (8,8)\}$

Operations on Relation

* Intersection :- It consist of ordered pairs common to both relations.

$x(R \cap S)y = xRy \wedge xSy$

Ex. $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$

$R = \{(1,1), (2,2), (3,3)\}$

$S = \{(1,1), (1,2), (1,3), (1,4)\}$

Find $R \cap S$, $R \cup S$, $R - S$, $\sim R$

$\Rightarrow R \cap S = \{(1,1)\}$

* Union :- It consist of ordered pairs of relation R or relation S

$x(R \cup S)y = xRy \vee xSy$

$\Rightarrow R \cup S = \{(1,1), (2,2), (3,3), (1,2), (1,3), (1,4)\}$

* Difference :- All ordered pairs of R which are not ordered pairs of S

$x(R-S)y = xRy \wedge x\not{S}y$

$R - S = \{(2,2), (3,3)\}$

$S - R = \{(1,2), (1,3), (1,4)\}$

* Compliment :- $\sim R$ or R^c or $\bar{R}$
set of order pairs which belongs to both ~~relations~~ sets but they are not present in R

$x(\sim R)y = x\not{R}y$

$\sim R = (A \times B) - R$

$(A \times B) = \{(1,1), (1,2), (1,3), (1,4),$
$(2,1), (2,2), (2,3), (2,4),$
$(3,1), (3,2), (3,3), (3,4),$
$(4,1), (4,2), (4,3), (4,4)\} -$
$\{(1,1), (2,2), (3,3)\}$

$\sim R = \{(1,2), (1,3), (1,4), (2,1), (2,3),$
$(2,4), (3,1), (3,2), (3,4), (4,1),$
$(4,2), (4,3), (4,4)\}$

$g: Y \to Z$

Composite function :-

Ex. Let $X = \{1,2,3\}$ $Y = \{p,q\}$ also let $Z = \{a,b\}$
$f: X \to Y$ be $f = \{(1,p),(2,p),(3,q)\}$
and $g: Y \to Z$
$g = \{(p,b),(q,b)\}$
find gof

$\Rightarrow$ $f(1) = p$, $f(2) = p$, $f(3) = q$
$g(p) = b$, $g(q) = b$
$f: X \to Y$, $g: Y \to Z$
$gof: X \to Z$

$gof(1) = g(f(1)) = g(p) = b$
$gof(2) = g(f(2)) = g(p) = b$
$gof(3) = g(f(3)) = g(q) = b$

$gof: \{(1,b),(2,b),(3,b)\}$

Ex. $X = \{1,2,3\}$ and f, g, h be functions from X to X given by
$f = \{(1,2),(2,3),(3,1)\}$
$g = \{(1,2),(2,1),(3,3)\}$
$h = \{(1,1),(2,2),(3,1)\}$
find fog, gof, $fohog$.

$fog = \{(1,3),(2,2),(3,1)\}$

$gof = \{(1,1),(2,3),(3,2)\}$

$fohog = (foh)og \Rightarrow \{(1,2),(2,3),(3,2)\}$ og

$\Rightarrow \{(1,3),(2,2),(3,2)\}$

Ex. Let $f(x) = x+2$, $g(x) = x-2$, $h(x) = 3x$
find gof, fog, gog, foh, hog

$gof(x) = g(f(x)) = g(x+2) = (x+2)-2 = x$

$fog(x) = f(g(x)) = f(x-2) = (x-2)+2 = x$

$fof(x) = f(f(x)) = f(x+2) = (x+2)+2 = x+4$

$gog(x) = g(g(x)) = g(x-2) = (x-2)-2 = x-4$

$foh(x) = f(h(x)) = f(3x) = 3x+2$

$hog(x) = h(g(x)) = h(x-2) = 3(x-2) = 3x-6$

Ex. Let $f(x) = 5x+1$, $g(x) = 3x-2$

Find $(f \circ g)(x)$, $(g \circ f)(x)$

$\Rightarrow$

$f \circ g(x) = f(g(x))$

$= f(3x-2)$

$= \cancel{3(5x+1)-2}$

$= \cancel{15x+3-2}$

$= 5(3x-2)+1$

$= 15x-10+1$

$= 15x-9$

$g \circ f(x) = g(f(x))$

$= g(5x+1)$

$= 3(5x+1)-2$

$= 15x+3-2$

$= 15x-1$

Ros, RoR, RoR^2, RoR^3 — RoR^{-1}, R^{-1}, RoR, $R^{-1}oR$, $R = \{\ \}$, $R^{-1} = \{\ \}$, $R \circ R^{-1} =$

$R = \{(1,2), (3,4), (2,2)\}$

$S = \{(4,2), (2,5), (3,1)\}$

$R \circ S = \{(1,5), (3,2), (2,5)\}$

Composition of Binary Relation:-

Ex. Let $R = \{(1,2), (3,4), (2,2)\}$ and $S = \{(4,2), (2,5), (3,1), (1,3)\}$.

Find. RoS, SoR, Ro(SoR), (RoS)oR, RoR, SoS, and, RoRoR, RoR^{-1}

$\Rightarrow$ $RoS = \{(1,5), (3,2), (2,5)\}$

$SoR = \{(4,2), (3,2), (1,4)\}$

$Ro(SoR) = \{(3,2)\} = (RoS)oR$

$RoR = \{(1,2), (2,2)\}$

$SoS = \{(4,5), (3,3), (1,1)\}$

$RoRoR = \{(1,2), (2,2)\}$

$R^{-1} = \{(2,1), (4,3), (2,2)\}$

$RoR^{-1} = \{(1,1), (3,3), (2,1), (2,2)\}$

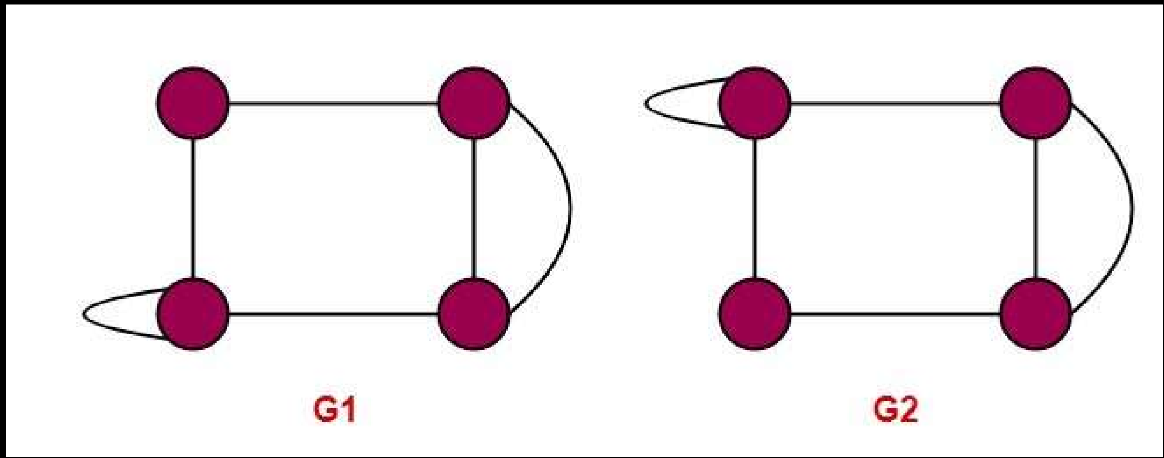
Isomorphism

Graph Isomorphism Conditions-

- For any two graphs to be isomorphic, following 4 conditions must be satisfied-
-
- Number of vertices in both the graphs must be same.
- Number of edges in both the graphs must be same.
- Degree sequence of both the graphs must be same.
- If a cycle of length k is formed by the vertices $\{ v_1 , v_2 , \ldots , v_k \}$ in one graph, then a cycle of same length k must be formed by the vertices $\{ f(v_1) , f(v_2) , \ldots , f(v_k) \}$ in the other graph as well.

Graph Isomorphism Conditions-

- If a cycle of length k is formed by the vertices { v_1 , v_2 , ….. , v_k } in one graph, then a cycle of same length k must be formed by the vertices { $f(v_1)$, $f(v_2)$, ….. , $f(v_k)$ } in the other graph as well.


G1
G2

Important Points-

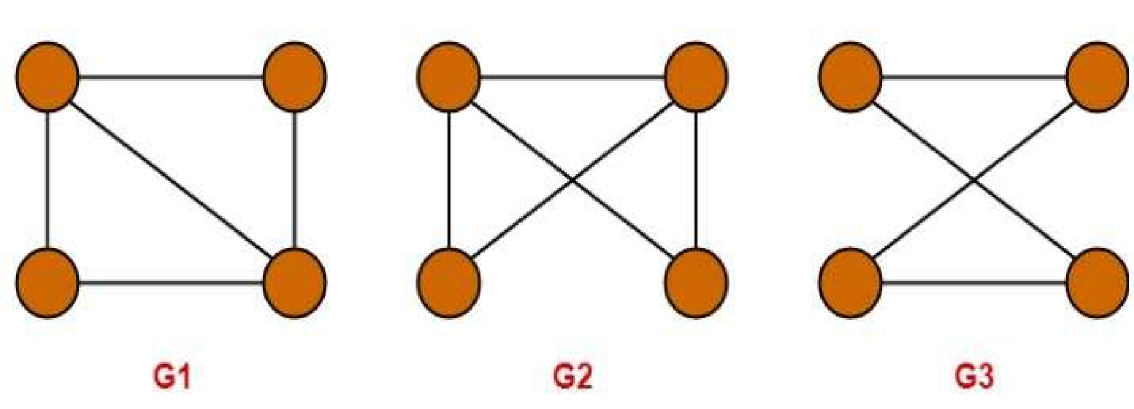
- The above 4 conditions are just the necessary conditions for any two graphs to be isomorphic.
- They are not at all sufficient to prove that the two graphs are isomorphic.
- If all the 4 conditions satisfy, even then it can’t be said that the graphs are surely isomorphic.
- However, if any condition violates, then it can be said that the graphs are surely not isomorphic.

Sufficient Conditions-

- The following conditions are the sufficient conditions to prove any two graphs isomorphic.
- If any one of these conditions satisfy, then it can be said that the graphs are surely isomorphic.

Sufficient Conditions-

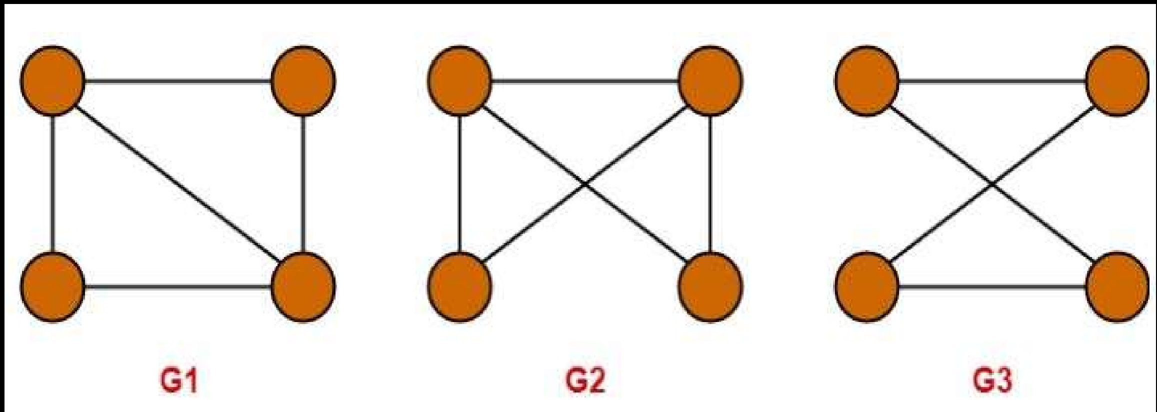
- Two graphs are isomorphic if and only if their complement graphs are isomorphic.
- Two graphs are isomorphic if their adjacency matrices are same.
- Two graphs are isomorphic if their corresponding sub-graphs obtained by deleting some vertices of one graph and their corresponding images in the other graph are isomorphic



Condition-01:

Number of vertices in graph G1 = 4
Number of vertices in graph G2 = 4
Number of vertices in graph G3 = 4

Here,
All the graphs G1, G2 and G3 have same number of vertices.
So, Condition-01 satisfies.



Condition-02:

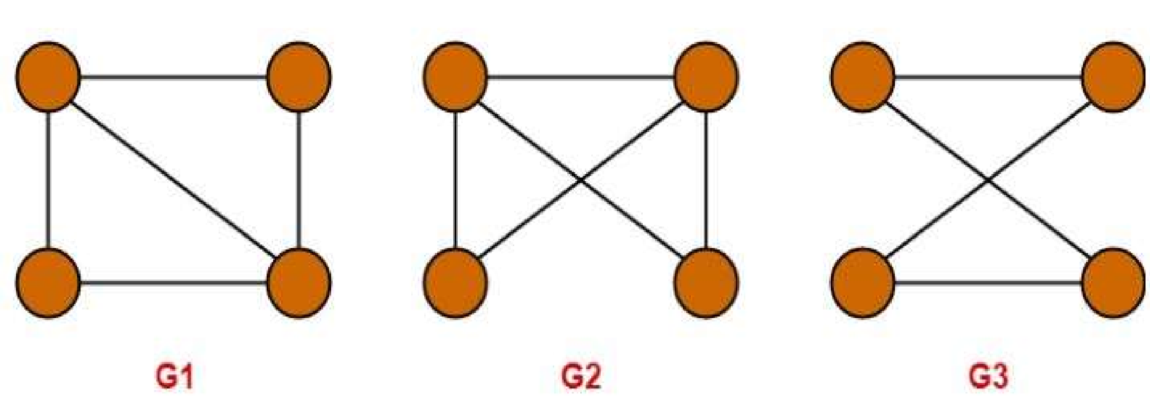
Number of edges in graph G1 = 5
Number of edges in graph G2 = 5
Number of edges in graph G3 = 4

Here,
The graphs G1 and G2 have same number of edges.
So, Condition-02 satisfies for the graphs G1 and G2.
However, the graphs (G1, G2) and G3 have different number of edges.
So, Condition-02 violates for the graphs (G1, G2) and G3.

Since Condition-02 violates for the graphs (G1, G2) and G3, so they can not be isomorphic.
∴ G3 is neither isomorphic to G1 nor G2.

Since Condition-02 satisfies for the graphs G1 and G2, so they may be isomorphic.
∴ G1 may be isomorphic to G2.

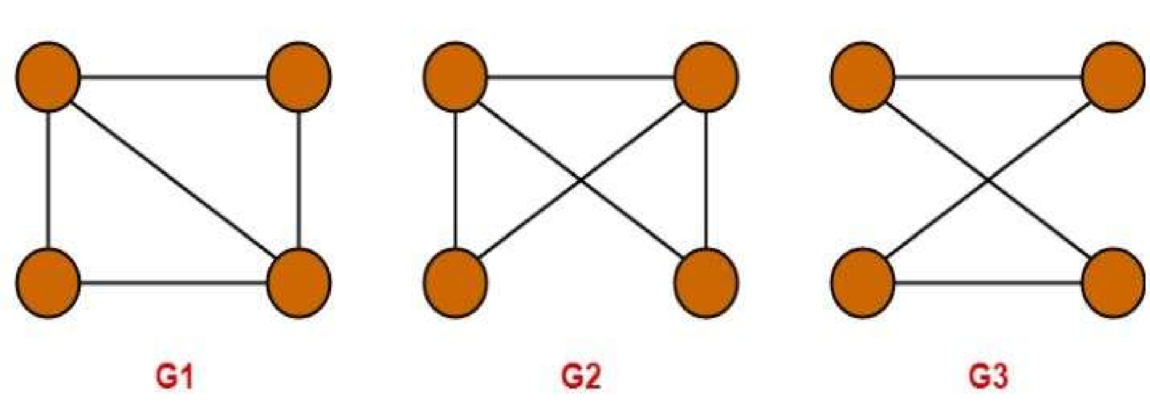
Now, let us continue to check for the graphs G1 and G2



Condition-03:

Degree Sequence of graph G1 = { 2 , 2 , 3 , 3 }
Degree Sequence of graph G2 = { 2 , 2 , 3 , 3 }

Here,
Both the graphs G1 and G2 have same degree sequence.
So, Condition-03 satisfies.



Condition-04:

Both the graphs contain two cycles each of length 3 formed by the vertices having degrees { 2 , 3 , 3 }

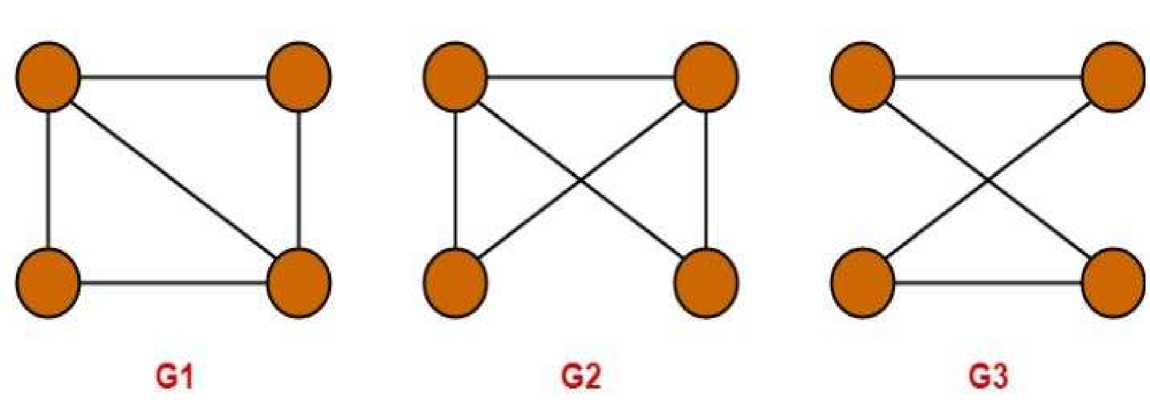
It means both the graphs G1 and G2 have same cycles in them.

So, Condition-04 satisfies.

Thus,

All the 4 necessary conditions are satisfied.

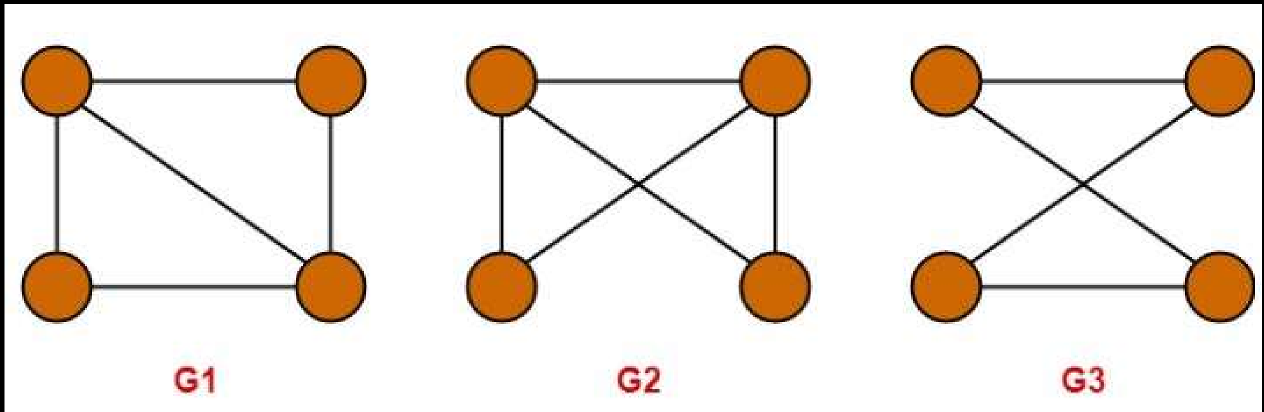
So, graphs G1 and G2 may be isomorphic.

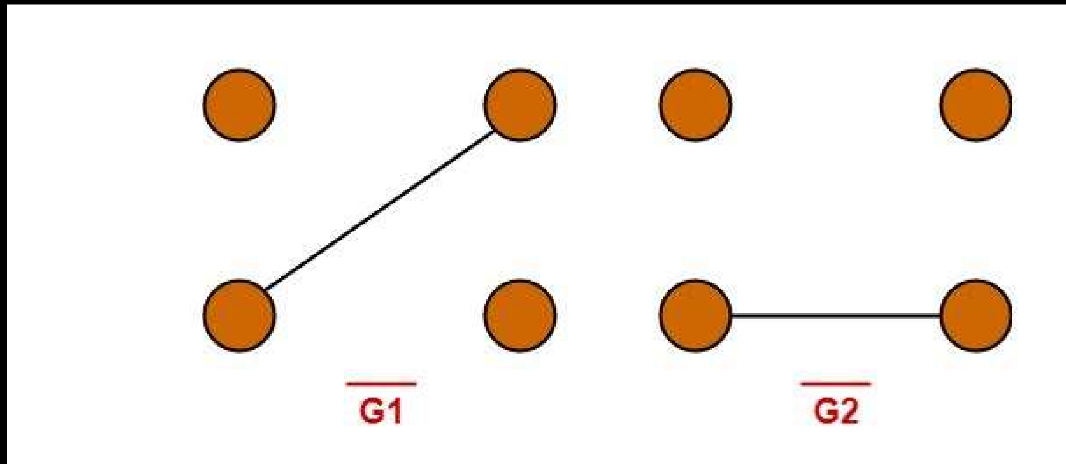


Checking Sufficient Condition-

We know that two graphs are surely isomorphic if and only if their complement graphs are isomorphic.

