

Supplementary File for Incentivizing High Quality Crowdsourcing Information using Bayesian Inference and Reinforcement Learning

Anonymous Authors¹

1. Basic Lemmas

Lemma 1. If $x \sim \text{Bin}(n, p)$, $\mathbb{E}t^x = (1 - p + tp)^n$ holds for any $t > 0$, where $\text{Bin}(\cdot)$ is the binomial distribution.

Proof.

$$t^x = e^{x \log t} = m_x(\log t) = (1 - p + pe^{\log t})^n \quad (1)$$

where $m_x(\cdot)$ denotes the moment generating function. \square

Lemma 2. For given $n, m \geq 0$, if $0 \leq p \leq 1$, we can have

$$\begin{aligned} \sum_{x=0}^n \sum_{w=0}^m C_n^x C_m^w p^{x+w} (1-p)^{y+z} \times \\ B(x+z+1+t, y+w+1) = \\ \int_0^1 [(2p-1)x+1-p]^n [(1-2p)x+p]^m x^t dx \end{aligned}$$

Proof. By the definition of the beta function (Olver, 2010),

$$B(x, y) = \int_0^{+\infty} u^{x-1} (1+u)^{-(x+y)} du \quad (2)$$

we can have

$$\begin{aligned} \sum_{x,w} C_n^x C_m^w p^{x+w} (1-p)^{y+z} B(x+z+1+t, y+w+1) \\ = \int_0^{+\infty} \mathbb{E}u^x \cdot \mathbb{E}u^z \cdot u^t \cdot (1+u)^{-(n+m+2+t)} du \end{aligned} \quad (3)$$

where we regard $x \sim \text{Bin}(n, p)$ and $z \sim \text{Bin}(m, 1-p)$. Thus, according to Lemma 1, we can obtain

$$\begin{aligned} \int_0^{+\infty} \mathbb{E}u^x \cdot \mathbb{E}u^z \cdot u^t \cdot (1+u)^{-(n+m+3)} du \\ = \int_0^{+\infty} \frac{[1-p+up]^n \cdot [p+(1-p)u]^m \cdot u^t}{(1+u)^{n+m+2+t}} du. \end{aligned} \quad (4)$$

For the integral operation, substituting u with $v-1$ at first and then v with $(1-x)^{-1}$, we can conclude Lemma 2. \square

Lemma 3. $\sum_{n=0}^N C_N^n \cdot x^n = (1+x)^N$.

Lemma 4. $\sum_{n=0}^N C_N^n \cdot n \cdot x^n = N \cdot x \cdot (1+x)^{N-1}$.

Lemma 5. $\sum_{n=0}^N C_N^n \cdot n \cdot x^{N-n} = N \cdot (1+x)^{N-1}$.

Lemma 6. $\sum_{n=0}^N C_N^n \cdot n^2 \cdot x^n = Nx(1+Nx)(1+x)^{N-2}$.

Lemma 7. $\sum_{n=0}^N C_N^n \cdot n^2 \cdot x^{N-n} = N(x+N)(1+x)^{N-2}$.

Lemma 8. If $0 < x < 1$, we can have

$$\sum_{n=0}^{\lfloor N/2 \rfloor} C_N^n \cdot x^n \geq (1 - e^{-cN}) \cdot (1+x)^N$$

$$\sum_{n=\lfloor N/2 \rfloor + 1}^N C_N^n \cdot x^{N-n} \geq (1 - e^{-cN}) \cdot (1+x)^N.$$

where $c = 0.5(1-x)^2(1+x)^{-2}$.

Proof. To prove the lemmas above, we firstly define

$$F_t(x) = \sum_{n=0}^N C_N^n n^t x^n \quad (5)$$

Then, Lemma 3 can be obtained by expanding $(1+x)^N$. Lemma 4 can be proved as follows

$$\begin{aligned} F_1(x) &= \sum_{n=0}^N C_N^n (n+1) x^n - (1+x)^N \\ \sum_{n=0}^N C_N^n (n+1) x^n &= \frac{d}{dx} [x F_0(x)] \\ &= Nx(1+x)^{N-1} + (1+x)^N. \end{aligned} \quad (6)$$

Lemma 5 can be obtained as follows

$$\begin{aligned} \sum_{n=0}^N C_N^n n x^{N-n} &= x^N \sum_{n=0}^N C_N^n n \left(\frac{1}{x}\right)^n \\ &= x^N \cdot N \cdot \frac{1}{x} \cdot \left(1 + \frac{1}{x}\right)^{N-1}. \end{aligned} \quad (7)$$

