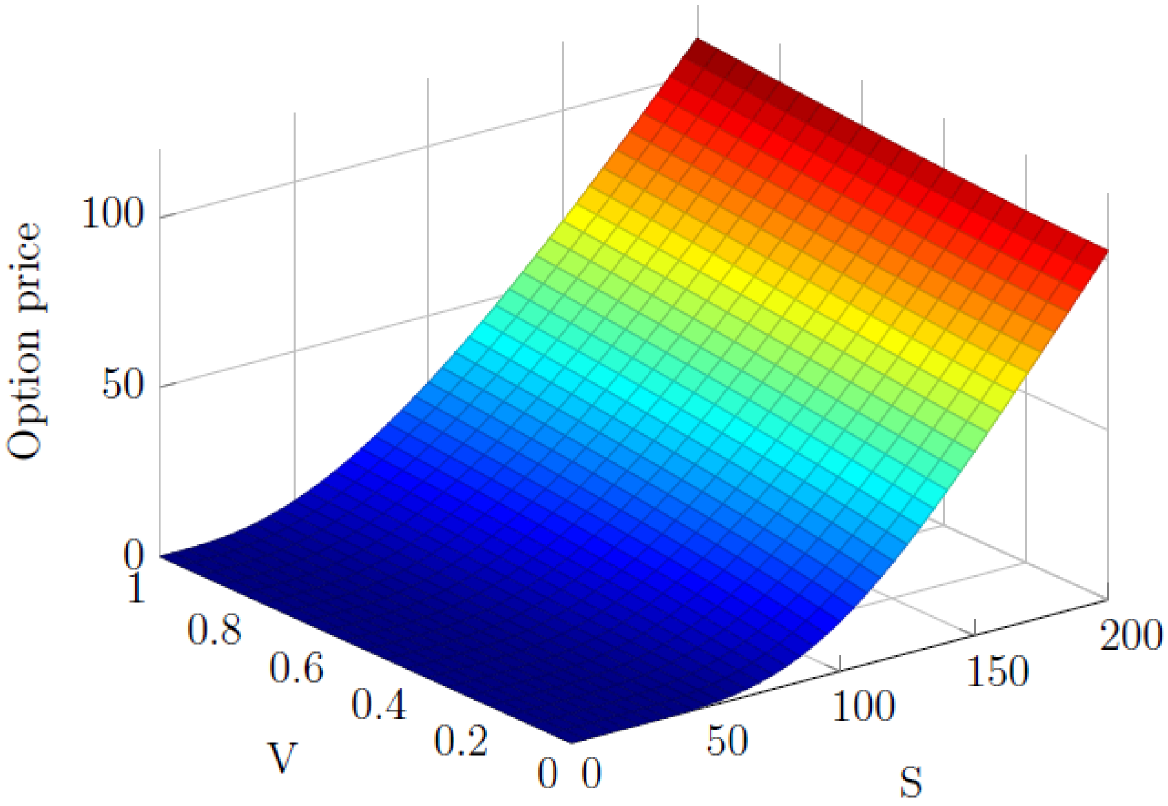
Advanced Volatility and associated topics



Introduction

In this section, we discuss details of volatility. Volatility is one of the most interesting and important aspects of Quantitative Finance. Volatility is the key parameter determining the price of an option, yet it is also the hardest to measure. There are many types of volatility; the precise nature and difference is very important - it is crucial that we know which volatility we are talking about. This adds to the difficulty of volatility considerations.

In the Black-Scholes model, the SDE for the stock has two parameters, μ and σ but later the drift disappears even though the stock depends on it. Some find this counter-intuitive that the value of a call option does not depend on whether the underlying stock is more likely to go up than it is to go down. Recall this is a consequence of hedging. Hence the importance of modelling the volatility 'correctly' if in the business of derivative pricing. If not things become increasingly complex! Suppose we are concerned with stock selection,then the drift comes back in.

The Black-Scholes model is very elegant but it does not perform well in practice. A basic assumption of the framework is a constant geometric Brownian motion for the underlying

$$\frac{dS}{S} = \mu dt + \sigma dW_t$$

and leads to a partial differential equation for which either an analytical solution exists or can be treated numerically.

Thus far the role of σ is that of a parameter. It is the most important parameter when pricing an option, and is also the most difficult to measure. In comparing the solution to reality, the important question arising is "How plausible is a constant volatility?". Recapping the model for a Call option $C(S,t)$, the pricing equation and terminal condition in turn

$$\begin{aligned} \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC &= 0 \\ C(S,T) &= \max(S-E,0). \end{aligned}$$

The solution is

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 = d_1 - \sigma\sqrt{T-t}.$$

This pricing formula relates the option price C to six arguments; the variables S, t and the parameters r, σ, E and T.

Quantity	Observable?	
C_M	yes	quoted market option price
S	yes	today's spot price
t	yes	today's date
T	yes	expiration
E	yes	strike price
r	yes	today's interest rate
σ	No	

So in derivatives the volatility is the most important parameter.

Drift is not important.

It doesn't matter if e.g. it is doubling in price or halving in price. It is the level of noise/randomness that affects the price. But it is very hard to measure - volatility cannot be seen/observed. It is how much randomness there is in a stock price in an instant in time. For these reasons different types of volatility needs to be discussed.

Spot Volatility

- Define logarithmic returns $R_t := \log\left(\frac{S_{t+\delta t}}{S_t}\right)$. Recall these are popular for many reasons, both theoretic and algorithmic.

- The mean and variance in turn are

$$m_t = \mathbb{E}\left[R_t\right], \quad v_t = \mathbb{E}\left[(R_t - m_t)^2\right]$$

- The classical Black-Scholes stock price process $S_t,\ t \geq 0$ given by GBM is

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

- Using Itô to calculate $d(\log S_t)$ gives the solution as

$$\log S_t = \log S_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t$$

- Now look at the mean and variance of the logarithmic return $\log\left(\frac{S_t}{S_0}\right)$

$$\mathbb{E}\left[\log\left(\frac{S_t}{S_0}\right)\right] =: \eta t := \left(\mu - \frac{1}{2}\sigma^2\right) t$$

$$\mathbb{V}\left[\log\left(\frac{S_t}{S_0}\right)\right] =: s^2\,(t) := \sigma^2 t$$

- This gives one definition of volatility σ. It is a measure of the variance of the logarithmic returns, such that the square of the volatility gives the rate of increase of the log-returns:

$$\frac{d}{dt}s^2(t) = \sigma^2.$$

Equivalently, if we denote the quadratic variation of a stochastic process Y_t by $[Y]_t$, then we have

$$[\log S]_t = \sigma^2 t,$$

so the squared volatility is the rate of change of the quadratic variation of the log-stock price.

Deterministic Volatility Models

Simplest generalisation of the Black-Scholes constant volatility paradigm is to allow the volatility to be a deterministic function of time so that the stock price SDE becomes

$$dS_t = \mu S_t dt + \sigma\left(t\right) S_t dW_t,$$

and the modified Black-Scholes equation becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\left(t\right) S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

for the option price $V\left(t, S\right)$, where $V\left(T, S\right) = h\left(S\right).$

By the Feynman-Kac theorem the option pricing function is given by the risk-neutral expectation

$$V\left(t, S\right) = \mathbb{E}^{\mathbb{Q}}\left[\left. e^{-r(T-t)}\, h\left(S_T\right)\right| S_t = x\right],$$

where under the $\mathbb{Q}$ measure, S_t follows GBM above with μ replaced by r :

$$dS_t = \mu S_t dt + \sigma(t) S_t dW_t^{\mathbb{Q}},$$

where $W_t^{\mathbb{Q}}$ is a $\mathbb{Q}$ Brownian motion. Under $\mathbb{Q}$, the terminal log-stock price is now given by

$$\log S_T = \log S_t + r(T-t) - \frac{1}{2}\int_t^T \sigma^2(s)\,ds + \int_t^T \sigma(s)\,dW_s^{\mathbb{Q}}$$

Hence, under $\mathbb{Q}$, given $S_t = x$, $\log S_T$ is normally distributed:

$$\log S_T \sim N\left(\log x + \left(r - \tfrac{1}{2}\overline{\sigma}^2\right)(T-t), \overline{\sigma}^2(T-t)\right),$$

where $\overline{\sigma}^2$ is the root mean square (RMS) volatility, given by

$$\overline{\sigma}^2(T-t) = \int_t^T \sigma^2(s)\,ds$$

It follows that one simply prices the option using the Black-Scholes formula with the volatility σ replaced by

$$\overline{\sigma}_t^2 = \frac{1}{(T-t)}\int_t^T \sigma^2(s)\,ds$$

Thus, in all BS pricing formulas for European, path-independent options, just replace σ by $\overline{\sigma}_t$.

For example, the price of a vanilla call at time t is given by

$$C\left(\overline{\sigma}_t^2, S, t\right) = SN(d_1) - Ee^{-r(T-t)}N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\log(S/E) + (r + \frac{1}{2}\overline{\sigma}_t^2)(T-t)}{\overline{\sigma}_t\sqrt{T-t}}, \\ d_2 &= d_1 - \overline{\sigma}_t\sqrt{T-t}, \\ \overline{\sigma}_t^2 &= \frac{1}{(T-t)}\int_t^T \sigma^2(s)\,ds. \end{aligned}$$

Local Volatility

- Further generalisation of Black-Scholes:

$$dS_t = \mu S_t dt + \sigma(t, S_t) S_t dW_t.$$

- The deterministic function

$$(t, S) \to \sigma(t, S_t)$$

is called *local volatility*.

- Option price $V(t, S_t)$ for terminal payoff $h(S_T)$ satisfies the BSE

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(t, S) S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV &= 0 \\ V(T, S_T) &= h(S_T). \end{aligned}$$

- The model is also referred to as 'restricted stochastic' volatility model, since the volatility path $\sigma_t = \sigma(t, S_t)$ is stochastic, but the only source of randomness enters through the state variable S.

Special cases are

i. $\sigma = \sigma(t)$ is a function of time alone: There is a term-structure, but the model fails to predict smiles and skews. As discussed above.

ii. $\sigma = \sigma(S)$ is a function of the stock alone: An important family are constant elasticity of variance (CEV) models of the form

$$dS_t = \mu S_t dt + \sigma S_t^{\alpha} dW_t$$

In the previous table we note that C_M, S, t, T, E and r are observables. For a plain vanilla option we define the *implied volatility* denoted σ_i to be that value of the unobservable σ, which gives the market price of the option when substituted into the Black-Scholes option pricing formula. All the other observables are fixed in this process. It is described as the market's view of the future actual volatility during the life of the option.

The implied volatility according to the Black-Scholes model should be independent of both strike and expiration; in reality it depends on both.

Consider the following example: A trader can see on their screen that a certain call option with one year until expiry and a strike of £100 is trading at £10.45 with the underlying at £100 and a short-term interest rate of 5%. Can we use this information in some way?

We can take invert this relationship between volatility and an option price by asking "What volatility must I use to get the correct market price?"

This is called the implied volatility. If $C_{\textsf{BS}}$ denotes the Black-Scholes theoretical price then solving

$$C_M(S,t) = C_{\textsf{BS}}(S,T,r,E,\sigma_i(E,T))$$

for σ_{i} becomes a root-finding problem. Use of e.g. Newton-Raphson method will work.

For implied volatility to be a useful concept means there should be a unique implied vol. This is only true if the options vega υ, where

$$\upsilon(S,t) = \frac{\partial V}{\partial \sigma}(S,t;\sigma;E,T;r;....),$$

does not change sign for any value of S, t or other parameters, besides σ, involved. While this is true for European calls and puts it is true for all options.

Smiles

According to the classical Black-Scholes analysis

$$\frac{dS}{S} = \mu dt + \sigma dW$$

so that σ is a property of S alone. For a vanilla call or put option the strike E and the expiry T are properties only of the option. Thus the volatility (which in the Black-Scholes model should be the same thing as the implied volatility) σ should be independent of both strike E and the expiry T or a vanilla put or call.

In practice, we find the implied volatility for a vanilla call (or put) depends on both the strike and the expiry, the so-called *smile*. This implies that there is something wrong with the Black-Scholes model.

The dependence of implied volatility on expiry could imply a term structure for volatility, $\widehat{\sigma}(t)$, rather than a constant volatility σ. This is not a serious problem; we know how to deal with time dependent volatilities; we just replace σ^2 in the Black-Scholes formulae by

$$\frac{1}{T-t}\int_t^T \widehat{\sigma}^2(s)\,ds.$$

The dependence on strike is a serious problem. It is quite inconsistent with the Black-Scholes analysis.

One way of explaining the smile effect is to assume that the volatility is a function of both spot price and time;

$$\sigma = \sigma(S,t).$$

This is not the only possible explanation.

Volatility Surfaces

One means of implementing a no-arbitrage model is to assume that $\sigma = \sigma(S_t, t)$. The stock price process dynamics follow

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma(S_t, t)\, dW_t$$

The Black-Scholes equation then becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(S, t)\, S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

and its solution can be written in the form

$$V(S, t) = e^{-r(T-t)} \int_0^\infty p\left(S, t; S', T\right) V\left(S', T\right) dS'.$$

$V(S', T)$ is the payoff and $p(S, t; S', T)$ is the risk-neutral probability density associated with the Kolmogorov equations

$$\begin{aligned} \frac{\partial p}{\partial T} &= \frac{1}{2}\frac{\partial^2}{\partial S'^2}\left(\sigma^2\left(S', t\right) S'^2 p\right) - \frac{\partial}{\partial S'}\left(rS'p\right), \\ -\frac{\partial p}{\partial t} &= \frac{1}{2}\sigma^2\left(S, t\right) S^2 \frac{\partial^2 p}{\partial S^2} + rS\frac{\partial p}{\partial S}. \end{aligned}$$

That is $p(S, t; S', T)$ can be viewed in two ways (given that the above analysis only makes sense if $t < T$):

If S and t are fixed (today's spot price and date) then we can regard $p(S, t; S', T)$ as the probability density that at time $T > t$ the spot price will be S'. This is a conditional probability density for future values, S' and T, given that the present values are S and $t < T$.

If S' and T are fixed (some given value of the spot price and date, say) then $p(S, t; S', T)$ is the probability density that at time $t < T$ the spot price was S; again this is a conditional probability function; the probability that the spot price was S at time t given that the spot price is S' at time T.

Dupire's method

Suppose now that we want to find $\sigma\left(S,t\right)$ from market data. In fact we shall find $\sigma\left(E,T\right)$. More correctly, we find $\sigma\left(S,t;E,T\right)$ with the usual notation of the function arguments.

Using

$$\frac{\partial p}{\partial T}=\frac{1}{2}\frac{\partial^2}{\partial S'^2}\left(\sigma^2\left(S',t\right)S'^2p\right)-\frac{\partial}{\partial S'}\left(rS'p\right)$$

and that

$$C\left(S,t\right)=e^{-r(T-t)}\int_E^\infty p\left(S,t;S',T\right)\left(S'-E\right)dS'.$$

Then

$$\frac{\partial C}{\partial T}=-rC+e^{-r(T-t)}\int_E^\infty\frac{\partial p}{\partial T}\left(S'-E\right)dS'.$$

Also (from Leibniz)

$$\begin{aligned}\frac{\partial C}{\partial E} &= -e^{-r(T-t)}\int_E^\infty p\left(S,t;S',T\right)dS' \\ \frac{\partial^2 C}{\partial E^2} &= e^{-r(T-t)}p\left(S,t;E,T\right)\end{aligned}$$

Now use the fact that

$$\frac{\partial p}{\partial T} = \frac{1}{2}\frac{\partial^2}{\partial S'^2}\left(\sigma^2\left(S',t\right)S'^2 p\right) - \frac{\partial}{\partial S'}\left(rS'p\right)$$

where as earlier,

$$p\left(S,t;S',T\right)dS'$$

denotes the risk-neutral probability of a spot price in $(S', S' + dS')$ at time $T > t$, contingent on the spot price being S at t.

Recall that p is contingent on today's information, (S,t) ; in general

$$p\left(S_1,t_1;S',T\right) = p\left(S_2,t_2;S',T\right)$$

for say today, (S_1,t_1) , and tomorrow (S_2,t_2) .

