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**States:**

**Initial State:** You start with 0 points and three dice to roll.

**Intermediate States:** After each roll, the state can be represented by a combination of the dice values (e.g., (3, 5, 2)) and the current score.

**Terminal State:** The game ends once you decide to stick. The total points represent the final score.

**Actions:**

**Stick:** Accept the values shown on the dice.

**Reroll:** Roll the dice again. Here you have further decisions to make on which dice to hold and which to reroll.

**Transition Model:**

Rolling the dice can be modelled probabilistically since there's a  $1/6$  chance for each face of a die to show up.

Choosing to stick will deterministically transition to the terminal state.

**Reward Function:**

**Rerolling:** Deducts 1 point from the score.

**Sticking:** The sum of the dice values is added to the score. If two or more dice have the same value, they are flipped and then the sum is calculated.

Given the defined MDP structure, value iteration or policy iteration can be used to derive the optimal strategy. Here's how:

- Initialization: Initialize the value function arbitrarily (e.g., all zeros) for all states.
- Iteration: For each state, compute the expected value of each action (stick or reroll) and update the state's value based on the action with the maximum expected value.
- Convergence: Continue this process until the value function converges.

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**State Representation:** Each state can be represented as a set of dice values. Expected Value Calculation: For each case, we need to calculate the expected adhesion value and the expected value of re-rolling (this will be based on potential future situations and a -1 penalty). Policy Derivation: For each case, the action with the highest expected value will be the most appropriate action. The important part will be in calculating the expected value for re-registration. We will have to take into account:

All possible combinations of dice to hold and re-roll.

The probabilities of getting from the current state to each potential future state.

The value of every potential future situation.

Given this knowledge and approach, it is possible to derive the optimal strategy for any dice roll, even in complex situations such as 3, 4, 5.

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### Rules of the game:

1. Starting      Situation: You start at 0 points.
2.      . Dice Roll: You roll three fair six-sided dice.
3.      Selection: You can choose one of two options:
  1.      Stick: Accept the values indicated by the dice. If two or more dice show the same value, they are reversed: a value of 1 becomes a 6, a value of 2 becomes a value of 5, a value of 3 becomes a value of 4, and vice versa. The total is added to your current score, which becomes your final score.
  2.      Reroll: You can roll the dice again. If you want, you can keep some dice at their current value and roll others again. You pay 1 point to roll again. That is, your score can be negative during or at the end of the game. You then make that choice again.

### Highlights:

1.      The game's highest score is 18, which is achieved by discarding three 1 values on the first throw.
2.      The re-roll penalty ensures that you don't roll an infinite number of dice in order to get that score.
3.      The optimal decision is made regardless of your current score. For example, if you started with a score of 0 or had a score of -19 and you saw that all three dice came up with a 6, rolling the dice again will give you a better final score.
4.      We don't know what the outcome of each shot will be, but we know exactly the probability of each outcome. By modeling these probabilities, we can mathematically calculate the best strategy.

Summary: The goal of the game is to get the highest score by rolling the dice. However, the re-throw option deducts 1 point from your score. When two or more dice show the same value, they are reversed. You must make a strategic choice between rolling the dice again or accepting values. This selection should be made based on the expected value of the current dice values.

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### Annotation

Of course, let's try to make you understand the concept better by going through an example.

Let's say your first roll of the dice had results (3, 5, 6). Now we are in the decision-making phase.

At this stage, you need to:

### Calculating the value of the Stick option:

If at least two of the dice have the same value, we turn them upside down. This is not the case in this case.

Hence, the sum of the dice:  $3 + 5 + 6 = 14$  points.

Calculating the expected value of the Reroll option:

If we choose to roll the dice again, we lose 1 point.

But in the option to reroll these dice, we also have the option to keep some dice and reroll others.

For example, if we calculate the probability of holding the dice, which is 6, and rerolling the other two dice:

The probability that the first dice will land 1:  $1/6$ , and in this case its value returns to 6.