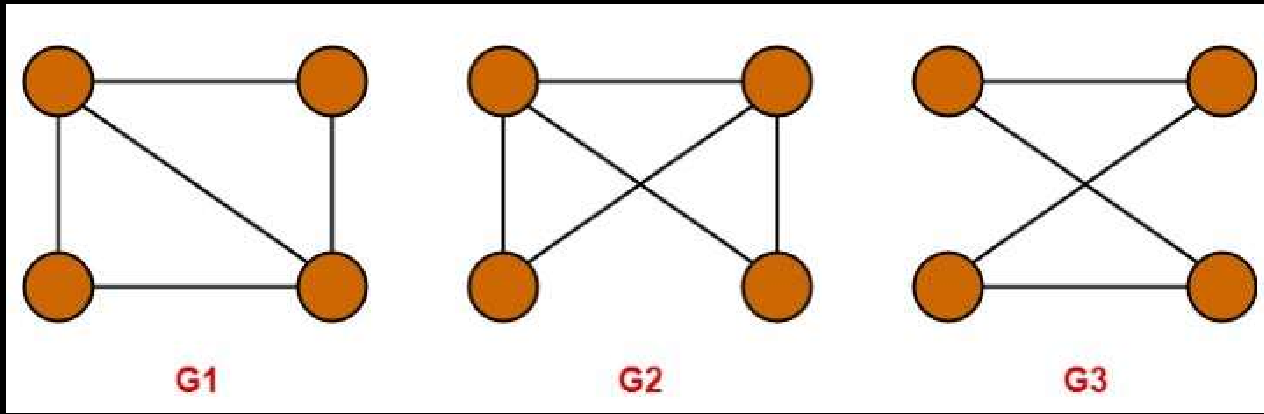
So, let us draw the complement graphs of G1 and G2.
The complement graphs of G1 and G2 are-

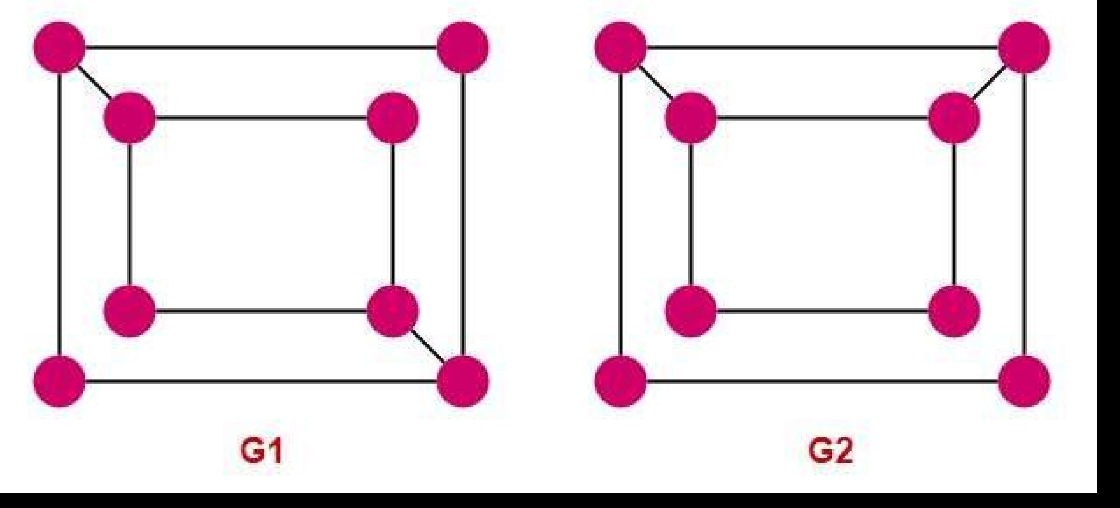


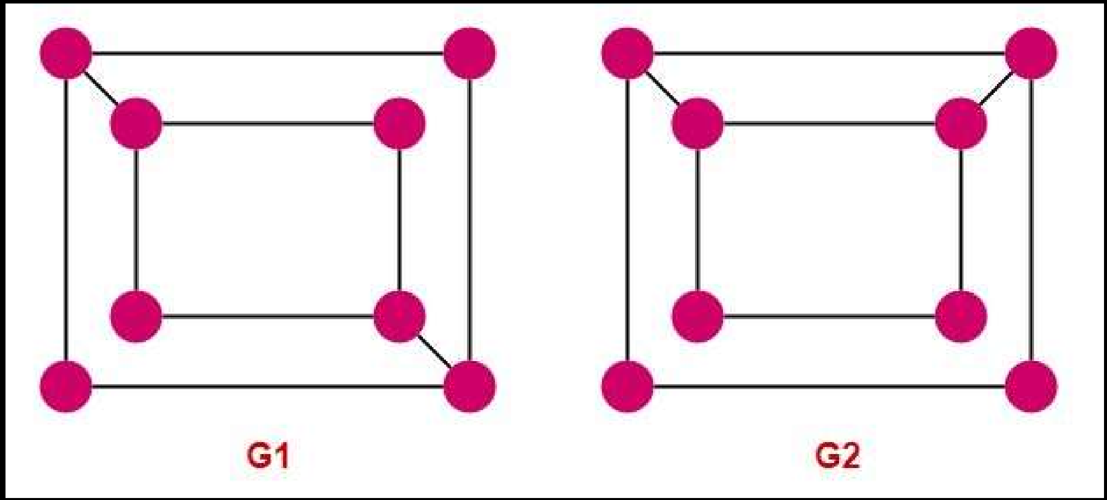


Clearly, Complement graphs of G1 and G2 are isomorphic.

∴ **Graphs G1 and G2 are isomorphic graphs.**


G1
G2
G3


G1
G2

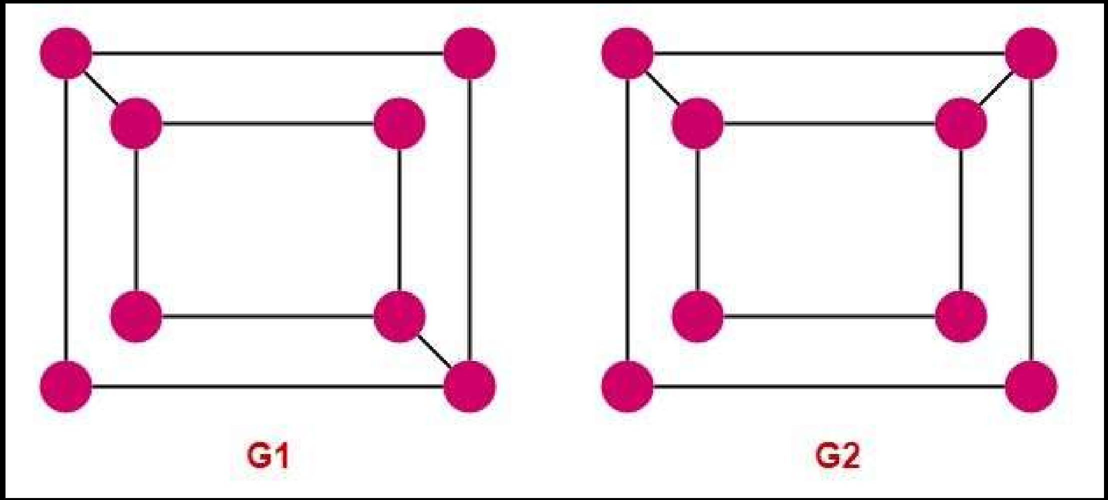


Checking Necessary Conditions-

Condition-01:

Number of vertices in graph G1 = 8
Number of vertices in graph G2 = 8

Here,
Both the graphs G1 and G2 have
same number of vertices.
So, Condition-01 satisfies.



Checking Necessary Conditions-

Condition-01:

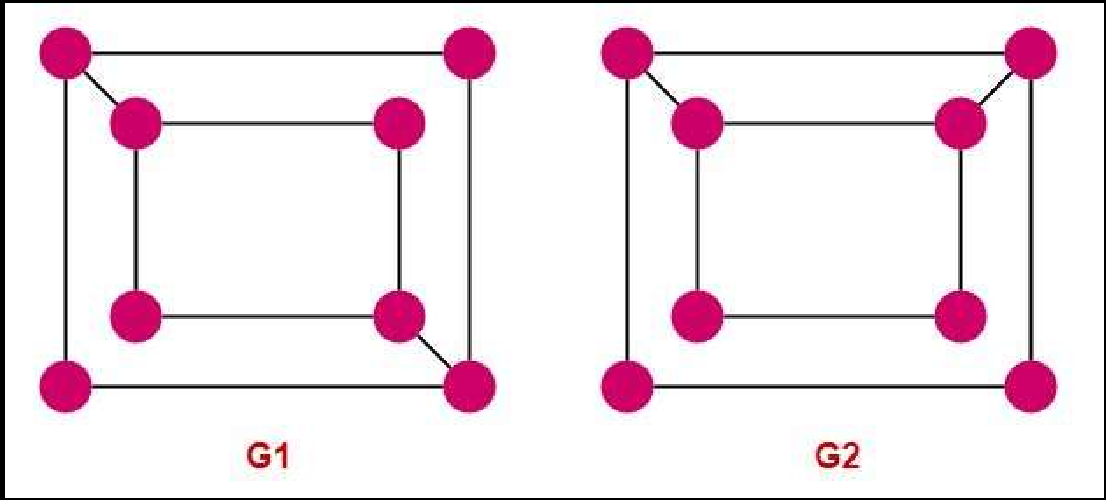
Number of vertices in graph G1 = 8
Number of vertices in graph G2 = 8

Here,
Both the graphs G1 and G2 have same number of vertices.
So, Condition-01 satisfies.

Condition-02:

Number of edges in graph G1 = 10
Number of edges in graph G2 = 10

Here,
Both the graphs G1 and G2 have same number of edges.
So, Condition-02 satisfies.



Condition-03:

Degree Sequence of graph G1 = { 2 , 2 , 2 , 2 , 3 , 3 , 3 , 3 }

Degree Sequence of graph G2 = { 2 , 2 , 2 , 2 , 3 , 3 , 3 , 3 }

Here,

Both the graphs G1 and G2 have same degree sequence.

So, Condition-03 satisfies.

Checking Necessary Conditions-

Condition-01:

Number of vertices in graph G1 = 8

Number of vertices in graph G2 = 8

Here,

Both the graphs G1 and G2 have same number of vertices.

So, Condition-01 satisfies.

Condition-02:

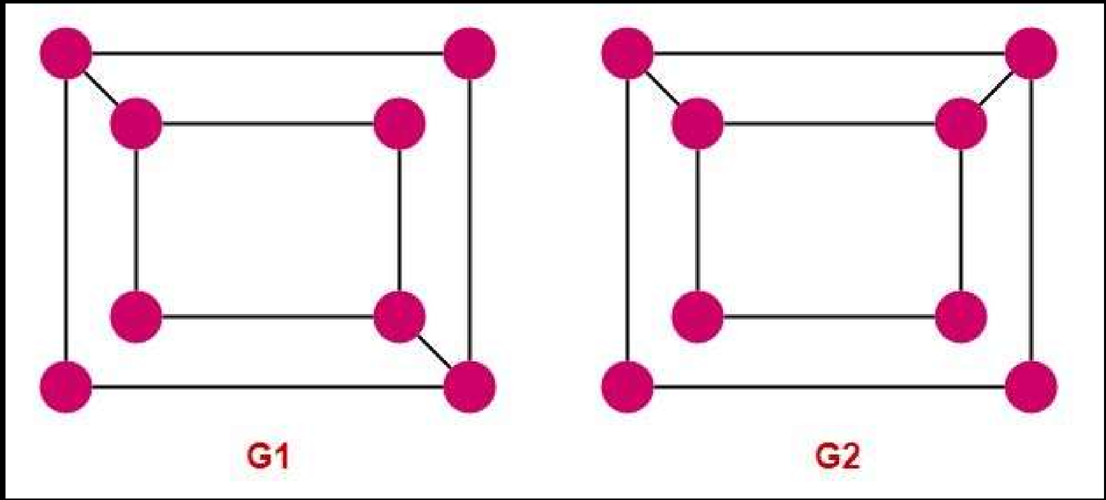
Number of edges in graph G1 = 10

Number of edges in graph G2 = 10

Here,

Both the graphs G1 and G2 have same number of edges.

So, Condition-02 satisfies.



Checking Necessary Conditions-

Condition-01:

Number of vertices in graph G1 = 8
Number of vertices in graph G2 = 8

Here,
Both the graphs G1 and G2 have same number of vertices.
So, Condition-01 satisfies.

Condition-02:

Number of edges in graph G1 = 10
Number of edges in graph G2 = 10

Here,
Both the graphs G1 and G2 have same number of edges.
So, Condition-02 satisfies.

Condition-03:

Degree Sequence of graph G1 = { 2 , 2 , 2 , 2 , 3 , 3 , 3 , 3 }
Degree Sequence of graph G2 = { 2 , 2 , 2 , 2 , 3 , 3 , 3 , 3 }

Here,
Both the graphs G1 and G2 have same degree sequence.
So, Condition-03 satisfies.

Condition-04:

In graph G1, degree-3 vertices form a cycle of length 4.
In graph G2, degree-3 vertices do not form a 4-cycle as the vertices are not adjacent.

Here,
Both the graphs G1 and G2 do not contain same cycles in them.
So, Condition-04 violates.

Since Condition-04 violates, so given graphs can not be isomorphic.

① **Number of permutations of r objects among n distinct objects.**

Suppose we are given with n distinct objects and wish to arrange r of the objects denoted by

$$P(n, r) = {}^{n}P_{r} = \frac{n!}{(n-r)!}$$

② **Number of permutations of n objects (with duplication)**

The number of permutations that can be formed from a collection of n objects of which n_1 are of one type, n_2 are of second type .... n_k are of k^{th} type with $n_1 + n_2 + \cdots + n_k = n$, then the number of permutations of n objects are $$\frac{n!}{n_1! * n_2! * \cdots * n_k!}$$

③ **Circular permutations:** The permutations in a circle are called circular permutations.

The total no. of ways of arranging the n persons in a ~~circle~~ circle is $(n-1)!$

Ex: How many ways are there to sit 10 boys and 10 girls around a circular table?

Sol: Total no of persons $= 10+10 = 20$

$n = 20$

Circular permutation $= (n-1)! = (20-1)! = 19!$ ways

→ How many ways are there where 3 persons st around a table.

Sol: $n=3$

$\therefore$ Total no of ways $= (n-1)! = (3-1)! = 2! = 2$ way

→ How many words of 3 distinct letters can be formed from the letters of the word PASCAL.

Sol: $n=6$

no. of ways of 3 distinct letters of the word PASCAL

$$\Rightarrow p(n,r) = p(6,3) = \frac{6!}{(6-3)!} = \frac{6!}{3!} = 120$$

→ Prove the following identities for $r \geq 0$

n is an integer. $\boxed{\frac{10!}{9!} = 10}$

(i) $p(n+1,r) = (n+1).p(n,r-1)$

Sol: $$\frac{p(n+1,r)}{p(n,r-1)} = \frac{\frac{(n+1)!}{\cancel{(n+1-r)!}}}{\frac{n!}{\cancel{(n-r+1)!}}} = \frac{(n+1)!}{n!} = n+1$$

(ii) $p(n,r) = (n-r+1)\,p(n,r-1)$

Sol: $$\frac{p(n,r)}{p(n,r-1)} = \frac{\frac{\cancel{n!}}{(n-r)!}}{\frac{\cancel{n!}}{(n-r+1)!}} = \frac{(n-r+1)!}{(n-r)!} = n-r+1$$

(iii) $p(n+1,r) = \left(\frac{n+1}{n+1-r}\right) p(n,r)$

Sol: $$\frac{p(n+1,r)}{p(n,r)} = \frac{\frac{(n+1)!}{(n+1-r)!}}{\frac{n!}{(n-r)!}} = \frac{(n+1)!}{(n+1-r)!} \times \frac{(n-r)!}{n!}$$

$$= \frac{(n+1)!}{n!} \times \frac{(n-r)!}{(n-r+1)!}$$

$$= \frac{(n+1)}{}$$

Permutations

A permutation of n object taken r at a time, also called an r-permutation of n objects is an ordered selection or arrangement of r of the objects.

→ Suppose, we are given with n distinct objects and wish to arrange r of these objects in a line. Since there are n ways of choosing the first object and after this is done, $n-1$ ways of choosing the second object and finally $n-r+1$ ways of choosing r^{th} object. It follows by the product rule of counting that the number of different arrangements or permutations is $n(n-1)(n-2)\ldots(n-r+1)$, we denote this number by $p(n,r)$ and is referred to as the number of permutations of size r of n objects.

$$\therefore\ p(n,r) = n(n-1)(n-2)\ldots(n-r+1)$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots 2.1}{(n-r)(n-r-1)\ldots 2.1}$$

$$\boxed{{}^{n}P_{r} = \frac{n!}{(n-r)!}}$$

Examples:

① Q) How many different strings of length 4 can be formed using the letters of the word FLOWER?

Sol: All 6 letters are distinct

$$P(6,4) = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6\times5\times4\times3\times2\times1}{2\times1} = 360$$

(2) Find the value of n in each of the following cases:

(i) $P(n,2) = 90$

$n(n-1) = 90$

$n^2 - n - 90 = 0$

$n^2 + 9n - 10n - 90 = 0$

$n(n-10) + 9(n-10) = 0$

$\boxed{n=10}$

(ii) $P(n,3) = 3P(n,2)$

$n - 2 = 3$

$\boxed{n=5}$

(iii) $P(n,4) = 42P(n,2)$

$(n-2)(n-3) = 42$

$n^2 - 5n - 36 = 0$

$n^2 - 9n + 4n - 36 = 0$

$(n+4)(n-9) = 0$

$\boxed{n=9}$

(3) Ten different paintings are to be put in n rooms so that no room gets more than one painting. Find the number of ways of accomplishing this if (i) $n=12$ (ii) $n=5$.

(i) $n = 12$

12 rooms

10 paintings

10 paintings are to be kept in 12 rooms

$\therefore$ $^{12}P_{10}$

$P(12,10)$

(ii) $n = 5$

10 paintings

5 rooms

10 paintings are to be kept in 5 rooms. As there are 5 rooms you can select 5 paintings from 10.

$\therefore$ $^{10}P_5$

$P(10,5)$

(4) Find the number of distinguishable permutation of the letters in the following words.

(i.) BASIC

$$\frac{5!}{1!1!1!1!1!} = 120$$

(ii) PASCAL

$$\frac{6!}{2!1!1!1!1!} = 360$$

(iii) BANANA

$$\frac{6!}{3!2!1!} = 60$$

(iv) ENGINEERING

3E
3N
2G
2I
1R

$$\frac{11!}{3!3!2!2!1!} = 277200$$

(v) MATHEMATICS

2M 1C
2A 1S
2T
1H
1E

$$\frac{11!}{2!2!2!1!1!1!1!1!} = 4989600$$

(5.) How many different arrangements of letters in the word BOUGHT can be formed if the vowels must be kept next to each other.

Sol: • BOUGHT ⟹ 2 vowels & 4 consonants

As two vowels O & U are distinct characters their swapping creates new variety. So their arrangement is possible in $(2!) = 2$ ways.

~~OU~~ ⟹ B$\boxed{OU}$GHT $\frac{5!}{1!1!1!1!1!} = 120$

another possibility $\boxed{UO}$ ⟹ 120

$\boxed{\therefore \text{ Total } 120+120 = 240}$ letters can be formed if the vowels must be kept next to each other.

(6.) Find the number of permutations of all letters of the word BASEBALL if the words are to begin & end with vowel.

Sol: There are 3 vowels in the word BASEBALL

2 A's and 1 E.

Five consonants 2 B's, 2 L's & one S.

If we take 2 A's as these are identical their swapping creates no new variety. So arrangement can be done in 1 ways.

A _ _ _ _ _ _ A $\therefore \frac{6!}{2!\times 2!} = \frac{720}{4} = 180$

If two vowels are different A & E then their swapping creates arrangement in 2 ways.

A _ _ _ _ _ _ E ⟹ 180

E _ _ _ _ _ _ A ⟹ 180

$\boxed{\therefore \text{ Total } 180+180+180 = 540}$ letters can be formed, which all begin & end with vowels

(7.) How many 9 letter words can be formed by using the letters of the word ~~du~~ DIFFICULT

Sol:

$$\frac{9!}{2!\,2!\,1!\,1!\,1!\,1!\,1!} = 90720$$

2 F
2 I
1 → DCULT

(8.) (i) Find the number of permutations of the letters of the word MASSASAUGA. (ii) In how many of these All Four A's are together? (iii) How many of them begin with S.

Sol: (i)

$$\frac{10!}{4!\,3!\,1!\,1!\,1!} = 25,200$$

4 A
3 S
1 M U G

(ii) All A's as single letter

$\underline{AAAA}$, S, S, S, M, U, G

$$\frac{7!}{1!\,3!\,1!\,1!\,1!} = 840$$

(iii) For the permutations beginning with S.

there are 9 open positions to fill 2S, 4A, 1 MUG

$$\Rightarrow \frac{9!}{2!\,4!\,1!\,1!\,1!} = 7560.$$

(9) How many positive integers n can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want n to exceed 5,000,000?

$\boxed{\text{Ans: } 720}$

Sol: Here $n = n_1 n_2 n_3 n_4 n_5 n_6 n_7$

n: total 7 digits

If we want n to exceed 5,000,000 then n_1 value should be 5 or 6 or 7

if $n_1 = 5 \Rightarrow \frac{6!}{2!\,1!\,1!\,1!\,1!} = 360$

if $n_1 = 6 \Rightarrow \frac{6!}{1!\,2!\,2!\,1!} = 180$

if $n_1 = 7 \Rightarrow \frac{6!}{1!\,2!\,2!\,1!} = 180$

∴ By sum rule, the no. of n's are 360+180+180=720

(10.) How many ways are there to distribute 10 different books among 15 people if no person is to receive more than 1 book?

Sol: $^{15}P_{10}$ ways

(11.) In how many ways can n persons be seated at a round table if arrangements are considered the same when one can be obtained from the other by rotation?

Sol: Let that one person can be seated anywhere Then the remaining $(n-1)$ persons can be seated in $(n-1)!$ ways.

(12.) It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible.

Sol: 5 men seated in odd places in $5!$ ways

4 women may be seated in even places in $4!$ ways

$\therefore$ The total no. of arrangements is

$5! \times 4! = 120 \times 24 = 2880$ ways.

(13) In how many ways can 6 men and 6 women be seated in a row such that (i) if any person may sit next to any other? (ii) if men & women must occupy alternate seats?

Sol: (i) no distinction is made between men & women. So, 12 persons in all can be seated in $12!$ ways $= 479001600$

(ii) men occupy odd & women occupy even $\Rightarrow 6! \times 6! = 518400$

women occupy odd & men occupy even $\Rightarrow 518400$.

$\therefore$ Total no of ways $= 518400 + 518400 = 1036800$

(14) Find the value of n so that $2P(n,2)+50 = P(2n,2)$

$$2\times \frac{n!}{(n-2)!} + 50 = \frac{(2n)!}{(2n-2)!}$$

$$\Rightarrow 2\times \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} + 50 = \frac{2n(2n-1)\cancel{(2n-2)!}}{\cancel{(2n-2)!}}$$

$$2n(n-1)+50 = 2n(2n-1)$$

$$2n^2 - \cancel{2n} + 50 = 4n^2 - \cancel{2n}$$

$$2n^2 = 50$$

$$n^2 = 25$$

$$n = \pm 5$$

$\therefore n=5$ [since n cannot be negative].

15. Find the number of permutations of the letters of the word MISSISSIPPI. How many of these begin with an I? How many of these begin and end with S?

Sol: Total No. of permutations MISSISSIPPI

4 I
4 S
2 P
1 M

$$\Rightarrow \frac{11!}{4!\,4!\,2!\,1!} = 34650$$

begin with I

3 I
4 S
2 P
1 M

$$\frac{10!}{3!\,4!\,2!\,1!} = 12600$$

begin & end with S

4 I
2 S
2 P
1 M

$$\frac{9!}{4!\,2!\,2!}$$

(16) How many different signals each consisting of six flags hung in a vertical line, can be formed from four identical red flags and 2 identical blue flags?

Sol: $\frac{6!}{4!\times 2!} = 15$

(17) In how many ways can the symbols a, b, c, d, e, e, e, e, e be arranged so that no e is adjacent to another e?

_a_b_c_d_e

As e can't be adjacent 5 e's has to occupy all odd places.

abcd can be arranged in $4! = 24$ ways.

(18) consider the permutations of the letters a, c, f, g, i, t, w, x. (i) How many of these start with t. (ii) How many of these start with t and end with c

t_______ (7)

Sol: (i) start with t $= 7!$

(ii) start with t and end with $c = 6!$ t______c (6)

(19) Five red pen's, two black pens and three blue pens are arranged in a row. If the pens of the same colour are not distinguishable, how man different arrangement are possible.

Sol: $$\frac{10!}{5!\,2!\,3!} = 2520$$

(20) In how many ways can seven books be arranged on a shelf if (a) any arrangement is allowed? (b) three particular books must always be next to eachother? (c) Two particular books must occupy the ends.