Substituting for $\frac{\partial p}{\partial T}$ and using integration by parts we arrive, after lengthy calculation, at

$$\frac{\partial C}{\partial T} = -rC + e^{-r(T-t)}\left(\sigma\left(E,T\right)^2 E^2 p\left(S,t;E,T\right) + r\int_E^\infty S' p dS'\right).$$

Later we will use

$$\int_E^\infty S' p dS' = \int_E^\infty \left(S' - E\right) p dS' + E\int_E^\infty p dS'$$

and from earlier we note that

$$\frac{\partial C}{\partial E} = e^{-r(T-t)}\int_E^\infty p dS', \quad \frac{\partial^2 C}{\partial E^2} = e^{-r(T-t)} p\left(S,t;E,T\right),$$

where the second expression gives

$$p\left(S,t;E,T\right) = e^{r(T-t)}\frac{\partial^2 C}{\partial E^2}.$$

Now use the Kolmogorov equation to express $\frac{\partial p}{\partial T}$ in $\frac{\partial C}{\partial T}$

$$\begin{aligned}
\frac{\partial C}{\partial T} &= -rC + e^{-r(T-t)} \int_E^\infty \frac{\partial p}{\partial T} \left(S' - E \right) dS' \\
&= -rC + \\
& \quad e^{-r(T-t)} \int_E^\infty \left(\frac{1}{2} \frac{\partial^2}{\partial S'^2} \left(\sigma^2 \left(S', t \right) {S'}^2 p \right) - \frac{\partial}{\partial S'} \left(r S' p \right) \right) \left(S' - E \right) dS'
\end{aligned}$$

The integral

$$\int_E^\infty \left(\frac{1}{2} \frac{\partial^2}{\partial S'^2} \left(\sigma^2 \left(S', t \right) {S'}^2 p \right) - \frac{\partial}{\partial S'} \left(r S' p \right) \right) \left(S' - E \right) dS' =$$

$$\int_E^\infty \frac{1}{2} \frac{\partial^2}{\partial S'^2} \left(\sigma^2 \left(S', t \right) {S'}^2 p \right) \left(S' - E \right) dS'$$

$$- \int_E^\infty \frac{\partial}{\partial S'} \left(r S' p \right) \left(S' - E \right) dS'$$

In what follows, we assume p decays sufficiently fast.

$$\frac{1}{2}\int_E^\infty \frac{\partial^2}{\partial S'^2}\left(\sigma^2\left(S',T\right)S'^2p\right)\left(S'-E\right)dS' :$$

$$\begin{array}{ll} v = S' - E & u' = \frac{\partial^2}{\partial S'^2}\left(\sigma^2\left(S',T\right)S'^2p\right) \\ v' = 1 & u = \frac{\partial}{\partial S'}\left(\sigma^2\left(S',T\right)S'^2p\right) \end{array}$$

$$\begin{aligned} & \frac{1}{2}\int_E^\infty \frac{\partial^2}{\partial S'^2}\left(\sigma^2\left(S',T\right)S'^2p\right)\left(S'-E\right)dS' \\ = \; & \underbrace{\frac{1}{2}\frac{\partial^2}{\partial S'^2}\left(\sigma^2\left(S',T\right)S'^2p\right)\left(S'-E\right)\Big|_E^\infty}_{=0} - \frac{1}{2}\int_E^\infty \frac{\partial}{\partial S'}\left(\sigma^2\left(S',T\right)S'^2p\right)dS' \end{aligned}$$

$$= \frac{1}{2}\sigma^2\left(E,T\right)E^2p\left(S,t;E,T\right) = \frac{1}{2}\sigma^2\left(E,T\right)E^2e^{r(T-t)}\frac{\partial^2 C}{\partial E^2}$$

Similarly $\int_E^\infty \frac{\partial}{\partial S'}(rS'p)(S'-E)\,dS'$:

$$\begin{array}{ll} v = S' - E & u' = \frac{\partial}{\partial S'}(rS'p) \\ v' = 1 & u = rS'p \end{array}$$

$$\begin{aligned} & \int_E^\infty \frac{\partial}{\partial S'}\left(rS'p\right)\left(S'-E\right)dS' \\ = \; & \underbrace{rS'p\left(S'-E\right)\Big|_E^\infty}_{=0} - r\int_E^\infty S'pdS' \end{aligned}$$

Now using

$$\int_E^\infty S'pdS' = \int_E^\infty \left(S'-E\right)pdS' + E\int_E^\infty pdS'$$

- For the first integral term it is the expected payoff (i.e. option price without discount factor), i.e. $e^{r(T-t)}C$.

- The second integral term from earlier $\frac{\partial C}{\partial E} = -e^{-r(T-t)} \int_E^\infty p dS'$ gives $-e^{r(T-t)} \frac{\partial C}{\partial E}$

Hence

$$\begin{aligned}
\int_E^\infty \frac{\partial}{\partial S'} \left(r S' p \right) \left(S' - E \right) dS' &= r e^{r(T-t)} C - r E e^{r(T-t)} \frac{\partial C}{\partial E} \\
&= -r e^{r(T-t)} \left(C - E \frac{\partial C}{\partial E} \right)
\end{aligned}$$

Putting everything together

$$\begin{aligned}\frac{\partial C}{\partial T} &= -rC + e^{-r(T-t)}\left(\frac{1}{2}\sigma^2(E,T)\,E^2 e^{r(T-t)}\frac{\partial^2 C}{\partial E^2} + re^{r(T-t)}\left(C - E\frac{\partial C}{\partial E}\right)\right)\\ &= -rC + \frac{1}{2}\sigma^2(E,T)\,E^2\frac{\partial^2 C}{\partial E^2} + rC - rE\frac{\partial C}{\partial E}\end{aligned}$$

Hence

$$\frac{\partial C}{\partial T} = \frac{1}{2}\sigma^2(E,T)\,E^2\frac{\partial^2 C}{\partial E^2} - rE\frac{\partial C}{\partial E}.$$

We can now, in principle solve for

$$\sigma^2(E,T) = \frac{\frac{\partial C}{\partial T} + rE\frac{\partial C}{\partial E}}{\frac{1}{2}E^2\frac{\partial^2 C}{\partial E^2}}.$$

If we now retrace our calculations we find that, because the call value C is a function of today's spot, S, today's date, t, the call's strike E and the call's maturity, T, $C = C\left(S, t; E, T\right)$, what we have called $\sigma\left(E, T\right)$ is actually

$$\sigma^2\left(S, t; E, T\right) = \frac{\frac{\partial C(S,t;E,T)}{\partial T} + rE\frac{\partial C(S,t;E,T)}{\partial E}}{\frac{1}{2}E^2\frac{\partial^2 C(S,t;E,T)}{\partial E^2}}.$$

Recall that, in practice, when we compute $\sigma\left(S, t; E, T\right)$, today's spot price S and date t are fixed. We can vary only the strike E and the maturity T. That is, we have found a *local volatility surface* $\sigma\left(E, T\right)$, or more correctly $\sigma\left(S, t; E, T\right)$, as it is conditional on today's spot price S and date t.

Practical problems with this approach

- requires continuum of strikes and maturities (interpolation, extrapolation)

- numerical differentiation is ill conditioned

- the denominator $\frac{\partial^2 C}{\partial E^2}$ tends to zero for $E \to \infty$

The last problem can be circumvented to some extent by switching from quoted prices to implied volatilities.

Implied and local volatility

If we use implied volatilities σ_i, repeated application of the implicit function theorem gives

$$\sigma^2(E,T) = \frac{\sigma_\mathrm{i}^2 + 2\sigma_\mathrm{i}(T-t)\frac{\partial\sigma_\mathrm{i}}{\partial T} + 2r\sigma_\mathrm{i}E(T-t)\frac{\partial\sigma_\mathrm{i}}{\partial E}}{\left(1+Ed_1\sqrt{T-t}\frac{\partial\sigma_\mathrm{i}}{\partial E}\right)^2 + \sigma_\mathrm{i}(T-t)E^2\left(\frac{\partial^2\sigma_\mathrm{i}}{\partial E^2} - d_1\left(\frac{\partial\sigma_\mathrm{i}}{\partial E}\right)^2\sqrt{T-t}\right)}$$

where, as usual,

$$d_1 = \frac{\log(S/E) + (r+\frac{1}{2}\sigma_\mathrm{i}^2)(T-t)}{\sigma_\mathrm{i}\sqrt{T-t}}$$

Finding Roots

A fundamental problem in numerical analysis consists of obtaining the zero of a function. Given a function $y = f(x)$ obtain the root of $f(x) = 0$, i.e. find the value of $x = c$ which satisfies $f(c) = 0$. e.g.

$$f(x) = x - \sin x.$$

Four broad categories of root finding:

(i) Methods which do not use derivatives of the function

(ii) Methods which do use $f'(x)$

(iii) Methods for polynomials

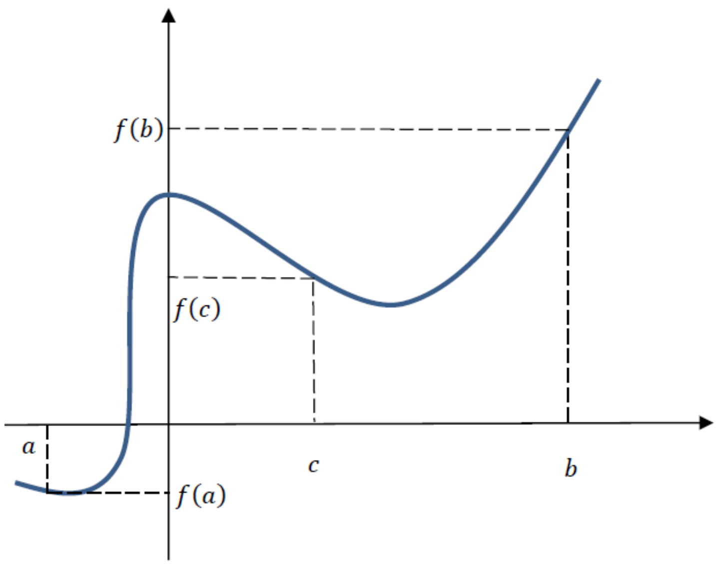
(iv) Methods which deal with complex roots

Bisection

The simplest method is that of bisection. The following theorem, from calculus class, insures the success of the method.

Intermediate Value Theorem

Suppose $f(x)$ is continuous on $[a,\ b]$ then for any y s.t y is between $f(a)$ and $f(b)$ there $\exists\ c \in [a,\ b]$ s.t $f(c) = y$.



Example 1

The function $f(x) = \frac{1}{x}$ is not continuous at 0. Thus if $0 \in [a,\ b]$, we *cannot* apply the IVT. In particular, if $0 \in [a,\ b]$ it happens to be the case that for *every* y between $f(a),\ f(b)$ there is no $c \in [a,\ b]$ such that $f(c) = y$.

In particular, the IVT tells us that if $f(x)$ is continuous and we know $a,\ b$ such that $f(a),\ f(b)$ have different sign, then there is some root in $[a,\ b]$. This is a fundamental test we can apply.

Example 2 Show that the function $g(x) = x^3 + 2x^2 + 5x - 1$ has a root lying between 0 and 1.

We note $f(0) = -1;\ f(1) = 7$. The sign change confirms that $\exists$ a root α s.t. $\alpha \in (0, 1)$.

Once location of a root α is established then a reasonable estimate of α is $c = \dfrac{a+b}{2}$. We can check whether $f(c) = 0$. If this does not hold then one and only one of the two following options holds:

1. $f(a)$, $f(c)$ have different signs.

2. $f(c)$, $f(b)$ have different signs.

We now choose to recursively apply bisection to either $[a, c]$ or $[c, b]$, respectively, depending on which of these two options hold.

Whichever an interval is chosen,the new interval containing the root can be further subdivided. If the first interval is $|b-a|$, then the second is half the length and so on.

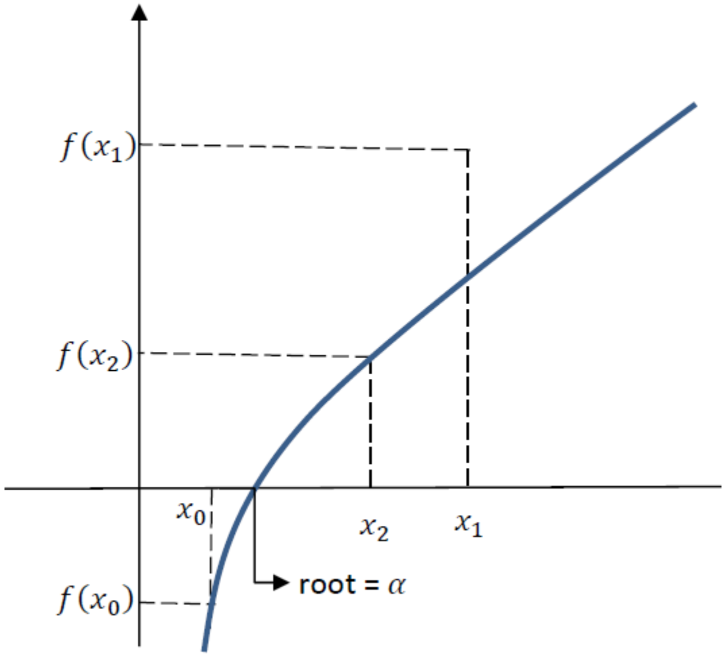
After n steps of bisection the interval containing the root will be reduced in size to

$$\frac{|b-a|}{2^n}$$

where in the earlier example the value $b = 1$ and $a = 0$.

If the size of the interval becomes smaller than some specified tolerance, t, then the calculation stops and convergence has been attained.

Theorem 2 (Bisection Method Theorem)

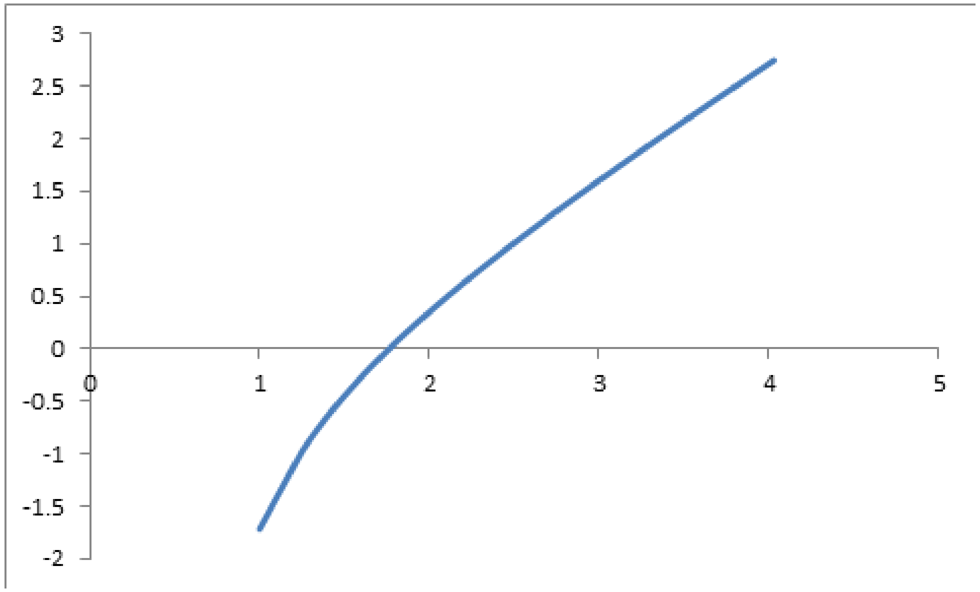


If $f(x)$ is a continuous function on $[a,\ b]$ such that $f(a)f(b) < 0$, then after n steps, the method will return c such that

$$|c - \alpha| \leq \frac{|b-a|}{2^n}$$

where α is some approximate root of f.

Example Consider $f(x) = x - e^{1/x}$. There is a root in $[1,\ 2]$.