The probability that the first dice will land 2 is:  $1/6$ , and in this case its value returns to 5.

The probability that the first dice will land 3:  $1/6$ , and in this case its value returns to 4.

The probability that the first dice will land 4, 5 or 6 is calculated along with the other dice.

In this way, we can find the expected value of the re-throw by calculating all the possible outcomes.

If this value is **greater than the value of the "Hold" option, we must re-roll, otherwise we must hold the dice.**

Calculating all the probabilities on this example can be quite complicated, but that's the general idea. Do you really want to calculate this example in detail, or do you want to know the general approach?

#### **Annotation-2**

I want to calculate it in detail. For example, I started by throwing 2,3,4. The total was 9. I haven't reached 18 yet. I should try to throw it again, right?

#### **ChatGPT**

Yes, that's right. But to mathematically verify this decision as well, let's calculate the expected value of not rerolling the dice (when you hold the dice) and the expected value of rerolling the dice.

Calculating the value of the Stick option:

The value of the dice (2, 3, 4). We do not reverse the values as no two dice are the same.

Total value:  $2 + 3 + 4 = 9$

#### **Calculating the expected value of the Reroll option:**

In this case, you need to decide which dice to keep and which to reroll. We need to evaluate all the combinations to find the combination with the best expectation.

But for the sake of simplicity, let's decide to reroll all the dice and calculate the expectation of this scenario.

The probability of each of the dice coming up is 1:  $1/6$ . If it comes to 1, its value returns to 6.

The probability of each of the dice coming up with 2:  $1/6$ . If it comes to 2, its value returns to 5.

...

The probability of each of the dice coming up with 6:  $1/6$ .

Now, we have to calculate the probabilities of all the dice coming together:

The probability that all three dice will land 1:  $(1/6) * (1/6) * (1/6) = 1/216$

The probability that all three dice will land 2:  $(1/6) * (1/6) * (1/6) = 1/216$

...

The probability that two dice will land 1 and the other will be 2:  $3 * (1/6) * (1/6) * (1/6) = 3/216$  (3 represents combinations of each dice being 1 or 2: (1,1,2), (1,2,1), (2,1,1))

By calculating such combinations for all dice values, we can find the expected value of the reroll. However, before making this calculation, keep in mind the re-throw penalty of -1 point.

As a result of these calculations, if the expected value of the re-throw is greater than 9, we must re-throw. If not, we must hold the dice.

To make this calculation we must take into account all the combinations and probabilities. This can be a bit complicated and time-consuming. If you still want to do this calculation, I can help you step by step.

### **Annotation-3**

Well, I'm trying my second shot chance. With -1, my score dropped to 8. I tried the second shot 1,3,6 and it came.

### **ChatGPT**

Then we are now in the decision-making phase with these new dice values.

#### **Calculating the value of the Stick option:**

Dice values (1, 3, 6).

We just reverse the value of "1" (the opposite of 1 is 6).

Total value:  $6 + 3 + 6 = 15$

#### **Calculating the expected value of the Reroll option:**

If we choose to re-throw, we lose 1 more point, which reduces our total score to 7. This time, however, we also need to decide which dice to keep and which to re-roll.

The dice values this time around are pretty good (15 in total), so it seems logical to keep these dice. However, in order to mathematically confirm this decision as well, we need to see which dice we can achieve a higher expectation by holding.

For example, we must calculate the probability of holding the dice, which is 6, and rerolling the other two dice. However, since the dice values are already quite high in this case (15), we can assume that it makes more sense to keep these dice.

As a result, holding the dice on this roll (scoring 15 points) seems to be the most logical option. However, you can create an optimal policy by using techniques such as the Markov decision process (MDP) to make such decisions automatically with an overall strategy. In this example, holding the dice seems to be the best strategy to get the best result with the given dice values (1, 3, 6). This makes your total score  $8 + 15 = 23$ , but don't forget the -1 point re-throw penalty, which makes your final score 22.

\*\*\*\*\*We move on to the code \*\*\*\*\*

