IIT ROORKEE

NPTEL ONLINE CERTIFICATION COURSE

Lecture 6: Introduction to Probability

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Lecture objectives

- Comprehend the different ways of assigning probability
- Understand and apply marginal, union, joint, and conditional probabilities
- Solve problems using the laws of probability including the laws of addition, multiplication and conditional probability
- Revise probabilities using Bayes' rule

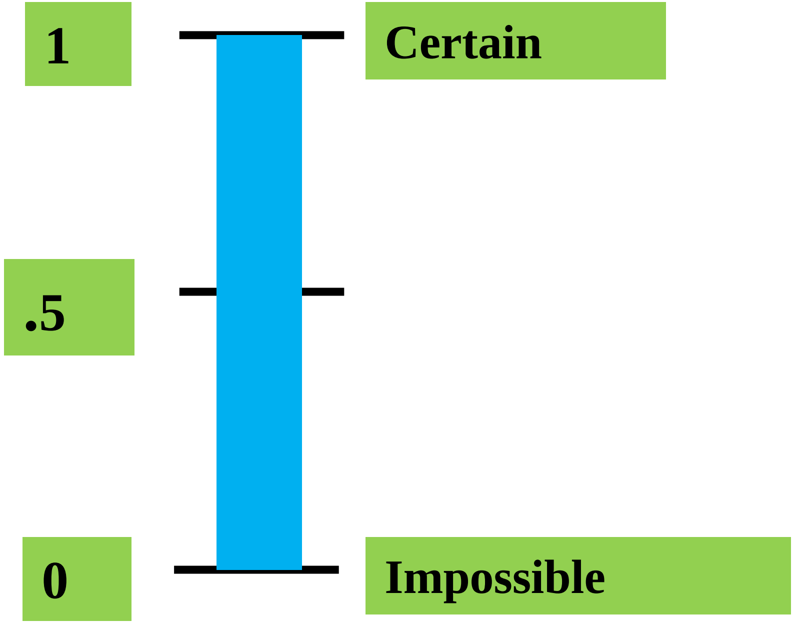
Probability

- Probability is the numerical measure of the likelihood that an event will occur.

- The probability of any event must be between 0 and 1, inclusively
  - $0 \leq P(A) \leq 1$ for any event A.

- The sum of the probabilities of all mutually exclusive and collectively exhaustive events is 1.
  - $P(A) + P(B) + P(C) = 1$
  - A, B, and C are mutually exclusive and collectively exhaustive

Range of Probability



Methods of Assigning Probabilities

- Classical method of assigning probability (rules and laws)
- Relative frequency of occurrence (cumulated historical data)
- Subjective Probability (personal intuition or reasoning)

Classical Probability

- Number of outcomes leading to the event divided by the total number of outcomes possible
- Each outcome is equally likely
- Determined *a priori* -- before performing the experiment
- Applicable to games of chance
- Objective -- everyone correctly using the method assigns an identical probability

Classical Probability

$$P(E) = \frac{n_e}{N}$$

Where:

N = total number of outcomes

n_e = number of outcomes in E

Relative Frequency Probability

- Based on historical data
- Computed after performing the experiment
- Number of times an event occurred divided by the number of trials
- Objective -- everyone correctly using the method assigns an identical probability

Relative Frequency Probability

$$P(E) = \frac{n_e}{N}$$

Where:

N = total number of trials

n_e = number of outcomes producing E

Subjective Probability

- Comes from a person's intuition or reasoning
- Subjective -- different individuals may (correctly) assign different numeric probabilities to the same event
- Degree of belief
- Useful for unique (single-trial) experiments
  - New product introduction
  - Initial public offering of common stock
  - Site selection decisions
  - Sporting events

Probability - Terminology

- Experiment
- Event
- Elementary Events
- Sample Space
- Unions and Intersections
- Mutually Exclusive Events
- Independent Events
- Collectively Exhaustive Events
- Complementary Events

Experiment, Trial, Elementary Event, Event

- Experiment: a process that produces outcomes
  - More than one possible outcome
  - Only one outcome per trial
- Trial: one repetition of the process
- Elementary Event: cannot be decomposed or broken down into other events
- Event: an outcome of an experiment
  - may be an elementary event, or
  - may be an aggregate of elementary events
  - usually represented by an uppercase letter, e.g., A, E_1