Use the bisection method to show that the root of $f(x) = x - e^{1/x}$ in the interval $[1,\ 2]$ is 1.763 (correct to 3 decimal places).

$$\left.\begin{array}{c} f(x_0) = f(1) = 1 - e^1 < 0 \\ f(x_1) = f(2) > 0 \end{array}\right| \quad \rightarrow \quad x_2 = \frac{x_0 + x_1}{2} = 1.5$$

$$\left.\begin{array}{c} f(1.5) = 1.5 - e^{2/3} = f(x_2) = -0.4477 \\ f(x_0)\, f(x_2) > 0 \end{array}\right| \quad \therefore \quad \text{root in } [x_2,\ x_1] \text{ i.e. in } [1.5,\ 2]$$

$$
\begin{aligned}
x_3 &= \frac{x_2 + x_1}{2} = \frac{1.5 + 2}{2} = 1.75 \\
f(x_3) &= 1.75 - e^{1/1.75} = -0.0208 \\
f(x_3)\, f(x_2) &> 0 \therefore \text{root in } [x_3,\ x_1] \text{ i.e. in } [1.75,\ 2]
\end{aligned}
$$

$$
\begin{aligned}
x_4 &= \frac{1.75 + 2}{2} = \frac{3.75}{2} = 1.875 \\
f(x_4) &= f(1.875) = 0.1704 \\
f(x_3)\, f(x_4) &< 0 \therefore \text{root in } [x_3,\ x_4] \text{ i.e. in } [1.75,\ 1.875]
\end{aligned}
$$

$$
\begin{aligned}
x_5 &= \frac{x_3 + x_4}{2} = \frac{1.75 + 1.875}{2} = 1.8125 \\
f(x_5) &= f(1.8125) = 0.0763 \\
f(x_3)\, f(x_5) &< 0
\end{aligned}
$$

Continuing in this way we have we find the root in $[x_9,\ x_{10}]$ i.e. in $[1.761718,\ 1.7636715]$

$$
\begin{aligned}
x_{11} &= \frac{x_9 + x_{10}}{2} = 1.76269 \\
&= 1.763 \text{ to 3 decimal places}
\end{aligned}
$$

Newton's Method

Newton's method is an *iterative* method for root finding. That is, starting from some guess at the root, x_0, one iteration of the algorithm produces a number x_1, which is supposed to be closer to a root; guesses x_2, x_3, ..., x_n follow identically.
We know from Taylor that

$$f(x+h) = f(x) + f'(x)h + O\left(h^2\right).$$

This approximation is better when $f''(.)$ is "well-behaved" between x and $x+h$. Newton's method attempts to find some h such that

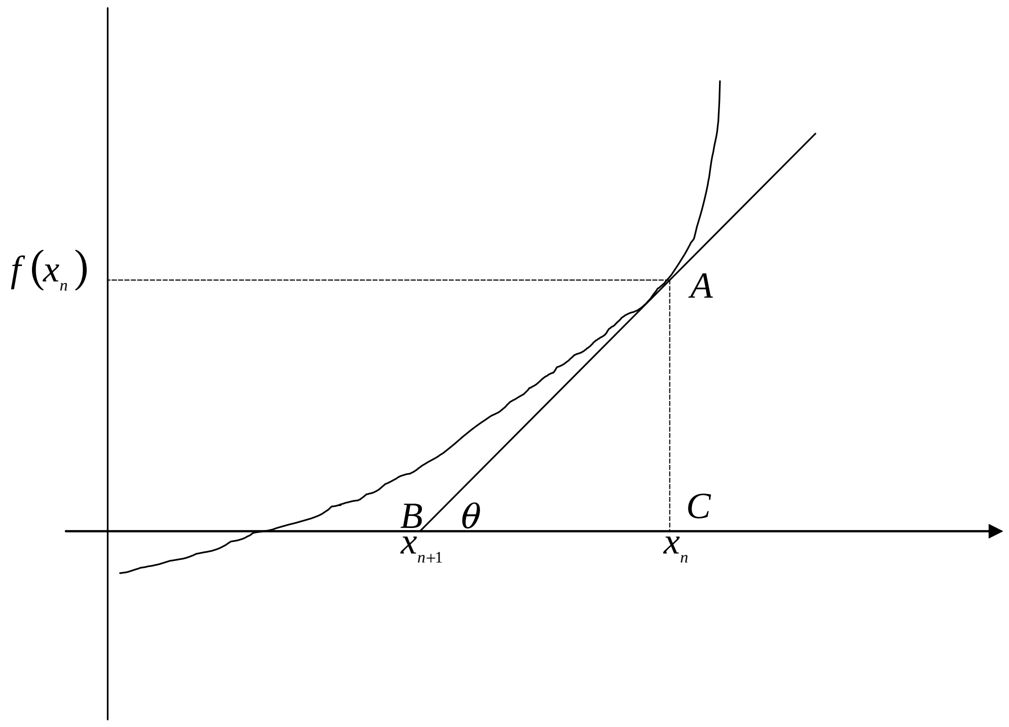
$$0 = f(x+h) = f(x) + f'(x)h.$$

This is easily solved as

$$h = \frac{-f(x)}{f'(x)}$$

An iteration of Newton's method, then, takes some guess x_n and returns x_{n+1} defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$



From above we see that $\tan\theta = \frac{AC}{BC} = \frac{f(x_n)}{(x_n - x_{n+1})}$

But

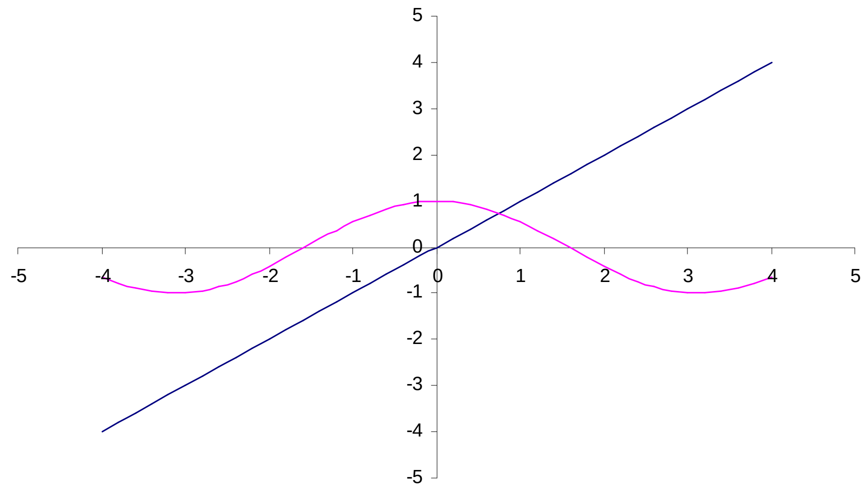
$$\begin{aligned} \tan\theta &= f'(x_n) \\ f'(x_n) &= \frac{f(x_n)}{(x_n - x_{n+1})} \end{aligned}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This is the *Newton-Raphson Technique.*

Example: Solve for roots, the function $f(x) = x - \cos x$.

Start by considering $x = \cos x$. That is draw $y = x$ and $y = \cos x$ to obtain an initial guess for the root(s).



Clearly the diagram above shows that there is only one root $\alpha \in (0, 1)$.

We use the Newton formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots.$$

where $n = 0$ is the initial guess. $f(x_n) = x_n - \cos x_n \longrightarrow f'(x_n) = 1 + \sin x_n$.

x	0	1
$f(x)$	-1	0.75

so numerically we also see that $f(0)\, f(1) < 0 \implies \alpha \in (0,1)$. NR formula for this function becomes

$$x_{n+1} = x_n - \frac{x_n - \cos x_n}{1 + \sin x_n}, \quad x_0 = 1$$

$$\begin{aligned}
x_1 &= x_0 - \frac{x_0 - \cos x_0}{1 + \sin x_0} = 0.75036 \\
x_2 &= 0.75036 - \frac{0.75036 - \cos 0.75036}{1 + \sin 0.75036} = 0.73911 \\
x_3 &= 0.73911 - \frac{0.73911 - \cos 0.73911}{1 + \sin 0.73911} = 0.73909 \\
x_4 &= 0.73909 - \frac{00.73909 - \cos 0.73909}{1 + \sin 0.73909} = 0.73909
\end{aligned}$$

which gives the root $\alpha \approx 0.73909$.

Problems

As mentioned above, convergence is dependent on $f(x)$, and the initial estimate x_0. A number of conceivable problems might come up. We illustrate them here.

Example Consider Newton's method applied to the function $f(x) = \frac{\ln x}{x}$, with initial estimate $x_0 = 3$. Note that $f(x)$ is continuous on $\mathbb{R}^+$. It has a single root at $x = 1$. Our initial guess is not too far from this root. However, consider the derivative:

$$f'(x) = \frac{1 - \ln x}{x^2}$$

If $x > e^1$, then $1 - \ln x < 0$, and so $f'(x) < 0$. However, for $x > 1$, we know $f(x) > 0$. Thus taking

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} > x_n$$

The estimates will diverge from the root $x = 1$.

Fourier Transforms

If $f = f(x)$ then consider

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{ix\xi} dx.$$

If this integral converges, it is called the *Fourier Transform* of $f(x)$. Similar to the case of Laplace Transforms, it is denoted as $\mathcal{F}(f)$, i.e.

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} f(x)\, e^{ix\xi} dx = \widehat{f}(\xi).$$

The *Inverse Fourier Transform* is then

$$\mathcal{F}^{-1}\left(\widehat{f}(\xi)\right) = \int_{-\infty}^{\infty} \widehat{f}(\xi)\, e^{-ix\xi} d\xi = f(x).$$

The convergent property means that $\widehat{f}(\xi)$ is bounded and we have

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

Functions of this type $f(x) \in L_1(-\infty, \infty)$ and are called *square integrable.*

We know from integration (basic property of Riemann integral) that

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

Hence

$$\begin{aligned} \left|\widehat{f}(\xi)\right| &= \left|\int_{\mathbb{R}} f(x)\,e^{ix\xi}dx\right| \\ &\leq \int_{\mathbb{R}} \left|f(x)\,e^{ix\xi}\right| dx \end{aligned}$$

and Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$ implies that $\left|e^{i\theta}\right| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$, therefore

$$\left|\widehat{f}(\xi)\right| \leq \int_{\mathbb{R}} |f(x)|\,dx < \infty.$$

In addition to the boundedness of $\widehat{f}(\xi)$, it is also continuous (requires a $\delta - \epsilon$ proof).

Note: If $f(x)$ represents the probability density of some random variable X then the Fourier transform is the characteristic function of $f(x)$, i.e.

$$\widehat{f}(\xi) = \mathbb{E}\left[e^{i\xi x}\right].$$

Example: Obtain the Fourier transform of $f\ (x) = e^{-|x|}$

$$\begin{aligned}
\widehat{f}(\xi) &= \mathcal{F}(f) = \int_{-\infty}^{\infty} f\ (x)\, e^{ix\xi} dx \\
&= \int_{-\infty}^{\infty} e^{-|x|} e^{ix\xi} dx \\
&= \int_{-\infty}^{0} e^{-|x|} e^{ix\xi} dx + \int_{0}^{\infty} e^{-|x|} e^{ix\xi} dx \\
&= \int_{-\infty}^{0} e^{x} e^{ix\xi} dx + \int_{0}^{\infty} e^{-x} e^{ix\xi} dx = \\
& \int_{-\infty}^{0} \exp\left[(1+i\xi)\, x\right] dx + \int_{0}^{\infty} \exp\left[-(1-i\xi)\, x\right] dx \\
&= \frac{1}{(1+i\xi)} \exp\left[(1+i\xi)\, x\right]\Big|_{-\infty}^{0} - \frac{1}{(1-i\xi)} \exp\left[-(1-i\xi)\, x\right]\Big|_{0}^{\infty} \\
&= \frac{1}{(1+i\xi)} + \frac{1}{(1-i\xi)} = \frac{2}{\left(1+\xi^{2}\right)}
\end{aligned}$$

Our interest in differential equations continues, hence the reason for introducing this transform. We now look at obtaining Fourier transforms of derivative terms. We assume that $f \ (x)$ is continuous and $f(x) \to 0$ as $x \to \pm\infty$. Consider

$$\mathcal{F}\left\{f'(x)\right\} = \int_{\mathbb{R}} f'(x)\, e^{ix\xi} dx$$

which is simplified using integration by parts

$$f(x)\, e^{ix\xi}\Big|_{-\infty}^{\infty} - i\xi \int_{\mathbb{R}} f \ (x)\, e^{ix\xi} dx$$

so

$$\mathcal{F}\left\{f'(x)\right\} = -i\xi \int_{\mathbb{R}} f(x)\, e^{ix\xi} dx = -i\xi \widehat{f}(\xi).$$

We can obtain the Fourier transform for the second derivative by performing integration by parts (twice) to give

$$\mathcal{F}\left\{f''(x)\right\} = (-i\xi)^2 \mathcal{F}\{f(x)\} = -\xi^2 \widehat{f}(\xi).$$

$$\boxed{\mathcal{F}\{f'(x)\} = -i\xi \widehat{f}(\xi)}$$

$$\boxed{\mathcal{F}\{f''(x)\} = -\xi^2 \widehat{f}(\xi)}$$

Example: Solve the diffusion equation problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0 \\ u(x, 0) &= e^{-|x|}, \quad -\infty < x < \infty \end{aligned}$$

Here $u = u(x, t)$, so we begin by defining

$$\mathcal{F}\{u(x, t)\} = \int_{-\infty}^{\infty} u(x, t)\, e^{ix\xi} dx = \widehat{u}(\xi, t).$$

Now take Fourier transforms of our PDE, i.e.

$$\mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} = \mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

to obtain

$$\frac{d\widehat{u}}{dt} = -\xi^2 \widehat{u}(\xi, t).$$

We note that the second order PDE has been reduced to a first order equation of type variable separable. This has general solution

$$\widehat{u}(\xi, t) = Ce^{-\xi^2 t}.$$

We can find the constant C by transforming the initial condition

$$\begin{aligned}\mathcal{F}\left\{u\left(x,0\right)\right\} &= \mathcal{F}\left\{e^{-|x|}\right\} \\ \widehat{u}\left(\xi,0\right) &= \int_{-\infty}^{\infty} e^{-|x|}e^{ix\xi}dx = \frac{2}{\left(1+\xi^2\right)}.\end{aligned}$$

Applying this to the solution $\widehat{u}\left(\xi,t\right)$ gives

$$\widehat{u}\left(\xi,0\right) = C = \frac{2}{\left(1+\xi^2\right)},$$

hence

$$\widehat{u}\left(\xi,t\right) = \frac{2}{\left(1+\xi^2\right)}e^{-\xi^2 t}.$$

We now use the inverse transform to get $u(x,t) = \mathcal{F}^{-1}(\widehat{u}(\xi,t))$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \widehat{u}(\xi,t)\, e^{-ix\xi} d\xi \\
&= 2\int_{-\infty}^{\infty} \tfrac{1}{(1+\xi^2)} e^{-\xi^2 t} e^{-ix\xi} d\xi \\
&= 2\int_{-\infty}^{\infty} \frac{1}{\left(1+\xi^2\right)} e^{-\xi^2 t} \left(\cos x\xi - i \sin x\right) d\xi \\
&= 2\int_{-\infty}^{\infty} \tfrac{1}{(1+\xi^2)} e^{-\xi^2 t} \cos x\xi \ d\xi - 2i\int_{-\infty}^{\infty} \tfrac{1}{(1+\xi^2)} e^{-\xi^2 t} \sin x\xi \ d\xi.
\end{aligned}$$

This now simplifies nicely because $\dfrac{1}{\left(1+\xi^2\right)} e^{-\xi^2 t} \sin x\xi$

is an odd function, hence

$$\int_{-\infty} \frac{1}{\left(1+\xi^2\right)} e^{-\xi^2 t} \sin x\xi \ d\xi = 0.$$

Therefore

$$u\left(x,t\right)=2\int_{-\infty}^{\infty}\frac{1}{\left(1+\xi^{2}\right)}e^{-\xi^{2}t}\cos x\xi\ d\xi.$$

In order to solve this we now need to use *Residues* (Complex Analysis).

Complex Variables

In the following sections we shall begin our study of analytic functions of a complex variable. Complex variable theory is one of the most beautiful branches of pure mathematics but it also has important applications in applied mathematics. More excitingly, complex variables are now used in derivative pricing, when solving the pricing equations via the Fourier Transform approach.

In what follows we shall convey some of the basic ideas of complex analysis without emphasis on any rigor.

Review of Complex Numbers

A complex number $z = x + iy$, is a pair (x, y) of real numbers.

$x =$ real part $= \mathrm{Re}\, z$; $y =$ imaginary part $= \mathrm{Im}\, z$

Operations on complex numbers:

1. Addition: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
2. Multiplication: $(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$

The set of all complex numbers defined by $\mathbb{C}$ is called a *field*, i.e. addition and multiplication are associative and commutative

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

$$z_1(z_2z_3) = (z_1z_2)z_3$$

distributive

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3$$

zero:

$$(0,0) + (x,y) = (x,y)$$

identity:

$$(1,0) \text{ s.t. } (1,0)\cdot(x,y) = (x,y)$$

Non-zero complex numbers have inverses, i.e. given $(x,y) \neq (0,0)$ $\exists$ (x',y') s.t.

$$(x,y)\cdot\left(x',y'\right) = (1,0)$$

In fact

$$\left(x',y'\right) = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$$

Look at the complex numbers $(x, 0)$,

$$\begin{aligned} (x_1, 0) + (x_2, 0) &= (x_1 + x_2, 0) \\ (x_1, 0) \cdot (x_2, 0) &= (x_1 x_2, 0) \end{aligned}$$

So $\{(x, 0) \in \mathbb{C}\}$ is a subfield of $\mathbb{C}$.