Sol: (a) 7 books can be arranged in $7!$ ways $= 5040$

(b) Assuming 3 books as 1 book. Now we have 5 books arranged in $5!$ ways $= 120$

The three books can be arranged in $3 \times 2 = 6$ ways

$\therefore$ Required number of ways $= 120 \times 6 = 720$

(c) two particular books must occupy ends.

$\bar{B}$ - - - - - - - $\bar{B}$ remaining 5 books are arranged in $5!$ ways $= 120$

5 books

2 books at end can be arranged in $2!$ ways $= 2$

$\therefore$ The required number of arrangement $= 120 \times 2 = 240$

(21) In how many ways can 10 books be arranged in a shelf such that a particular pair of books are never together.

Sol:

10 books can be arranged in $10!$ ways

If ~~given~~ pair of books to be together, considering them as one, total sets $= 9$.

No. of ways of arranging 9 books is $9!$ ways.

And those two books are arranged in $2!$ ways.

$\therefore$ Total ways that the pair of books is together $= 9! \times 2!$

No. of ways that are not together

$\Rightarrow 10! - 9! \times 2!$

$\Rightarrow 9!(10-2)$

$\Rightarrow 8 \times 9!$ ways

(22) There are two books of five volumes each and two books of two volumes each. In how many ways can these books be arranged in a shelf such that volumes of same books remain together?

2 books of 5 volumes each $= 5 \times 5!$

2 books of 2 volumes each $= 2 \times 2!$

Total there are 4 sets of books arranged in $4!$ ways.

$\therefore$ Total required number of arrangement is

$4! \times 5! \times 5! \times 2! \times 2!$

$= 1382400$ ways

Combinations

→ Combinations: An ordered selection of r elements of a set containing n distinct elements is called an r-combination of n elements and is denoted by $C(n,r)$ or nC_r or $\binom{n}{r}$

$$C(n,r) = \frac{n!}{(n-r)!\,r!} \quad \text{for } 0 \le r \le n$$

(1) How many committees of 5 with a given chair-person can be selected from 12 persons.

Sol:

Total no of persons = 12

each committee consists of 5 persons

1 person is selected as chairperson.

The chairperson can be selected among 12 person in 12 ways.

out of 5, 1 is chair person

The remaining 4 persons in the committee can be selected in $C(11,4)$ ways.

∴ possible no of such committees is

$$12 \times C(11,4) = 12 \times \frac{11!}{4!\,7!} = 12 \times 330 = 3960.$$

(2) In how many different ways can a committee of 5 teachers and 4 students can be selected from 9 teachers and 15 students?

Sol: 5 teachers can be selected from 9 teachers in $9c_5$ ways.

4 students can be selected from 15 students in $15c_4$ ways

∴ No of ways of selecting 5 teachers & 4 students is $\boxed{9c_5 * 15c_4}$

(3) How many different seven-person committees can be formed each containing 3 women & 4 men from an available set of 20 women & 30 men

Sol: 3 women can be selected by $C(20,3)$

4 men can be selected by $C(30,4)$

$\therefore$ The different seven-person committees can be formed by $C(20,3) * C(30,4)$

(4) A bag contains 5 red marbles and 6 white marbles find the number of ways that 4 marbles can be drawn from the bag if the 4 marbles must be of the same colour.

Sol: 4 red marbles can be drawn by $C(5,4)$

4 white marbles can be drawn by $C(6,4)$

$\therefore$ The no. of ways to draw the 4 marbles of same colour:

$C(5,4) + C(6,4)$.

(5) There are 21 consonants and 5 vowels in the English alphabets. Consider only 8 letter words with 3 different vowels and 5 different consonants.

(a) How many of such words can be formed

Total 21 consonants & 5 vowels.

3 different vowels can be selected in ${}^5C_3$

5 different consonants can be selected in ${}^{21}C_5$

8 letter word with all distinct letters can be formed in $8!$ ways

$\therefore$ ${}^5C_3 * {}^{21}C_5 * 8!$ words can be formed

(b) How many begin with 'a' and end with 'b'

a - - - - - b a is vowel ${}^4C_2$

b is consonant ${}^{20}C_4$

Remaining 6 are arranged in $6!$ ways

$\therefore$ ${}^4C_2 * {}^{20}C_4 * 6!$

(c) How many contain the letters a, b, c

Sol: a is vowel, so a is selected in $4c_2$ ways

b, c are consonants, so they are selected in $19c_3$

Among ~~even~~ 8 letter word a, b, c are 3 letters

So totally 8 letter word is arranged is 8! ways

$\therefore 4c_2 * 19c_3 * 8!$

(6) Find the number of 5-letter words which contains three different consonants and ~~two~~ two different vowels of the English alphabet.

Sol: 3 different consonants can be selected in $21c_3$ ways

2 different vowels can be selected in $5c_2$ ways

5 letter words is arranged in 5! ways.

$\therefore 21c_3 * 5c_2 * 5!$

(7) Find the number of committees of 5 that can be selected from 7 men and 5 women if the committee is to consist of atleast 1 man and atleast 1 woman.

Sol: From given data totally there are 12 persons the number of committees of 5 that can be formed from 12 is

$12c_5$

(or) 7M 5F committe of 5 can be selected as

$7c_1 \times 5c_4 + 7c_2 \times 5c_3 + 7c_3 \times 5c_2 + 7c_4 \times 5c_1$

$7 \times 5 + 21 \times 10 + 35 \times 10 + 35 \times 5 = 35 + 210 + 350 + 175 = 770$

7 men $\Rightarrow 7c_5$ committees consisting of 5 men

5 women $\Rightarrow 5c_5 = 1$ committee consisting of 5 women

$\therefore$ Number of committees containing atleast one man and woman is

$12c_5 - 7c_5 - 5c_5 = \frac{12!}{7!\,5!} - \frac{7!}{5!\,2!} - 1$

$= 792 - 21 - 1 = 770$

(8) At a certain college, the housing office has decided to appoint, for each floor one male & one female residential advisor. How many different pairs of advisors can be selected for a seven floor building from 12 male ~~mah~~ and 15 female candidates?

Sol: Total 7 floor's, for each floor one male & female to be selected

7 candidates can be selected from 12 male in $^{12}C_7$ ways

7 Candidates can be selected from 15 female in $^{15}C_7$ ways.

∴ The total no. of possible pairs of advisors are

$$^{12}C_7 \times {}^{15}C_7 = \frac{12!}{7!5!} \times \frac{15!}{7!8!} = 792 \times 6435 = 5096520.$$

(9) A certain question paper contains two parts A and B each containing 4 questions. How many different ways a student can answer 5 questions by selecting atleast 2 questions from each pair?

Sol: 3 Q's from part A & 2 Q's from part B is done in

$$^4C_3 \times {}^4C_2 = 4 \times 6 = 24 \text{ ways}$$

2Q's from part A & 3Q's from part B is done in

$$^4C_2 \times {}^4C_3 = 6 \times 4 = 24 \text{ ways.}$$

∴ The total no of ways a student can answer 5 questions under the given restrictions is $24+24 = 48$ ways.

(10) Find the number of ways of selecting 4 persons out of 12 persons to a party if two of them will not attend the party together.

Sol: Since two particular persons x and y will not attend the party together, ~~only~~ there are 2 possible cases.

(i) only one of them (i.e., either x or y) is invited

(ii) None of them can be invited.

The number of ways of selecting the invitees with ~~x~~ invited is

$$^{10}C_3 = \frac{10!}{7!\,3!} = \frac{\overset{4}{\cancel{8}}.\overset{3}{\cancel{9}}.10}{\cancel{2}.\cancel{3}} = 120$$

The number of ways of selecting the invitees with y invited is

$$^{10}C_3 = 120$$

If both x and y are not invited then the no of ways of choosing invitees is $^{10}C_4 = 210$

∴ The total no. of invitees selected $= 120 + 120 + 210 = 450$ ways

(11) A party is attended by n persons. If each person in the party shake hands with all the others in the party. Find the number of hand shakes.

sol: 2 persons should be there to determine a handshake

$\therefore$ If each person shakes hands with all the other persons the total number of handshakes is equal to the number of combinations of two persons that can be selected from the n persons.

$$\therefore {}^{n}C_2 = \frac{n!}{(n-2)!\,2!} = \frac{1}{2}n(n-1)$$

(12) How many bytes contain (i) exactly two 1's (ii) exactly four 1's (iii) exactly six 1's (iv) atleast six 1's?

sol: (i) Two positions in 1 byte can be selected in ${}^{8}C_2 = \frac{8!}{6!2!} = \frac{78}{2}$

$= 28$

(ii) ${}^{8}C_4 = 70$ ways

(iii) ${}^{8}C_6 = 28$ ways

(iv) Number of bytes that contain at-least six 1's is given by ${}^{8}C_6 + {}^{8}C_7 + {}^{8}C_8 = 28 + 8 + 1 = 37$ ways

Binomial and Multinomial Theorems

One of the basic properties of ^nC_r is that it is the coefficient of $x^r y^{n-r}$ in the expansion of the expression $(x+y)^n$, where x and y are any real numbers.

$$\boxed{(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}}$$

This result is known as Binomial Theorem for a positive integral index.

The numbers in ^nC_r for $r=0,1,2....n$ in the above result are known as the binomial coefficients.

Multinomial Theorem:

For positive integers n and t, the coefficient of $x_1^{n_1} x_2^{n_2} .... x_t^{n_t}$ in the expansion of $(x_1+x_2+\cdots+x_t)^n$ is $\frac{n!}{n_1!n_2!....n_t!}$ where each n_i is a non-negative integer $\le n$

and $n_1+n_2+n_3+\cdots+n_t = n$

The expression $\frac{n!}{n_1!n_2!..n_t!}$ is also written as

$\binom{n}{n_1\ n_2\ n_3 \cdots n_t}$ and is called multinomial coefficient

① Find the coefficient of x^9y^3 in the expansion of $(x+2y)^{12}$

<u>Sol</u>: By binomial theorem $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$

$$(x+2y)^{12} = \sum_{r=0}^{12} \binom{12}{r} x^r (2y)^{12-r}$$

$$= \sum_{r=0}^{12} \binom{12}{r} 2^{12-r} x^r y^{12-r}$$

The coefficient of x^9y^3 in the above expansion corresponds to $r=9$

$$^{12}C_9 \, 2^{12-9} x^9 y^3 = \frac{12!}{3!\,9!} \cdot 2^3$$

$$= \frac{10 \cdot 11 \cdot 12}{2 \cdot 3} \cdot 2^3$$

$$= 220 \times 8$$
$$= 1760$$

$\therefore$ The coefficient of x^9y^3 is 1760.

② Find the coefficient of x^5y^2 in the expansion $(2x-3y)^7$

<u>Sol</u>: $$(2x-3y)^7 = \sum_{r=0}^{7} \binom{7}{r} (2x)^r (-3y)^{7-r}$$

$$= \sum_{r=0}^{7} \binom{7}{r} 2^r (-3)^{7-r} x^r y^{7-r}$$

$r=5$ gives x^5y^2

$\therefore r=5$

$$^7C_5 \, 2^5 (-3)^2 = \frac{7!}{2!\,5!} \cdot 2^5 (-3)^2 = \frac{\overset{3}{6} \cdot 7}{2} \cdot 2^5 \cdot (-3)^2$$

$= 21 \times 32 \times 9 = 6048$

$\therefore$ 6048 is the coefficient of x^5y^2

③ Find the coefficient of x^9y^3 in the expansion $(2x-3y)^{12}$

Sol:
$$(2x-3y)^{12} = \sum_{r=0}^{12} \binom{12}{r} (2x)^r (-3y)^{12-r} = \sum_{r=0}^{12} \binom{12}{r} 2^r (-3)^{12-r} x^r y^{12-r}$$

$r=9$

$$^{12}C_9 \, 2^9 (-3)^3 = \frac{12!}{9!\,3!} \times 2^9 \times (-3)^3$$

$$= \frac{12\times 11\times 10}{2\cdot 3} \times 2^9 \times (-3)^3$$

$$= -2^{10} \times 3^3 \times 11 \times 10 = 1946$$

④ Evaluate (i) $\binom{12}{5\ 3\ 2\ 2}$ (ii) $\binom{7}{2\ 3\ 2}$

$$= \frac{12!}{5!\,3!\,2!\,2!} = 166320 \qquad = \frac{7!}{2!\,3!\,2!} = 210$$

(iii) $\binom{8}{4\ 2\ 2\ 0} = \frac{8!}{4!\,2!\,2!} = 420$ (iv) $\binom{10}{5\ 3\ 2\ 2}$

Sum of $n_1+n_2+n_3+n_4 \neq 10$

$\therefore$ It is meaningless.

⑤ Find the coefficient of xyz^5 in the expansion of $(x+y+z)^7$

Sol:
$$\binom{7}{n_1, n_2, n_3} x^{n_1} y^{n_2} z^{n_3}$$
Given xyz^5

$\therefore n_1 = 1, n_2 = 1, n_3\ 5$

$$\binom{7}{1,1,5} x^1 y^1 z^5 = \frac{7!}{1!\,1!\,5!} xyz^5 = 6\times 7\ xyz^5$$

$$= 42\ xyz^5$$

$\therefore$ 42 is the coefficient of xyz^5

⑥ Find the term which contains x^{11} and y^4 in the expansion of $(2x^3-3xy^2+z^2)^6$

sol:

$$\binom{6}{n_1\ n_2\ n_3}(2x^3)^{n_1}(-3xy^2)^{n_2}(z^2)^{n_3}$$

$$= \binom{6}{n_1\ n_2\ n_3} 2^{n_1}(-3)^{n_2}\ x^{3n_1+n_2}\ y^{2n_2}\ z^{2n_3}$$

Given x^{11} & y^4

we have $x^{3n_1+n_2}$ y^{2n_2}

$3n_1+n_2 = 11$ $\quad$ $2n_2 = 4$

$3n_1+2 = 11$ $\quad$ $n_2 = 2$

$n_1 = 3$

$n_1+n_2+n_3 = 6$ $\quad$ $\therefore n_3 = 1$

$$\binom{6}{3\ 2\ 1} 2^3(-3)^2\ x^{11}y^4z^2$$

$$= \left\{\frac{6!}{3!\,2!\,1!} \times 8 \times 9\right\} x^{11}y^4z^2$$

$= 4320\, x^{11}y^4z^2.$

Determine the coefficient of

(i) xyz^2 in the expansion of $(2x-y-z)^4$

(ii) $a^2b^3c^2d^5$ in the expansion of $(a+2b-3c+2d+5)^{16}$

(iii) xyz^{-2} in the expansion of $(x-2y+3z^{-1})^4$

(iv) x^3z^4 in the expansion of $(x+y+z)^7$

(v) $x^3y^3z^2$ in the expansion of $(2x-3y+5z)^8$

(vi) $w^3x^2yz^2$ in the expansion of $(2w-x+3y-2z)^8$

(vii) $x_1^2x_3x_4^3x_5^4$ in the expansion of $(x_1+x_2+x_3+x_4+x_5)^{10}$

Combinations with repetitions:

Suppose we wish to select a combination of r object with repetition from a set of n distinct objects. The number of such selections is given by

$$\boxed{C(n+r-1,\ r) \equiv C(r+n-1,\ n-1)}$$

The following are the other interpretations of this number

(i) $C(n+r-1, r) = C(r+n-1, n-1)$ represents the number of ways in which r identical objects can be distributed among n distinct containers.

(ii) $C(n+r-1, r) = C(r+n-1, n-1)$ represents the number of non-negative integer solutions of the equation $x_1+x_2+\cdots+x_n = r$

Note: A non-negative integer solution of the equation $x_1+x_2+\cdots+x_n = r$ is an n-tuple, where $x_1, x_2 \cdots x_n$ are non-negative integers whose sum is r.

→ In how many ways can we distribute 10 identical marbles among 6 distinct containers.

10 identical marbles $\Rightarrow r=10$

6 distinct containers $\Rightarrow n=6$

$$C(10+6-1, 10) = {}^{15}C_{10} = \frac{15!}{10!\cdot 5!} = \frac{11\cdot \cancel{12}^{3}\cdot 13\cdot \cancel{14}^{7}\cdot \cancel{15}^{3}}{1\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdot\cancel{5}} = 3003$$

→ Find the number of non-negative integer solutions of the equation $x_1+x_2+x_3+x_4+x_5=8$.

$r=8,\ n=5$

$C(5+8-1,8)=12_{C_8}=495$.

→ Find the number of distinct terms in the expansion of $(x_1+x_2+x_3+x_4+x_5)^{16}$.

By multinomial theorem $\binom{16}{n_1\ n_2\ n_3\ n_4\ n_5}(x_1)^{n_1}x_2^{n_2}x_3^{n_3}x_4^{n_4}x_5^{n_5}$

$n_1+n_2+n_3+n_4+n_5=16$

$n=5,\ r=16$

$C(5+16-1,16)=20_{C_{16}}=4845$.

→ Find the number of non negative integer solutions of the inequality $x_1+x_2+x_3+x_4\cdots+x_6<10$

$x_1+x_2+x_3+\cdots x_6<10 \Rightarrow x_1+x_2+\cdots x_6\leq 9$

$\Rightarrow x_1+x_2+\cdots x_6\leq 9-x_7$

$x_1+x_2+\cdots x_7=9$

$n=7,\ r=9$

$C(7+9-1,9)={}^{15}C_9=\frac{15!}{9!\,6!}=5005$.

→ In how many ways can 20 similar books be placed on 5 different shelves

$C(5+20-1,20)=24_{C_{20}}$

→ Find the number of ways of placing 8 identical balls in 5 numbered boxes.

$$C(5+8-1, 8) = {}^{12}C_8 = \frac{12!}{4!\,8!} = \frac{\cancel{9}^{3}\cdot\cancel{10}^{5}\cdot 11\cdot \cancel{12}^{3}}{1\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4}}$$

$= 495$

→ Determine the number of nonnegative integer solutions of the equation $x_1+x_2+x_3+x_4=7$

$n=4, \; r=7$

$$C(4+7-1, 7) = {}^{10}C_7 = \frac{10!}{3!\,7!} = \frac{\cancel{8}^{4}\cdot\cancel{9}^{3}\cdot 10}{1\cdot\cancel{2}\cdot\cancel{3}} = 120$$

→ Find the number of distinct terms in the expansion of $(w+x+y+z)^{10}$.

$$\left.\begin{array}{l}\binom{10}{n_1,n_2,n_3,n_4} w^{n_1}x^{n_2}y^{n_3}z^{n_4} \\ n_1+n_2+n_3+n_4=10\end{array}\right\}$$

$n=4, \; r=10$

$$C(4+10-1, 10) = {}^{13}C_{10} = \frac{13!}{10!\,3!} = \frac{11\cdot\cancel{12}^{2}\cdot 13}{\cancel{2}\cdot\cancel{3}}$$

$= 286$

→ How many integer solutions are there to $x_1+x_2+x_3+x_4+x_5=20$ where each $x_i \geq 2$?

Given $x_i \geq 2$

$x_1 = y_1+2, \; x_2 = y_2+2, \; x_3 = y_3+2, \; x_4 = y_4+2 \; x_5 = y_5+2$

$y_1+y_2+y_3+y_4+y_5 = 20-10$

$y_1+y_2+y_3+y_4+y_5 = 10$

$n=5, \; r=10$

$C(5+10-1, 10) = {}^{14}C_{10}$

→ Find the number of integer solutions of

$x_1+x_2+x_3+x_4+x_5=30$ where $x_1 \geq 2, x_2 \geq 3, x_3 \geq 4, x_4 \geq 2$
$x_5 \geq 0$

Sol: $x_1 \geq 2 \Rightarrow y_1+2$, $x_2 \geq 3 \Rightarrow y_2+3$, $x_3 \geq 4 \Rightarrow y_3+4$

$x_4 \geq 2 \Rightarrow y_4+2$ $x_5 \geq 0 \Rightarrow y_5$.

$(y_1+2)+(y_2+3)+(y_3+4)+(y_4+2)+y_5=30$

$y_1+y_2+y_3+y_4+y_5=30-11$

$y_1+y_2+y_3+y_4+y_5=19$

$n=5, r=19$

$C(5+19-1, 19) = {}^{23}C_{19} = \dfrac{23!}{19!4!} = 8855$

→ How many integer solutions are there to

$x_1+x_2+x_3+x_4+x_5=20$, where $x_1 \geq 3, x_2 \geq 2, x_3 \geq 4, x_4 \geq 6, x_5 \geq 0$?

Sol: $x_1 \geq 3 \Rightarrow y_1+3$, $x_2 \geq 2 \Rightarrow y_2+2$, $x_3 \geq 4 \Rightarrow y_3+4$

$x_4 \geq 6 \Rightarrow y_4+6$, $x_5 \geq 0 \Rightarrow y_5+0$

$y_1+y_2+y_3+y_4+y_5 = 20-(3+2+4+6+0)$
$= 20-15$

$y_1+y_2+y_3+y_4+y_5=5$

$n=5, r=5$

$C(5+5-1, 5) = {}^{9}C_5$

→ Find the number of non negative integer solutions of the inequality $x_1+x_2+x_3+x_4+x_5 \leq 19$

$x_1+x_2+x_3+x_4+x_5 = 19-x_6$ $n=6$ $C(6+19-1, 19)$

$x_1+x_2+x_3+x_4+x_5+x_6=19$ $r=19$ $= {}^{24}C_{19}$

→ Determine the integer solutions of the equation $x_1+x_2+x_3+x_4=32$, where $x_1,x_2 \geq 5$, $x_3,x_4 \geq 7$

Sol: $x_1 = y_1+5$, $x_2 = y_2+5$, $x_3 = y_3+7$, $x_4 = y_4+7$

I) $y_1+y_2+y_3+y_4 = 32-5-5-7-7$

$y_1+y_2+y_3+y_4 = 8$

$n=4$, $r=8$

$C(4+8-1, 8) = {}^{11}C_8$

→ Find the number of ways of placing 20 identical balls into 5 boxes with atleast one ball put into each box.

Sol: If one ball is kept in each box then 5 balls are exhausted.

remaining 15 balls can be placed into 5 boxes with $n=5$ $r=15$

$C(5+15-1, 15) = {}^{19}C_{15}$ ways.

→ In how many ways can we distribute 7 apples and 6 oranges among 4 children so that each child gets atleast 1 apple?

Sol: Suppose if we give 1 apple to each child then 4 apples are exhausted and remaining 3 apples distributed among 4 children as $r=3$, $n=4$

$C(4+3-1, 3) = {}^{6}C_3$ ways

6 oranges can distributed to 4 children in $r=6$, $n=4$

$C(4+6-1, 6) = {}^{9}C_6$ ways.

∴ By product rule the fruit distribution is

${}^{6}C_3 \times {}^{9}C_6 = \frac{6!}{3! \times 3!} \times \frac{9!}{6! \, 3!} = 20 \times 84 = 1680$ ways

→ A total of Rs. 10,000/- is to be distributed to four persons A, B, C, D in multiples of Rs. 1000/- In how many ways can the distribution be done.

(i) If there is no restriction

(ii) If every one of these persons should receive atleast Rs. 1000/-?

(iii) If every one should receive atleast Rs. 1000/- and A in particular should receive atleast Rs 5000/-.

Sol: (i) No restriction.