An Example Experiment

- Experiment: randomly select, without replacement, two families from the residents of Tiny Town
- Elementary Event: the sample includes families A and C
- Event: each family in the sample has children in the household
- Event: the sample families own a total of four automobiles

Tiny Town Population		
Family	Children in Household	Number of Automobiles
A	Yes	3
B	Yes	2
C	No	1
D	Yes	2

Sample Space

- The set of all elementary events for an experiment
- Methods for describing a sample space
  - roster or listing
  - tree diagram
  - set builder notation
  - Venn diagram

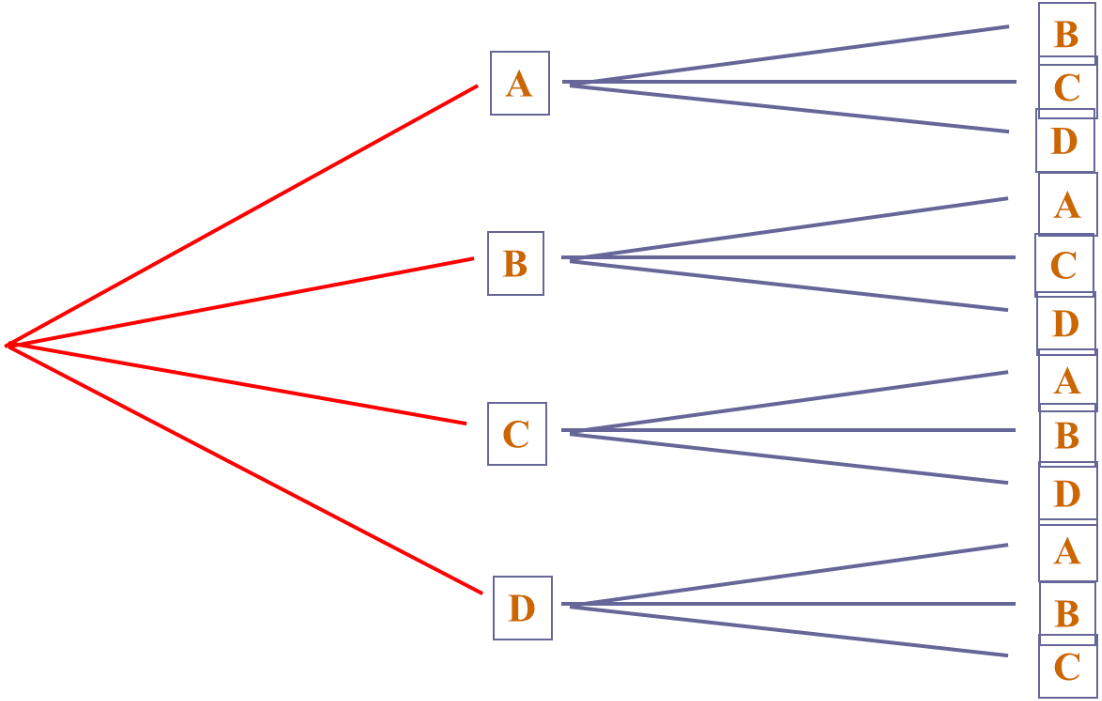
Sample Space: Roster Example

- Experiment: randomly select, without replacement, two families from the residents of Tiny Town
- Each ordered pair in the sample space is an elementary event, for example -- (D,C)

Family	Children in Household	Number of Automobiles
A	Yes	3
B	Yes	2
C	No	1
D	Yes	2

Listing of Sample Space
(A,B), (A,C), (A,D), (B,A), (B,C), (B,D), (C,A), (C,B), (C,D), (D,A), (D,B), (D,C)

Sample Space: Tree Diagram for Random Sample of Two Families



Sample Space: Set Notation for Random Sample of Two Families

- S = {(x,y) | x is the family selected on the first draw, and y is the family selected on the second draw}
- Concise description of large sample spaces

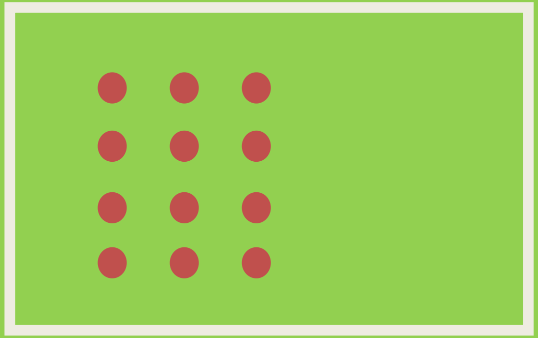
Sample Space

- Useful for discussion of general principles and concepts

Listing of Sample Space

(A,B), (A,C), (A,D),
(B,A), (B,C), (B,D),
(C,A), (C,B), (C,D),
(D,A), (D,B), (D,C)