In fact it is the same as $\mathbb{R}$

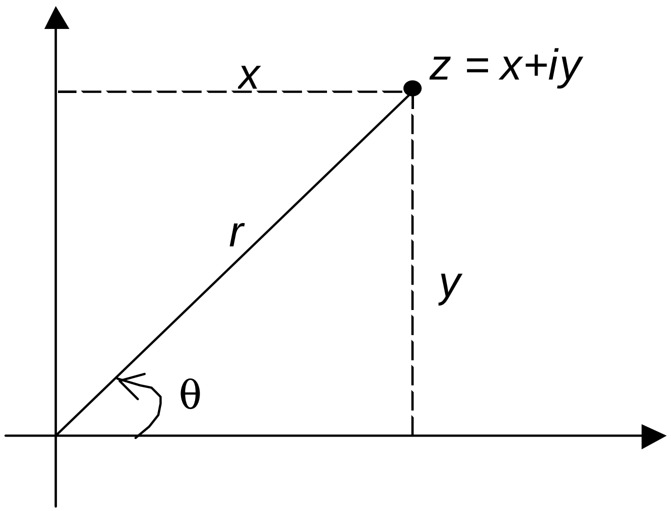
$$x \in \mathbb{R} \longmapsto (x, 0) \in \mathbb{C}.$$

Geometrical Representation

There is a 1-1 correspondence between $\mathbb{C}$ and $\mathbb{R}^2$

$z = x + iy = (x, y) \longleftrightarrow$ the point with coordinates (x, y) .

$\mathbb{R}^2$ is called an *Argand diagram* or the *Complex Plane.*



This is polar coordinate form (r, θ)

$$
\begin{aligned}
x &= r\cos\theta;\ y = r\sin\theta \\
r &= \sqrt{x^2 + y^2}
\end{aligned}
$$

so

$$
\begin{aligned}
\sin\theta &= \frac{y}{\sqrt{x^2 + y^2}} = y/r \\
\cos\theta &= \frac{x}{\sqrt{x^2 + y^2}} = x/r
\end{aligned}
$$

giving us an alternative representation of complex numbers, i.e.

$$
\begin{aligned}
z &= x + iy = z = r(\cos\theta + i\sin\theta) \\
&= re^{i\theta}
\end{aligned}
$$

The final form is Euler's identity/formula and called the mod-arg form of z.

$r =$ modulus of $z = \text{mod}\, z = |z|$

$\theta = \arg z =$ argument of z.

θ is only determined by x and y up to the addition of an integer multiple of 2π.

e.g. $z = -1 - i$; $x = -1 = y \longrightarrow r = \sqrt{2}$

To find θ solve

$$-1 = \sqrt{2}\cos\theta \text{ or } -1 = \sqrt{2}\sin\theta$$

to get $\theta = -\frac{3\pi}{4}$ or $\frac{5\pi}{4}$.

The values of $\arg z$ are $-\frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{13\pi}{4},\ \frac{21\pi}{4}, \ldots., -\frac{11\pi}{4},\ -\frac{19\pi}{4}$, i.e. $-\frac{3\pi}{4} + 2n\pi;\ n \in \mathbb{Z}$.

The *Principal Value* of $\arg z$ is the argument θ which satisfies $-\pi < \theta \leq \pi$.

So we see that the angle is not unique, there are many values for the argument. The set

$$\{\theta + 2n\pi : n \in \mathbb{Z}\}$$

is written Arg$z$.

Examples: $z = 1 - i$

$$\begin{aligned} |z| &= \sqrt{2} \\ \arg z &= \arctan \frac{-1}{1} = -\frac{\pi}{4} \in (-\pi, \pi] \end{aligned}$$

$\mathrm{Arg} z = \left\{ ..., -\frac{9\pi}{4}, -\frac{\pi}{4}, \frac{7\pi}{4}, \frac{15\pi}{4}, ... \right\}$

$$z = -\sqrt{3} + i$$

$$\begin{aligned} |z| &= 2 \\ \mathrm{Arg} z &= \left\{ \ldots, -\frac{7\pi}{6}, \frac{5\pi}{6}, \frac{17\pi}{6}, \ldots\ldots \right\} \\ \arg z &= \frac{5\pi}{6} \end{aligned}$$

$$\mathrm{Arg} z = \left\{ \ldots, -\frac{9\pi}{4}, -\frac{\pi}{4}, \frac{7\pi}{4}, \frac{15\pi}{4}, \ldots\ldots \right\}$$

For any $z \in \mathbb{C}$, the expression

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \tan z = \frac{\sin z}{\cos z}$$

defines the generalized circular functions, and

$$\sinh z = \frac{e^{z} - e^{-z}}{2}, \quad \cosh z = \frac{e^{z} + e^{-z}}{2} \quad \text{and} \quad \tanh z = \frac{\sinh z}{\cosh z}$$

the generalized hyperbolic function.

Using Euler's formula with positive and negative components we have

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}$$

Adding gives

$$2\cos\theta = e^{i\theta} + e^{-i\theta} \Rightarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and subtracting gives

$$2i \sin\theta = e^{i\theta} - e^{-i\theta} \Rightarrow \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

We can extend these results to consider other functions:

$$\begin{aligned} \cos\mathrm{ec}\ z &= \frac{1}{\sin z}, \quad \sec = \frac{1}{\cos z}, \quad \cot z = \frac{1}{\tan z} \\ \cosh\mathrm{ec}\ z &= \frac{1}{\sinh z}, \quad \sec = \frac{1}{\cosh z}, \quad \cot z = \frac{1}{\tanh z} \end{aligned}$$

We can also obtain a relationship between circular and hyperbolic functions:

$$\sin(iz) = \frac{1}{2i}\left(e^{-z} - e^{z}\right)$$

we know $1/i = -i$ hence

$$\sin(iz) = -i.\frac{1}{2}\left(e^{-z} - e^{z}\right) = i.\frac{1}{2}\left(e^{z} - e^{-z}\right)$$

so

$$\sin(iz) = i \sinh z.$$

Similarly it can be shown that

$$\begin{aligned} \sinh(iz) &= i \sin z \\ \cos(iz) &= \cosh z \\ \cosh(iz) &= \cos z \end{aligned}$$

Example:

Let $z = x+iy$ be any complex number, find all the values for which $\cosh z = 0$.

We use the hyperbolic identity

$$\cosh(a+b) = \cosh a \cosh b + \sinh a \sinh b$$

to give

$$\begin{aligned} \cosh z &= \cosh\,(x+iy) = \cosh x \cosh iy + \sinh x \sinh iy \\ &= \cosh x \cos y + i \sinh x \sin y \end{aligned}$$

i.e.

$$\cosh x \cos y + i \sinh x \sin y = 0$$

so equating real and imaginary parts we have two equations

$$\cosh x \cos y = 0$$

$$\sinh x \sin y = 0$$

From the first we know that $\cosh x \neq 0$ so we require $\cos y = 0 \Rightarrow y = \frac{\pi}{2} + n\pi \quad \forall n \in \mathbb{Z}$.

Putting this in the second equation gives

$$\sinh x \sin (2n+1)\frac{\pi}{2} = 0$$

where

$$\sin (2n+1)\frac{\pi}{2} = \cos n\pi = (-1)^n$$

so

$$\sinh x = 0$$

which has the solution $x = 0$. Therefore the solution to our equation $\cosh z = 0$ is

$$z_n = i\,(2n+1)\,\frac{\pi}{2}, \quad n \in \mathbb{Z}$$

De Moivres Theorem

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= \left(e^{i\theta}\right)^n \\ &= e^{in\theta} \\ &= \cos n\theta + i\sin n\theta\end{aligned}$$

Similarly

$$(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta.$$

It is quite common to write $\cos\theta + i\sin\theta$ as cis.

If

$$z = e^{i\theta} = \cos\theta + i\sin\theta \quad \text{then} \quad \frac{1}{z} = e^{-i\theta} = \bar{z} = \cos\theta - i\sin\theta.$$

So

$$\begin{aligned}\cos\theta &= \operatorname{Re} z = \frac{1}{2}(z+\overline{z}) = \frac{1}{2}\left(z+\frac{1}{z}\right)\\ \sin\theta &= \operatorname{Im} z = \frac{1}{2i}(z-\overline{z}) = \frac{1}{2i}\left(z-\frac{1}{z}\right).\end{aligned}$$

Also $z^n = e^{in\theta} \longrightarrow$

$$\begin{aligned}z^n + z^{-n} &= (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta)\\ &= 2\cos n\theta\end{aligned}$$

$\therefore$rearranging gives

$$\cos n\theta = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right).$$

Similarly

$$\sin n\theta = \frac{1}{2}\left(z^n - \frac{1}{z^n}\right)$$

Finding Roots of Complex Numbers

Consider a number w, which is an n^{th} root of the complex number z. That is, if $w^n = z$, and hence we can write

$$w = z^{1/n}.$$

We begin by writing in polar/mod-arg form

$$z = r\left(\cos\theta + i\sin\theta\right).$$

hence

$$z^{1/n} = r^{1/n}\left(\cos\theta + i\sin\theta\right)^{1/n}$$

and then by DMT we have

$$z^{1/n} = r^{1/n}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right) \quad k = 0, 1, \ldots.., n-1.$$

Any other values of k would lead to repetition.

This method is particularly useful for obtaining the $n-$ roots of unity. This requires solving the equation

$$z^n = 1.$$

There are only two real solutions here, $z = \pm 1$, which corresponds to the case of even values of n. If n is odd, then there exists one real solution, $z = 1$. Any other solutions will be complex. Unity can be expressed as

$$1 = \cos 2k\pi + i \sin 2k\pi$$

which is true for all $k \in \mathbb{Z}$. So $z^n = 1$ becomes

$$r^n \left(\cos n\theta + i \sin\left(n\theta\right)\right) = \cos 2k\pi + i \sin 2k\pi.$$

The modulus and argument for $z = 1$ is one and zero, in turn. Equating the modulus and argument of both sides gives the following equations

$$r^n = 1 \quad \text{and} \quad n\theta = 2k\pi$$

Therefore

$$\begin{aligned} z &= \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = 1 \\ &= \exp\left(\frac{2k\pi i}{n}\right) \quad k = 0, ..., n-1 \end{aligned}$$

If we set $\omega = \exp\left(\frac{2k\pi i}{n}\right)$ then the $n-$ roots of unity are $1, \omega, \omega^2, ..., \omega^{n-1}$.

These roots can be represented geometrically as the vertices of an $n-$ sided regular polygon which is inscribed in a circle of radius 1 and centred at the origin. Such a circle which has equation given by $|z| = 1$ and is called the *unit circle*.

More generally the equation

$$|z - z_0| = R$$

represents a circle centred at z_0 of radius R. If $z_0 = a + ib$, then

$$\begin{aligned} |z - z_0| &= |(x, y) - (a, b)| \\ &= |(x - a) + i(y - b)| \end{aligned}$$

and

$$\begin{aligned} |(x-a)+i\,(y-b)|^2 &= R^2 \\ (x-a)^2+(y-b)^2 &= R^2 \end{aligned}$$

which is the Cartesian form for a circle, centred at (a,b) with radius R.

The *unit circle* is defined as

$$|z| = 1$$

and the *unit disk* is $|z| \le 1$.

If

$$\begin{aligned} |z| &< 1 \text{ then the disk is the } \textit{open unit disk} \\ |z| &\le 1 \text{ then the disk is the } \textit{closed unit disk} \end{aligned}$$

These are examples of open and closed disks

$$|z-z_0| < \delta;\ |z-z_1| \le \epsilon$$

Consider the *annulus* (ring shaped region)

$$r < |z - z_0| < R.$$

For the special case $r = 0$, i.e.

$$0 < |z - z_0| < R,$$

we call this the *punctured disk* of radius R around the point z_0.

Definition 1

The open disc centre $z_0 \in \mathbb{C}$ and radius $r > 0$ is the set $N_r(z_0)$ given by

$$N_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

Definition 2

The closed disc centre $z_0 \in \mathbb{C}$ and radius $r > 0$ is the set $\overline{N_r}(z_0)$ given by

$$\overline{N_r}(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$$

Applications

Example 1

Calculate the indefinite integral $\displaystyle\int \cos^4\theta \; d\theta$.

We begin by expressing $\cos^4\theta$ in terms of $\cos n\theta$ (for different n).

$$
\begin{aligned}
\cos\theta &= \frac{1}{2}\left(z+\frac{1}{z}\right) \Rightarrow 2^4\cos^4\theta = \left(z+\frac{1}{z}\right)^4 \therefore \\
2^4\cos^4\theta &= z^4+4z^3\frac{1}{z}+6z^2\frac{1}{z^2}+4z\frac{1}{z^3}+\frac{1}{z^4} \text{ using Pascals triangle} \\
&= z^4+4z^2+6+4\frac{1}{z^2}+\frac{1}{z^4} \\
&= \left(z^4+\frac{1}{z^4}\right)+4\left(z^2+\frac{1}{z^2}\right)+6
\end{aligned}
$$

We know

$$\tfrac{1}{2}\left(z^n+\frac{1}{z^n}\right) = \cos n\theta$$

$$2^4\cos^4\theta = 2.\tfrac{1}{2}\left(z^4+\frac{1}{z^4}\right)+4.2.\tfrac{1}{2}\left(z^2+\frac{1}{z^2}\right)+6$$

hence

$$\begin{aligned}2^4\cos^4\theta &= 2\cos 4\theta + 8\cos 2\theta + 6\\ \cos^4\theta &= \frac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right)\therefore\end{aligned}$$

Now integrating

$$\begin{aligned}\int \cos^4\theta d\theta &= \frac{1}{8}\int\left(\cos 4\theta + 4\cos 2\theta + 3\right)\ d\theta\\ &= \frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta + K\end{aligned}$$

Example 2

As another application , express $\cos 4\theta$ in terms of $\cos^n \theta$.

We know from De Moivres theorem that

$$\cos 4\theta = \mathrm{Re}\,(\cos 4\theta + i \sin 4\theta)$$

So

$$\cos 4\theta = \mathrm{Re}\,(\cos \theta + i \sin \theta)^4\,,$$

and put $c \equiv \cos \theta, \quad is \equiv i \sin \theta,$ to give

$$\begin{aligned}
\cos 4\theta &= \mathrm{Re}\left(c^4 + 4c^3\,(is) + 6c^2\,(is)^2 + 4c\,(is)^3 + (is)^4\right) \\
\cos 4\theta &= \mathrm{Re}\left(c^4 + i4c^3 s - 6c^2 s^2 - i4cs^3 + s^4\right) \\
\cos 4\theta &= c^4 - 6c^2 s^2 + s^4
\end{aligned}$$

Now $s^2 = 1 - c^2, \ \therefore$

$$\begin{aligned} \cos 4\theta &= c^4 - 6c^2\left(1 - c^2\right) + \left(1 - c^2\right)^2 = 8c^4 - 8c^2 + 1 \Rightarrow \\ \cos 4\theta &= 8\cos^4\theta - 8\cos^2\theta + 1. \end{aligned}$$

Example 3

Find the square roots of -1 , i.e. solve $z^2 = -1$. The complex number -1 has a modulus of one and argument π, so

$$-1 = \cos\left(\pi + 2k\pi\right) + i\sin\left(\pi + 2k\pi\right).$$

Hence,

$$\begin{aligned} (-1)^{1/2} &= \left(\cos\left(\pi + 2k\pi\right) + i\sin\left(\pi + 2k\pi\right)\right)^{1/2} \\ &= \cos\left(\frac{\pi + 2k\pi}{2}\right) + i\sin\left(\frac{\pi + 2k\pi}{2}\right) \end{aligned}$$

for $k = 0, 1$:

$$(-1)^{1/2} = \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) = 0 + i$$

$$(-1)^{1/2} = \cos\left(\frac{3\pi}{2}\right) + i \sin\left(\frac{3\pi}{2}\right) = 0 - i$$

Therefore the square roots of -1 are $z_0 = i$ and $z_1 = -i$.