Rs. 10,000/- treated as 10 units where each unit is 1000/-

$\therefore r = 10$, 4 persons $n = 4$

$$C(4+10-1, 10) = {}^{13}C_{10} = \frac{13!}{3!\,10!} = \frac{11 \cdot \cancel{12}^{2} \cdot 13}{\cancel{2} \cdot \cancel{3}}$$

$$= 143 \times 2$$

$$= 286$$

(ii) every one of person should receive atleast 1000/-

Assuming that every person A, B, C, D received 1000 each ie, 4 units Consumed

then remaining are 6 units to be distributed among 4

$r = 6$, $n = 4$

$$C(4+6-1, 6) = {}^{9}C_{6} = \frac{9!}{3!\,6!} = \frac{7 \cdot \cancel{8}^{4} \cdot \cancel{9}^{3}}{\cancel{2} \cdot \cancel{3}}$$

$$= 28 \times 3 = 84$$

(iii) every one should receive 1000/-
so 4 units consumed for A, B, C, D
and A should receive 5000/-
Already 1 consumed by A so again 4 units to A

So out of 10 8 units are consumed remaining 2 units
$C(4+2-1, 2) = {}^{5}C_{2} = 10$

→ How many ways are there to place 12 marbles of the same size in five different jars.

(a) if the marbles are all black

$n=5, r=12$ $C(5+12-1, 12) = {}^{16}C_{12}$

(b) if the marbles are all of different colours.

$n^r = 5^{12}$ ways.

→ A total amount of RS. 1500 is to be distributed to 3 poor students A, B, C of a class. In how many ways the distribution can be made in multiples of Rs. 100/-

(i) if every one of these must get atleast Rs 300

(ii) if A must get atleast Rs 500 and B & C must get atleast Rs. 400 each.

(i) If we take 100/- as 1 unit we have 15 units to distribute

If 3 units distributed to 3 student

remaining 6 units distributed to 3 student:

in $C(3+6-1, 6) = {}^{8}C_{6} = 28$

(ii) A → 5 units
B - 4 units
C - 4 units
Total 13 units distributed

remaining 2 units to be distributed to 3 student.

$n=3, r=2$

$C(3+2-1, 2) = {}^{4}C_{2} = 6$ ways.

The principle of Inclusion-Exclusion:

$$|\bar{A}| = |U| - |A| \rightarrow (1)$$

$$|A \cup B| = |A| + |B| - |A \cap B| \rightarrow (2)$$

$$|\bar{A} \cap \bar{B}| = |\overline{A \cup B}| = |U| - |A \cup B|$$

$$|\bar{A} \cap \bar{B}| = |U| - |A| - |B| + |A \cap B| \longrightarrow (3)$$

Expressions (2) & (3) are equivalent to one another. Either of these is referred to as ~~Addition rule~~ Addition principle (Rule) or the principle of inclusion-exclusion for two sets.

If A & B are disjoint sets then

$$|A \cup B| = |A| + |B| - |\phi| = |A + B|$$

This is known as principle of disjunctive counting for two sets.

→ In a class of 52 students, 30 are studying C++, 28 are studying pascal and 13 are studying both languages. How many in this class are studying atleast one of these languages? How many are studying neither of these languages.

$|U| = 52$, $|A| = 30$ $|B| = 28$, $|A \cap B| = 13$

$|A \cup B| = |A| + |B| - |A \cap B| = 30 + 28 - 13 = 45$

$|\overline{A \cup B}| = |U| - |A \cup B| = 52 - 45 = 7$

∴ 45 students of the class study atleast one of the two languages and 7 students of the class study neither of these languages

→ In a sample of 100 logic chips, 23 have a defect D_1, 26 have defect D_2, 30 have a defect D_3. 7 have defects D_1 and D_2, 8 have defects D_1 and D_3, 10 have defects D_2 & D_3 and 3 have all the three defects. Find the number of chips having (i) atleast one defect (ii) no defect.

$|U| = 100$, $|A| = 23$, $|B| = 26$, $|C| = 30$

$|A \cap B| = 7$, $|A \cap C| = 8$, $|B \cap C| = 10$, $|A \cap B \cap C| = 3$

The set of chips having atleast one defect is $A \cup B \cup C$,

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$= 23 + 26 + 30 - 7 - 8 - 10 + 3 = 57$

The set of chips with no defect is $(\overline{A \cup B \cup C})$

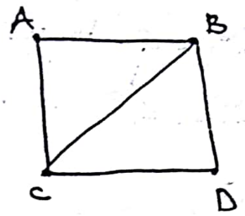
$|\overline{A \cup B \cup C}| = |U| - |A \cup B \cup C| = 100 - 57 = 43.$

UNIT-5 GRAPH THEORY

→ **Graph:-**

A graph G is a pair of sets (V, E) where V = set of vertices (nodes) and
E = set of edges (lines).

Ex:-



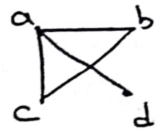
$V = \{A, B, C, D\}$
$E = \{(A,B), (A,C), (B,C), (B,D), (C,D)\}$.

⊛ $E \subseteq V \times V$.

→ **Types of graphs:-**

(i) **Undirected graph:** An undirected graph G is a pair of sets (V, E) where V is set of vertices and E is set of unordered pair of vertices such that $E \subseteq V \times V$

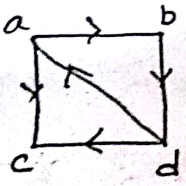
Ex:-



$V = \{a, b, c, d\}$
$E = \{(a,b), (a,c), (a,d), (b,c)\}$.

(ii) **Directed graph:-** In a digraph the elements of E are ordered pairs of vertices. In this case an edge (u, v) is said to be from u to v. $E \subseteq V \times V$.

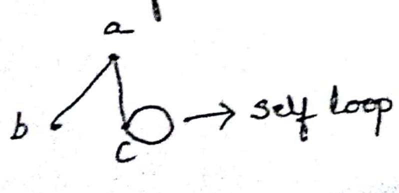
Ex:-



$V = \{a, b, c, d\}$
$E = \{(a,b), (a,c), (d,a), (b,d), (d,c)\}$.

(iii) **Self Loop:-** An edge E is drawn between a vertex and itself.

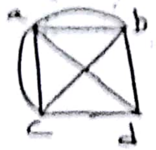
Ex:-



$V = \{a, b, c\}$
$E = \{(a,b), (a,c), (c,c)\}$.

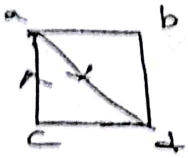
(iv) Multigraph:- A graph containing more than one edge between a pair of vertices is called as a multigraph.

Ex:-



(v) Mixed graph:- A graph containing both directed and undirected edges is called a Mixed graph.

Ex:-

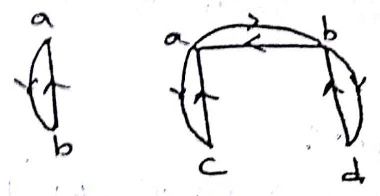


$V = \{a, b, c, d\}$

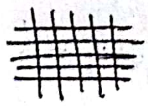
$E = \{(a,b), (c,a), (a,d), (b,d), (c,d)\}$

(vi) Symmetric directed graph:- A graph $G(V,E)$ such that for every directed edge $\{u,v\} \in E$ - there exist a directed edge $\{v,u\} \in E$.

Ex:-

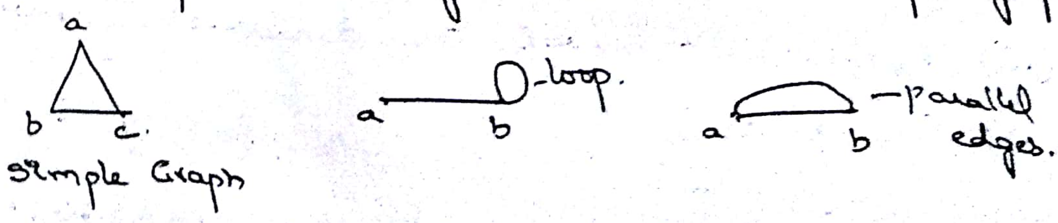


(vii) Finite and Infinite Graph:- A graph with finite number of vertices and finite number of edges is called finite graph otherwise it is called infinite graph.

Ex:- infinite graph. 

(viii) Simple Graph:- A graph $G(V,E)$ with no self loops and parallel edges is called simple graph.

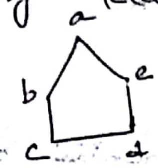
Ex:-



→ **Order of the graph:-** A graph $G(V,E)$ with V as set of vertices then $|V(G)|$ or $|V|$ denotes number of vertices is called order of the graph.

→ **Size of the graph:-** The number of edges in a graph G is called the size of the graph. Denoted by $|E(G)|$ or $|E|$.

Ex:-



$|V| = 5$

$|E| = 5$

→ **Degree:-** Degree of a vertex in an undirected graph is the number of edges incident with it.

(*) A loop at a vertex contributes twice to the degree of that vertex.

(*) The degree of a vertex v is denoted by $deg(v)$.

(*) A vertex with degree 'zero' is called an Isolated vertex.

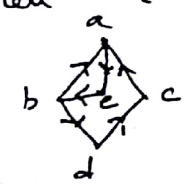
(*) **Pendant vertex:** A vertex of degree 1 is called pendant vertex.

Ex:-

(*) **Indegree:** In a digraph, the number of vertices incident to a vertex is called indegree of the vertex. Denoted by $deg^+(v)$

(*) **Out degree:** The number of vertices incident from a vertex is called outdegree of the vertex. $deg^-(v)$.

Ex:-



$deg^+(a) = 2$

$deg^-(a) = 1$ $\Rightarrow deg(a) = deg^+(a) + deg^-(a)$

$= 2+1 = 3.$

(*) A loop contributes ~~degree~~ one edge for in-degree and one for out-degree.

→ **Neighbours (adjacent):** If there is an edge incident from u to v, or incident on u and v then u and v are said to be adjacent.

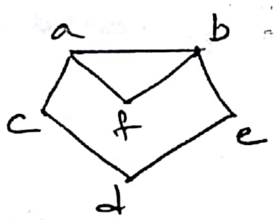
→ **Minimum degree of a graph:-** It is represented by $\delta(G)$. It is the minimum of all the degrees of vertices in a graph G.

→ **Maximum degree of a graph:** Represented by $\Delta(G)$. It is the maximum of all the degrees of vertices in a graph G.

→ **Degree Sequence:** If $v_1, v_2 \ldots v_n$ are the vertices of a graph G, then the sequence $\{d_1, d_2 \ldots d_n\}$ where d_i = degree of v_i is called the degree sequence of G. such that

$$d_1 \leq d_2 \leq d_3 \leq \ldots \leq d_n.$$

Ex:



$d(a) = 3$

$d(b) = 3$

$d(c) = 2$

$d(d) = 2$

$d(e) = 2$

$d(f) = 2$

degree sequence = $\{2, 2, 2, 2, 3, 3\}$.

→ First theorem of graph Theory:

Sum of degrees Theorem:

Statement:- If a graph G contain a vertex set $V = \{v_1, v_2, \ldots v_n\}$ then $\sum_{i=1}^{n} deg(v_i) = 2 \cdot |E|$.

Proof: suppose the graph G contains n edges says $\{e_1, e_2 \ldots e_n\}$

let an edge e_i $i=1$ to n connects two vertices say u and v.

Then e_i contributes a degree 1 to u and a degree 1 to v.

∴ Every edge contributes a degree 2 to the total degree of the graph.

As there are n edges in the graph and each edge contributes 2 to the total degree

∴ Total degree $= 2+2+ \ldots + n$ times

$$\sum_{i=1}^{|V|} deg(v_i) = 2n$$
$$= 2|E|.$$

Examples:- How many vertices will the graph G contains if (a) 16 edges, all vertices of degree 2.

$$\sum_{i=1}^{|V|} deg(v_i) = 2|E|$$

$$\sum_{i=1}^{|V|} 2 = 2 \times 16$$

$$2(|V|) = 32$$

$$|V| = 16.$$

(2) 21 edges, 3 vertices of degree 4 and other vertices of degree 3.

$$\sum_{i=1}^{3} 4 + \sum_{4}^{|V|} 3 = 2 \times 21$$

$$4(3) + 3(|V|-3) = 42$$

$$3|V| = 39$$

$$|V| = 13.$$

(*) In a non-directed graph G. if $k = \delta(G)$ and $m = \Delta(G)$ then

$$k.|V| \leq 2.|E| \leq m.|V|$$

→ Theorem 2:

Statement: In a graph G there are even number of odd degree vertices.

Proof: Given $\sum_{i=1}^{|V|} deg(v_i) = 2|E|.$

Let V_o denotes the set of vertices of odd degree.

V_e denotes the set of vertices of even degree.

$$\therefore V = V_o + V_e$$

$$V_o \cap V_e = \phi$$

$$|V| = |V_o| + |V_e|$$

$$\sum_{i=1}^{|V|} \left(deg(V_{oi}) + deg(V_{ei})\right) = \text{even}.$$

⇒ sum of odd no's + sum of even no's = even.

⇒ sum of odd no's + even = even

⇒ sum of odd no's = even.

∴ The number of odd no's are even to become the sum of odd number as even.

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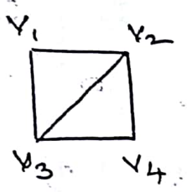
→ Representation of Graphs:-

(i) Adjacency matrix:- Let $G(V,E)$ be a graph with n vertices ordered from $v_1, \dots, v_n$ then the matrix A_G or $A(G)$ of order $n \times n$ is defined by $A_G = [a_{ij}]_{n\times n}$ where

$$a_{ij} = 1 \text{ if } v_i \text{ is adjacent to } v_j$$
$$= 0 \text{ otherwise.}$$

Ex:-



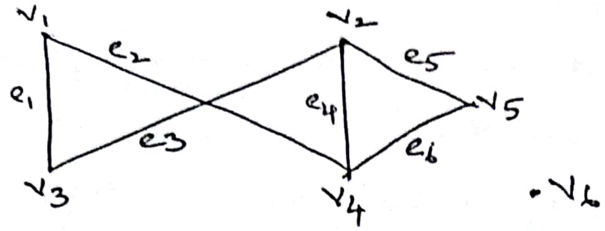
$$A_G = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

- ⊛ If G is simple, then diagonal elements are zero
- ⊛ It is easy and simple to implement.
- ⊛ If G is simple, the degree of vertex v_i is given by the sum of the elements in row i of A_G.
- ⊛ The drawback is the matrix takes $O(n^2)$ space in memory.

(ii) Incidence Matrix:- Let $G(V,E)$ be a graph with n vertices ordered from $v_1, v_2 \dots v_n$ and m edges ordered from $e_1, e_2 \dots e_m$ then the matrix $I(G)$ or I_G of order $n \times m$ is defined by $I_G = [a_{ij}]_{n\times m}$ where

$$a_{ij} = 1 \text{ if } v_i \text{ is incident on } e_j$$
$$= 0 \text{ otherwise.}$$

Ex:-



$$I_G = \begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline v_1 & 1 & 1 & 0 & 0 & 0 & 0 \\ v_2 & 0 & 0 & 1 & 1 & 1 & 0 \\ v_3 & 1 & 0 & 1 & 0 & 0 & 0 \\ v_4 & 0 & 1 & 0 & 1 & 0 & 1 \\ v_5 & 0 & 0 & 0 & 0 & 1 & 1 \\ v_6 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

(*) If any ~~row~~ contains all zero's then that vertex is called isolated vertex.

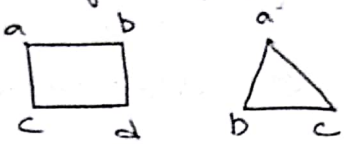
(iii) path Matrix :- Let $G(V,E)$ be a simple (digraph) G with no parallel edges with n vertices ordered from $v_1, v_2 \dots v_n$ then the $n \times n$ matrix $P(G)$ is defined by

$$P = [a_{ij}]_{n \times n} \text{ where}$$

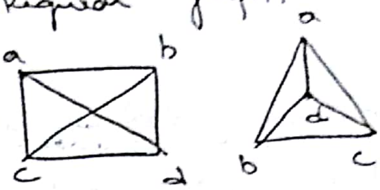
$$a_{ij} = 1 \text{ if } v_i \text{ has a path to } v_j$$
$$= 0 \text{ otherwise.}$$

2). K-Regular Graph:- A graph G is K-Regular if all its vertices are of same degree K.

Ex:- 2-Regular graph.

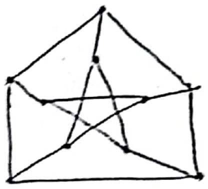


3-Regular graph.



(*) 3-Regular graph are called cubic graphs.

(*) A 3-Regular graph with 10 vertices and 15 edges is called the peterson graph.

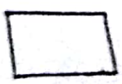


(*) A cubic graph with $8 = 2^3$ vertices is called the three-dimensional hypercube and denoted by Q_3.

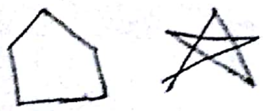
(*) In general, for any +ve integer K, a loop-free K-regular graph with 2^K vertices is called K-dimensional hypercube or K-Cube and denoted by Q_K. and contains $K \cdot 2^{K-1}$ edges.

3) Cycle Graph:- A graph G is said to be cyclic if all the edges form one cycle of length 'n' and is denoted by C_n.

Ex:- C_4 :

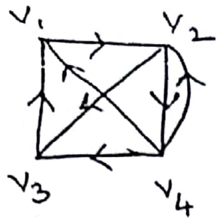


C_5 :



(*) A Cycle graph C_n of order n has n vertices and n edges.

Ex:-



$$P_G = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

(*) A path matrix with all 1's is called a fully connected graph.

(*) If a column has all 0's then the indegree of that vertex is zero.

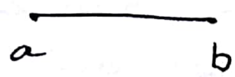
(*) A path matrix cannot completely define a graph.

→ Types of Graphs:-

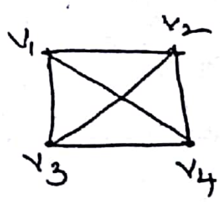
1) Complete graph (K_n): A graph with n vertices is said to be complete graph if it contains n mutually adjacent vertices.

Ex:-

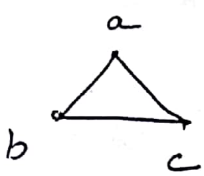
K_2 :



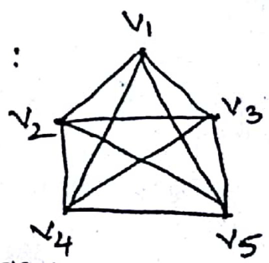
K_4 :



K_3 :



K_5 :



(*) The degree of a complete graph is n-1 of the order of the graph.

(*) K_5 graph is called Kuratowski's first graph.

4) Wheel graph:- A wheel graph is obtained by adding a single new vertex to each vertex of a cycle graph of order $n-1$. It is denoted by W_n.

Ex:- W_4 : => $n=4$

cycle graph of order $n-1 : 3 \Rightarrow C_3$

W_4.

(*) Here W_n has n vertices $2(n-1)$ edges.

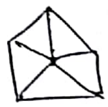
Second way of representing wheel graph:-

draw a wheel graph by adding a new vertex to cycle graph of order n.

W_5 : C_5 :

C_5 :

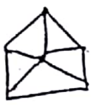
add vertex



W_n has $n+1$ vertices and $2n$ edges.

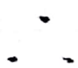
W_6 : C_5

W_6 :



5) Null graph:- A null graph of order n is a graph with n vertices and no edges.

Ex:-

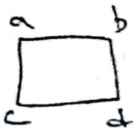


3 vertices

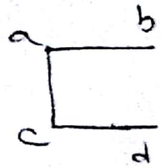
6) Path graph:- A path graph is obtained by removing any one edge from C_n.

Ex:- P_4 :

C_4 :



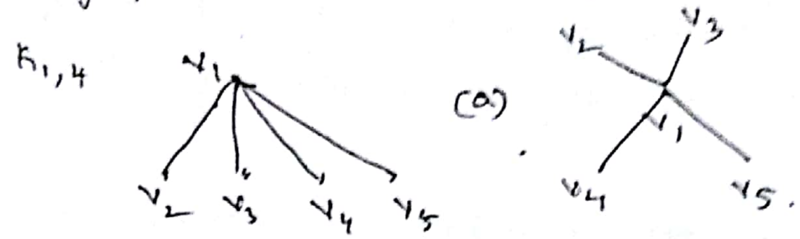
$c-\{b,d\}$



$\Rightarrow P_4$

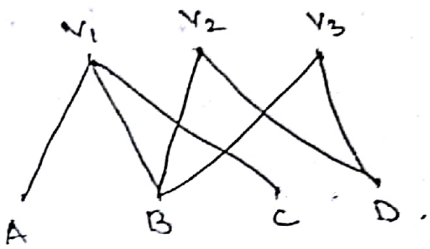
7) Star graph:- A graph of the form $K_{1,n}$ is called a star graph.

Ex: $K_{1,4}$



8) Bipartite Graph:- A graph in which the vertex set can be partitioned into two sets m and n in such a way that edge in G joins a vertex in m to a vertex in n.

Ex:-



(*) Two set m and n are such that $m \cap n = \phi$.

$m \cup n$ = vertex set.

(*) No edge that joins two vertices both of which are in m or n.

9) Complete Bipartite Graph:- A bipartite graph in which every vertex of M is adjacent to every other vertex of N is called complete bipartite graph.

(*) It is denoted as $K_{m,n}$. It has $m+n$ vertices and mn edges.

(*) The number of edges in bipartite graph is $\leq (n^2/4)$.

→ **Isomorphism:-**

Two graphs G and G' are said to be isomorphic if there is a function $f: V(G) \to V(G')$ from vertices of G to vertices of G' such that

(i) f is one-one fn.

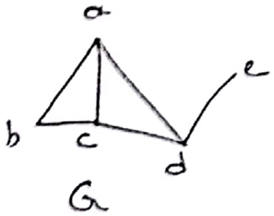
(ii) f is onto

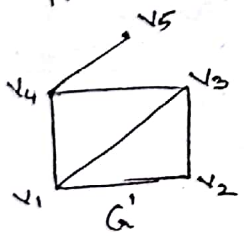
(iii) for each pair of vertices u and v, an edge $\{u,v\} \in E(G)$ iff $\{f(u), f(v)\} \in E(G')$.

Any function with the above 3 properties is called an isomorphism from G to G'.

The condition (iii) says that vertices u and v are adjacent in G iff $f(u)$ and $f(v)$ are adjacent in G'.

Ex:-





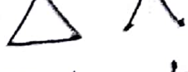
(i) one-one
$a \to v_1$
$b \to v_2$
$c \to v_3$
$d \to v_4$
$e \to v_5$

(ii) range = codomain.

(iii) $a - b \iff v_1 - v_2$
⋮
for every pair of vertices in G there is a corresponding edge in G'.

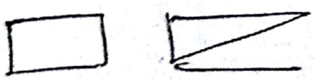
⊛ **Observations:-**

(i) $|V(G)| = |V(G')|$



Not isomorphic $G \ncong G'$

(ii) $|E(G)| = |E(G')|$



$G \ncong G'$

(iii) degree sequences of G and G' are same.

$G \neq G'$

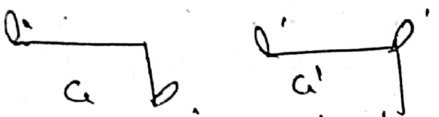
(iv) if u-u is a loop then f(u)-f(u) is a loop in G'.

(v) If $v_0 - v_1 - \cdots v_n$ is a path in G then $f(v_0) - f(v_1) - f(v_n)$ is a path in G'.