Venn Diagram

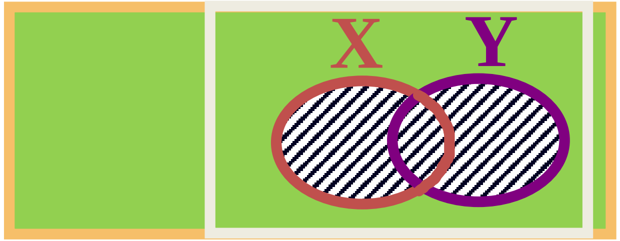


Union of Sets

- The union of two sets contains an instance of each element of the two sets.

$$X = \{1,4,7,9\}$$
$$Y = \{2,3,4,5,6\}$$
$$X \cup Y = \{1,2,3,4,5,6,7,9\}$$

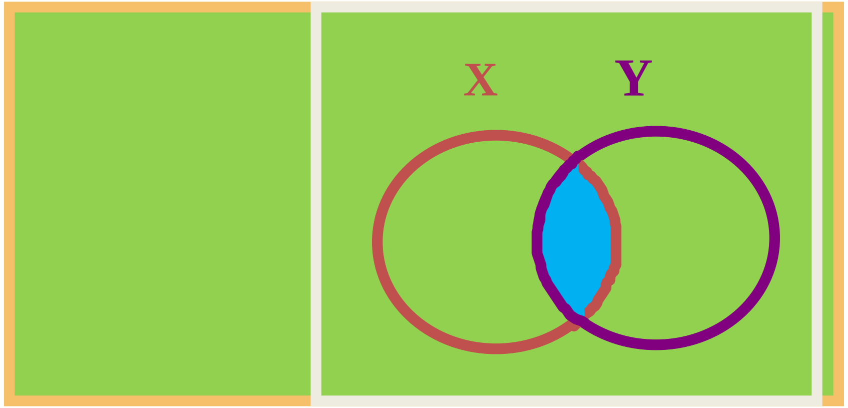
$$C = \{IBM, DEC, Apple\}$$
$$F = \{Apple, Grape, Lime\}$$
$$C \cup F = \{IBM, DEC, Apple, Grape, Lime\}$$



Intersection of Sets

- The intersection of two sets contains only those element common to the

$$X = \{1,4,7,9\}$$
$$Y = \{2,3,4,5,6\}$$
$$X \cap Y = \{4\}$$



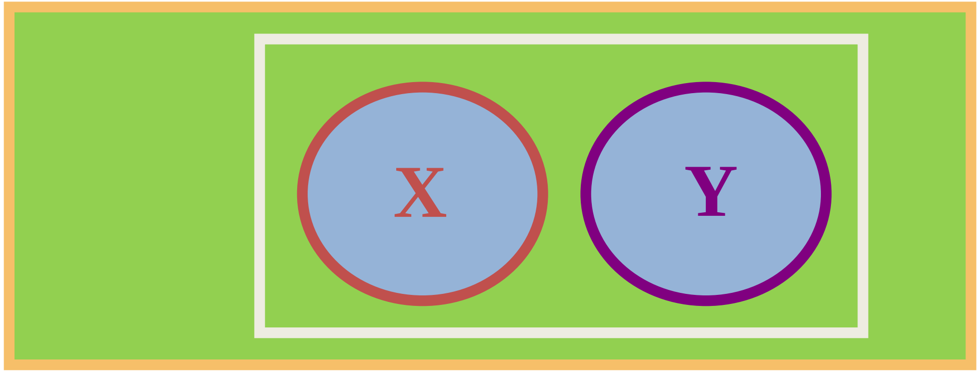
$$C = \{IBM, DEC, Apple\}$$
$$F = \{Apple, Grape, Lime\}$$
$$C \cap F = \{Apple\}$$