Example 4

Find the fifth roots of -1 , i.e. solve $z^5 = -1$. The complex number -1 has a modulus of one and argument π, so

$$\begin{aligned}(-1)^{1/5} &= (\cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi))^{1/5} \\ &= \cos\left(\frac{\pi + 2k\pi}{5}\right) + i \sin\left(\frac{\pi + 2k\pi}{5}\right)\end{aligned}$$

for $k = 0, 1, 2, 3, 4$:

$$z_0 = \cos\left(\frac{\pi}{5}\right) + i \sin\left(\frac{\pi}{5}\right)$$

$$z_1 = \cos\left(\frac{3\pi}{5}\right) + i \sin\left(\frac{3\pi}{5}\right)$$

$$z_2 = \cos(\pi) + i \sin(\pi)$$

$$z_3 = \cos\left(\frac{7\pi}{5}\right) + i \sin\left(\frac{7\pi}{5}\right)$$

$$z_4 = \cos\left(\frac{9\pi}{5}\right) + i \sin\left(\frac{9\pi}{5}\right)$$

Example 5

Find all $z \in \mathbb{C}$ such that $z^3 = 1 + i$. So we wish to find the cube roots of $(1 + i)$. The argument of this complex number is $\theta = \arctan 1 = \pi/4$. The modulus of $(1+i)$ is $r=\sqrt{2}$. We can express $(1+i)$ compactly in $r\exp(i\theta)$ as

$$1+i=\sqrt{2}\exp\left(i\frac{\pi}{4}\right)$$

So

$$(1+i)^{1/3}=2^{1/6}\exp\left(i\frac{\pi(8k+1)}{12}\right)$$

for $k=0,1,2$.

$$z_0=2^{1/6}\exp\left(i\frac{\pi}{12}\right)$$

$$z_1=2^{1/6}\exp\left(i\frac{9\pi}{12}\right)$$

$$z_2=2^{1/6}\exp\left(i\frac{17\pi}{12}\right)$$

Example 6: We can apply Euler's formula to integral problems. Consider the earlier example

$$\int e^x \cos x dx$$

which was simplified using the integration by parts method. We know $\operatorname{Re} e^{i\theta} = \cos\theta$, so the above becomes

$$\begin{aligned}
\int e^x \operatorname{Re} e^{ix} dx &= \int \operatorname{Re} e^{(i+1)x} dx = \operatorname{Re} \tfrac{1}{1+i} e^{(i+1)x} \\
&= e^x \operatorname{Re} \tfrac{1}{1+i} \left(e^{ix}\right) = e^x \operatorname{Re} \tfrac{1-i}{(1+i)(1-i)} \left(e^{ix}\right) \\
&= \tfrac{1}{2} e^x \operatorname{Re} (1-i) \left(e^{ix}\right) = \tfrac{1}{2} e^x \operatorname{Re} \left(e^{ix} - i e^{ix}\right) \\
&= \tfrac{1}{2} e^x \operatorname{Re} (\cos x + i \sin x - i \cos x + \sin x) \\
&= \tfrac{1}{2} e^x (\cos x + \sin x)
\end{aligned}$$

Exercise: Repeat this method of working for evaluating

$$\int e^x \sin x dx$$

Functions

Polynomial Functions: A polynomial function of z has the form

$$f(z) = a_0 + a_1 z + a_2 z^2 + O\left(z^3\right) = \sum_{n=0}^{\infty} a_n z^n$$

and is of degree n. The domain is the set $\mathbb{C}$ of all complex numbers. So for example a 3rd degree polynomial is $2 - z + a_2 z^2 + 3z^3$.

Rational Functions: A rational function has the form

$$R(z) = \frac{P_1(z)}{P_2(z)}$$

where $P_1,\ P_2$ are polynomials. The domain is the set $\mathbb{C}-$zeroes of $P_2(z)$. For example

$$f(z) = \frac{2z+3}{z^2 - 3z + 2} = \frac{2z+3}{(z-1)(z-2)}$$

and domain is $\mathbb{C} - \{1, 2\}$.

Power Series:

$$\begin{aligned}
\exp(\pm z) &= 1 \pm z + \frac{1}{2!}z^2 \pm \frac{1}{3!}z^3 + \ldots.. = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!} \\
\sinh z &= z + \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \ldots.. = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \\
\cosh z &= 1 + \frac{1}{2!}z^2 + \frac{1}{4!}z^4 + \ldots.. = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \\
\sin z &= z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \ldots.. = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \\
\cos z &= 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \ldots.. = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}
\end{aligned}$$

Functions of a Complex Variable

A rule which assigns to every complex number

$$z = x + iy = re^{i\theta}$$

in some region D, a unique complex number

$$w = u + iv = \rho e^{i\phi}.$$

w is called a *function of a complex variable*.

So $w = f(z)$

$$= u(x, y) + iv(x, y)$$

So we see that

$$\begin{aligned} \mathsf{Re}\, w &= u(x, y) \\ \mathsf{Im}\, w &= v(x, y) \end{aligned}$$

e.g.

$$\begin{aligned} w &= z^2 \\ &= (x+iy)^2 \\ &= x^2 - y^2 + 2xyi \end{aligned}$$

Here

$$\begin{aligned} u(x,y) &= x^2 - y^2 \\ v(x,y) &= 2xy \end{aligned}$$

Note

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -2y = -\frac{\partial v}{\partial x} \end{aligned}$$

Exponential Function:

$$\begin{aligned} w &= f(z) = e^z \\ &= e^{x+iy} = e^x e^{iy} \end{aligned}$$

$\mathrm{Re}\, e^z : u(x, y) = e^x \cos y$

$\mathrm{Im}\, e^z : v(x, y) = e^x \sin y$

$|\exp z| = e^x$ and y is the argument.

Logarithmic Function:

If

$$e^w = z$$

we say w is a logarithm and we write

$$w = \mathbf{Log} z$$

which is not unique, for suppose

$$w = \mathbf{Log} z = u + iv$$

or

$$\begin{aligned} e^{u+iv} &= z \\ e^u e^{iv} &= z \\ e^u (\cos v + i \sin v) &= z \end{aligned}$$

therefore

$$e^u = |z| \Longrightarrow u = \ln |z|$$

and

$$v = \arg z + 2n\pi$$

Thus we can write

$$\mathbf{Log} z = \ln |z| + i\,(\arg z + 2n\pi)$$

$\mathbf{Log} z$ has infinitely many vales. If we take the principal value of $\arg z$ then the corresponding value of $\mathbf{Log} z$ is called the principal value of $\mathbf{Log} z$ and written $\log z$ where

$$\log z = \ln |z| + i \arg z$$

and $\log z$ is now a function.

Example: $z = -1 + i\sqrt{3}$

$$|z| = 2;\;\; \arg z = \arctan\left(-\sqrt{3}\right) = -\frac{\pi}{3} = \frac{2\pi}{3}$$

hence

$$
\begin{aligned}
\mathbf{Log} z &= \ln|2| + i\left(\frac{2\pi}{3} + 2n\pi\right);\ n \in \mathbb{Z} \\
\log z &= \ln|2| + i\frac{2\pi}{3}
\end{aligned}
$$

Power Series

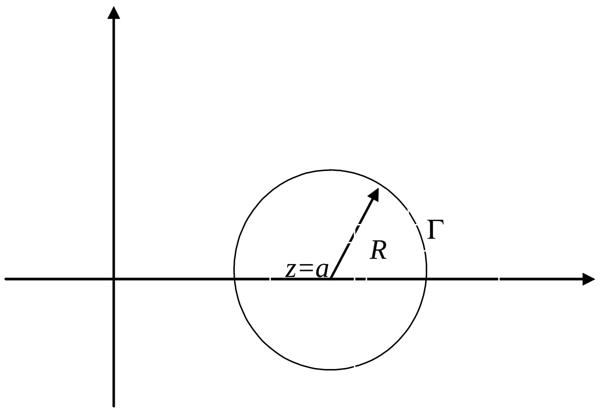
We define a *power series* in $(z-a)$ or about $z=a$ as

$$a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots\ldots + a_n(z-a)^n + \ldots = \sum_{n=0}^{\infty} a_n(z-a)^n \tag{$\dagger$}$$

This infinite series converges for $z=a$ (plus other points).

$\exists\ R \in \mathbb{Q}^+$ s.t. $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges $|z-a| < R$ and diverges $|z-a| > R$.

The special case $|z-a| = R$ may or may not converge.



(†) will converge at all points in Γ and diverge everywhere outside Γ. On Γ we do not know (needs additional work).

The special cases $R = a$ and $R = \infty$ correspond in turn to

$$\begin{aligned} R &= a \text{ corresponds to converges at } z = a \text{ only} \\ R &= \infty \text{ corresponds to converges } \forall \text{ finite values of } z \end{aligned}$$

$R -$ *radius of convergence*

Γ$-$ *circle of convergence*

Various Tests

Absolute Convergence

If $\sum_{n=1}^{\infty} |u_n|$ converges then $\sum_{n=1}^{\infty} u_n$ converges

<u>**Comparison Test**</u>:

If $\sum_{n=1}^{\infty} |v_n|$ converges and $|u_n| \leq |v_n|$ then $\sum_{n=1}^{\infty} u_n$ converges absolutely.

If $\sum_{n=1}^{\infty} |v_n|$ diverges and $|u_n| \geq |v_n|$ then $\sum_{n=1}^{\infty} |u_n|$ diverges but $\sum_{n=1}^{\infty} u_n$ may or may not converge.

Ratio Test:

If

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = L$$

then $\sum_{n=1}^{\infty} u_n$

$$\begin{cases} \text{converges (absolutely)} & L < 1 \\ \text{diverges} & L > 1 \\ \text{test fails} & L = 1 \end{cases}$$

$\mathbf{p}-$ series Test:

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for any constant $p > 1$ and diverges for $p \leq 1$.

Example 1: Show that

$$\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}$$

converges absolutely for $|z| \leq 1$.

If $|z| \leq 1$ then

$$\begin{aligned} \left|\frac{z^n}{n(n+1)}\right| &= \frac{|z^n|}{n(n+1)} \leq \frac{1}{n(n+1)} \\ &\leq \frac{1}{n^2} \end{aligned}$$

and by the $p-$ series test for $p = 2$ we know that it converges, hence comparison test implies convergence.

Let's re-do but using the Ratio test

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right|$$

where

$$u_n = \frac{z^n}{n\,(n+1)}; \quad u_{n+1} = \frac{z^{n+1}}{(n+1)\,(n+2)}$$

so

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{z^{n+1}}{(n+1)(n+2)}}{\frac{z^n}{n(n+1)}} \right| \\
&= \lim_{n\to\infty} \left| \frac{z^{n+1}}{n+2} \frac{n}{z^n} \right| \\
&= \lim_{n\to\infty} \underbrace{\frac{n}{n+2}}_{<1} \underbrace{|z|}_{\leq 1} \\
&< 1
\end{aligned}$$

so we have convergence by the Ratio test.

Example 2: Calculate the radius of convergence of

$$\sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{4^n\,(n+1)^3}$$

For $z=-2$ this converges.

Use the Ratio test with $u_n = \frac{(z+2)^{n-1}}{4^n(n+1)^3}$; $\quad u_{n+1} = \frac{(z+2)^n}{4^{n+1}(n+2)^3}$

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(z+2)^n}{4^{n+1}(n+2)^3}}{\frac{(z+2)^{n-1}}{4^n(n+1)^3}}\right| \\
&= \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^3\left|\frac{z+2}{4}\right| \\
&\approx \frac{|z+2|}{4}\begin{cases} <1 \text{ abs cgce} \\ =1 \text{ test fails} \\ >1 \text{ diverges}\end{cases}
\end{aligned}$$

Therefore $|z+2| < 4$ gives $R = 4$, which is the region of convergence. Circle centred at $(-2, 0)$ and radius 4.

We also see that $z = -2$ is included in $|z+2| < 4$.

What about the boundary of the circle $|z+2| = 4$? The Ratio test does not assist here.

So try the Comparison Test – look at

$$\left|\frac{(z+2)^{n-1}}{4^n (n+1)^3}\right|$$

the numerator becomes, using $|z+2| = 4$, 4^{n-1}

$$\left|\frac{4^{n-1}}{4^n (n+1)^3}\right| = \left|\frac{1}{4(n+1)^3}\right| \leq \frac{1}{n^3}$$

which converges (from comparison test for $p = 3$).

So the series is absolutely convergent for $|z+2| \leq 4$, i.e. region of circle centre -2 and radius 4, including the boundary.

Differentiation in The Complex Plane

Recall that for a real variable

$$\frac{df}{dx} = \lim_{\delta x \longrightarrow 0} \frac{f(x+\delta x) - f(x)}{\delta x}.$$

For functions of complex variables there are an infinite number of paths along which $\delta z \longrightarrow 0$ and so as many values of

$$f'(z) = \lim_{\delta z \longrightarrow 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

are possible.

If all these limits are the same we say that $f(z)$ is *differentiable* and the derivative is the value of the limit. So in other words if the derivative exists it must be independent of the way in which δz tends to zero.

Another definition for $f'(z)$ at the point z_0 is

$$f'(z_0) = \lim_{z \longrightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Holomorphic Functions

Consider a region $\mathbb{U}$. If the derivative $f'(z)$ exists at all points in $\mathbb{U}$ then the function is said to be *Holomorphic* in $\mathbb{U}$. This is a relatively new term, some of the older books use the synonyms *regular* and *analytic.* We then write $f(z) \in H(\mathbb{U})$.

A function $f(z)$ is said to be *holomorphic at a point* z_0 if $\exists$ a neighbourhood

$$|z - z_0| < \delta$$

at all points of which $f'(z)$ exists.

If a function is holomorphic everywhere we simply say $f(z)$ is a holomorphic function, i.e. $f(z) \in H(\mathbb{C})$.

Example: Show that $f(z) = \overline{z}$ is not differentiable at any point.

$$z = x + iy \text{ and } \delta z = \delta x + i\delta y$$

then

$$z + \delta z = (x + \delta x) + i (y + \delta y) .$$

$$\begin{aligned} f'(z) &= \lim_{\delta z \longrightarrow 0} \frac{f(z + \delta z) - f(z)}{\delta z} \\ &= \lim_{\delta z \longrightarrow 0} \frac{\overline{(z + \delta z)} - \overline{z}}{\delta z} \\ &= \lim_{\delta z \longrightarrow 0} \frac{\delta x - i\delta y}{\delta x + i\delta y} \end{aligned}$$

Now consider the limits.

First let $\delta y \longrightarrow 0$ and then $\delta x \longrightarrow 0$

$$\lim_{\delta z \longrightarrow 0} \frac{\delta x - i\delta y}{\delta x + i\delta y} = \lim_{\delta x \longrightarrow 0} \frac{\delta x}{\delta x} = 1$$

now $\delta x \longrightarrow 0$ and then $\delta y \longrightarrow 0$

$$\lim_{\delta z \longrightarrow 0} \frac{\delta x - i\delta y}{\delta x + i\delta y} = \lim_{\delta x \longrightarrow 0} \frac{-i\delta y}{i\delta y} = -1$$

as the results differ $f(z) = \overline{z}$ is not differentiable at any point.

A point at which $f(z)$ is not differentiable is called a *singularity*, or a *singular point* of $f(z)$.

As we cannot test all the paths as $\delta z \longrightarrow 0$, this provides us with a way to establish non-differentiability - by simply finding two paths which give different limits.

Example: Prove (using the definition) that

$$f(z) = \begin{cases} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

is not holomorphic at $z = 0$.

Let's use the definition $f'(z_0) = \lim_{z \longrightarrow z_0} \frac{f(z)-f(z_0)}{z-z_0}$, which becomes

$$f'(0) = \lim_{z \longrightarrow 0} \frac{f(z) - f(0)}{z - 0}$$

$$\lim_{z \longrightarrow 0} \frac{x^3 (1+i) - y^3 (1-i)}{\left(x^2+y^2\right)(x+iy)}$$

Let $z \longrightarrow 0$ along the line $y = mx$ and examine the limit

$$\frac{x^3 (1+i) - m^3 x^3 (1-i)}{\left(x^2+m^2x^2\right)(x+imx)} = \frac{(1+i) - m^3 (1-i)}{\left(1+m^2\right)(1+im)}$$

hence

$$\lim_{x \longrightarrow 0} \frac{(1+i) - m^3 (1-i)}{\left(1+m^2\right)(1+im)}$$

has many values depending on m which implies that $f'(0)$ does not exist.

If $f(z)$ is a function of z, e.g. z^2, e^z, $\sin z$ then differentiate in the normal/real way and the various rules (e.g. product/quotient) apply.

Example: If $f(z) = \operatorname{cosec} z$

$$f'(z) = -\cos\mathrm{ec} z \cot z = -\frac{1}{\sin z} \cdot \frac{\cos z}{\sin z} = -\frac{\cos z}{\sin^2 z}$$

which has singularities where $\sin z = 0 \iff z = n\pi : n \in \mathbb{Z}$.

As an **exercise** verify these singularities by solving $\sin z = 0$ for $z = x + iy$.

The Cauchy-Riemann Equations

We need a way of showing that a function is differentiable as the definition of differentiability is really only useful for establishing non-differentiability.

Let

$$z = x + iy; \quad f(z) = u(x, y) + iv(x, y).$$

If $f(z)$ is differentiable at a given point z then the ratio $\dfrac{f(z+\delta z) - f(z)}{\delta z} \longrightarrow f'(z)$ no matter how $\delta z \longrightarrow 0$

$$f'(z) = \lim_{\delta z \longrightarrow 0} \tfrac{u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y) - u(x,y) - iv(x,y)}{\delta x + i\delta y}.$$

We consider this in two steps:

1. Let $\delta z \longrightarrow 0$ horizontally i.e. $\delta y = 0$ and $\delta x \longrightarrow 0$

$$\begin{aligned}
f'(z) &= \lim_{\delta x \longrightarrow 0} \frac{u(x+\delta x, y) + iv(x+\delta x, y) - u(x,y) - iv(x,y)}{\delta x} \\
&= \lim_{\delta x \longrightarrow 0} \left(\frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i\frac{v(x+\delta x, y) - v(x,y)}{\delta x} \right) \\
&= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.
\end{aligned}$$

2. Let $\delta z \longrightarrow 0$ vertically i.e. $\delta x = 0$ and $\delta y \longrightarrow 0$

$$\begin{aligned}
f'(z) &= \lim_{\delta y \longrightarrow 0} \frac{u(x, y+\delta y) + iv(x, y+\delta y) - u(x,y) - iv(x,y)}{i\delta y} \\
&= \lim_{\delta y \longrightarrow 0} \left(\frac{1}{i}\frac{u(x, y+\delta y) - u(x,y)}{\delta y} + \frac{v(x, y+\delta y) - v(x,y)}{\delta y} \right) \\
&= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}
\end{aligned}$$

If $f'(z)$ exists these two limits must be equal and hence

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

equating real and imaginary parts (in turn) gives

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

These are the *Cauchy-Riemann Equations.*

They are necessary for differentiability but not sufficient, i.e. if C-R equations are satisfied, $f(z)$ may or may not be differentiable. We can say with certainty that if the conditions are not satisfied then the function is non-differentiable.