For Lemma 6, we can have

$$\begin{aligned} F_2(x) &= \sum_{n=0}^N C_N^n (n+2)(n+1) x^n - 3F_1(x) - 2F_0(x) \\ &= [x^2 F_0(x)]' - 3F_1(x) - 2F_0(x) \end{aligned} \quad (8)$$

Thus, we can have

$$F_2(x) = Nx(1 + Nx)(1 + x)^{N-2} \quad (9)$$

which concludes Lemma 6. Then, Lemma 7 can be obtained by considering Equation 10.

$$\sum_{n=0}^N C_N^n n^2 x^{N-n} = x^N \sum_{n=0}^N C_N^n n^2 \left(\frac{1}{x}\right)^n. \quad (10)$$

For Lemma 8, we can have

$$\sum_{n=0}^{\lfloor N/2 \rfloor} C_N^n x^n = (1+x)^N \sum_{n=0}^{\lfloor N/2 \rfloor} C_N^n p^n (1-p)^{N-n} \quad (11)$$

where $p = x(1+x)^{-1}$. Let $X \sim \text{Bin}(N, p)$, we can have

$$\sum_{n=0}^{\lfloor N/2 \rfloor} C_N^n p^n (1-p)^{N-n} \geq 1 - P(X \geq N/2). \quad (12)$$

Since $x < 1$, $p < 0.5$ and $Np < N/2$. Considering Hoeffding's inequality, we can get

$$P(X \geq N/2) \leq \exp \left[-\frac{N(1-x)^2}{2(1+x)^2} \right] \quad (13)$$

which concludes the first inequality in Lemma 8. Similarly, for the second inequality, we can have

$$\sum_{n=K}^N C_N^n x^{N-n} = (1+x)^N \sum_{n=K}^N C_N^n (1-p)^n p^{N-n} \quad (14)$$

where $K = \lfloor N/2 \rfloor + 1$. Suppose $Y \sim \text{Bin}(N, 1-p)$, we can have

$$\sum_{n=K}^N C_N^n (1-p)^n p^{N-n} \geq 1 - P(Y \leq N/2). \quad (15)$$

Considering Hoeffding's inequality, we can also get

$$P(Y \leq N/2) \leq \exp \left[-\frac{N(1-x)^2}{2(1+x)^2} \right] \quad (16)$$

which concludes the second inequality in Lemma 8. \square

Lemma 9. For any $x, y \geq 0$, we can have

$$(1+x)^y \leq e^{xy}.$$

Proof. Firstly, we can know $(1+x)^y = e^{y \log(1+x)}$. Let $f(x) = x - \log(x)$. Then, we can have $f(0) = 0$ and $f'(x) \geq 0$. Thus, $x \geq \log(1+x)$ and we can conclude Lemma 9 by taking this inequality into the equality. \square

Lemma 10.

$$g(x) = \frac{e^x}{e^x + 1}$$

is a concave function when $x \in [0, +\infty)$.

Proof. $g'(x) = (2 + t(x))^{-1}$, where $t(x) = e^x + e^{-x}$. $t'(x) = e^x - e^{-x} \geq 0$ when $x \in [0, +\infty)$. Thus, $g'(x)$ is monotonically decreasing when $x \in [0, +\infty)$, which concludes Lemma 10. \square

Lemma 11. For $x \in (-\infty, +\infty)$,

$$h(x) = \frac{1}{e^{|x|} + 1}$$

satisfies

$$h(x) < e^x \text{ and } h(x) < e^{-x}.$$

Proof. When $x \geq 0$, we can have

$$h(x) < \frac{1}{e^x} = e^{-x} \leq e^x. \quad (17)$$

When $x \leq 0$, we can have

$$h(x) = \frac{e^x}{e^x + 1} < e^x \leq e^{-x}. \quad (18)$$

\square

Lemma 12. If $\lambda = p/(1-p)$ and $0.5 < p < 1$, then

$$\sum_{n=\lfloor N/2 \rfloor}^N C_N^n \lambda^{m-n} p^n (1-p)^m \leq [4p(1-p)]^{N/2}$$

$$\sum_{n=0}^{\lfloor N/2 \rfloor} C_N^n \lambda^{n-m} p^n (1-p)^m \leq [4p(1-p)]^{N/2}$$

where $m = N - n$.