(vi) if $v_0 - v_1 - \cdots - v_n - v_0$ is a cycle in G then $f(v_0) - f(v_1) \cdots - f(v_n) - f(v_0)$ is a cycle in G'.

(vii) Number of cycles of length of n are equal.

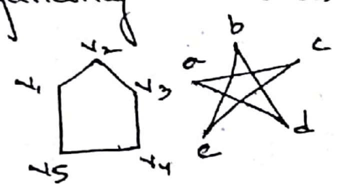
Ex:



No: of cycle of length 1 in G = 2
" " " " " G' = 2

(viii) Adjacency matrices are equal.

Ex:

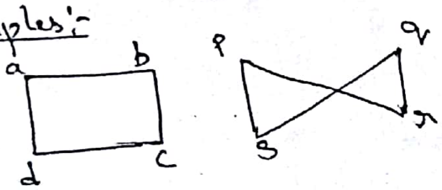


$v_1 - e$
$v_2 - b$
$v_3 - d$
$v_4 - a$
$v_5 - c$.

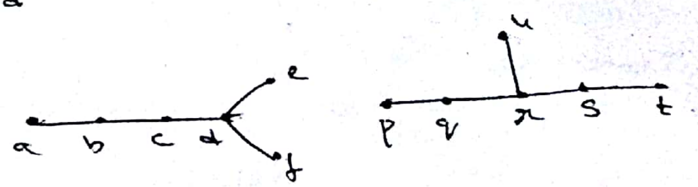
$\therefore G \cong G'$

Examples:-

1)



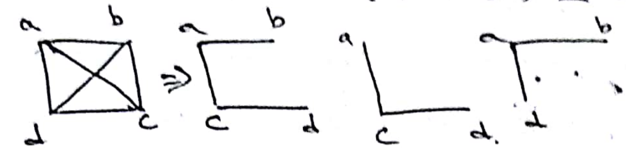
2)



→ **Subgraph:-** Let G be a graph, we say H as a subgraph of G iff (i) $V(H) \subseteq V(G)$

(ii) $E(H) \subseteq E(G)$.

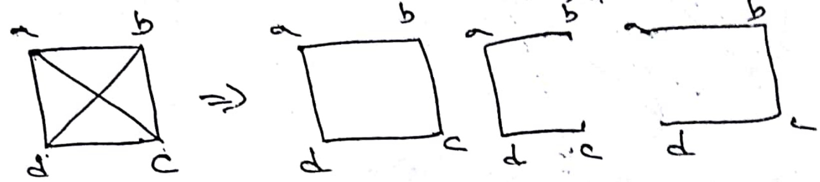
Ex:-



→ **Spanning subgraph:-** we say a graph H as a spanning subgraph of G iff (i) $V(H) = V(G)$

(ii) $E(H) \subseteq E(G)$.

Ex:-

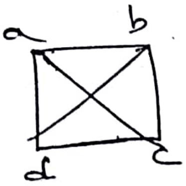


→ **Induced subgraph:-** Let G be a graph, let W be a vertex set subset of G, then subgraph induced by vertex set W is given by

(i) $V(H) = W$

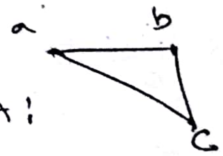
(ii) $E(H)$ = All edges that are adjacent within vertices of W.

Ex:-



$W = \{a, b, c\}$

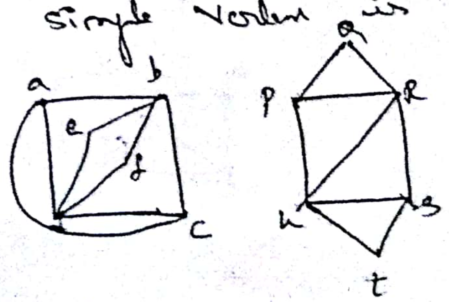
H:

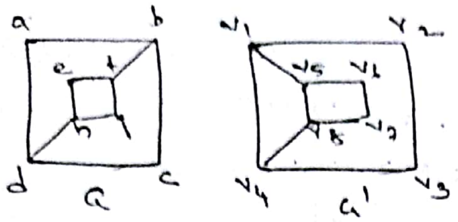


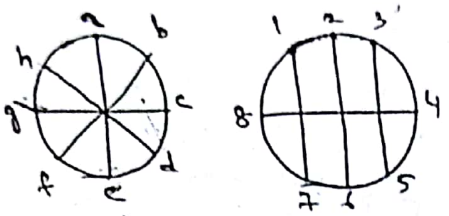
Note:-

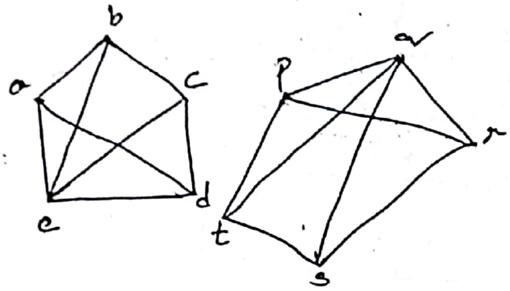
1. Every graph is a subgraph of itself.
2. Null graph is a subgraph of any graph.
3. A simple vertex is a subgraph of a graph.

Ex:-



2) 

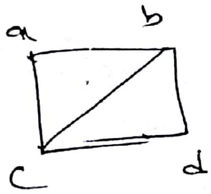
3) 

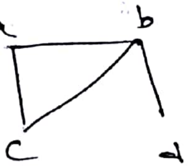
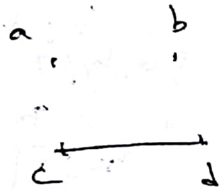
4) 

→ **Complement of a subgraph (H):-**

The complement of subgraph H in graph G is given by $\bar{H}$ as $V(\bar{H}) = V(H)$

$E(\bar{H}) = \{(x,y) \mid (x,y) \notin H$ and $(x,y) \in G\}$

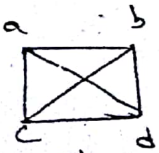
Ex:- 

H:  $\bar{H}$: 

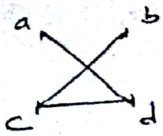
→ **Complement of a graph:-**

The complement of a graph G of order n is equivalent to complement of G in K_n.
(or) let $G(V,E)$ be graph then $\bar{G}$ is given by

$V(\bar{G}) = V(G)$
$E(\bar{G}) = \{(x,y) \mid (x,y) \notin G\}$

K4 

Ex:-  order of G is 4

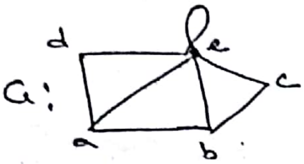
$\bar{G}$: 

→ Operations on a graph:-

(1) deletion of a vertex:-

Let G be a graph with vertex set $\{v_1, v_2 --- v_n\}$ then graph with $G - v_i$ is given by removing the vertex v_i from G and all edges incident to v_i.

Ex:-

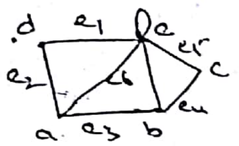


$G - \{a, b\}$:

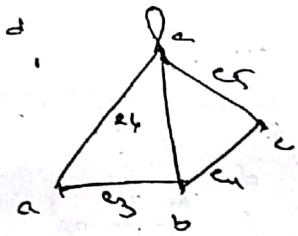
(2) Deletion of an edge:-

Let G be a graph with edge set $\{e_1, e_2 -- e_n\}$ then graph with $G - e_i$ is given by removing the edge e_i from G without any change on vertices.

Ex:-



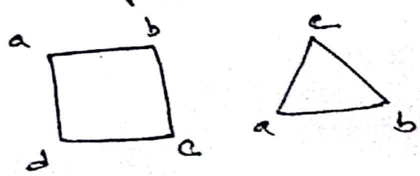
$G - \{e_1, e_2\}$:



(3) Union of two graphs:-

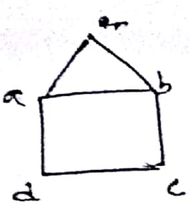
Let G_1 and G_2 be two graphs with $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ then union of G_1 and G_2 is given by $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

Ex:-



$V_1 \cup V_2 = \{a, b, c, d, e\}$

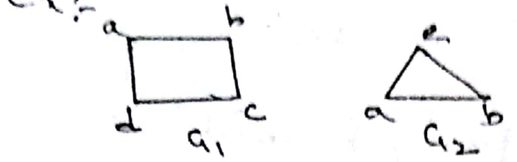
$G_1 \cup G_2$:



4) <u>Intersection of two graphs</u>:- Let $G_1(V_1, E_1)$, $G_2(V_2, E_2)$ be two graphs then intersection $G_1 \cap G_2$ is given by

$$G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2).$$

Ex:-



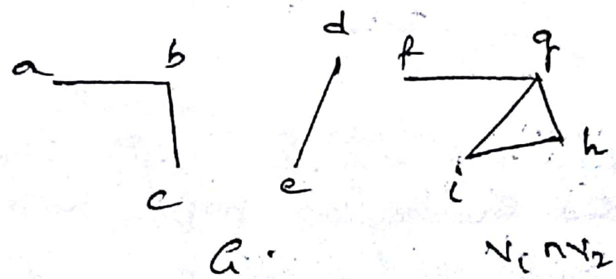
$G_1 \cap G_2$:

→ <u>Components of a graph</u>:-

A <u>disconnected graph</u>: A graph that is not connected is said to be disconnected graph.

* The vertices in a disconnected graph can be partitioned into disjoint non-empty sets $V_1, V_2, \ldots V_n$ then the graphs $H_1, H_2 \ldots H_n$ induced by sets $V_1, V_2 \ldots V_n$ resp. are called components of a graph.
* Each component is a connected graph.
* The no of components in a graph G is denoted by $C(G)$.
* In a connected graph $C(G) = 1$.

Ex:-



$$V_1 \cap V_2 \cap V_3 = \phi$$

$$C(G) = 3.$$

$$V_1 \cup V_2 \cup V_3 = V.$$

→ Trees:-

A simple graph G is a tree iff each pair of vertices is connected by a unique simple path.

(or)

A graph G is a tree iff it is connected and has no cycles..

Ex:

(1)

(2)

→ A tree with n vertices has n-1 edges.

→ **Rooted tree:** D is a directed graph and G is its underlying graph.

D is called directed tree whenever G is a tree.

* A directed ~~tree~~ T is called a Rooted tree if

(i) T contains a unique tree, called the root, whose in-degree is equal to 0 and

(ii) the in-degree of all other vertices of T are equal to 1

Ex:

→ **Binary tree:-** A tree in which every nonleaf node has atmost 2 childs.

 ,

→ Construction of spanning Tree:

* Two Techniques to construct a spanning tree
  (i) DFS (depth first search).
  (ii) BFS (Breadth first search)

1) DFS Algorithm | procedure for DFS Technique:

Let $G = (V, E)$ be a connected graph of order n. with vertices labelled $v_1, v_2 \ldots v_n$ in some specified order.

Let v be a variable stands for vertex being considered.

We use stacks to store the vertices.

step 1: Assign the first vertex v_1 to v and initialize T as the tree consisting of just this vertex.

step 2: select the next vertex(v_k) such that it is adjacent to v [in some specified order] and is not already included in T.

step 3: Attach the edge $\{v, v_k\}$ to T

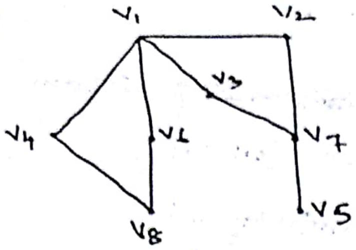
step 4: Assign now v_k to v as current a top of stack.

step 5: Return to step 2.

step 6: if $v = v_1$, the tree T is spanning tree.

step 7: if $v \neq v_1$, back track from v, if u is the parent of the vertex assigned to v in T, then assign u to v and return to step 2.

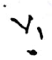
Example:

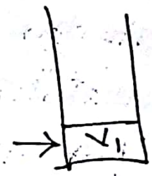


1) Initially the stack is empty.

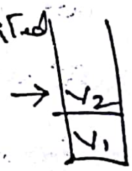
2) Start at a vertex V_1 and follow the alphabetical order or numerical order, to insert adjacent vertex into stack.

3) push V_1 into stack as it is empty and add into empty spanning tree T as vertex not visited.

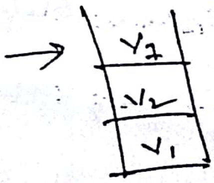


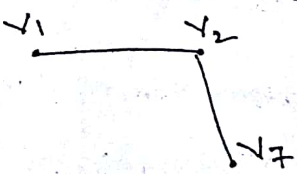


4) as V_1 as adjacent vertices V_2, V_3, V_4, V_6 a numerical order is V_2 insert V_2 to stack now top of stack is V_2 and add edge between V_1 and V_2, V_2 not visited

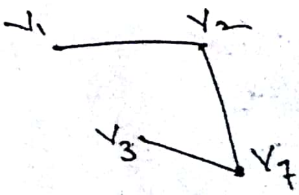


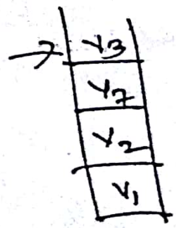
5) V_2 is top, V_2 as adjacent vertices V_7 so push V_7 into stack.



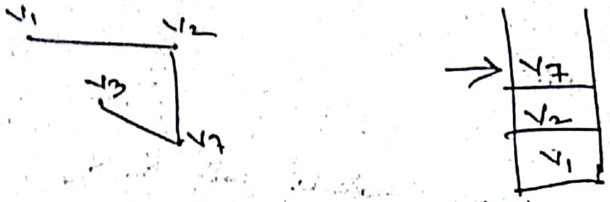


6) Now V_7 is top, V_7 as V_3, V_5 so push V_3

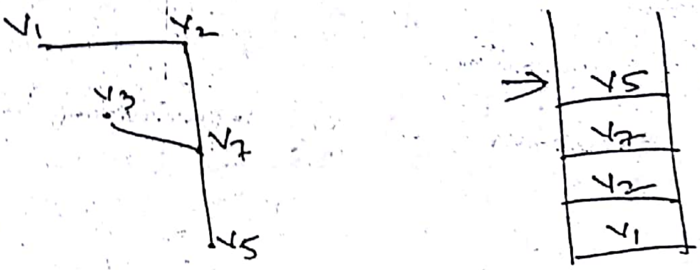




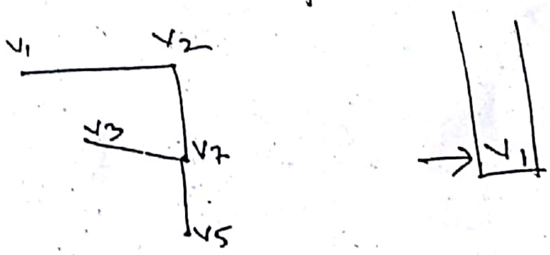
7) Now V_3 is top, V_3 has no new vertices to add which are adjacent.
So pop V_3 from stack, to go its parent node



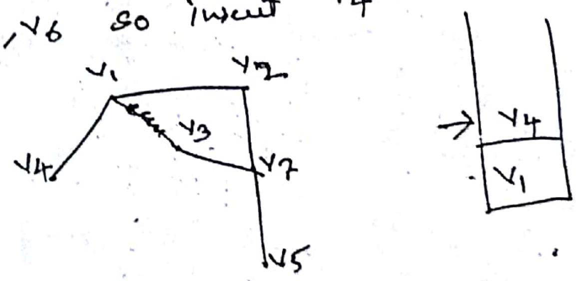
8) Now top is V_7, as V_7 as new V_5 vertex adj to V_7 add V_5 to stack.



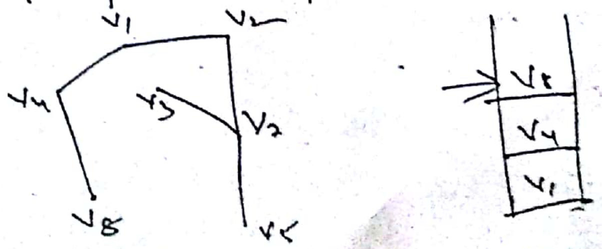
9) no new vertices adj to V_5 – pop V_5, pop V_7, pop V_2



10) V_1 is top of stack, V_1 as new vertices adj to it.
V_4, V_6 so insert V_4



11) V_4 is top, V_4 adj are V_8 push V_8



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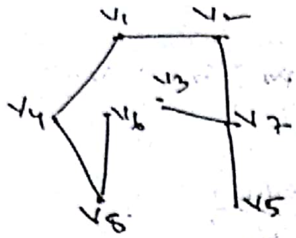
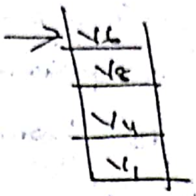
12) V_8 is top, insert V_6 to stack. since V_6 adj. to V_8

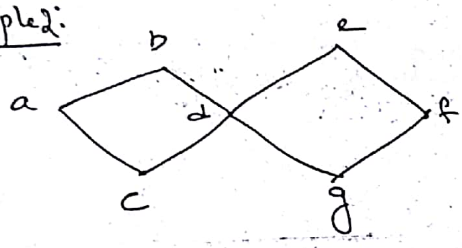


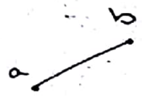


13) no more new vertices adj to V_6, pop V_6, pop V_8 pop V_4, and pop V_1 all vertices are in the spanning tree obtained and visited, and stack is empty.

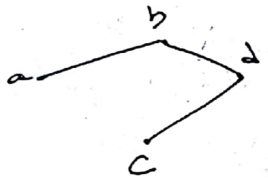
The obtained is a spanning tree using DFS.

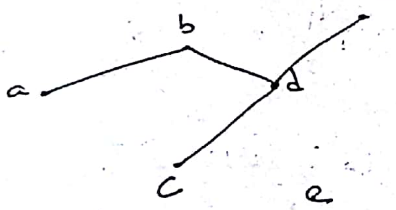
Example2:

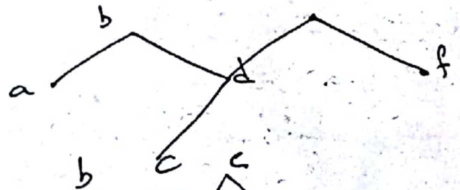


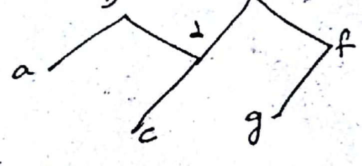
1) 

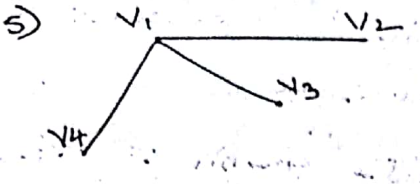
2) 

3) 

4) 

5) 

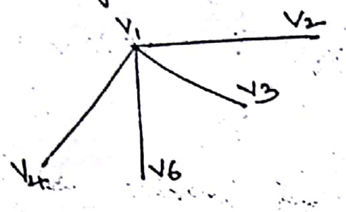
6) 

5) 

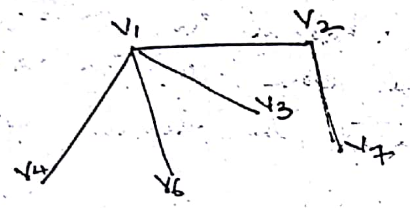
delete V_4 and insert V_8 into Q as V_4 is ad. to V_1.

V_6	V_7	V_8
V_1	V_2	V_4

6) Now V_6 delete V_6, add to V_1 and insert edges



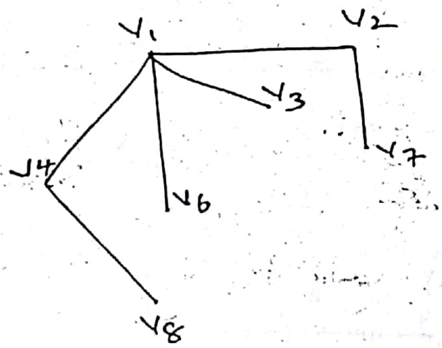
V_7	V_8
V_2	V_4

7) 

Now V_7 is front delete V_7 add vertices adjacent to V_7 into Q

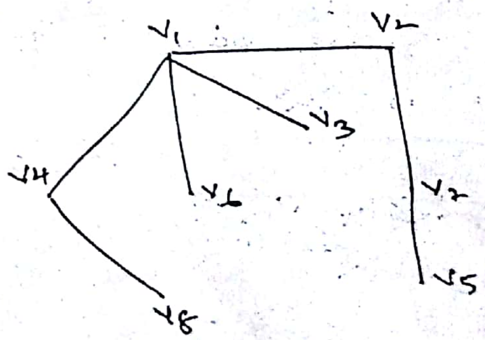
V_8	V_5
V_4	V_7

8) V_8 is front delete V_8 add to V_4 as it is inserted into Q when V_4 is deleted.



V_5
V_7

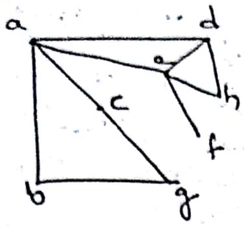
9) Now add V_5 to V_7, Now



Now Q is empty as all vertices are in tree.

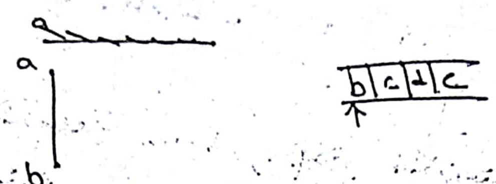
∴ The obtained is a spanning tree using BFS.

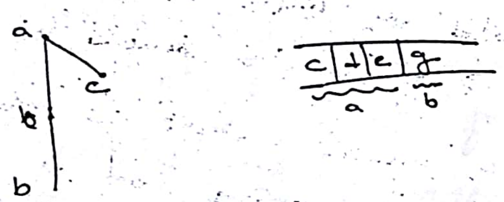
Example :-

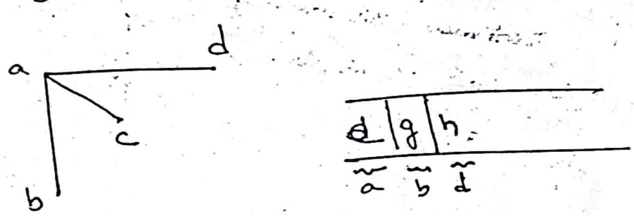


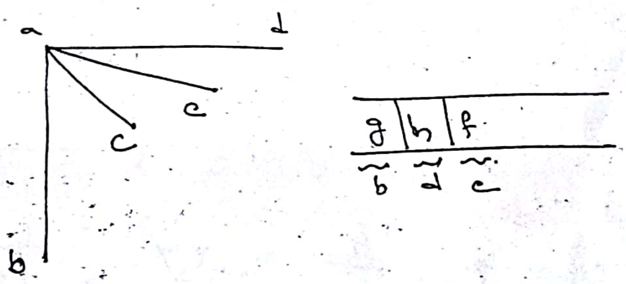
BFS :

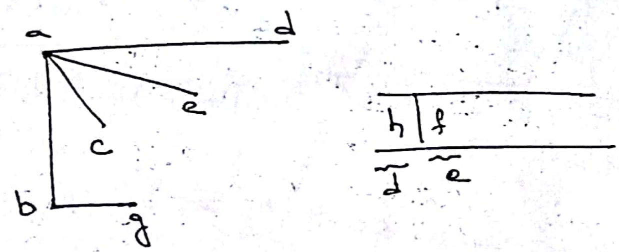
1) a

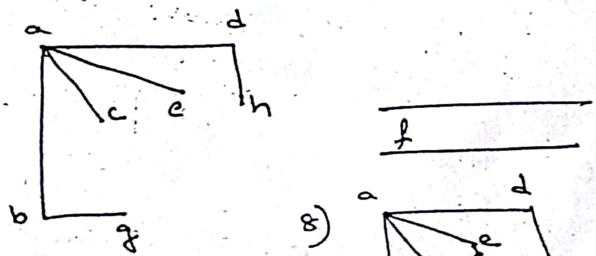
2)


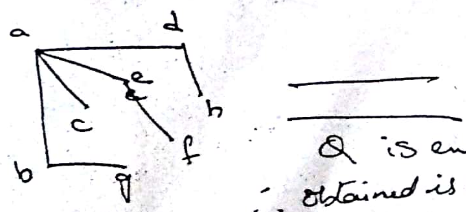
3)


4)


5)


6)


7)


8)


Q is empty
∴ obtained is ST.