Example: Show that the functions

$$u = \frac{x}{x^2 + y^2}, \; v = -\frac{y}{x^2 + y^2}$$

satisfy the Cauchy-Riemann equations everywhere except at $(0, 0)$.

This can be done simply by verifying

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

for the given $u(x, y)$ and $v(x, y)$:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}; \quad \frac{\partial v}{\partial x} = \frac{2xy}{\left(x^2 + y^2\right)^2} \\ \frac{\partial u}{\partial y} &= \frac{-2xy}{\left(x^2 + y^2\right)^2}; \quad \frac{\partial v}{\partial y} = \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2} \end{aligned}$$

so C-R equations are satisfied. The partial derivatives are continuous everywhere except at $(0,0)$, where they do not exist.

$$\begin{aligned} f(z) &= u + iv \\ &= \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} \\ &= \frac{1}{\left(x^2+y^2\right)^2}(x - iy) \end{aligned}$$

further simplification gives

$$\begin{aligned} f(z) &= \frac{1}{|z|^2}\overline{z} \\ &= \frac{1}{z\overline{z}}\overline{z} \\ &= \frac{1}{z} \end{aligned}$$

A function $\Theta(x, y)$ is called *harmonic* if it satisfies Laplace's Equation

$$\frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2} = 0$$

The real and imaginary parts of a Holomorphic function satisfy Laplace's Equation.

This is very easy to verify, for if $f(z) = u + iv \in H(\mathbb{C})$ then the C-R equations are satisfied.

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} && (1) \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} && (2) \end{aligned}$$

and we can differentiate these partially. Differentiate (1) w.r.t x, and (2) w.r.t y

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{3}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \tag{4}$$

(3) $+$ (4) gives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Similarly differentiating (1) and (2) wrt to y and x respectively gives

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

So we see that both real and parts of a holomorphic function are harmonic.

They are sometimes called *harmonic conjugates.* Given one harmonic function we can use the C-R equations to find a conjugate harmonic function.

Consider the following

$$u\left(x,y\right)=e^{x^2-y^2}\sin 2xy$$

does this satisfy the PDE above?

$$\begin{aligned}
\frac{\partial u}{\partial x} &= 2e^{x^2-y^2}\left(y\cos 2xy+x\sin 2xy\right)\\
\frac{\partial^2 u}{\partial x^2} &= e^{x^2-y^2}\sin 2xy\left(4x^2-4y^2+2\right)+\\
&\quad 8xye^{x^2-y^2}\cos 2xy
\end{aligned}$$

$$\frac{\partial^2 u}{\partial y^2}=e^{x^2-y^2}\left(-4x^2\sin 2xy-8xy\cos 2xy+4y^2\sin 2xy-2\sin 2xy\right)$$

so clearly $u_{xx}+u_{yy}=0$.

Complex Integration

A complex integral is an integral taken along a curve (contour) in the complex plane. We will denote this by γ.

We base our definition of such an integral on real integrals to avoid unnecessary work.

Consider first the type of curve along which we will integrate. A curve can be written in parametric form if it can be expressed as

$$z = z(t) : a \leq t \leq b$$

where t is a real parameter with initial and final points $z(a)$ and $z(b)$, in turn.

Examples:

1. The circle centre 0, radius r starting and finishing at the point A $(|z| = r)$. Positively described means anti-clockwise,

$$\begin{aligned} z &= re^{it} \\ &= r(\cos t + i \sin t) : 0 \leq t \leq 2\pi \end{aligned}$$

2. Semi-circle, centre 0, radius 1 lying in the right hand half of the plane. The initial point is A $(z = -i)$ and final point B $(z = i)$

$$z = e^{it} : -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

3. Circle centre $z_0 = x_0 + iy_0$ of radius r

$$z = z_0 + re^{it} : 0 \leq t \leq 2\pi$$

4. The positive real axis starting at 0

$$z = t : 0 \leq t < \infty$$

What about the real axis from -3 to 2?

5. The imaginary axis from $z = -2i$ to $z = -5i$

$$z = it : -2 \leq t \leq 5$$

6. The line segment from $a + ic$ to $b + ic$

$$z = t + ic : a \leq t \leq b$$

There is a useful general method for obtaining this, if we wish to express the line segment from z_1 to z_2 :

$$z = z_1 + t\,(z_2 - z_1) : 0 \leq t \leq 1$$

7. The line from a to $a + ib$

$$z = a + it : 0 \le t \le b$$

8. The line from -1 to $1 + 2i$

$$z = t + i\,(1 + t) : -1 \le t \le 1$$

If γ is parameterized by

$$z = z\,(t) : a \le t \le b$$

then $-\gamma$ can be expressed as

$$z = z\,(-\theta) : -b \le t \le -a$$

The contour γ is called *closed* if $z\,(a) = z\,(b)$, i.e. the starting point and end point are the same. For example, consider the circle earlier

$$z = e^{it} : 0 \le t \le 2\pi$$

here we see $z(0) = z(2\pi) = 1$

A closed contour which does not cross itself is called a *simple closed* contour.

Integration Along a Contour

Let γ be a contour defined by $\gamma(t) = z(t) : t \in [a, b]$. Let $f(z)$ be a continuous function on γ and $z'(t)$ is also continuous on γ, then

$$\int_\gamma f(z)\, dz = \int_a^b f(z(t)) \frac{dz}{dt} dt$$

Simple Example: γ is the first quadrant of the unit circle, i.e. centre 0 from 1 to i. Evaluate

$$\int_\gamma z dz$$

First define γ :

$$\begin{aligned} z &= e^{it} \quad 0 \leq t \leq \pi/2 \\ \frac{dz}{dt} &= ie^{it} \end{aligned}$$

therefore

$$\begin{aligned}
\int_\gamma z dz &= \int_0^{\pi/2} e^{it} i e^{it} dt \\
&= \int_0^{\pi/2} e^{2it} i dt = \frac{1}{2i} e^{2it} i \Big|_0^{\pi/2} \\
&= \frac{e^{i\pi} - 1}{2} = -1
\end{aligned}$$

Example 2: Let γ be the straight line from 1 to $2+i$. Evaluate $\int_\gamma (1+2z)\,dz$

$$\begin{aligned}
\gamma &: \quad z = t + (t-1)\,i \;:\; 1 \le t \le 2 \\
\frac{dz}{dt} &= 1 + i
\end{aligned}$$

hence

$$\begin{aligned}\int_\gamma (1+2z)\,dz &= \int_1^2 1+2\left(t+(t-1)\,i\right)(1+i)\,dt \\ &= (1+i)\int_1^2 \left(1-2i+2\,(1+i)\right)dt \\ &= 3+5i\end{aligned}$$

If $\dfrac{dz}{dt}$ is continuous except at $t = c_1,\ c_2,\ \ldots..,\ c_n$ we can define

$$\int_\gamma f\,(z)\,dz = \int_a^{c_1} f\,(z\,(t))\,\tfrac{dz}{dt}dt + \int_{c_1}^{c_2} f\,(z\,(t))\,\tfrac{dz}{dt}dt + \ldots\ldots + \int_{cn}^{b} f\,(z\,(t))\,\tfrac{dz}{dt}dt$$

Example 1: Let γ be the line from -1 to 0 together with the line from 0 to i

$$\gamma(t) \quad : \quad \begin{cases} z = t & -1 \leq t \leq 0 \\ z = it & 0 \leq t \leq 1 \end{cases}$$

$$\frac{dz}{dt} \quad = \quad \begin{cases} 1 & -1 \leq t < 0 \\ i & 0 \leq t \leq 1 \end{cases}$$

Then $\int_\gamma z dz =$

$$\begin{aligned} & \int_{-1}^{0} t \cdot 1 dt + \int_{0}^{1} it \cdot i dt \\ = \quad & -1 \end{aligned}$$

Example 2:

Consider the following parametrization:

$$\gamma(t) = \begin{cases} t & t \in [0,1] \\ (t-1)\,i + (2-t)\,i & t \in [1,2] \\ (3-t)\,i & t \in [2,3] \end{cases}$$

Evaluate $\int_\gamma \mathrm{Re}\,(z)\,dz =$

$$\begin{aligned} & \int_0^1 \mathrm{Re}\,(t)\,dt + \int_1^2 \mathrm{Re}\,((t-1)\,i + (2-t)\,i) \cdot (i-1)\,dt + \\ & \int_2^3 \mathrm{Re}\,((3-t)\,i) \cdot (-i)\,dt \\ = \; & \int_0^1 t\;dt + \int_1^2 (2-t) \cdot (i-1)\;\;dt + \int_2^3 0\;dt \\ = \; & \left.\frac{t^2}{2}\right|_0^1 + (i-1)\left.\left(2t - \frac{t^2}{2}\right)\right|_1^2 = \frac{1}{2}i \end{aligned}$$

Properties of the Integral

$$\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz$$

As an example consider $\int_{\gamma} z\ dz$ where γ is the straight line from 0 to $1+i$

$$\begin{aligned} \gamma(t) &: \quad z = t + it \qquad 0 \le t \le 1 \\ z' &= 1 + i \end{aligned}$$

$$\begin{aligned} \int_{\gamma} z\ dz &= \int_0^1 (t+it)(1+i)\,dt \\ &= i \end{aligned}$$

Now consider

$$-\gamma(t) : z = -t - it \qquad -1 \le t \le 0$$

$$
\begin{aligned}
\int_{-\gamma} z\ dz &= \int_{-1}^{0} (-t - it)(-1 - i)\, dt = -i \\
&= -\int_{\gamma} z\ dz
\end{aligned}
$$

If γ has initial and final point z_1 and z_2, in turn and if $f(z) = \frac{dF}{dz}$ on γ then

$$
\begin{aligned}
\int_{\gamma} f(z)\, dz &= \int_{\gamma} \frac{dF}{dz} dz \\
&= F(z_2) - F(z_1)
\end{aligned}
$$

for suppose $\gamma : z = z(t) \quad a \leq t \leq b$, i.e. $z_1 = z(a)$ & $z_2 = z(b)$ then

$$
\begin{aligned}
\int_{\gamma} f(z)\, dz &= \int_{\gamma} \frac{dF}{dz} dz \\
&= \int_{a}^{b} \frac{dF(z(t))}{dz} \frac{dz}{dt} dt \\
&= F(z(t))\big|_{a}^{b} = F(z(b)) - F(z(a)) \\
&= F(z_2) - F(z_1)
\end{aligned}
$$

Note:

1. If $z_2 = z_1$, i.e. γ is a closed contour then $\int_\gamma f(z)\,dz = F(z_1) - F(z_1) = 0$

2. These results mean that if $f(z)$ can be integrated directly then we do not need to parameterize γ.

Example 1: γ is the circular contour joining 1 to i.

Then $\displaystyle\int_\gamma z^2 dz = \left.\frac{z^3}{3}\right|_\gamma = \left.\frac{z^3}{3}\right|_1^i - \tfrac{1}{3}\left(1+i\right).$

Note that this answer only depends on the initial and final points of γ, not on γ itself.

Example 2: Let γ be the line from 1 to $2+i$

$$\begin{aligned}\int_\gamma (1+2z)\,dz &= \left. z+z^2\right|_\gamma \\ &= \left. z+z^2\right|_1^{2+i} = 5i+3\end{aligned}$$

Cauchy's Theorem

If γ is any closed contour and if $f(z)$ is differentiable inside and on γ then

$$\int_\gamma f(z)\,dz = 0$$

There are many versions of this theorem which make different assumptions about the contour γ.

As an example, a polynomial $P(z)$ is differentiable everywhere. The exponential, circular and hyperbolic functions are holomorphic on $\mathbb{C}$. Therefore given a closed contour γ

$$\int_\gamma P(z)\,dz = \int_\gamma e^z dz = \int_\gamma \sin z dz = \ldots\ldots = \int_\gamma \cosh z dz = 0$$

e.g. if C is the unit circle $|z| = 1$ then

$$\int_C \left(z^2 + 6z - 3\right) dz = 0.$$

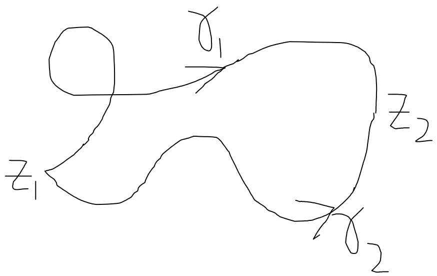
A rational function $R(z) = P_1(z) / P_2(z)$ is holomorphic everywhere except at the zeroes of $P_2(z)$. Therefore $\int_\gamma P_1(z) / P_2(z) \; dz = 0$ on any closed contour which does not contain or pass through any zero of $P_2(z)$.

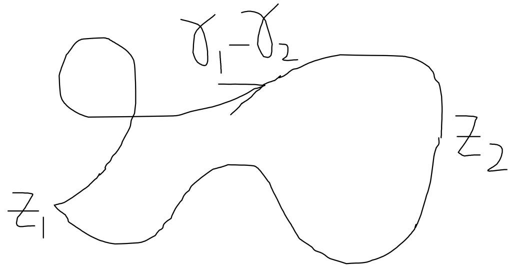
Corollary to Cauchy's Theorem

If γ_1 and γ_2 are any two contours with the same initial and final points and if $f(z)$ is differentiable inside and on $\gamma_1 - \gamma_2$ then

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz$$

i.e. the integral does not depend on the contour, only on the initial and final points.





$\gamma_1 - \gamma_2$ is closed now and $f(z)$ is differentiable inside and on $\gamma_1 - \gamma_2$ (given).

Therefore (by Cauchy)

$$\begin{aligned}
\int_{\gamma_1-\gamma_2} f(z)\,dz &= 0\\
\int_{\gamma_1} f(z)\,dz + \int_{-\gamma_2} f(z)\,dz &= 0\\
\int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz &= 0
\end{aligned}$$

hence $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$.

Example: Evaluate $\int_\gamma \frac{dz}{z^2+2z+2}$ where γ is the semi-circle joining -1 and 1 in the upper $1/2-$ plane.

This contour is not closed. How ever by introducing the line segment L which goes from -1 to 1 along the real axis $\gamma - L$ is now closed. $\frac{1}{z^2+2z+2}$ is holomorphic except where $z^2+2z+2=0$, i.e. $z=-1\pm i$, which lies outside $\gamma - L$.

It follows by the Corollary to Cauchy's Theorem that

$$\int_\gamma \frac{dz}{\left(z^2+2z+2\right)} = \int_L \frac{dz}{\left(z^2+2z+2\right)}$$

So we solve along L by parameterizing:

$$L : z\left(t\right) = t;\ -1 \le t \le 1$$

$$\begin{aligned}\int_\gamma \frac{dz}{z^2+2z+2} &= \int_{-1}^{1} \frac{dt}{t^2+2t+2} \\ &= \int_{-1}^{1} \frac{dt}{(t+1)^2+1}\end{aligned}$$

Use a substitution $u = t + 1$, which gives

$$\begin{aligned}\int_0^2 \frac{du}{u^2+1} &= \tan^{-1} u\Big|_0^2 \\ &= \tan^{-1} 2\end{aligned}$$

Example: By integrating e^{-z^2} around the rectangle with sides $y = 0$, $y = b$, $x = \pm R$, $\int_\gamma \frac{dz}{z^2+2z+2}$ where γ is the semi-circle joining -1 and 1 in the upper $1/2 -$ plane.

This contour is not closed. How ever by introducing the line segment L which goes from -1 to 1 along the real axis $\gamma - L$ is now closed. $\frac{1}{z^2+2z+2}$ is holomorphic except where $z^2 + 2z + 2 = 0$, i.e. $z = -1 \pm i$, which lies outside $\gamma - L$.