Proof. For the first inequality, we can have

$$\sum_{n=\lfloor N/2 \rfloor}^N C_N^n \lambda^{m-n} p^n (1-p)^m \quad (19)$$

$$= \sum_{n=\lfloor N/2 \rfloor}^N C_N^n p^m (1-p)^n \leq \sum_{m=0}^{\lfloor N/2 \rfloor} C_N^m p^m (1-p)^n$$

According to the inequality in (Arratia and Gordon, 1989), we can have

$$\sum_{m=0}^{\lfloor N/2 \rfloor} C_N^m p^m (1-p)^n \leq \exp(-ND) \quad (20)$$

where $D = -0.5 \log(2p) - 0.5 \log(2(p-1))$, which concludes the first inequality in Lemma 12.

For the second inequality, we can have

$$\begin{aligned} & \sum_{n=0}^{\lfloor N/2 \rfloor} C_N^n \lambda^{n-m} p^n (1-p)^m \\ &= \frac{1}{[p(1-p)]^N} \sum_{n=0}^{\lfloor N/2 \rfloor} C_N^n [p^3]^n [(1-p)^3]^m \\ &= \frac{[p^3 + (1-p)^3]^N}{[p(1-p)]^N} \sum_{n=0}^{\lfloor N/2 \rfloor} C_N^n x^n (1-x)^m \end{aligned} \quad (21)$$

where $x = p^3/[p^3 + (1-p)^3]$. By using Equation 20, we can have

$$\begin{aligned} & \sum_{n=0}^{\lfloor N/2 \rfloor} C_N^n \lambda^{n-m} p^n (1-p)^m \\ & \leq \frac{[p^3 + (1-p)^3]^N}{[p(1-p)]^N} [x(1-x)]^{N/2} \\ & = [4p(1-p)]^{N/2} \end{aligned} \quad (22)$$

which concludes the second inequality of Lemma 12. \square

2. H function analysis

Here, we present our analysis on the H function defined in the proof of Proposition 1. Firstly, we can have:

Lemma 13. $H(m, 0.5; M, t) = 2(t+1)^{-1}$.

Lemma 14. $H(m, p; M, t) = H(n, \bar{p}; M, t)$.

Lemma 15. As a function of m , $H(m, p; M, t)$ is logarithmically convex.

Proof. Lemma 13 can be proved by integrating $2x^t$ on $[0, 1]$. Lemma 14 can be proved by showing that $H(n, \bar{p}; M, t)$ has the same expression as $H(m, p; M, t)$. Thus, in the following proof, we focus on Lemma 15. Fixing p , M and t , we denote $\log(H)$ by $f(m)$. Then, we compute the first-order derivative as

$$H(m)f'(m) = 2^{M+1} \int_0^1 \lambda u^n (1-u)^m x^t dx \quad (23)$$

where $u = (2p-1)x + 1 - p$ and $\lambda = \log(1-u) - \log(u)$. Furthermore, we can solve the second-order derivative as

$$\begin{aligned} & 2^{-2(M+1)} H^2(m) f''(m) = \\ & \int_0^1 g^2(x) dx \int_0^1 h^2(x) dx - \left(\int_0^1 g(x) h(x) dx \right)^2 \end{aligned} \quad (24)$$

where the functions $g, h : (0, 1) \rightarrow \mathbb{R}$ are defined by

$$g = \lambda \sqrt{u^n(1-u)^m}, \quad h = \sqrt{u^n(1-u)^m}. \quad (25)$$

By the Cauchy-Schwarz inequality,

$$\int_0^1 g^2(x) dx \int_0^1 h^2(x) dx \geq \left(\int_0^1 g(x) h(x) dx \right)^2 \quad (26)$$

we can know that $f''(m) \geq 0$ always holds, which concludes that f is convex and H is logarithmically convex. \square

Then, for the case that $t = 1$ and $M \gg 1$, we can further derive the following three lemmas for $H(m, p; M, 1)$:

Lemma 16. The ratio between two ends satisfies

$$\log \frac{H(0, p; M, 1)}{H(M, p; M, 1)} \approx \begin{cases} \log(M) + \epsilon(p) & p > 0.5 \\ 0 & p = 0.5 \\ -\log(M) - \epsilon(\bar{p}) & p < 0.5 \end{cases}$$

where $\epsilon(p) = \log(2p-1) - \log(p)$ and $\epsilon(p) = 0$ if $p = 0.5$.

Lemma 17. The lower bound can be calculated as

$$\log H(m, p) \gtrsim \begin{cases} H(0, p) - \underline{k}m & 2m \leq M \\ H(M, p) - \underline{k}n & 2m > M \end{cases}$$

where $\underline{k} = \log(\max\{p/(\bar{p} + M^{-1}), \bar{p}/(p + M^{-1})\})$.