Mutually Exclusive Events

- Events with no common outcomes
- Occurrence of one event precludes the occurrence of the other event

$$C = \{IBM, DEC, Apple\}$$
$$F = \{Grape, Lime\}$$
$$C \cap F = \{\ \}$$

$$X = \{1,7,9\}$$
$$Y = \{2,3,4,5,6\}$$
$$X \cap Y = \{\ \}$$



$$P(X \cap Y) = 0$$

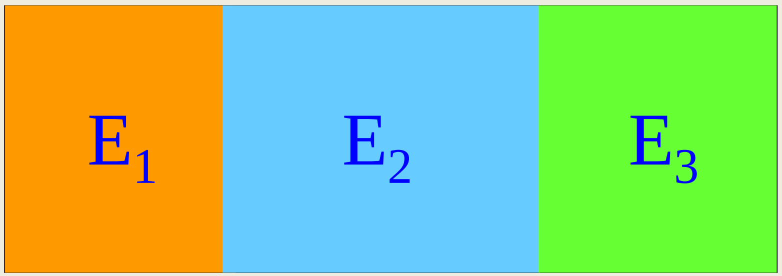
Independent Events

- Occurrence of one event does not affect the occurrence or nonoccurrence of the other event
- The conditional probability of X given Y is equal to the marginal probability of X.
- The conditional probability of Y given X is equal to the marginal probability of Y.

$$\boxed{P(X|Y) = P(X) \text{ and } P(Y|X) = P(Y)}$$

Collectively Exhaustive Events

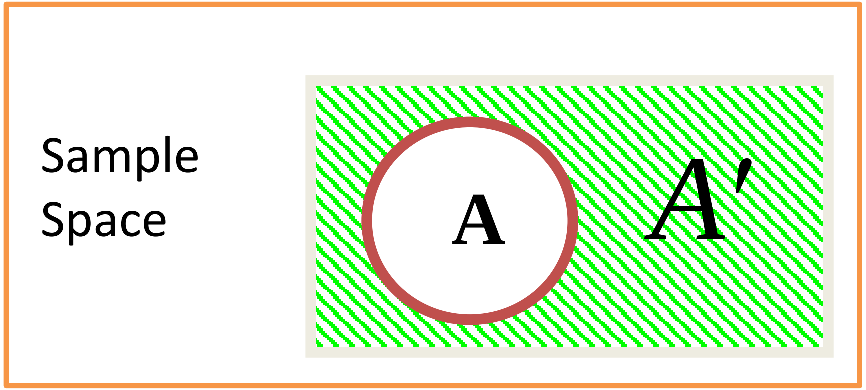
- Contains all elementary events for an experiment



Sample Space with three collectively exhaustive events

Complementary Events

- All elementary events not in the event ‘A’ are in its complementary event.



$$P(Sample\ Space) = 1$$

$$P(A') = 1 - P(A)$$

Counting the Possibilities

- mn Rule
- Sampling from a Population with Replacement
- Combinations: Sampling from a Population without Replacement

mn Rule

- If an operation can be done m ways and a second operation can be done n ways, then there are mn ways for the two operations to occur in order.
- This rule is easily extend to k stages, with a number of ways equal to

$n_1.n_2.n_3..n_k$

- Example: Toss two coins . The total umber of simple events is 2 x 2 =4

Sampling from a Population with Replacement

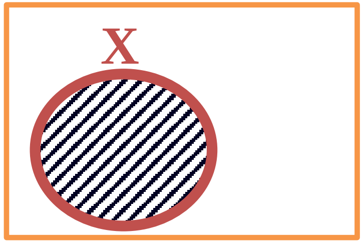
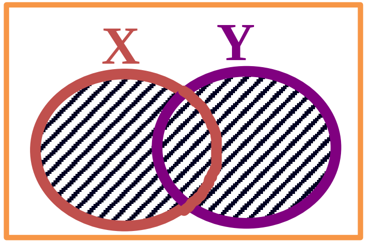
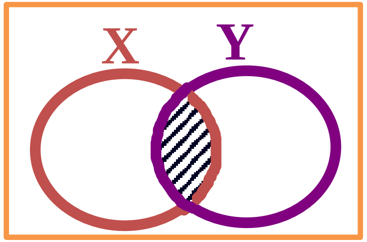
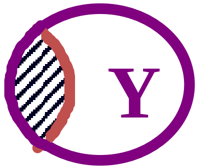
- A tray contains 1,000 individual tax returns. If 3 returns are randomly selected with replacement from the tray, how many possible samples are there?
- $(N)^n = (1,000)^3 = 1,000,000,000$

Combinations

- A tray contains 1,000 individual tax returns. If 3 returns are randomly selected without replacement from the tray, how many possible samples are there?