It follows by the Corollary to Cauchy's Theorem that

$$\int_\gamma \frac{dz}{\left(z^2 + 2z + 2\right)} = \int_L \frac{dz}{\left(z^2 + 2z + 2\right)}$$

So we solve along L by parameterizing:

$$L : z\,(t) = t;\; -1 \leq t \leq 1$$

$$\begin{aligned}\int_\gamma \frac{dz}{z^2+2z+2} &= \int_{-1}^{1} \frac{dt}{t^2+2t+2} \\ &= \int_{-1}^{1} \frac{dt}{(t+1)^2+1}\end{aligned}$$

Use a substitution $u = t + 1$, which gives

$$\begin{aligned}\int_0^2 \frac{du}{u^2+1} &= \tan^{-1} u\Big|_0^2 \\ &= \tan^{-1} 2\end{aligned}$$

An Extension of Cauchy's Theorem

When there is a simple type of singularity of $f(z)$ on C, let $f(z)$ be regular in and on C except for a single singularity at $z = a$ which is on C.

With centre $z = a$ and radius δ draw an arc of a circle to indent the contour C at a forming a new contour Γ. Since $f(z)$ is holomorphic in and on Γ so by Cauchy's Theorem

$$\int_{\Gamma} f(z)\, dz = 0$$

Let $\delta \longrightarrow 0$

$$\int_{C} f(z)\, dz + \lim_{\delta \longrightarrow 0} \int_{\text{indent}} f(z)\, dz = 0$$

Cauchy's Integral Formula

The following important result is due to Cauchy, and is also a Theorem.

Let C be a simple closed contour and suppose that f is holomorphic in and on C. If $z = \xi$ is a point inside C then

$$f(\xi) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - \xi} dz,$$

the integral being taken in the positive (anti-clockwise) sense.

It is possible to deduce from Cauchy's integral formula that f is differentiable at ξ and that the derivative of f to all orders n, can be computed by formally differentiating with respect to z under the integral sign. Thus

$$f^{(n)}(\xi) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - \xi)^{n+1}} dz \quad n \in \mathbb{N}$$

Now for some examples.

Example 1

Evaluate $\oint_C \frac{e^z}{z} dz$, where

$$C : z(\theta) = e^{i\theta},\ 0 \leq \theta \leq 2\pi$$

by using Cauchy's integral formula.

Let $f(z) = e^z$. Then f is a holomorphic function we may apply Cauchy's integral formula in the form

$$f(0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - 0} dz$$

It follows that

$$\begin{aligned} 1 &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-0} \\ \oint_C \frac{e^z}{z}dz &= 2\pi i \end{aligned}$$

Example 2

Evaluate

$$\oint_C \frac{e^z}{(z-1)(z-3)}dz$$

taken round the circle C given by $|z| = 2$ in the positive (anti-clockwise) sense. What is the value of the integral taken around the circle $|z| = 1/2$ in the positive sense?

Put

$$f(z) = e^z/(z-3)$$

Then f is holomorphic in a domain which contains the circle $|z| = 2$ and its interior (but not, of course, the point $z = 3$). Cauchy's integral formula is applicable and we have

$$f(1) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-1)}dz = \frac{1}{2\pi i}\oint_C \frac{e^z/(z-3)}{(z-1)}dz$$

where $f(1) = -e/2$

We conclude that

$$\oint_C \frac{e^z}{(z-1)(z-3)}dz = 2\pi i f(1) = -\pi e i$$

By Cauchy's theorem the integral taken round the circle $|z| = 1/2$ in the positive sense is zero because the integrand is holomorphic in a domain which contains the circle and its interior.

Taylor's Theorem

If $f(z)$ be holomorphic in the a neighbourhood of $z = a$ then it has a power series expansion

$$f(z) = \sum_{0}^{\infty} a_n (z-a)^n$$

with a non-zero radius of convergence R.

Example: Expand

$$f(z) = \frac{1}{(z-1)(z-2)}$$

about the origin and the point at infinity.

1. About $z = 0$. $f(z)$ has singularities at $z = 1, 2$. Since $z = 0$ is a regular point there is a Taylor expansion about $z = 0$ of the form $\sum_{0}^{\infty} a_n z^n$

convergent for $|z| < 1$

$$\begin{aligned}
\frac{1}{(z-1)(z-2)} &\equiv \frac{1}{(z-2)} - \frac{1}{(z-1)} = \frac{1}{-2(1-z/2)} + \frac{1}{(1-z)} \\
&= \frac{1}{-2(1-z/2)} + \frac{1}{1-z} = -\frac{1}{2}\sum_0^\infty \left(\frac{z}{2}\right)^n + \sum_0^\infty z^n \\
&= \sum_0^\infty \left(1 - \frac{1}{2^{n+1}}\right) z^n
\end{aligned}$$

Or Note $f(z) = \sum_0^\infty a_n z^n$ where $a_n = \dfrac{f^{(n)}(0)}{n!}$. We know from above that $f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}$

$$\begin{aligned} f'(z) &= \frac{-1}{(z-2)^2} + \frac{1}{(z-1)^2}; \quad f''(z) = \frac{2}{(z-2)^3} - \frac{2}{(z-1)^3} \\ f'''(z) &= \frac{-2\times 3}{(z-2)^4} + \frac{2\times 3}{(z-1)^4}; \\ f^{(n)}(z) &= \frac{(-1)^n\, n!}{(z-2)^{n+1}} + \frac{(-1)^{n+1}\, n!}{(z-1)^{n+1}} \to \\ f^{(n)}(0) &= n!\left(1 - \frac{1}{2^{n+1}}\right) \therefore \frac{f^{(n)}(0)}{n!} = 1 - \frac{1}{2^{n+1}}. \end{aligned}$$

About $z =$ point at infinity.

$$f(z) = \frac{1}{(z-1)(z-2)}$$

z is point at infinity $\Longrightarrow \frac{1}{z} = 0$ so let $t = 1/z$, so

$$f(t) = \frac{1}{\left(\frac{1}{t} - 1\right)} - \frac{1}{\left(\frac{1}{t} - 2\right)} = \frac{t^2}{(1-t)(1-2t)}$$

$t=0$ is a regular point therefore we can expand in a Taylor series valid for $|t| < 1/2$.

$$
\begin{aligned}
f(t) &= \frac{t^2}{(1-t)(1-2t)} = \frac{1}{2} + \frac{3t/2 - 1/2}{(1-t)(1-2t)} \\
&= \frac{1}{2} - \frac{1}{1-t} + \frac{1}{2(1-2t)} \\
&= \frac{1}{2} - \sum_0^\infty t^n + \frac{1}{2}\sum_0^\infty (2t)^n = \frac{1}{2} + \sum_0^\infty \left(2^{n-1} - 1\right) t^n \\
&= \sum_1^\infty \left(2^{n-1} - 1\right) t^n = \sum_1^\infty \left(2^{n-1} - 1\right) \frac{1}{z^n} \quad |z| > 2
\end{aligned}
$$

To expand $f(z)$ about $z=3$ put $t=z-3$ and expand in powers of t (about $t=0$).

Laurent's Theorem

Let $f(z)$ be holomorphic in the annulus $b < |z-a| < c$ then it has a power series expansion

$$\sum_{-\infty}^{\infty} A_n (z-a)^n$$

Here $A_n = \dfrac{1}{2\pi i}\oint_C \dfrac{f(z)}{(z-a)^{n+1}} dz$. C is any circle $|z-a| = R$ where $b < R < c$ and the expansion is valid for any z in the annulus.

The Nature of Singularities

If $f(z)$ has an isolated singularity at $z = a$ then by Laurent's theorem

$$f(z) = \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + ... + \frac{b_n}{(z-a)^n} + a_0 + a_1(z-a) + ...$$

If there is no singularity at $z = a$ then by Taylor's Theorem

$$f(z) = \sum_0^\infty A_n (z-a)^n$$

Hence $\displaystyle\sum_1^\infty \frac{b_r}{(z-a)^r}$ has been produced by the singularity .

We call this part of the expansion the **Principal Part** (PP) or Laurent Part.

We define the type of singularity according to the shape of the *principal part*:-

a) If the PP has an infinite number of terms, we say that $f(z)$ has an Isolated Essential Singularity (IES) at $z = a$.

b) If the PP has a finite number of terms we say that $f(z)$ has a pole at $z = a$. The **ORDER** of the pole equals the highest power of $\dfrac{1}{z-a}$ which occurs.

Example:

1. If

$$f(z) = \underbrace{\frac{6}{(z-2)^6} + \frac{3}{(z-2)^2} + \frac{1}{(z-2)}}_{\text{PP}} + 4 + 2(z-2)^2 + \ldots$$

$f(z)$ has a 6^{th} order pole at $z = 2$.

2.

$$\begin{aligned} f(z) &= \frac{e^z}{z} = \frac{1}{z}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\right) \\ &= \underbrace{\frac{1}{z}}_{\text{PP}} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \ldots \end{aligned}$$

First order pole - also called simple pole at $z = 0$.
We can also examine the point at infinity. Put $t = 1/z$ to get

$$\begin{aligned} te^{1/t} &= t\left(1 + \frac{1}{t} + \frac{1}{2!t^2} + \frac{1}{3!t^3} + \ldots\right) \\ &= t + 1 + \frac{1}{2!t} + \frac{1}{3!t^2} + \ldots \end{aligned}$$

PP has an infinite number of terms there IES at $t = 0$ i.e. $z =$ point at infinity.

3.

$$f(z) = \frac{e^z}{z} \text{ at } z = 1$$

So put $t = z - 1$ expand in powers of t.

$$\begin{aligned}
\frac{e^{t+1}}{t+1} &= e(1+t)^{-1} e^t \\
&= e\left(1 - t + t^2 - t^3 + ...\right)\left(1 + t + \frac{t^2}{2!} + ...\right) \\
&= A_0 + A_1 t + A_2 t^2 + ...
\end{aligned}$$

If $f(z)$ has a pole of order n at $z = a$

$$f(z) = \frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + ... + \frac{b_n}{(z-a)^n} + a_0 + a_1(z-a) + ...$$

In the residue theorem (next) we find that the coefficient of $\frac{1}{(z-a)}$ i.e. b_1 is very important. b_1 is called the **residue** (poles only) of $f(z)$ at $z = a$.

To Find The Residue

1. Use the definition of the residue - expand $f(z)$ in powers of $(z-a)$ and pick out the coefficient of $\dfrac{1}{z-a}$.

   (a) For the simple pole

$$f(z) = \frac{b_1}{(z-a)} + a_0 + a_1 (z-a) + ...$$

$$(z-a) f(z) = b_1 + a_0 + a_1 (z-a)^2 + ...$$

$$b_1 = \lim_{z \longrightarrow a} (z-a) f(z)$$

(b) For an n^{th} order pole

$$\begin{aligned} f(z) &= \frac{b_n}{(z-a)^n} + ... + \frac{b_1}{(z-a)} + a_0 + a_1(z-a) + ... \\ (z-a)^n f(z) &= b_n + b_{n-1}(z-a) + ... + b_1(z-a)^{n-1} + a_0(z-a)^n + .. \end{aligned}$$

Differentiate $(n-1)$ times

$$\frac{d^{n-1}}{dz^{n-1}}\left((z-a)^n f(z)\right) = b_1(n-1)! + A_0(z-a) + A_1(z-a)^2 + ...$$

Thus

$$b_1 = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\lim_{z\to a}\left[(z-a)^n f(z)\right]$$

Examples

1.

$$f(z) = \frac{e^{1/z}}{(z-2)^2 (z+1)}$$

$z = -1$ is a simple pole

$$\text{residue} = \lim_{z \to -1} (z+1) \frac{e^{1/z}}{(z-2)^2 (z+1)} = \frac{1}{9e}$$

$z = 2$ is a double pole

$$\text{residue} = \lim_{z \to 2} \frac{d}{dz} \left[(z-2)^2 \frac{e^{1/z}}{(z-2)^2 (z+1)} \right] = -\frac{7e^{1/2}}{36}$$

Also examine $z = 0$ by expanding in powers of z

$$\begin{aligned}
f(z) &= \frac{1}{4} e^{1/z} (1+z)^{-1} \left(1 - \frac{z}{2}\right)^{-2} \\
&= \frac{1}{4} \left(1 + \frac{1}{z} + \frac{1}{2! z^2} + \ldots\right) \left(1 - z + z^2 - \ldots\right) (1 + z..) \\
&= \ldots \frac{B_n}{z^n} + \ldots. + \frac{B_1}{(z-a)^2} + A_0 + A_1 z + \ldots.
\end{aligned}$$

Hence there is an IES at $z = 0$.

2.

$$\begin{aligned}
f(z) &= \frac{1}{z^2 \sin \pi z} = \frac{1}{z^2 \left(\pi z - \frac{\pi^3 z^3}{3!} + \ldots\right)} \\
&= \frac{1}{\pi z^3 \left(1 - \frac{\pi^2 z^2}{3!} + \frac{\pi^4 z^4}{5!}\right)} \\
&= \frac{1}{\pi z^3} \left[1 + \frac{\pi^2 z^2}{3!} - \frac{\pi^4 z^4}{5!} + \frac{\pi^4 z^4}{3!} + \ldots\right]
\end{aligned}$$

Hence a 3rd order pole at $z = 0$ and residue $= \pi/6$.

The Residue Theorem

If $f(z)$ is holomorphic in and on a simple closed curve C apart from a number of poles in C then

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of residues of } f(z) \text{ at all its poles in } C.$$

Example: Show that

$$\int_{-\infty}^{\infty} \frac{x+3}{\left(x^2+9\right)^2} dx = \frac{\pi}{18}$$

Start by constructing a suitable contour. C consists of the straight line from $-R$ to R and the semi-circular contour of radius R.

$$\int_C \frac{z+3}{\left(z^2+9\right)^2} dz$$

There are singularities at $z = \pm 3i$. Let $R \to \infty$.

$$\int_{-\infty}^{\infty} \frac{x+3}{\left(x^2+9\right)^2} dx + \lim_{R \longrightarrow \infty} \int_0^{\pi} \frac{R\, e^{i\theta} + 3}{\left(R^2 e^{2i\theta} + 9\right)^2} i\, R\, e^{i\theta} d\theta = 2\pi i \times \text{residue at } 3i$$

Now using the earlier result

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx$$

we have

$$\begin{aligned} \left| \int_0^{\pi} \right| &\leq \int_0^{\pi} \frac{(R+3)\, R}{\left(R^2 - 9\right)^2} d\theta \\ &= \pi \times \frac{R^2 + 3R}{R^4 + 81 - 18R^2} = \frac{\pi}{R^2} \longrightarrow 0 \text{ as } R \to \infty \end{aligned}$$

$$\int_{-\infty}^{\infty} \frac{x+3}{\left(x^2+9\right)^2} dx = 2\pi i \times \text{residue at } 3i$$

$$\text{Residue at } 3i =$$

$$\lim_{z \to 3i} \frac{d}{dz} \left[(z - 3i)^2 \frac{z+3}{\left(z^2+9\right)^2} \right] = \frac{1}{36i}$$

$$\int_{-\infty}^{\infty} \frac{x+3}{\left(x^2+9\right)^2} dx = 2\pi i \times \frac{1}{36i} = \frac{\pi}{18}.$$

Stochastic Volatility Models

An observation when pricing derivatives is the fact that volatility of an asset price is anything but constant. We have seen in the much celebrated Black-Scholes framework that the assumptions do not consider these market features. Volatility does not behave how the Black–Scholes equation would like it to behave; it is not constant, it is not predictable, it's not even directly observable. Volatility is difficult to forecast - although not impossible.

This makes it a prime candidate for modelling as a random (stochastic) variable. There are many economic, empirical, and mathematical reasons for choosing a model with such a form. Empirical studies have shown that an asset's log-return distribution is non-Gaussian. It is characterised by heavy tails and high peaks (leptokurtic). There is also empirical evidence and economic arguments that suggest that equity returns and implied volatility are negatively correlated (also termed 'the leverage effect').

These reasons has been cited as evidence for non-constant volatility.

Stochastic volatility models were first introduced by Hull and White (1987), Scott (1987) and Wiggins (1987) to overcome the drawbacks of the Black and Scholes (1973) and Merton (1973) model. So it seems plausible to model volatility as a stochastic process. The method gives more parameters to fit, hence popular for calibration purposes.

These are systems of bi-variate SDEs. We continue to assume that S satisfies GBM

$$dS = \mu S dt + \sigma S dW_1,$$

but we further assume that volatility σ satisfies an arbitrary SDE

$$d\sigma = p(S, \sigma, t)dt + q(S, \sigma, t)dW_2.$$

Here both drift and diffusion are arbitrary, with $q(S, \sigma, t)$ being volatility of the volatility (vol of vol).

The two increments dW_1 and dW_2 have a correlation of ρ

$$\mathbb{E}^{\mathbb{P}}\left[dW_1 dW_2\right] = \rho dt.$$

Here $\mathbb{P}$ represents the physical measure. The choice of functions $p(S, \sigma, t)$ and $q(S, \sigma, t)$ is crucial to the evolution of the volatility, and thus to the pricing of derivatives.

The value of an option with stochastic volatility is a function of three variables, $V(S, \sigma, t)$.