→ Minimal spanning tree:

Let G be a graph and a positive real number is associated with each edge of G. Then G is called weighted graph and no. is called cost of the edge e.

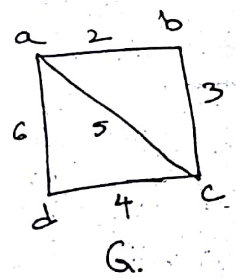
T be a spanning tree of this graph (G). The sum of the weights of all the branches of T is called the weight of T.

If G is weighted graph, then weight of an edge e of G is denoted by wt(e) and weight of a spanning tree T of G is denoted by wt(T).

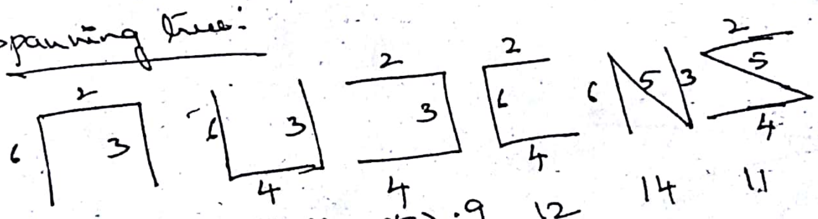
A spanning tree whose weight is the least is called a minimal spanning tree of the graph.

This tree is not unique.

Ex:

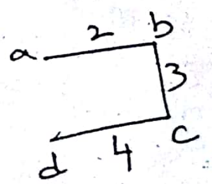


spanning tree:



C(T₁) = 11 C(T₂) = 13 C(T₃) : 9

∴ 3rd S.T as minimum cost of 3.

∴ 

as the least cost.

→ Algorithms for minimal spanning tree:

Construction of minimal spanning tree (MST) in 2 ways.

(i) Kruskal's Algorithm

(ii) Prim's Algorithm.

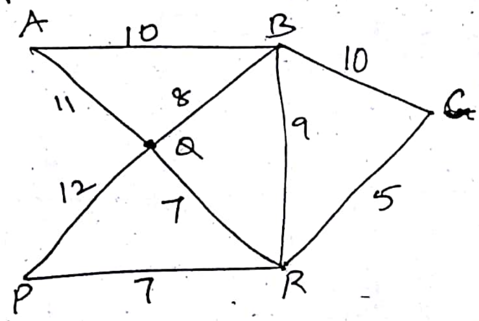
(1) Kruskals Algorithm:

step 1: list all the edges with their cost in ascending order.

step 2: start with a smallest weighted edge, proceed sequentially by selecting one edge at a time such that no cycle is formed.

step 3: Stop the process of step 2 when n-1 edges are selected. These n-1 edges constitute a MST of G.

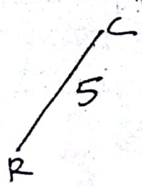
Ex:



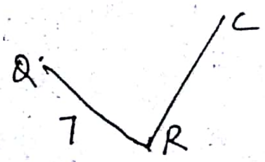
edge	CR	RQ	PR	QB	BR	BC	AB	AQ	PQ
cost	5	7	7	8	9	10	10	11	12
	✓	✓	✓	✓	✗	✗	✓	✗	✗

Select CR

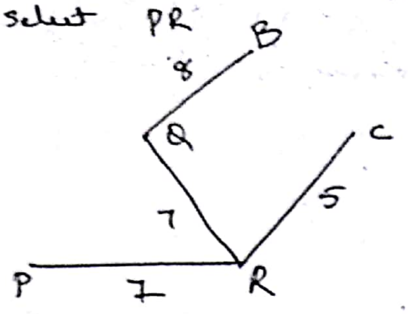
1)



2) RQ

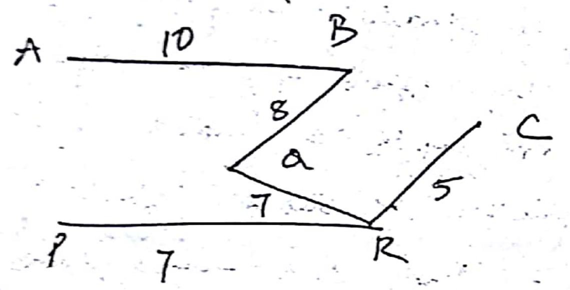


3) select PR



4) ~~No~~ BR, BC is not selected as it forms cycle.

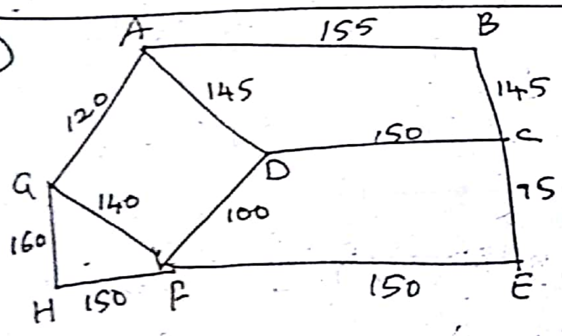
5) AB is selected



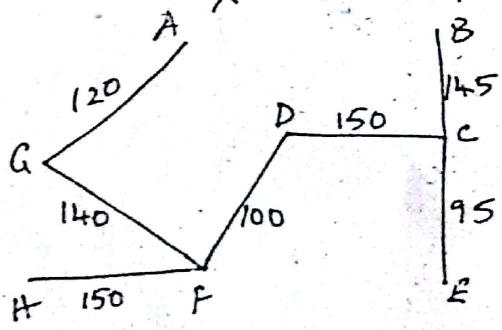
6) AQ, PQ, forms cycle so not selected.

$\therefore$ $10 + 8 + \cancel{77} + 5 = 37$. is the cost of MST.

2)



edge	CE	DF	AG	FG	AD	BC	CD	EF	FH	AB	GH
cost	95	100	120	140	145	145	150	150	150	155	160
	✓	✓	✓	✓	✗	✓	✓	✗	✓		



$\therefore$ 900 is the total ~~cost~~ cost of MST

Scanned by CamScanner

(2) Prim's Algorithm:

Step 1: Given a graph G, construct a table with vertices as labels of rows and columns. In which the weights of all edges are shown.

Indicate the weights of the non-existing edges as ∞.

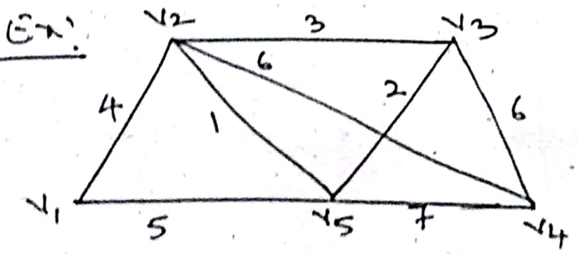
Step 2: Start from vertex v_1 and connect it to its nearest neighbor in the v_1-row say v_k.

Now consider the edge $\{v_1, v_k\}$ and connect it to its closest neighbour to a vertex other than v_1 and v_k, that has the smallest entry among all entries in v_1 and v_k rows). Let this vertex be v_m.

Step 3: Start from the vertex v_m and repeat the process of step 2. Stop the process when all the n vertices have been connected by n-1 edges.

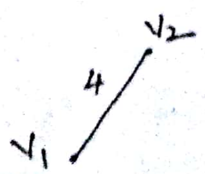
These n-1 edges constitute a minimal spanning tree.

Ex:

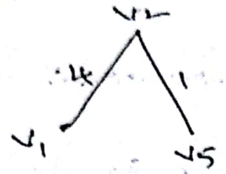


$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}
\begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} - & 4 & \infty & \infty & 5 \\ 4 & - & 3 & 6 & 1 \\ \infty & 3 & - & 6 & 2 \\ \infty & 6 & 6 & - & 7 \\ 5 & 1 & 2 & 7 & - \end{bmatrix} \end{array}$$

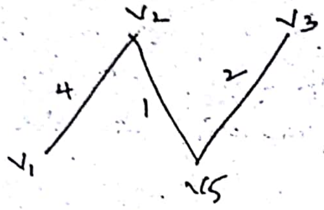
Start at a vertex v_1 and pick the smallest entry



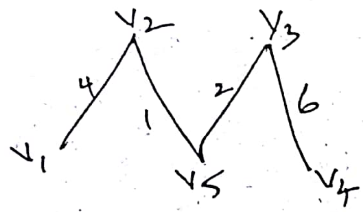
Next now examine the all entries in V_1 and V_2 except other than V_1 and V_2, this corresponds to V_5 with value 1 in $V_2 - V_5$ entry.



Now compare V_1, V_2, V_5 entries find the smallest entries, it is $V_5 - V_3$ is 2

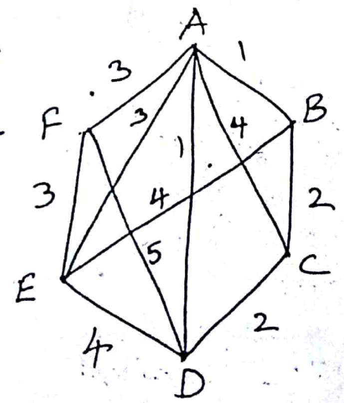


Now compare all rows V_1, V_2, V_3, V_5 which as the lowest cost, it is $(V_2 - V_4)$ or $(V_3 - V_4)$ any one can be selected as they are having same cost.
6 6



$\therefore$ $4+1+2+6 = 13$ is total cost of MST.

Ex:

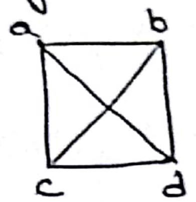


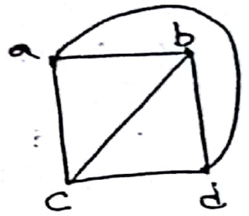
Graph Theory - 2

→ **planar graph:**

A graph G is called a planar graph if it can be drawn on a plane without edge crossing ie the edges meet only at vertices.

Ex:

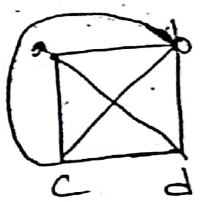




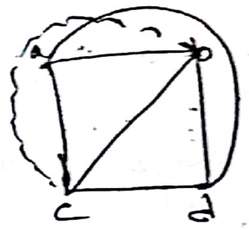
→ **Non-planar graph:**

A graph which cannot be represented by a plane drawing in which the edges meet only at vertices is called a non-planar graph.

Ex:



-



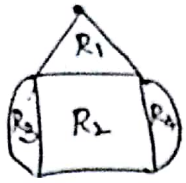
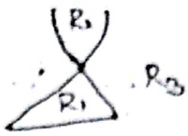
- is not planar graph.

Note: K_2, K_3, K_4 are planar.

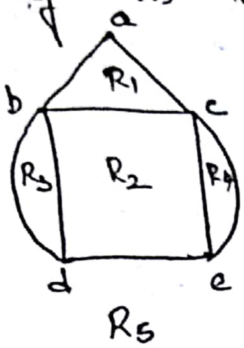
Note: K_5 (Kuratowski's first graph) is non-planar.

Note: $K_{3,3}$ (Kuratowski's second graph) is non-planar.

Region of a graph: A region of a planar graph is defined to be an area of the plane ie., bounded by edges and cannot be further divided.

Ex:  , 

→ <u>Degree of a region:</u> The degree of a region is the length of its boundary.

<u>Ex:</u> 

$R_1 \Rightarrow a-b-c-a = 3 \Rightarrow deg(R_1)$

$R_2 \Rightarrow b-c-e-d-b = 4 \Rightarrow deg(R_2)$

$R_3 \Rightarrow b-d-b = 2 \Rightarrow deg(R_3)$

$R_4 \Rightarrow c-e-c = 2 \Rightarrow deg(R_4)$

$R_5 \Rightarrow a-b-d-e-c-a = 5 \Rightarrow deg(R_5)$

→ A graph with region one is called Tree. 

→ Every planar graphs satisfies $|V|-|E|+|R|=2$.

→ <u>Theorem:</u> Euler's theorem of planar graph.

<u>Statement:</u> In a connected planar graph with cardinality of $|V|$ vertices, $|E|$ edges and $|R|$ regions then $|V|-|E|+|R|=2$.

<u>Proof:</u> By mathematical induction on $|R|$.

<u>Basic step:</u> $|R|=1$

A graph with $|R|=1$ is called Tree.

In a tree, we have $|E|=|V|-1$

$|V|-|E|+|R|=2$

$|V|-(|V|-1)+|R|$

$=2.$

Inductive Hypothesis: suppose the result is true for any connected planar graph with k- regions with inductive hypothesis.

Say $|V|-|E|+|R|=2$ for $1 \leq |R| \leq k$

Inductive step: Let $|R|=k+1$

Remove an edge from connected planar graph with $k+1$ regions.

i.e., an edge common to two regions.

$\therefore$ The no of regions decreases by 1.

$\therefore$ The no of regions $|R(G')| = |R(G)|-1$
$= k+1-1 = k$.

no of edges $|E(G')| = |E(G)|-1$

no of vertices $|V(G')| = |V(G)|$

By I.H, the result is true for G'.

Since $|R(G')| = k$.

$\therefore$ we have $|V(G')| - |E(G')| + |R(G')| = 2$

$$|V(G)| - |E(G)|+1 + |R(G)|-1 = 2$$

$$|V(G)| - |E(G)| + |R(G)| = 2$$

$\therefore$ Hence the result.

*Note: In a connected simple planar graph with $|E|$ edges and $|R|$ regions, then $|R| \leq \frac{2}{3}|E|$.

*Note: In a connected planar simple graph with $|E|$ edges and $|V|$ vertices, we have $|E| \leq 3|V|-6$.

→ Dual of a planar graph:

Consider a connected planar graph G and a plane drawing thereof. Let $R_1, R_2, R_3 \ldots$ be the regions. Let us construct a graph G^* following the procedure:

(1) choose one point inside each of the regions $R_1, R_2, \ldots$. Denote these points by $v_1^*, v_2^*, \ldots$. These are the vertices of G^*.

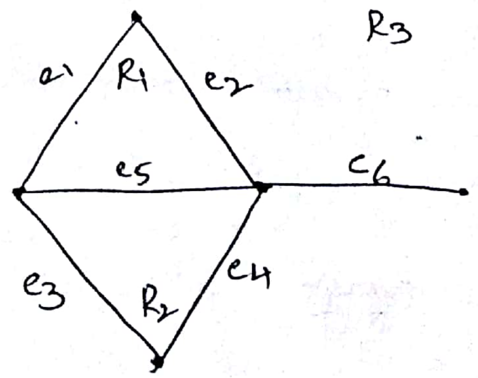
(2) If two regions are R_i and R_j are adjacent draw a line e_k^* joining the points v_i^* and v_j^* that intersects the common edge e_k exactly once

(3) If there is more than one edge common to R_i and R_j, draw one line e_p^* b/w the points v_i^* and v_j^* for each common edge e_p, intersecting e_p exactly once.

(4) For each edge e_i lying entirely in one region, say R_i, draw a loop e_i^* at the point v_i^* intersecting e_i exactly once.

The graph G^* so constructed is called a geometric dual or just a dual of G.

Ex:-



(3)

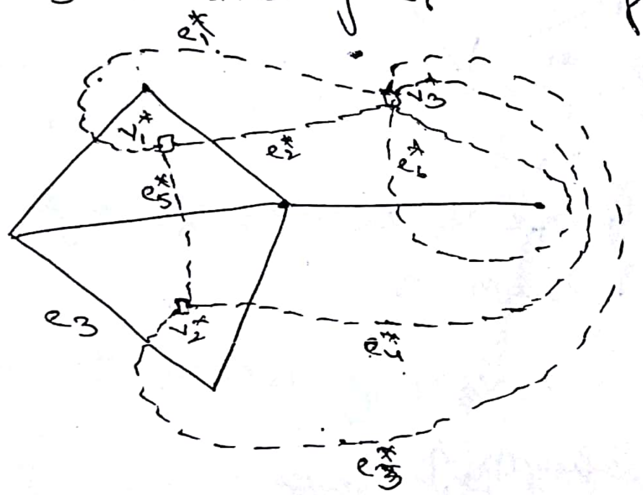
1) choose 3 points v_1^*, v_2^*, v_3^* inside the regions R_1, R_2, R_3.

2) R_1 and R_2 have e_5 common edge, draw a line e_5^* joining v_1^* and v_2^* that cross e_5 once.

3) R_1 & R_3 two edges e_1, e_2 draw two line e_1^*, e_2^* b/w v_1^* & v_3^* with e_1^* and e_2^* crossing their edges exactly once.

4) repeat the same process as above.

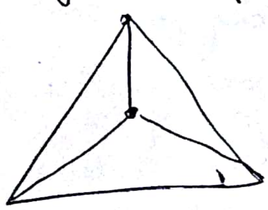
5) the e_6 is completely contained in R_3, draw a loop e_6^* at v_3^* intersecting e_1 exactly once.

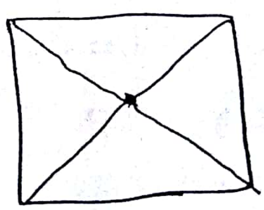


c) The construction of G^* is now complete. with v_1^*, v_2^*, v_3^* are the vertices and $e_1^* --- e_6^*$ as edges.

Ex:- Draw the dual graphs of

1) 

2) 

3) 

→ Euler and Hamilton graphs :-

* **Walk** :- Consider a graph G having atleast one edge.

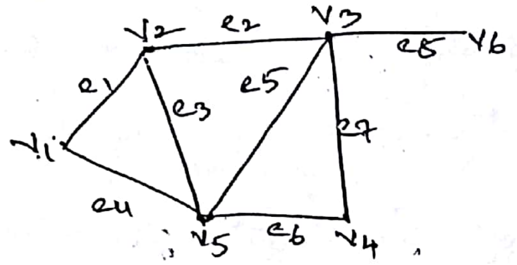
A sequence of vertices and edges of the form

$$v_i e_j v_{i+1} e_{j+1} v_{i+2} \ldots e_k v_m$$

which begins and ends with vertices is called a walk

* In a walk, a vertex or an edge (or both) can appear more than once.

Ex:-



A sequence : $v_1 e_1 v_2 e_2 v_3 e_8 v_6$ is a walk of length 3.

A sequence : $v_1 e_4 v_5 e_6 v_4 e_7 v_3 e_8 v_6$ is a walk of length 4

A sequence : $v_1 e_1 v_2 e_3 v_5 e_3 v_2 e_2 v_3$ is a walk of length 4 with edge e_3 and vertex v_2 is repeated.

* **Closed walk**: A walk that begins and ends at the same vertex is called a closed walk, ie, the terminal vertices are coincident.

Ex: from the above graph $v_1 e_1 v_2 e_3 v_5 e_4 v_1$ is a closed walk of length 3.

* **Open walk:** A walk that begins and ends at two different vertices is called an open walk.

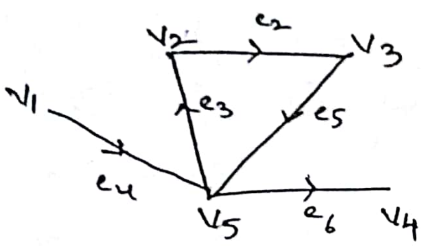
Ex: from the example:

$v_1 e_1 v_2 e_2 v_3 e_5 v_5$ is an open walk of length 3.

* **Trail:**

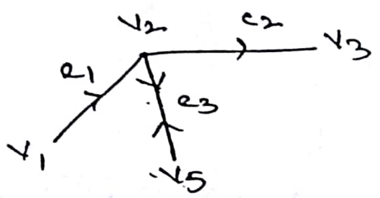
An open walk with no edges appears more than once, then the walk is called a Trial.

Ex: $v_1 e_4 v_5 e_3 v_2 e_2 v_3 e_5 v_5 e_6 v_4$



It is a trial.

Ex:

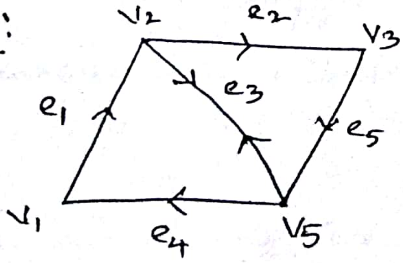


Not a trial.

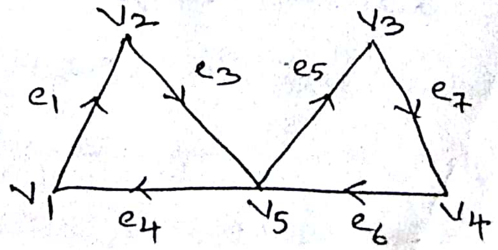
Since $v_1 e_1 v_2 e_3 v_5 e_3 v_2 e_2 v_3$.

* **Circuit:** A closed walk in which no edge appears more than once is called a circuit.

Ex:



Not a circuit

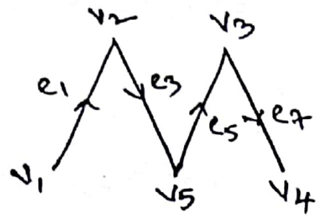


circuit.

* **Path** :-

A trail in which no vertex appears more than once is called a path.

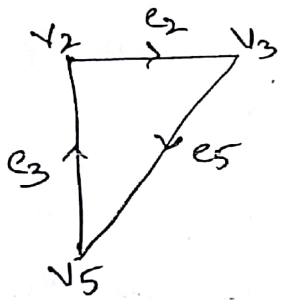
Ex:



$v_1 e_1 v_2 e_3 v_5 e_5 v_3 e_7 v_4$ is a path without vertex repitition.

* **Cycle** :- A circuit in which the terminal vertex does not appear as an internal vertex (also) and no internal vertex is repeated is called a cycle.

Ex:



$v_2 e_2 v_3 e_5 v_5 e_3 v_2$ is a cycle.

→ 1) A walk can be open or closed. In a walk (closed/open) a vertex and/or an edge can appear more than once.

2) A trail is an open walk in which vertex can appear more than once but an edge cannot appear more than once

3) A circuit is a closed walk in which a vertex can appear more than once but an edge cannot.

4) A path is an open walk in which neither vertex and/or edge can appear more than once.

5) A cycle is a closed walk in which neither vertex and/or edge can appear more than once.