Lemma 18. The upper bound can be calculated as

$$\log H(m, p) \lesssim \begin{cases} H(0, p) - \bar{k}m & 2m \leq M \\ H(M, p) - \bar{k}n & 2m > M \end{cases}$$

where $n = M - m$ and $\bar{k} = 2 \log(2 \cdot \max\{p, \bar{p}\})$.

Proof. By Lemma 13, $\log H(m, 0.5; M, 1) \equiv 0$, which proves the above three lemmas for the case that $p = 0.5$. Considering the symmetry ensured by Lemma 14, we thus focus on the case that $p > 0.5$ in the following proof and transform $H(m, p)$ into the following formulation

$$H(m, p) = \omega(p) \cdot \int_{\bar{p}}^p x^n (1-x)^m (x-1+p) dx \quad (27)$$

where $\omega(p) = 2^{M+1}/(2p-1)^2$. Then, we can solve $H(0, p)$ and $H(M, p)$ as

$$\begin{aligned} H(0, p) &= \omega(p) \int_{\bar{p}}^p x^M (x-\bar{p}) dx \\ &= \frac{(2p)^{M+1}}{(2p-1)(M+1)} - O\left(\frac{(2p)^{M+1}}{M^2}\right) \end{aligned} \quad (28)$$

$$\begin{aligned} H(M, p) &= \omega(p) \int_{\bar{p}}^p (1-x)^M (x-\bar{p}) dx \\ &= \frac{p(2p)^{M+1}}{(2p-1)^2(M+1)(M+2)} - O\left(\frac{(2\bar{p})^{M+1}}{M+2}\right). \end{aligned} \quad (29)$$

Using the Taylor expansion of function $\log(x)$, we can calculate the ratio in Lemma 16 as

$$\log \frac{H(0, p)}{H(M, p)} = \log(M) + \log \frac{2p-1}{p} + O\left(\frac{1}{M}\right) \quad (30)$$

which concludes Lemma 16 when $M \gg 1$.

Furthermore, we can solve $H(1, p)$ as

$$\begin{aligned} H(1, p) &= \omega(p) \int_{\bar{p}}^p x^{M-1} (x-\bar{p}) dx - H(0, p) \\ &= \frac{(2\bar{p} + M^{-1})(2p)^M}{(2p-1)(M+1)} - O\left(\frac{(2p)^{M+1}}{M^2}\right) \end{aligned} \quad (31)$$

The value ratio between $m = 0$ and $m = 1$ then satisfies

$$\log \frac{H(1, p)}{H(0, p)} = \log \frac{p}{\bar{p} + M^{-1}} + O\left(\frac{1}{M}\right). \quad (32)$$

By Rolle's theorem, there exists a $c \in [m, m+1]$ satisfying

$$\log H(1, p) - \log H(0, p) = f'(c) \quad (33)$$

where $f(m) = \log H(m, p)$. Meanwhile, Lemma 15 ensures that $f''(m) \geq 0$ always holds. Thus, we can have

$$\log H(m+1, p) - \log H(m, p) \geq \log \frac{H(1, 0)}{H(0, p)} \quad (34)$$

which concludes the first case of Lemma 17. Similarly, we compute the ratio between $m = M-1$ and M as

$$\log \frac{H(M, p)}{H(M-1, p)} = \log \frac{p}{\bar{p} + M^{-1}} + O\left(\frac{1}{M}\right). \quad (35)$$

Meanwhile, Rolle's theorem and Lemma 15 ensure that

$$\log H(m, p) - \log H(m-1, p) \leq \log \frac{H(M, 0)}{H(M-1, p)} \quad (36)$$

which concludes the second case of Lemma 17.

Lastly, we focus on the upper bound described by Lemma 18. According to the inequality of arithmetic and geometric means, $x(1-x) \leq 2^{-2}$ holds for any $x \in [0, 1]$. Thus, when $2m \leq M$ (i.e. $n \geq m$), we can have

$$H(m, p) \leq 2^{-2m} \omega(p) \cdot \int_{\bar{p}}^p x^{n-m} (x-1+p) dx \quad (37)$$

where the equality only holds when $m = 0$.