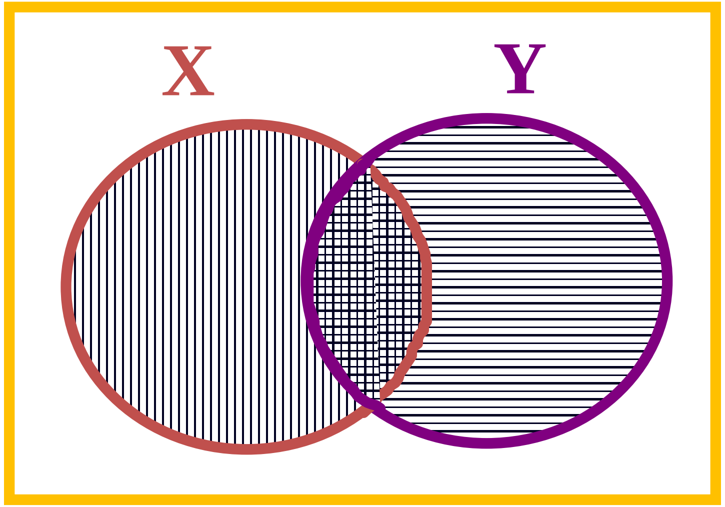
$$\boxed{\binom{N}{n} = \frac{N!}{n!(N-n)!} = \frac{1000!}{3!(1000-3)!} = 166,167,000}$$

Four Types of Probability

Marginal	Union	Joint	Conditional
$P(X)$	$P(X \cup Y)$	$P(X \cap Y)$	$P(X \mid Y)$
The probability of **X** occurring	The probability of **X or Y** occurring	The probability of **X and Y** occurring	The probability of **X occurring given that Y** has occurred
			

General Law of Addition

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$$



Design for improving productivity?

Problem

- A company conducted a survey for the American Society of Interior Designers in which workers were asked which changes in office design would increase productivity.
- Respondents were allowed to answer more than one type of design change.

Reducing noise would increase productivity	70 %
More storage space would increase productivity	67 %

Problem

- If one of the survey respondents was randomly selected and asked what office design changes would increase worker productivity,
  - what is the probability that this person would select reducing noise *or* more storage space?

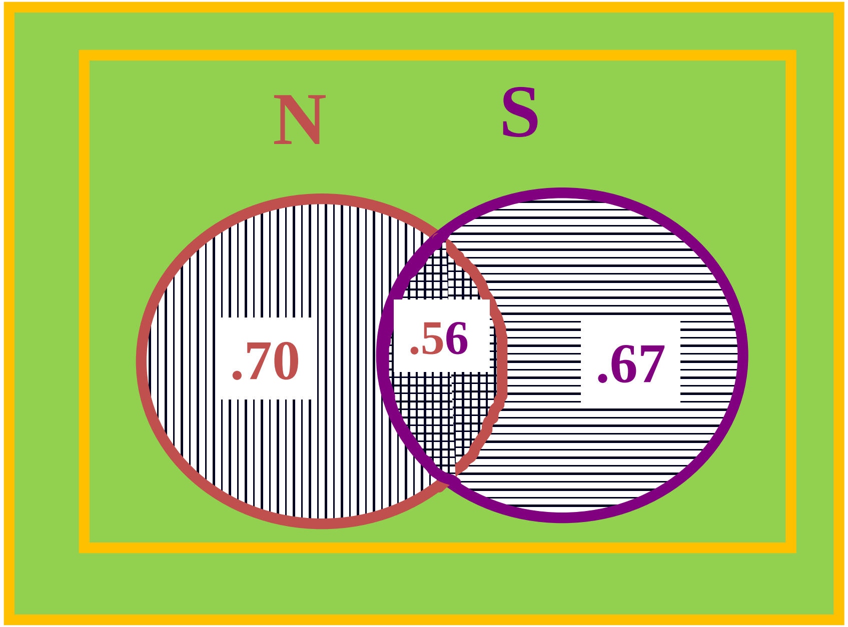
Solution

- Let *N* represent the event "reducing noise."
- Let *S* represent the event "more storage/ filing space."
- The probability of a person responding with *N or S* can be symbolized statistically as a union probability by using the law of addition.