→ Euler graph:

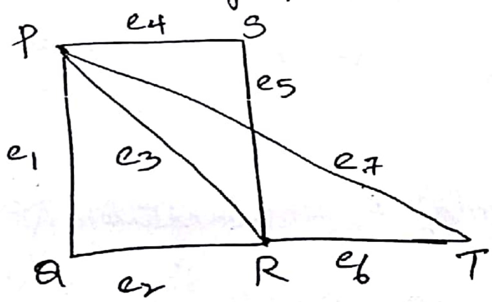
Euler circuit: If there is a circuit in G that contains all the edges of G, then that circuit is called an Euler circuit in G. (Eulerian line / Euler tour).

Euler Graph / Eulerian graph: A connected graph that contains an Euler circuit is called a Euler graph.

Euler trail: If there is a trail in G that contains all the edges of G, then that trail is called an Euler trail (unicursal line) in G.

Semi-Euler graph: A connected graph that contains an Euler trail is called a semi-Euler graph or unicursal graph.

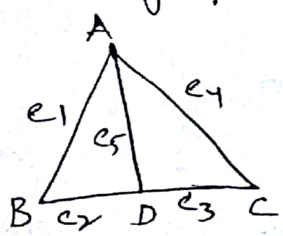
Ex:



Ex: $P e_1 Q e_2 R e_3 P e_4 S e_5 R e_6 T e_7 P$ is a Euler circuit in which all the edges of the graph are present in it.

∴ The graph is a Euler graph.

Ex:



It is not a Euler graph.

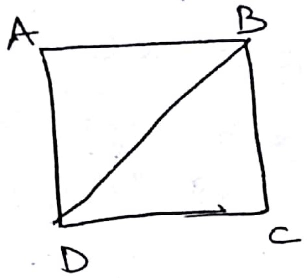
(Th) A connected graph G has a Euler circuit iff all vertices are of even degree.

→ Hamilton graph:

* Hamilton cycle: Let G be a connected graph. If there is a cycle in G that contains all the vertices of G, then that cycle is called Hamilton cycle in G.

* Hamilton graph: A graph that contains a Hamilton cycle is called a Hamilton graph (Hamiltonian graph).

Ex:

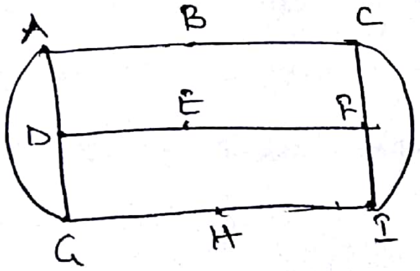


A-B-C-D-A.
It is a cycle which contains all vertices of G and ~~so~~ is a Hamilton graph.

** It does not require all edges ~~need~~ should be included in it.

* Hamilton path: A path in a connected graph which includes every vertex of a graph is called Hamiltonian path in the graph.

Ex:



A-B-C-F-E-D-G-H-I. all vertices are present in path and is called as Hamilton path.

(*) A simple connected graph with n vertices ($n \geq 3$) is hamilton if the degree of every vertex is $\geq n/2$.

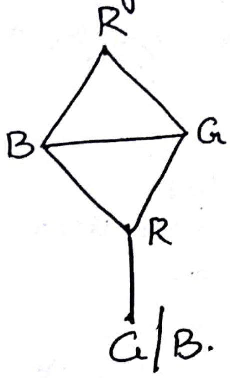
(*) G is hamilton if the sum of degree of every pair of non-adjacent vertices is $\geq n$. ($n \geq 3$).

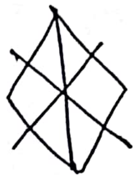
→ **Graph coloring:**

Given a planar / non planar graph G.

If we assign colors to its vertices in such a way that no two adjacent vertices have same color, then we say G is properly colored.

Ex:





* A graph can have more than one proper coloring.
* Two non-adjacent vertices in a properly colored graph can have the same colors.

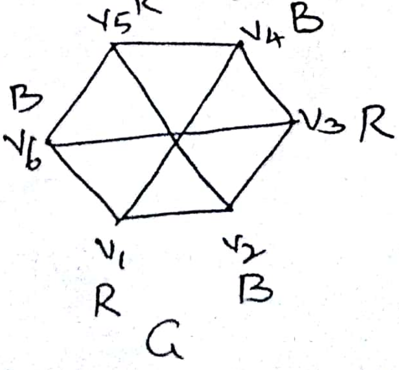
→ **Chromatic Number:**

A graph G is said to be K-colorable if we can properly color it with K colors.

The chromatic number of the graph is the minimum number of colors with which the graph can be properly colored.

The chromatic number of a graph G is usually denoted by $\chi(G)$.

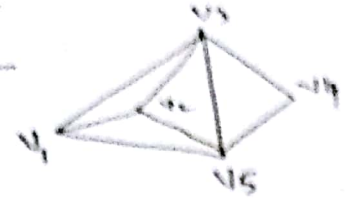
Ex:

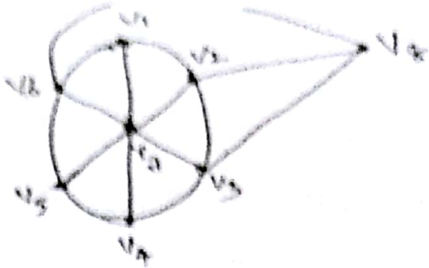


∴ The chromatic number of G

$\chi(G) = 2$.

Ex:





find the chromatic no. of the above graphs.

Ex: Find the chromatic no. of peterson graph and K_5 graph.

* A graph consisting of only isolated vertices is 1-chromatic.
* A graph with one or more edges is atleast 2-chromatic
* A graph G contains a graph G_1 as subgraph, then

$$\chi(G) \geq \chi(G_1)$$

* $\chi(K_n) = n$ for $n \geq 1$.
* If a graph G contains K_n as subgraph, then

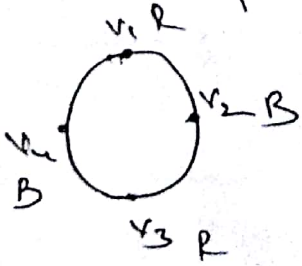
$$\chi(G) \geq n.$$

* If G is a graph of n vertices, then $\chi(G) \leq n$.
* A graph of order $(n \geq 2)$ consisting of a single cycle is

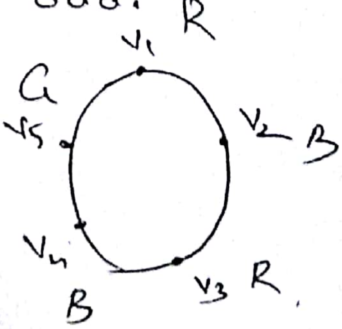
2-chromatic if n is even

3-chromatic if n is odd.

Ex:



2-chromatic (even vertices).

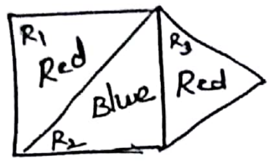


3-Chromatic (odd vertices)

→ Map coloring:

A planar graph divides a plane into regions. we say that these regions are properly colored if no two adjacent Regions have same color.
The properly coloring of regions is called map coloring.

Ex:



3 regions are there excluding the outer region. we used 2 colors for proper coloring.

* The vertices of every connected simple planar graph can be properly colored with five colors so, it is called as five color theorem.

* Every simple, connected planar graph is 4-colorable called as four color theorem.

Recurrence Relation

→ Sequences are generally defined by specifying their general terms. A sequence may be defined by indicating a relation connecting to its general terms with one or more of the preceding terms.

→ A sequence $\langle a_r \rangle$ may be defined by indicating a relation connecting to its general terms a_n with a_{n-1}, a_{n-2} etc.

→ The value a_n satisfies the recurrence relation is called its <u>"general solution"</u>.

→ If the values of some particular terms of the sequence are specified, then by using those values substituted in the general solution we obtain the <u>"particular solution"</u> that uniquely determines the sequences.

First order Recurrence Relation

→ A recurrence relation of the form

$$\boxed{a_n = c a_{n-1} + f(n)} \text{ for } n \geq 1 \rightarrow (1)$$

c is constant, $f(n)$ is a known function.

→ This relation is called linear recurrence relation of first order with constant coefficient.

→ If $f(n) = 0$, it is called homogeneous otherwise non homogeneous

$a_n = c a_{n-1} + f(n) \rightarrow (1) \quad n \geq 1$

(1) can be written as

$a_{n+1} = c a_n + f(n+1)$ for $n \geq 0 \rightarrow (2)$

for $n = 0, 1, 2 \ldots$ the relation yields.

$a_1 = c a_0 + f(1)$

$a_2 = c a_1 + f(2) = c\{c a_0 + f(1)\} + f(2)$

$= c^2 a_0 + c f(1) + f(2)$

$\vdots$

$a_n = c^n a_0 + c^{n-1} f(1) + c^{n-2} f(2) + \cdots + c f(n-1) + f(n)$

$$\boxed{a_n = c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k)}, \text{ for } n \geq 1 \rightarrow (3)$$

This is the general solution of the recurrence relation (2). which satisfies & equivalent to (1)

If $f(n) = 0$, it is homogeneous and (3) becomes

$$\boxed{a_n = c^n a_0} \text{ for } n \geq 1 \rightarrow (4)$$

(3) & (4) yield particular solution if a_0 is specified. The specified value is called initial condition.

1) Solve the recurrence relation $a_{n+1} = 4a_n$ for $n \geq 0$ given $a_0 = 3$

Sol:

$$a_{n+1} = 4a_n \quad , \quad n \geq 0$$

general solution is $a_n = c^n a_0$ given $a_0 = 3$

$$a_n = 4^n \times 3$$

$a_n = 3 \times 4^n$ is the particular solution

---

2) Solve $a_n = 7a_{n-1}$ $n \geq 1$, given $a_2 = 98$

$$a_n = 7a_{n-1}$$

general solution is $a_n = c^n a_0$

$$a_n = 7^n a_0$$

for $n = 2 \Rightarrow a_2 = 7^2 a_0$

$$98 = 7^2 a_0$$

$$a_0 = 2$$

$\therefore$ particular solution is $a_n = 7^n \times 2$ $n \geq 1$

---

3) Solve $a_n = n a_{n-1}$ for $n \geq 1$ given $a_0 = 1$

$a_1 = 1 \times a_0$

$a_2 = 2 \times a_1 = (2 \times 1) a_0$

$a_3 = 3 \times a_2 = (3 \times 2 \times 1) a_0$

$\vdots$

general solution $a_n = n!(a_0)$ $n \geq 1$

particular solution is $a_n = n! \Rightarrow a_0 = 1$

4) If a_n is a solution of $a_{n+1} = Ka_n, n \geq 0$ and :

$a_3 = \frac{153}{49}$, $a_5 = \frac{1377}{2401}$, what is k?

Sol: General solution $a_n = k^n a_0 \quad n \geq 1$

from this $a_3 = k^3 a_0$

$a_5 = k^5 a_0$

$$\frac{a_5}{a_3} = k^2$$

Given $a_3 = \frac{153}{49}$ and $a_5 = \frac{1377}{2401}$

After substituting a_3 & a_5 values in the above equation we get $k = \pm 3/7$

5) Find the recurrence relation for the sequence 2, 10, 50, 250, ...

$a_0 = 2$

$a_1 = 5 \times a_0 = 5 \times 2 = 10$

$a_2 = 5 \times a_1 = 5 \times 10 = 50$

$a_3 = 5 \times a_2 = 5 \times 50 = 250$

$\vdots$

$a_n = 5 \times a_{n-1}$

general solution is $a_n = 5^n a_0$ for $n \geq 1$

given $a_0 = 2$

$\boxed{a_n = 5^n \times 2} \rightarrow$ is the particular solution

Second order linear homogeneous Recurrence Relations

We now consider a method of solving recurrence relations of the form

$$C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = 0 \quad \text{for } n \geq 2 \rightarrow (1)$$

where C_n, C_{n-1} and C_{n-2} are real constants with $C_n \neq 0$. A relation of this type is called second-order linear homogeneous recurrence relation with constant coefficients.

We seek a solution of relation (1) in the form $a_n = Ck^n$ where $C \neq 0$ and $k \neq 0$. putting $a_n = Ck^n$ in (1) we get

$$C_n Ck^n + C_{n-1} Ck^{n-1} + C_{n-2} Ck^{n-2} = 0$$

(or) $\quad C_n k^2 + C_{n-1} k + C_{n-2} = 0$

Thus $a_n = Ck^n$ is a solution of (1) if k satisfies the quadratic equation (2). This quadratic equation is called the auxiliary equation or the characteristic equation for the relation (1).

The following cases arise:

Case 1: The two roots k_1 and k_2 of eqn (2) are real and distinct. Then we take

$$\boxed{a_n = Ak_1^n + Bk_2^n}$$

where A and B are arbitrary real constants.

Case 2: The two roots k_1 and k_2 of equation ②

are real and equal, with k as the common value, Then we take

$$a_n = (A + Bn)k^n$$

where A and B are arbitrary real constants

Case 3: The two roots k_1 and k_2 of eqn ② are Complex. Then k_1 and k_2 are complex conjugates of each other, so that if $k_1 = p + iq$, then $k_2 = p - iq$, and we take

$$a_n = r^n (A\cos n\theta + B\sin n\theta)$$

where A and B are arbitrary constants

$$r = |k_1| = |k_2| = \sqrt{p^2+q^2} \text{ and}$$

$$\theta = \tan^{-1}(q/p)$$

1 Solve the recurrence relation

$$a_n + a_{n-1} - 6a_{n-2} = 0 \text{ for } n \geq 2,$$

given that $a_0 = -1$ and $a_1 = 8$

Sol: Here the coefficients of a_n, a_{n-1} and a_{n-2} are $C_n = 1$, $C_{n-1} = 1$ and $C_{n-2} = -6$

∴ The characteristic equation is

$$k^2 + k - 6 = 0 \text{ or } (k+3)(k-2) = 0$$

$$k_1 = -3 \quad k_2 = 2$$

∴ Roots are real and distinct.

$\therefore$ The general solution is

$$a_n = Ak_1^n + Bk_2^n$$

$$a_n = A\times(-3)^n + B\times 2^n \rightarrow (i)$$

where A and B are arbitrary constants.

From this we get

$$a_0 = A\times(-3)^0 + B\times 2^0$$

$$a_0 = A+B$$

and $a_1 = -3A+2B$

Given $a_0 = -1$ & $a_1 = 8$

$$-1 = A+B$$

$$8 = -3A+2B$$

Solving this we get. $A=-2$ & $B=1$

substituting these values in (i)

we get $a_n = -2\times(-3)^n + 2^n$

(2) Solve the recurrence relation

$$a_n = 2(a_{n-1} - a_{n-2}) \text{ for } n \geq 2$$

given $a_0 = 1$ & $a_1 = 2$

Sol:

$$a_n - 2a_{n-1} + 2a_{n-2} = 0$$

$$k^2 - 2k + 2 = 0$$

The roots are

$$k = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$\therefore$ The roots are complex and

the general solution for a_n is

$$a_n = r^n \left[A \cos n\theta + B \sin n\theta \right] \rightarrow (1)$$

A & B are arbitrary constants,

$$r = |(1 \pm i)| = \sqrt{2}$$

$\tan\theta = 1/1 = 1$ where $\theta = \pi/4$

Thus

$$a_n = (\sqrt{2})^n \left[A \cos \frac{n\pi}{4} + B \sin \frac{n\pi}{4} \right] \rightarrow (2)$$

Given $a_0 = 1$ & $a_1 = 2$

$$a_0 = (\sqrt{2})^0 \left[A \cos \frac{0\pi}{4} + B \sin \frac{0\pi}{4} \right]$$

(1) $1 = A$

Substitute $a_1 = 2$ & $n = 1$

(2) $2 = \sqrt{2} \left[A \cos \frac{\pi}{4} + B \sin \frac{\pi}{4} \right] = A + B$

(1) $1 = A$

(2) $2 = A + B$

This gives $A = 1$ & $B = 1$

Substituting $A=1$ & $B=1$

in ② we get

$$a_n = (\sqrt{2})^n \left[\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4}\right]$$

Examples:

Solve the reccurence relation

① $a_n = 3a_{n-1} - 2a_{n-2}$ for $n \geq 2$ $a_1 = 5, a_2 = 3$

② $a_n - 6a_{n-1} + 9a_{n-2} = 0$ for $n \geq 2$ $a_0 = 5, a_1 = 12$

③ $D_n = bD_{n-1} - b^2 D_{n-2}$ for $n \geq 3$ $D_1 = b > 0$ and $D_2 = 0$

④ $2a_n = 7a_{n-1} - 3a_{n-2}$, $n \geq 2$ $a_0 = 2, a_1 = 5$

⑤ $a_n = 5a_{n-1} + 6a_{n-2}$, $n \geq 2$ $a_0 = 1, a_1 = 3$

⑥ $a_n + 5a_{n-1} + 5a_{n-2} = 0$, $n \geq 2$ $a_0 = 0, a_1 = 2\sqrt{5}$

$a_n - 6a_{n-1} + 9a_{n-2} = 0$ $a_0 = 5$ $a_1 = 12$

$k^2 - 6k + 9 = 0$

$(k-3)^2 = 0$

$k = 3, 3$

$a_n = (A + Bn)k^n$

$a_0 = 5$

$a_0 = (A + B \cdot 0)k^0$

$a_0 = A$

$$\boxed{a_n = (5-n)3^n}$$

$a_1 = 12$

$a_1 = (A+B)k^1$

$\overset{4}{\cancel{12}} = (A+B)\cancel{3}$

$A + B = 4$

$A = 5$

$A = 5$

$B = -1$

Third and Higher-order linear Homogeneous Recurrence Relations

Here, we illustrate the method of solving third and higher-order linear homogeneous recurrence relations with constant coefficients. These are of the form

$$C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} + C_{n-3} a_{n-3} + \cdots + C_{n-k} a_{n-k} = 0$$

$$\text{for } n \geq k \geq 3 \rightarrow (1)$$

where $C_n, C_{n-1} \cdots C_{n-k}$ are real constants with $C_n \neq 0$.

Examples

1.) Solve the recurrence relation

$$2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n \quad \text{for } n \geq 0$$

with $a_0 = 0,\ a_1 = 1,\ a_2 = 2$

Sol: Given relation can be written as

$$2a_n - a_{n-1} - 2a_{n-2} + a_{n-3} = 0 \quad \text{for } n \geq 3$$

This is a third-order relation of the form eq^n (1)

when $C_n = 2,\ C_{n-1} = 1,\ C_{n-2} = -2,\ C_{n-3} = 1$

we consider the characteristic equation

$$2k^3 - k^2 - 2k + 1 = 0 \Rightarrow (2k-1)(k^2-1) = 0$$

The roots are $k_1 = \frac{1}{2},\ k_2 = 1,\ k_3 = -1$ which are real and distinct.

$\therefore$ The general solution is

$$a_n = Ak_1^n + Bk_2^n + Ck_3^n$$

$$a_n = A\left(\tfrac{1}{2}\right)^n + B(1)^n + C(-1)^n \rightarrow (i)$$ where A, B, C are arbitrary constants

Given $a_0 = 0,\ a_1 = 1,\ a_2 = 2$

Substituting these values in (i) we get

$$0 = a_0 = A \times \left(\frac{1}{2}\right)^0 + B \times 1^0 + C \times (-1)^0$$

$$1 = a_1 = A \times \left(\frac{1}{2}\right)^1 + B \times 1^1 + C \times (-1)^1$$

$$2 = a_2 = A \times \left(\frac{1}{2}\right)^2 + B \times 1^2 + C \times (-1)^2$$

$$A + B + C = 0,\quad A + 2B - 2C = 2,\quad A + 4B + 4C = 8$$

By solving this we get $A = -\frac{8}{3},\ B = \frac{1}{6},\ C = \frac{5}{2}$

$$A = -\frac{8}{3} \times \left(\frac{1}{2}\right)^n + \frac{1}{6} \times (-1)^n + \frac{5}{2}.$$ is the required solution

2) Solve the recurrence relation

$$a_n + a_{n-1} - 8a_{n-2} - 12a_{n-3} = 0,\quad n \geq 3$$

with $a_0 = 1,\ a_1 = 5,\ a_2 = 1$

Sol:

$$k^3 + k^2 - 8k - 12 = 0 \Rightarrow (k+2)^2 (k-3) = 0$$

roots are $k_1 = k_2 = -2$ and $k_3 = 3$

The general solution for a_n is

$$a_n = (A + Bn)(-2)^n + C \times 3^n \quad \text{(i)}$$ where A, B, C are constants

Given $a_0 = 1,\ a_1 = 5,\ a_2 = 1$

Substituting in eq^n (i) we get

$$1 = a_0 = A + C$$

$$5 = a_1 = -2(A + B) + 3C$$

$$1 = a_2 = 4(A + 2B) + 9C$$

By solving this we get $A = 0,\ B = -1,\ C = 1$

Putting in eq (i) we get

$$a_n = (-n) \times (-2)^n + 3^n.$$

→ Solve the recurrance relation $D_n = bD_{n-1} - b^2 D_{n-2}$ for $n \geq 3$, given $D_1 = b > 0$ and $D_2 = 0$

Sol: The given relation is $D_n - bD_{n-1} - b^2 D_{n-2} = 0$

The characteristic equation is $k^2 - bk - b^2 = 0$

The roots are $k = \frac{b \pm \sqrt{b^2 - 4b^2}}{2} = \frac{b}{2}(1 \pm i\sqrt{3})$

The roots are imaginary. Therefore the general solution is $a_n = r^n \{A\cos n\theta + B\sin n\theta\}$

$r = \left|\frac{b}{2}(1 \pm i\sqrt{3})\right| = \frac{b}{2}\sqrt{1^2+3} = \frac{b}{2}\sqrt{4} = \frac{b}{2}\cdot 2 = b$

$r = b$

$\tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\therefore \theta = \pi/3$

$\therefore D_n = b^n \left[A\cos\frac{n\pi}{3} + B\sin\frac{n\pi}{3}\right]$

Given $D_1 = b$

$D_1 = b^1 \left[A\cos\frac{\pi}{3} + B\sin\frac{\pi}{3}\right]$

$b = b\left[\frac{1}{2}A + \frac{\sqrt{3}}{2}B\right]$

$1 = \frac{1}{2}A + \frac{\sqrt{3}}{2}B$

Given $D_2 = 0$

$D_2 = b^2\left[A\cos\frac{2\pi}{3} + B\sin\frac{2\pi}{3}\right]$

$0 = b^2\left[-\frac{1}{2}A + \frac{\sqrt{3}}{2}B\right]$

$0 = -\frac{1}{2}A + \frac{\sqrt{3}}{2}B$

$A = 1 \quad B = \frac{1}{\sqrt{3}}$

$\therefore D_n = b^n\left\{1\cdot\cos\frac{n\pi}{3} + \frac{1}{\sqrt{3}}\sin\frac{n\pi}{3}\right\}$

is the required solution.