$$\int_{\bar{p}}^p x^{n-m} (x-1+p) dx = \frac{(2p-1)p^\delta}{\delta} + \frac{\Delta}{\delta(\delta+1)} \quad (38)$$

where $\delta = n - m + 1$ and $\Delta = \bar{p}^{\delta+1} - p^{\delta+1} < 0$. Hence,

$$\log \frac{H(m, p)}{H(0, p)} \leq -2m[\log(2p) - \varepsilon(m)] + O\left(\frac{1}{M}\right) \quad (39)$$

where $\varepsilon(m) = -(2m)^{-1}[\log(n-m+1) - \log(M+1)]$. Since $\log(x)$ is a concave function, we can know that

$$\varepsilon(m) \leq (M)^{-1} \log(M+1) = O(M^{-1}) \quad (40)$$

which concludes the first case in Lemma 18. Similarly, for $2m > M$ (i.e. $n < m$), we can have

$$\log \frac{H(m, p)}{H(M, p)} \leq -2n[\log(2p) - \hat{\varepsilon}(n)] + O\left(\frac{1}{M}\right) \quad (41)$$

where $\hat{\varepsilon}(n) \leq O(M^{-1})$. Thereby, we can conclude the second case of Lemma 18. Note that the case where $p < 0.5$ can be derived by using Lemma 14. \square

For the case that $t = 0$ and $M \gg 1$, using the same method as the above proof, we can derive the same lower and upper bounds as Lemmas 18 and 17. On the other hand, for $t = 0$, Lemma 16 does not hold and we can have

Lemma 19. $H(m, p; M, 0) = H(n, p; M, 0)$

Proof. When $t = 0$,

$$H(m, p) = 2^{M+1} (2p-1)^{-1} \int_{\bar{p}}^p x^n (1-x)^m dx. \quad (42)$$

Then, substituting x as $1-v$ concludes Lemma 19. \square

3. Proof for Proposition 3

Denote the prior distribution of θ by π . Then,

$$P(\mathcal{L}|\alpha, \beta) = \prod_{j=1}^M P_{\theta}(\mathbf{x}_j) \int e^{[-M \cdot d_{KL}]} d\pi(\hat{\theta}) \quad (43)$$

$$d_{KL} = \frac{1}{M} \sum_{j=1}^M \log \frac{P_{\theta}(\mathbf{x}_j)}{P_{\hat{\theta}}(\mathbf{x}_j)} \rightarrow \text{KL}[P_{\theta}(\mathbf{x}), P_{\hat{\theta}}(\mathbf{x})] \quad (44)$$

where \mathbf{x}_j denotes the labels generated for task j . The KL divergence $\text{KL}[\cdot, \cdot]$, which denotes the expectation of the log-ratio between two probability distributions, is a constant for the given θ and $\hat{\theta}$. Thus, $\int e^{[-M \cdot d_{KL}]} d\pi(\hat{\theta}) = C_L(M)$. In addition, when $M \rightarrow \infty$, we can also have $\sum 1(\mathbf{x}_j = \mathbf{x}) \rightarrow M \cdot P_{\theta}(\mathbf{x})$, which concludes Proposition 3.

4. Proof for Proposition 4

Firstly, we introduce a set of variables to describe the real true labels and the collected labels. Among the n tasks of which the posterior true label is correct,

- x_0 and y_0 denote the number of tasks of which the real true label is 1 and 2, respectively.
- x_i and y_i denote the number of tasks of which worker i 's label is correct and wrong, respectively.

Also, among the remaining $m = M - n$ tasks,

- w_0 and z_0 denote the number of tasks of which the real true label is 1 and 2, respectively.
- w_i and z_i denote the number of tasks of which worker i 's label is correct and wrong, respectively.

Thus, we can have $x_i + y_i = n$ and $w_i + z_i = m$. Besides, we use ξ_i to denote the combination (x_i, y_i, w_i, z_i) .

To compute the expectation of m/M , we need to analyze the probability distribution of m . According to Equation ??, we can know that $P(m)$ satisfies

$$P(m) \approx \frac{C_M^m}{Z} \sum_{\xi_0, \dots, \xi_N} \prod_{i=0}^N P(\xi_i | m) B(\hat{\beta}) \prod_{i=1}^N B(\hat{\alpha}_i^*) \quad (45)$$

where $Z = C_p C_L [\prod_{\mathbf{x}} P_{\theta}(\mathbf{x})]^{M \cdot P_{\theta}(\mathbf{x})}$ is independent of ξ_i and m . Meanwhile, $\hat{\beta}_1 = x_0 + z_0 + 1$, $\hat{\beta}_2 = y_0 + w_0 + 1$, $\hat{\alpha}_{i1}^* = x_i + z_i + 2$ and $\hat{\alpha}_{i2}^* = x_i + z_i + 1$. When the m tasks