$$P(N \cup S)$$

General Law of Addition -- Example

$$P(N \cup S) = P(N) + P(S) - P(N \cap S)$$



$$P(N) = .70$$
$$P(S) = .67$$
$$P(N \cap S) = .56$$
$$P(N \cup S) = .70 + .67 - .56$$
$$= 0.81$$

Office Design Problem
Probability Matrix

Increase Storage Space

Noise Reduction	Yes	No	Total
Yes	.56	.14	.70
No	.11	.19	.30
Total	.67	.33	1.00

Joint Probability Using a Contingency Table

Event	Event: B_1	Event: B_2	Total
A_1	$P(A_1 \text{ and } B_1)$	$P(A_1 \text{ and } B_2)$	$P(A_1)$
A_2	$P(A_2 \text{ and } B_1)$	$P(A_2 \text{ and } B_2)$	$P(A_2)$
Total	$P(B_1)$	$P(B_2)$	1

Joint Probabilities

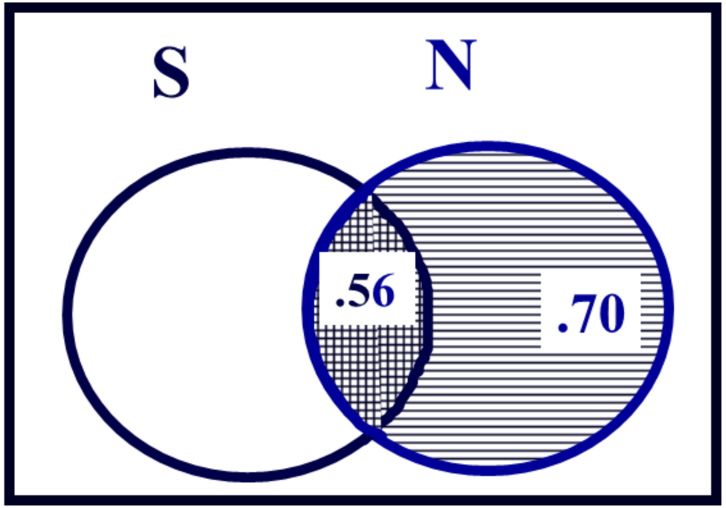
Marginal (Simple) Probabilities

Office Design Problem - Probability Matrix

		Increase Storage Space		
		Yes	No	Total
Noise Reduction	Yes	.56	.14	.70
	No	.11	.19	.30
	Total	.67	.33	1.00

$$
\begin{aligned}
P(N \cup S) &= P(N) + P(S) - P(N \cap S) \\
&= .70 + .67 - .56 \\
&= .81
\end{aligned}
$$

Law of Conditional Probability



$$P(N) = .70$$

$$P(N \cap S) = .56$$

$$P(S|N) = \frac{P(N \cap S)}{P(N)}$$

$$= \frac{.56}{.70}$$

$$= .80$$

Office Design Problem

		Increase Storage Space		
		Yes	No	Total
Noise Reduction	Yes	.56	.14	.70
	No	.11	.19	.30
	Total	.67	.33	1.00

$$P(\overline{N} \mid S) = \frac{P(\overline{N} \cap S)}{P(S)} = \frac{.11}{.67}$$

$$= .164$$

Problem

- A company data reveal that 155 employees worked one of four types of positions.
- Shown here again is the raw values matrix (also called a contingency table) with the frequency counts for each category and for subtotals and totals containing a breakdown of these employees by type of position and by sex.

Contingency Table

COMPANY HUMAN RESOURCE DATA

Type of Position		Sex: Male	Sex: Female	
	Managerial	8	3	11
	Professional	31	13	44
	Technical	52	17	69
	Clerical	9	22	31
		100	55	155

Solution

- If an employee of the company is selected randomly, what is the probability that the employee is female or a professional worker?

$$P(\mathrm{F} \cup \mathrm{P}) = P(\mathrm{F}) + P(\mathrm{P}) - P(\mathrm{F} \cap \mathrm{P})$$

$$P(\mathrm{F} \cup \mathrm{P}) = .355 + .284 - .084 = .555.$$

Problem

- Shown here are the raw values matrix and corresponding probability matrix for the results of a national survey of 200 executives who were asked to identify the geographic locale of their company and their company’s industry type.
- The executives were only allowed to select one locale and one industry type.