Third order

→ solve the recurrance relation

$a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ given $a_0 = 1, a_1 = 4, a_2 = 28$

Sol.:

$a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0$

$k^3 - 6k^2 + 12k - 8 = 0$

$(k-2)^3 = 0$

$k = k_1 = k_2 = k_3 = 2$ Roots are real & equal

∴ The general solution is

$a_n = (A + Bn + Cn^2)k^n \Rightarrow a_n = (A+Bn+Cn^2)2^n$

Given $a_0 = 1$

$a_0 = (A + B\cdot 0 + C\cdot 0)k^0$

$1 = A$

$a_1 = 4$

$a_1 = (A + B\cdot 1 + C\cdot 1^2)2^1$

$4 = (A+B+C)2$

$A+B+C = 2$

$a_2 = (A + 2B + 4C)2^2$

$28 = 4(A+2B+4C)$

By solving we get $A = 1$, $B = -1$, $C = 2$

Substituting A, B, C values we get the required solution as

$a_n = (1 + n(-1) + n^2\cdot 2)2^n$

$$\boxed{a_n = (1 - n + 2n^2)2^n}$$

→ Find the general solution of the recurrance relation

$a_n + a_{n-3} = 0 \quad n \geq 3$

sol:

$k^3 + 1 = 0$

$\Rightarrow (k+1)(k^2 - k + 1) = 0$

$k_1 = -1, \quad k_2 = \frac{1}{2}(1 + i\sqrt{3}), \quad k_3 = \frac{1}{2}(1 - i\sqrt{3})$

∴ general solution is

$a_n = A(-1)^n + r^n [B \cos n\theta + C \sin n\theta]$

$r = \frac{1}{2}\sqrt{1^2 + \sqrt{3}^2} = \frac{1}{2}\sqrt{1+3} = \frac{2}{2} = 1$

$r = 1$

$\tan\theta = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$

$\theta = \frac{\pi}{3}$

∴ The general solution is

$$\boxed{a_n = A(-1)^n + 1^n \left[B \cos \frac{n\pi}{3} + C \sin \frac{n\pi}{3}\right]}$$

Non-Homogeneous (or) Inhomogeneous Recurrence relations of second and higher order:

The recurrence relation which is of the form

$$c_n a_n + c_{n-1} a_{n-1} + c_{n-2} a_{n-2} + \cdots c_{n-k} a_{n-k} = f(n)$$

for $n \geq k \geq 2$ (1)

where $c_n, c_{n-1} \cdots c_{n-k}$ are real constants with $c_n \neq 0$ and $f(n)$ is a given real-valued function of n.

A general solution of the recurrance relation (1) is given by $\boxed{a_n = a_n^{(h)} + a_n^{(p)}}$

where $a_n^{(h)}$ is the general solution of the homogeneous part of the relation (1), namely the relation (1) with $f(n) = 0$. and

$a_n^{(p)}$ is any particular solution of relation (1).

Example: 1 Solve the recurrance relation

$$a_n + 4a_{n-1} + 4a_{n-2} = 8 \quad \text{for } n \geq 2 \text{ with}$$

$$a_0 = 1, \; a_1 = 2 \quad (1)$$

Sol: Given recurrence relation is a second order non-homogeneous recurrence relation.

$\therefore$ The general solution of relation (1) is

$$a_n = a_n^{(h)} + a_n^{(p)} \rightarrow (2)$$

To obtain $a_n^{(h)}$, substitute $f(n) = 0$ in the above recurrence relation (1)

$$a_n + 4a_{n-1} + 4a_{n-2} = 0 \rightarrow (3)$$

The characteristic equation of eq (3) is

$$k^2 + 4k + 4 = 0$$

$$a = 1,\ b = 4 \quad c = 4$$

$$k = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{4^2 - 4(1)(4)}}{2.1} = \frac{-4 \pm \sqrt{10}}{2} = \frac{-4}{2} = -2$$

$$\therefore k = k_1 = k_2 = -2$$

Therefore, the roots of the characteristic roots are real and equal.

$\therefore$ The general solution of $a_n^{(h)}$ is

$$a_n^{(h)} = (A + Bn)k^n$$

$$\boxed{a_n^{(h)} = (A + Bn)(-2)^n} \rightarrow (4)$$

This is the general solution when $f(n) = 0$

$a_n^{(P)}$: Given recurrance relation is

$$a_n + 4a_{n-1} + 4a_{n-2} = 8$$

The RHS part of the recurrance relation is constant. Hence the general solution is

$$a_n^{(P)} = A_0 \rightarrow (5)$$

equations (4) & (5) are substituted in eq (2)

$$a_n = a_n^{(h)} + a_n^{(P)} \Rightarrow \boxed{a_n = (A+Bn)(-2)^n + A_0}$$

for obtaining the value of A_0. substituted in eq(1)

$$a_n + 4a_{n-1} + 4a_{n-2} = 8$$

$$A_0 + 4A_0 + 4A_0 = 8$$

$$9A_0 = 8$$

$$A_0 = \frac{8}{9}$$

Given $a_0 = 1$ and $a_1 = 2$

$$a_n = (A + Bn)(-2)^n + \frac{8}{9}$$

$$a_0 = (A + B \cdot 0)(-2)^0 + \frac{8}{9}$$

$$a_0 = A + \frac{8}{9}$$

$$1 = A + \frac{8}{9}$$

$$A = 1 - \frac{8}{9}$$

$$A = \frac{1}{9}$$

$$a_1 = (A + B \cdot 1)(-2)^n + \frac{8}{9}$$

$$a_1 = (A + B)(-2)^1 + \frac{8}{9}$$

$$2 = (A + B)(-2) + \frac{8}{9}$$

$$2 - \frac{8}{9} = (A + B)(-2)$$

$$2 - \frac{8}{9} = \left(\frac{1}{9} + B\right)(-2)$$

$$\frac{10}{9} = \left[\frac{1}{9} + B\right](-2)$$

$$B = -\left[\frac{5}{9} + \frac{1}{9}\right] = -\frac{6}{9} = -\frac{2}{3}$$

$$\therefore \boxed{a_n = \left(\frac{1}{9} - \frac{2}{3}n\right)(-2)^n + \frac{8}{9}}$$

This is the required solution.

$$a_n = a_n^{(h)} + a_n^{(P)}$$

$a_n^{(P)}$ is calculated using following cases

Case 1: Suppose $f(n)$ is a polynomial of degree 'q' and 1 is not a root of characteristic equation of homogeneous part of the eq^n ① In this case $a_n^{(P)}$ is

$$\boxed{a_n^{(P)} = A_0 + A_1 n + A_2 n^2 + \cdots A_q n^q}$$

where $A_0, A_1, A_2 \cdots A_q$ are constants, that are evaluated by using the fact that $a_n = a_n^{(P)}$ satisfies the eq ①.

Case 2: Suppose $f(n)$ is a polynomial of degree q and 1 is a root of multiplicity m of the characteristic equation of the homogeneous part of eq^n ①. In this case $a_n^{(P)}$ is

$$\boxed{a_n^{(P)} = n^m \{A_0 + A_1 n + A_2 n^2 + \cdots A_q n^q\}}$$

Case 3: Suppose $f(n) = \alpha b^n$ where α is a constant and b is not a root of characteristic equation of homogeneous part of eq^n ① then $a_n^{(P)}$ is

$$\boxed{a_n^{(P)} = A_0 b^n}$$

Case 4 : Suppose $f(n) = \alpha \cdot b^n$ where α is constant and b is a root of multiplicity m of charecteristic equation of homogeneous part of eq ①

then $\boxed{a_n^{(P)} = A_0 n^m . b^n}$

$f(n)$	$a_n^{(P)}$
Constant	A_0
n	$A_0 + A_1 n$
n^2	$A_0 + A_1 n + A_2 n^2$
r^n	$A_0 \cdot r^n$

Ex:12 Solve the recurrance relation

$$a_n = 3a_{n-1} + 2n \quad \text{with } a_1 = 3$$

Sol: $a_n - 3a_{n-1} = 2n \rightarrow ①$

Given recurrance relation is first order non homogeneous & non linear recurrence relation.

The general solution is

$$a_n = a_n^h + a_n^P \rightarrow ②$$

To get $a_n^{(h)}$ substitute $f(n) = 0$ in ①

$$a_n - 3a_{n-1} = 0 \rightarrow ③$$

$a_n^{(h)} = c^n a_0 \Rightarrow \boxed{a_n^{(h)} = 3^n a_0}$

Given $a_n - 3a_{n-1} = 2n \rightarrow ①$

$a_n^{(p)} = A_0 + A_1 n \rightarrow ⑧$

Eqⁿ ⑧ is substituted in ①

$$[A_0 + A_1 n] - 3[A_0 + A_1(n-1)] = 2n$$

$$A_0 + A_1 n - 3A_0 - 3A_1 n + 3A_1 = 2n$$

$$-2A_0 - \underline{2A_1 n} + 3A_1 = \underline{2n}$$

nth term $\quad -\cancel{2}A_1 \cancel{n} = \cancel{2}\cancel{n}$

$$A_1 = -1$$

constant term

$$-2A_0 + 3A_1 = 0$$

$$\cancel{-2A_0 + 3A_1 = 0}$$

$$\cancel{2 + 3A_1 = 0}$$

$$\cancel{3A_1 = -2}$$

$$\cancel{A_1 =}$$

$$-2A_0 - 3 = 0$$

$$-2A_0 = 3$$

$$A_0 = \frac{-3}{2}$$

$$a_n^{(P)} = \frac{-3}{2} - 1 \cdot n$$

$$a_n = a_n^{(h)} + a_n^{(P)}$$

$$a_n = 3^n a_0 + \left[-\frac{3}{2} - n\right]$$

given $a_1 = 3$

$$a_1 = 3^1 a_0 + \left[-\frac{3}{2} - n\right]$$

$$3 = 3a_0 + \left[-\frac{3}{2} - 1\right]$$

$$3 = 3a_0 - \frac{5}{2}$$

$$3 + \frac{5}{2} = 3a_0$$

$$\frac{11}{2} = 3a_0$$

$$a_0 = \frac{11}{6}$$

$$\boxed{a_n = 3^n \frac{11}{6} - \frac{3}{2} - n}$$

Ex: Solve the recurrance relation $a_{n+2} - 10a_{n+1} + 21a_n = 3n^2 - 2$

Solution: Given $a_{n+2} - 10a_{n+1} + 21a_n = 3n^2 - 2$. $n \geq 0$

For homogeneous $n \geq 0$

$$k^2 - 10k + 21 = 0$$

$$(k-3)(k-7) = 0$$

$$\boxed{a_n = a_n^{(h)} + a_n^{(p)}}$$

$$k_1 = 3,\ k_2 = 7$$

The roots are real & distinct

$$\therefore a_n^{(h)} = A3^n + B7^n$$

RHS of the given relation is a polynomial of degree 2

$$\therefore a_n^{(p)} = A_0 + A_1 n + A_2 n^2$$

By substituting $a_n^{(p)}$ for a_n in the given relation

we get

$$\{A_0 + A_1(n+2) + A_2(n+2)^2\}$$
$$-10\{A_0 + A_1(n+1) + A_2(n+1)^2\}$$
$$+ 21\{A_0 + A_1 n + A_2 n^2\} = 3n^2 - 2$$

$$[A_0 + A_1 n + 2A_1 + A_2[n^2+4n+4]]$$
$$-10\{A_0 + A_1 n + A_1 + A_2[n^2+2n+1]\}$$
$$+21A_0 + 21A_1 n + 21A_2 n^2 = 3n^2 - 2$$

$$[A_0 + A_1 n + 2A_1 + A_2 n^2 + 4A_2 n + 4A_2$$
$$-10A_0 - 10A_1 n - 10A_1 - 10A_2 n^2 - 20nA_2 - 10A_2$$
$$+21A_0 + 21A_1 n + 21A_2 n^2 = 3n^2 - 2$$

$$A_0 + 2A_1 + 4A_2 - 10A_0 - 10A_1 - 10A_2 + 21A_0 = -2$$

$\rightarrow \boxed{12A_0 - 8A_1 - 6A_2 = -2}$ constant term

~~nth term~~

$$A_1 + 4A_2 - 10A_1 - 20A_2 + 21A_1 = 0$$

$\rightarrow \boxed{-16A_2 + 12A_1 = 0}$ n^{th} term

$$A_2 - 10A_2 + 21A_2 = 3$$

$\boxed{12A_2 = 3}$ n^2 term

By solving we get $A_2 = 4$, $A_1 = 16/3$, $A_0 = \frac{47}{9}$

$$a_n^{(p)} = \frac{47}{9} + \frac{16}{3}n + 4n^2$$

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$\therefore$ general solution is $\boxed{a_n = A3^n + B7^n + \left(\frac{47}{9} + \frac{16}{3}n + 4n^2\right)}$

→ Solve the recurrance relation

$a_n + 2a_{n-1} - 3a_{n-2} = 4n^2 - 5$ for $n \geq 2$

roots $k_1 = 1$ $k_2 = -3$

$$\boxed{a_n^{(h)} = A \cdot 1^n + B \times (-3)^n \qquad a_n^{(P)} = n^1 [A_0 + A_1 n + A_2 n^2]}$$

$= A_0 n + A_1 n^2 + A_2 n^3$

$A_2 = 1/3$ $A_1 = 5/6$ $A_0 = 1/4$

→ $a_n - 6a_{n-1} + 8a_{n-2} = 9$, $n \geq 2$ $a_0 = 10 \cdot a_1 = 25$

$$\boxed{a_n = 4 \times 4^n + 3 \times 2^n + 3}$$

→ $a_n - 4a_{n-1} + 4a_{n-2} = 2^n$, $n \geq 2$

→ $a_{n+2} - 6a_{n+1} + 9a_n = 3 \times 2^n + 7 \times 3^n$ for $n \geq 0$ $\cdot a_0 = 1$ $a_1 = 4$

$k^2 - 6k + 9 = 0$ $k = 3, 3$

$a_n^{(h)} = (A + Bn) \times 3^n$ $a_n^{(P)} = C \times 2^n + Dn^2 \times 3^n$

$C = 3$ $D = 7/18$

$$\boxed{A = -2 \quad B = 17/18}$$

Solve the recurrance relation $a_n - 2a_{n-1} + a_{n-2} = 5n$

Sol: for homogeneous part the characteristic equation is

$$k^2 - 2k + 1 = 0 \Rightarrow (k-1)^2 = 0$$

$$k = k_1 = k_2 = 1$$

The roots are real & equal. $\therefore \boxed{a_n^{(h)} = (A+Bn) \times 1^n}$

Since 1 is a root of the characteristic equation the RHS of the given relation is a polynomial degree 1.

$$\therefore a_n^{(P)} = n^2(A_0 + A_1 n) = A_0 n^2 + A_1 n^3$$

Substituting the above a_n in given relation.

$$A_0 n^2 + A_1 n^3 - 2[A_0(n-1)^2 + A_1(n-1)^3] + A_0(n-2)^2 + A_1(n-2)^3 = 5n$$

$$A_0 n^2 + A_1 n^3 - 2\{A_0(n^2 - 2n + 1) + A_1(n^3 - 3n^2 + 3n - 1)\} + A_0\{n^2 - 4n + 4\} + A_1(n^3 - 6n^2 + 12n - 8) = 5n$$

$$A_0 n^2 + A_1 n^3 - 2A_0 n^2 + 4A_0 n - 2A_0 - 2A_1 n^3 + 6A_1 n^2 - 6A_1 n + 2A_1 + A_0 n^2 - 4A_0 n + 4A_0 + A_1 n^3 - 6A_1 n^2 + 12A_1 n - 8A_1 = 5n$$

n^2 terms $A_0 - 2A_0 + 6A_1 + A_0 - 6A_1 = 0 \Rightarrow 0$

n^3 terms $A_1 - 2A_1 + A_1 = 0 \Rightarrow 0$

n terms $4A_0 - 6A_1 - 4A_0 + 12A_1 = 5 \Rightarrow 6A_1 = 5 \Rightarrow \boxed{A_1 = 5/6}$

Constant term $-2A_0 + 2A_1 + 4A_0 - 8A_1 = 0 \Rightarrow 2A_0 - 6A_1 = 0$

$2A_0 = 6 \cdot 5/6$

$\boxed{A_0 = \frac{5}{2}}$

$$a_n^{(P)} = \frac{5}{2}n^2 + \frac{5}{6}n^3$$

$$\therefore a_n = a_n^{(h)} + a_n^{(P)} \Rightarrow a_n = (A+Bn) + \frac{5}{2}n^2 + \frac{5}{6}n^3$$

→ solve the reccurrance relation $a_{n+2}+3a_{n+1}+2a_n=3^n$ for $n\geq 0$

given $a_0=0,\ a_1=1$.

Sol:

For homogeneous part the characteristic equation is

$$K^2+3K+2=0 \Rightarrow (K+2)(K+1)=0$$

$K_1=-2,\ K_2=-1$ ∴ roots are real & distinct.

$$\boxed{a_n^{(h)}=A(-2)^n+B(-1)^n}$$

Now, By considering R.H.S part 3^n, where 3 is not root

$$\therefore a_n^{(P)}=A_0\times 3^n$$

Substituting $a_n^{(P)}$ in given relation.

$$A_0\cdot 3^{n+2}+3A_0\,3^{n+1}+2A_0 3^n=3^n.$$

$$A_0\cdot 3^{n+2}+A_0\,3^{n+2}+2A_03^n=3^n$$

$$3^n\left[A_0\cdot 3^2+A_0 3^2+2A_0\right]=3^n$$

$$A_0\cdot 3^2+A_0 3^2+2A_0=\cancel{3}\ 1$$

$$9A_0+9A_0+2A_0=1$$

$$20A_0=1$$

$$\boxed{A_0=1/20}$$

∴ The general solution $a_n=a_n^{(h)}+a_n^{(P)}$

$$a_n=\left[A_0\times(-2)^n+B\times(-1)^n\right]+A_0\times 3^n$$

Given $a_0=0$ $\qquad$ $a_1=1$

$a_0=\left[A\times(-2)^0+B\times(-1)^0\right]+A_0\cdot 3^0$ $\qquad$ $a_1=\left[-2A-B\right]+\frac{3}{20}$

$1=-2A-B+\frac{3}{20}$

$0=\left[A+B\right]+\frac{1}{20}$ $\qquad$ $2A+B=\frac{3}{20}-1$

$A+B=-\frac{1}{20}$ $\qquad$ $A=\frac{-4}{5}$ $\quad$ $B=\frac{3}{4}$

$$\boxed{a_n=\left(\frac{-4}{5}\times(-2)^n+\frac{3}{4}(-1)^n\right)+\frac{1}{20}\cdot 3^n}$$

Q) solve the recurrance relation $a_n + 4a_{n-1} + 4a_{n-2} = 5\times(-2)^n$ $n \geq 2$

Sol: The characteristic eqⁿ is $k^2+4k+4=0$

$\Rightarrow (k+2)^2=0 \quad k_1=k_2=-2$

The roots are real and equal

$\therefore \boxed{a_n^{(h)} = (A+Bn)-2^n} \rightarrow$ For homogoneous part

If we obseve the RHS of the given relation, it contains $(-2)^n$ and -2 is repeated 2 times as a characteristic root.

$\therefore a_n^{(P)} = A_0 n^2 (-2)^n$

Substitute $a_n^{(P)}$ in given relation

$A_0 n^2 (-2)^n + 4\{A_0 (n-1)^2 (-2)^{(n-1)}\} + \{4[A_0 (n-2)^2 (-2)^{(n-2)}]\} = 5(-2)^n$

$\Rightarrow A_0 n^2 -2^n + 4\{A_0(n^2-2n+1)(-2)^{n-1}\} + 4\{A_0(n^2-4n+4)(-2)^{(n-2)}\} = 5\times -2^n$

devide the above eqⁿ by $(-2)^{n-2}$ we get

$A_0 \{\cancel{A_0} n^2 (-2)^2\} + 4\{A_0(n^2-2n+1)(-2)\} + 4\{A_0(n^2-4n+4)\} = 5\times -2^2$

$A_0\{4n^2 - 8\cancel{A_0}\{n^2-2n+1\} + 4\cancel{A_0}(n^2-4n+4)\} = 5\times -2^2$

$A_0\{\cancel{4n^2} - \cancel{8n^2} + \cancel{16n} - 8 + \cancel{4n^2} - \cancel{16n} + 16\} = 5\times -2^2$

$A_0\{8\} = 20 \Rightarrow A_0 = \frac{20}{8} = \frac{5}{2}$

$\therefore a_n^{(P)} = \frac{5}{2} n^2 (-2)^n$

$a_n = a_n^{(h)} + a_n^{(P)} \Rightarrow a_n = (A+Bn)-2^n + \frac{5}{2} n^2 (-2)^n$

$\Rightarrow \boxed{a_n = \left[A + Bn + \frac{5}{2} n^2\right] (-2)^n}$

Solve the following recurrence relations

(1) $a_{n+3} = 3a_{n+2} + 4a_{n+1} - 12a_n$ for $n \geq 0$ with $a_0 = 0$, $a_1 = -11$, $a_2 = -15$

$\boxed{a_n = 2\times(-2)^n + 2^n - 3\cdot 3^n}$

(2) $a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0$ for $n \geq 3$ with $a_0 = 1, a_1 = 4$, $a_2 = 8$

$\boxed{a_n = (5+3n)\times 2^n - 4\times 3^n}$

Find general solution for the following recurrence relation

(1) $a_n - 7a_{n-2} + 10a_{n-4} = 0$ $n \geq 4$ $\boxed{a_n = A(\sqrt{5})^n + B(-\sqrt{5})^n + C(\sqrt{2})^n + D(-\sqrt{2})^n}$

(2) $a_n - 9a_{n-1} + 27a_{n-2} - 27a_{n-3} = 0$ $n \geq 3$ $\boxed{a_n = (A + Bn + Cn^2)3^n}$

(3) $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$ $n \geq 3$ $\boxed{a_n = (A+Bn)\times 3^n + C\times 2^n}$

(4) $4a_{n+4} - 8a_{n+3} - 7a_{n+2} + 11a_{n+1} + 6a_n = 0$ $n \geq 4$

$\boxed{a_n = A\times(-1)^n + B\times 2^n + C\times\left(\frac{1}{2}\right)^n + D\times\left(\frac{3}{2}\right)^n}$

(5) $a_n - 8a_{n-2} - 9a_{n-4} = 0$ $n \geq 4$ $\boxed{a_n = A\times 3^n + B\times(-3)^n + \left[D\cos\frac{n\pi}{4} + C\sin\frac{n\pi}{4}\right]}$

(6) $a_{n+2} = 4(a_{n+1} - a_n)$ $n \geq 0$ $a_0 = 1$ $a_1 = 3$

$\boxed{a_n = 2^n + n(2^n - 1)}$

(7) $a_{n+2} + a_n = 0$ $n \geq 0$ $a_0 = 0$ $a_1 = 3$ $\boxed{a_n = 3\sin\left(\frac{n\pi}{2}\right)}$

(8) $a_{n+2}^2 - 5a_{n+1}^2 + 6a_n^2 = 0$ $n \geq 0$ $a_0 = a_1 = 1$

$\boxed{b_n = a_n^2}$ $a_n = \pm\sqrt{b_n}$ $\boxed{a_n = \frac{5}{3\sqrt{6}}\left\{(5+3\sqrt{6})^n - (5-3\sqrt{6})^n\right\}}$

(9) $a_n - 4a_{n-1} + 4a_{n-2} = 0$ $n \geq 2$ $a_0 = 5/2$ $a_1 = 8$

$\boxed{a_n = (5/2)2^n + (3/2)n\cdot 2^n}$

(10) $a_{n+2}^2 - 5a_{n+1}^2 + 4a_n^2 = 0$ for $n \geq 0$ given $a_0 = 4$ and $a_1 = 13$

$\boxed{a_n = \pm\sqrt{(51\times 4^n - 35)}}$