of which the posterior true label is wrong are given, we can know that $x_i \sim \text{Bin}(n, p_i)$ and $w_i \sim \text{Bin}(m, p_i)$, where $\text{Bin}(\cdot)$ denotes the binomial distribution. In addition, x_i and y_i are independent of w_i , z_i and $\xi_{k \neq i}$. Also, w_i and z_i are independent of x_i and y_i and $\xi_{k \neq i}$. Thus, we can further obtain $P(m) \approx \hat{Z}^{-1} \cdot C_M^m Y(m)$, where

$$Y(m) = e^{\log H(m, p_0; M, 0) + \sum_{i=1}^N \log H(m, p_i; M, 1)}$$

$$H(m, p; M, t) = \sum_{x=0}^n \sum_{w=0}^m 2^{M+1} C_n^x C_m^w \times \quad (46)$$

$$p^{x+w} (1-p)^{y+z} B(x+z+1+t, y+w+1)$$

and $\hat{Z} = 2^{-(N+1)(M+1)} Z$. Considering $\sum_{m=1}^M P(m) = 1$, we can know that $\hat{Z} \approx \sum_{m=1}^M C_M^m Y(m)$.

The biggest challenge of computing $P(m)$ exists in the analysis of function $H(m, p; M, t)$ which we put in the supplementary file because of the space limitation. Here, we directly use the obtained lower and upper bounds of the H function (Lemmas ?? and ??) and can have

$$\begin{cases} e^{C-K_l m} \lesssim Y(m) \lesssim e^{C-K_u m} & 2m \leq M \\ e^{C+\delta-K_l n} \lesssim Y(m) \lesssim e^{C+\delta-K_u n} & 2m > M \end{cases} \quad (47)$$

where $C = H(0, p_0; M, 0) + \sum_{i=1}^N H(0, p_i; M, 1)$ and

$$K_l = \sum_{i=0}^N \log \hat{\lambda}_i, \quad K_u = 2 \sum_{i=0}^N \log (2\hat{p}_i)$$

$$\delta = \Delta \cdot \log(M) + \sum_{i=1}^N (-1)^{1(p_i > 0.5)} \phi(\hat{p}_i)$$

$$\hat{\lambda}_i = \max \left\{ \frac{p_i}{\bar{p}_i + \frac{1}{M}}, \frac{\bar{p}_i}{p_i + \frac{1}{M}} \right\}, \quad \phi(p) = \log \frac{2p-1}{p}.$$

Besides, we set a convention that $\phi(p) = 0$ when $p = 0.5$. Thereby, the expectations of m and m^2 satisfy

$$\mathbb{E}[m] \lesssim \frac{\sum_{m=0}^M m e^{-K_u m} + \sum_{m=0}^M m e^{\delta-K_u n}}{\sum_{m=0}^k e^{-K_l m} + \sum_{m=k+1}^M e^{\delta-K_l n}} \quad (48)$$

$$\mathbb{E}[m^2] \lesssim \frac{\sum_{m=0}^M m^2 e^{-K_u m} + \sum_{m=0}^M m^2 e^{\delta-K_u n}}{\sum_{m=0}^k e^{-K_l m} + \sum_{m=k+1}^M e^{\delta-K_l n}} \quad (49)$$

where $k = \lfloor M/2 \rfloor$. By using Lemmas ??, ??, ?? and ??, we can know the upper bounds of the numerator in Equations 48 and 49 are $M(\varepsilon + e^\delta)(1 + \varepsilon)^{M-1}$ and $[M^2 \varepsilon^2 + M\varepsilon + e^\delta(M^2 + M\varepsilon)](1 + \varepsilon)^{M-2}$, respectively, where $\varepsilon = e^{-K_u}$. On the other hand, by using Lemma ??, we can obtain the lower bound of the denominator as $(1 + e^\delta)[1 - e^{-c(\omega)M}](1 + \omega)^M$, where $\omega = e^{-K_l}$ and $c(\omega) = 0.5(1 - \omega)^2(1 + \omega)^{-2}$. Considering $M \gg 1$, we can make the approximation that $e^{-c(\omega)M} \approx 0$ and $(1 + e^\delta)\varepsilon/M \approx 0$. Besides, $(1 + \omega)^M \geq 1$ holds because $\omega \geq 0$. In this case, Proposition 4 can be concluded by combining the upper bound of the numerator and the lower bound of the denominator.

References

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