Let's do the general theory first and then think about specific forms for p and q.

The pricing equation

The new stochastic quantity that we are modelling, the volatility, is not a traded asset. So as with the spot rate we cannot hold volatility. Thus, when volatility is stochastic we are faced with the problem of having a source of randomness that cannot be easily hedged away.

Because we have two sources of randomness we must hedge our option with two other contracts, one being the underlying asset as usual, but now we also need another option to hedge the volatility risk.

We therefore must set up a portfolio containing one option, with value denoted by $V(S, \sigma, t)$, a quantity $-\Delta$ of the asset and a quantity $-\Delta_1$ of another option with value $V_1(S, \sigma, t)$.

We have

$$\Pi = V - \Delta S - \Delta_1 V_1$$

The change in this portfolio in a time dt is given by

$$
\begin{aligned}
d\Pi \;=\; & \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \tfrac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} \right) dt \\
& - \Delta_1 \left(\frac{\partial V_1}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V_1}{\partial S \partial \sigma} + \tfrac{1}{2} q^2 \frac{\partial^2 V_1}{\partial \sigma^2} \right) dt \\
& + \left(\frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S} - \Delta \right) dS \\
& + \left(\frac{\partial V}{\partial \sigma} - \Delta_1 \frac{\partial V_1}{\partial \sigma} \right) d\sigma.
\end{aligned}
$$

where a higher dimensional form of Itô has been used on functions of S, σ and t.

To eliminate all randomness from the portfolio we must choose

$$
\frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S} - \Delta = 0,
$$

to eliminate the dS terms, which are the sources of randomness, and

$$\frac{\partial V}{\partial \sigma} - \Delta_1 \frac{\partial V_1}{\partial \sigma} = 0,$$

to get rid off $d\sigma$ terms.

Therefore our choice of delta terms to make the portfolio risk free become

$$\Delta_1 = \frac{\frac{\partial V}{\partial \sigma}}{\frac{\partial V_1}{\partial \sigma}}$$

and

$$\Delta = \frac{\partial V}{\partial S} - \frac{\frac{\partial V}{\partial \sigma}}{\frac{\partial V_1}{\partial \sigma}} \frac{\partial V_1}{\partial S}.$$

This leaves us with

$$
\begin{aligned}
d\Pi &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2}\right) dt \\
&\quad -\Delta_1 \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V_1}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V_1}{\partial \sigma^2}\right) dt \\
&= r\Pi dt = r\left(V - \Delta S - \Delta_1 V_1\right) dt,
\end{aligned}
$$

where we have used arbitrage arguments to set the return on the portfolio equal to the risk-free rate.

As it stands this is one equation in the two unknowns V and V_1.

This contrasts with the earlier Black–Scholes case with one equation in the one unknown - but presents the same type of problem when deriving the bond pricing equation.

Collecting all V terms on the left-hand side and all V_1 terms on the right-hand side we find that

$$\frac{\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma q S\frac{\partial^2 V}{\partial S\partial\sigma}+\frac{1}{2}q^2\frac{\partial^2 V}{\partial\sigma^2}+rS\frac{\partial V}{\partial S}-rV}{\frac{\partial V}{\partial\sigma}}$$

$$=\frac{\frac{\partial V_1}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V_1}{\partial S^2}+\rho\sigma q S\frac{\partial^2 V_1}{\partial S\partial\sigma}+\frac{1}{2}q^2\frac{\partial^2 V_1}{\partial\sigma^2}+rS\frac{\partial V_1}{\partial S}-rV_1}{\frac{\partial V_1}{\partial\sigma}}$$

We are lucky that the left-hand side is a functional of V but not V_1 and the right-hand side is a function of V_1 but not V.

Therefore both sides can only be functions of the independent variables, S, σ and t. So set both sides equal to

$$f\left(S,\sigma,t\right).$$

Thus we have

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma S q\frac{\partial^2 V}{\partial S\partial\sigma}+\frac{1}{2}q^2\frac{\partial^2 V}{\partial\sigma^2}+rS\frac{\partial V}{\partial S}-rV=-\left(p-\lambda q\right)\frac{\partial V}{\partial\sigma},$$

for some function $\lambda(S, \sigma, t)$.

Reordering this equation, we usually write

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma S q\frac{\partial^2 V}{\partial S\partial\sigma}+\frac{1}{2}q^2\frac{\partial^2 V}{\partial\sigma^2}+rS\frac{\partial V}{\partial S}+(p-\lambda q)\frac{\partial V}{\partial\sigma}-rV=0.$$

The function $\lambda(S, \sigma, t)$ is called the *market price of* (*volatility*) *risk*.

The market price of volatility risk

If we can solve the pricing equation on the previous slide then we have found the value of the option, and the hedge ratios.

But note that we find two hedge ratios, $\frac{\partial V}{\partial S}$ and $\frac{\partial V}{\partial \sigma}$.

• We have two hedge ratios because we have two sources of randomness that we must hedge away.

Because one of the modelled quantities, the volatility, is not traded we find that the pricing equation contains a market price of risk term.

What does this term mean?

Let's see what happens if we only hedge to remove the stock risk.

Suppose we hold one of the option with value V , and satisfying the pricing equation, delta hedged with the underlying asset only i.e. we have

$$\Pi = V - \Delta S.$$

The change in this portfolio value is

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} \right) dt \\ &\quad + \left(\frac{\partial V}{\partial S} - \Delta \right) dS + \frac{\partial V}{\partial \sigma} d\sigma. \end{aligned}$$

Because we are delta hedging the coefficient of dS is zero, leaving

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma q S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} \right) dt + \frac{\partial V}{\partial \sigma} d\sigma.$$

Now from the pricing PDE we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma S q \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2} = -rS\frac{\partial V}{\partial S} - (p - \lambda q) \frac{\partial V}{\partial \sigma} + rV.$$

We find that

$$d\Pi - r\Pi dt =$$

$$\begin{aligned} & \left(-rS\frac{\partial V}{\partial S} - (p - \lambda q) \frac{\partial V}{\partial \sigma} + rV \right) dt + \tfrac{\partial V}{\partial \sigma} d\sigma - r\left(V - \frac{\partial V}{\partial S} S \right) dt \\ = \quad & -(p - \lambda q) \frac{\partial V}{\partial \sigma} dt + \frac{\partial V}{\partial \sigma} (p dt + q dW_2) \end{aligned}$$

Now simplifying this last term gives

$$\lambda q \frac{\partial V}{\partial \sigma} dt + \frac{\partial V}{\partial \sigma} q dW_2$$

Observe that for every unit of volatility risk, represented by dW_2, there are λ units of extra return, represented by dt. Hence the name 'market price of risk.'

The return on this partially hedged portfolio in excess of the risk-free return is

$$q\frac{\partial V}{\partial \sigma}(\lambda dt + dW_2)$$

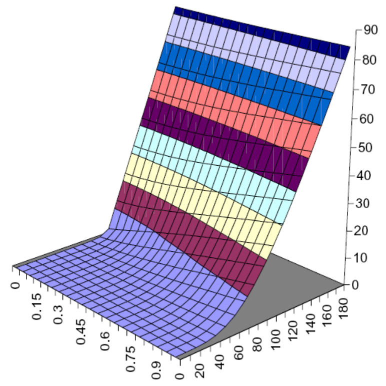
Returning to the pricing equation

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma S q\frac{\partial^2 V}{\partial S\partial\sigma}+\frac{1}{2}q^2\frac{\partial^2 V}{\partial\sigma^2}+rS\frac{\partial V}{\partial S}+(p-\lambda q)\frac{\partial V}{\partial\sigma}-rV=0.$$

The quantity $p-\lambda q$ is called the risk-neutral drift rate of the volatility.

Recall that the risk-neutral drift of the underlying asset is r and not μ.

When it comes to pricing derivatives, it is the risk-neutral drift that matters and not the real drift, whether it is the drift of the asset or of the volatility.



stochastic volatility: an example for particular value of $p, q, \ \rho$

The option price is shown for varying stock and volatility.

This is a snapshot at a fixed point in time. We notice it looks like a typical European option.

Note for larger σ we have greater curvature (i.e. larger diffusion).

In addition to the model for GBM we have SDE for volatility, where $v = \sigma^2$.

The equations look nicer expressed in terms of the variance (important quantity).

Many volatility models are of the form

$$dv = A(v)\,dt + cv^{\gamma} dW_2,$$

for some value γ and mean reverting drift $A(v)$, where the variance $v = \sigma^2$.

In the presence of a continuous dividend yield, the earlier PDE can be written as

$$\frac{\partial V}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2}+\rho\sqrt{v}Sq\frac{\partial^2 V}{\partial S\partial\sigma}+\frac{1}{2}q^2\frac{\partial^2 V}{\partial\sigma^2}+(r-D)\,S\frac{\partial V}{\partial S}+cv^{\gamma}\frac{\partial V}{\partial\sigma}-rV=0.$$

Popular Models

GARCH - diffusion: Generalized Autoregressive Conditional Heteroskedasticity.

A commonly used popular discrete time model in econometrics. It can be turned into the continuous time limit of many GARCH-processes by the following SDE

$$dv = (a - bv)\,dt + cv dW_2.$$

The popularity lies in the ease with which the positive valued parameters a, b and c can be estimated, hence allowing the pricing of options.

There is a mean reverting drift with speed b and mean rate a/b. c is the *vol of vol* which sets the scale for the random nature of volatility. In the case $a = b = 0$, the GARCH diffusion model reduces to the log-normal process without drift in the Hull and White (1987) model.

Given $v(0) > 0$,

$$v(t) = v(0)\, e^{-\left(b+\frac{1}{2}c^2\right)t+cW_t} + a\int_0^t e^{-\left(b+\frac{1}{2}c^2\right)(s-t)+c(W_t-W_s)}ds.$$

Heston:

He takes

$$dv = \gamma\,(m - v)\,dt + \xi\sqrt{v}dW_2$$

Also called the square root model because of the term in the diffusion - which gives a closed form solution, hence the popularity. This means it is easier to calibrate. Heston takes $\rho \neq 0$. In this model the the process is proportional to the square root of its level.

Must be comfortable with Complex Analysis Methods, as it requires the use of Fourier Transforms.

3/2 model:

Pronounced the *three-halves model* because of the $3/2$ power in the diffusion.

$$dv = v\,(a - bv)\,dt + cv^{3/2}dW_2$$

Again mean reverting - the existence of a Closed-form solution makes it a popular model. But note the mean reverting and volatility parameters are now stochastic.

See Alan Lewis' book on *Option valuation under stochastic volatility*, where he presents analytical solutions for this model.

This does a supposedly better job of calibrating than Heston, although Heston is more popular.

Hull & White

$$\frac{dv}{v} = \mu dt + \xi dW_2$$

No mean reversion. They take $\rho = 0$. Note the lognormal structure hence it can grow indefinitely.

Stein & Stein

$$d\sigma = -\theta (\sigma - m) dt + \xi dW_2$$

The model allows mean-reversion but σ can become negative. They take $\rho = 0$.

Ornstein-Uhlenbeck process:

This model is expressed in terms of the log of the variance.

Writing $y = \log x$

$$dy = (a - by)\, dt + c dW_2$$

Already seen the O-U-P interest rate model (looks very similar). This model matches data well.

This has a steady state distribution which is lognormal.

A closed form solution does not exist so requires numerical treatment.

The Heston Model

In his model the variance follows a mean-reverting square root process, first used by Cox-Ingersoll-Ross in 1985 to capture the dynamics of the spot rate where the mean reversion rate $m > 0$, and the speed $\gamma > 0$. The vol of vol $\xi > 0$.

$$dS = (\mu - D)\, S dt + \sqrt{v} S dW_1,$$

$$dv = \gamma\,(m - v)\, dt + \xi \sqrt{v} dW_2$$

Solving problems numerically is simple (FDM or Monte Carlo). In the case of MC take the stock drift as $(r - D)$.

In order for the mean-reverting square root dynamics for the variance to remain positive, there are a number of analytical results available. In particular is the Feller condition, i.e. if

$$\gamma m \geq \xi^2$$

then the variance process cannot become negative. If this condition is not satisfied then the origin is attainable and strongly rejecting so that the variance process may attain zero in finite time, without spending time at this point.

In deriving the PDE for Heston, he takes

$$f(S,v,t) = -\gamma(m-v) + \Lambda(S,v,t)\,\xi\sqrt{v}$$

giving the following pricing PDE for the option $U(S,v,t)$

$$\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\xi vS\frac{\partial^2 U}{\partial S\partial v} + \frac{1}{2}\xi^2 v\frac{\partial^2 U}{\partial v^2} +$$

$$\left(\gamma(m-v) - \Lambda(S,v,t)\,\sigma\sqrt{v}\right)\frac{\partial U}{\partial v} + rS\frac{\partial U}{\partial S} - rU = 0$$

Consider the pricing of a call option subject to the final condition $C(S,v,T) = \max(S_T - E, 0)$ with the following boundary conditions

$$\begin{aligned}
C(0,v,t) &= 0 \\
\lim_{S\to\infty}\frac{\partial C}{\partial S}(S,v,t) &= 1 \\
\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \gamma m\frac{\partial C}{\partial v} &= rC \\
\lim_{v\to\infty} C(S,v,t) &= S
\end{aligned}$$

How to use Heston

There are four parameters in the model, speed of mean reversion, level of mean reversion, volatility of volatility, correlation. That is $b,\ a/b, c, \rho$ respectively.

And also potentially a market price of volatility risk parameter.

The main four parameters can be chosen by matching data or by calibration.

Experience suggests that calibrated parameters are very unstable, and often unreasonable. (For example, the best fit to market prices might result in a correlation of exactly -1.)

Consider calibrating. Suppose

Parameters	Today	Next week
$a =$	14	-487
$b =$	29	$\sqrt{-12}$
$c =$	0.01	1000
$\rho =$	-0.6	-3

so a somewhat exaggerated sarcastic example, but nevertheless shows that when recalibrating it hasn't worked - the parameters which were fixed are totally different!

The Heston model with jumps

Increasingly popular are stochastic volatility with jumps models (SVJ).

Jump models require a parameter to measure probability of a jump (a Poisson process) and a distribution for the jumps.

Also have SVJJ - jumps in the stock and jumps in the volatility.

Pros: More parameters allow better fitting. The jump component of the model has most impact over short time scales.

Therefore use longer-dated options to fit the stochastic volatility parameters and the shorter-dated options to fit the jump component.

Cons: Mathematics slightly more complicated (and again we must work in the transform domain).

Hedging is even harder when the underlying stock process is potentially discontinuous.

People also looking at stochastic correlation models.

Whilst there is no such thing as the perfect model, you can always pretend to have the ideal one by introducing more parameters.

More parameters means more quantities to calibrate.

Case Study: The REGARCH model and its diffusion limit

REGARCH = Range-based Exponential GARCH

Although a closed form solution does not exist, a fairly nice model which looks very plausible.

'Range-based' refers to the use of the daily range, defined as the difference between the highest and lowest log asset price recorded throughout the day, rather than simply the closing prices.

'Exponential' refers to modelling the logarithm of the variance.

Diffusion limits exist for all GARCH-type of processes. That is, they can be expressed in continuous time using stochastic differential equations.

(This is achieved via ‘moment matching.’ The statistical properties of the discrete-time GARCH processes are recreated with the continuous-time SDEs.)

REGARCH is another econometrics discrete time model, but can be turned into the following three-factor model:

$$
\begin{aligned}
dS &= \mu S dt + \sigma_1 S dW_0 && \text{(a)} \\
d(\log \sigma_1) &= a_1 (\log \sigma_2 - \log \sigma_1) dt + b_1 dW_1 && \text{(b)} \\
d(\log \sigma_2) &= a_2 (c_2 - \log \sigma_2) dt + b_2 dW_2. && \text{(c)}
\end{aligned}
$$

This is a three-factor (higher dimensional) model, with two volatilities.

σ_1 represents the actual (short term) volatility of the asset returns, which is stochastic.

The σ_2 represents the (longer term) level to which σ_1 reverts, and is itself stochastic.

What are the dynamics of this model?

We have the usual GBM random walk for the stock given by (a) which has actual volatility σ_1. This is short term.

Note from (b) that the **log** of σ_1 mean reverts to $\log \sigma_2$. So rather than σ_2 being constant, it is fluctuating and σ_1 is chasing that.

From (c) we observe that σ_2 reverts to a constant mean c_2.

For pricing options we must replace these SDEs with the risk-neutral versions:

$$\begin{aligned} dS &= rSdt + \sigma_1 S dW_0 \\ d(\log \sigma_1) &= a_1 (\log \sigma_2 - \log \sigma_1 - \lambda b_1/a_1)\, dt + b_1 dW_1 \\ d(\log \sigma_2) &= a_2 (c_2 - \log \sigma_2 - \lambda b_2/a_2)\, dt + b_2 dW_2. \end{aligned}$$

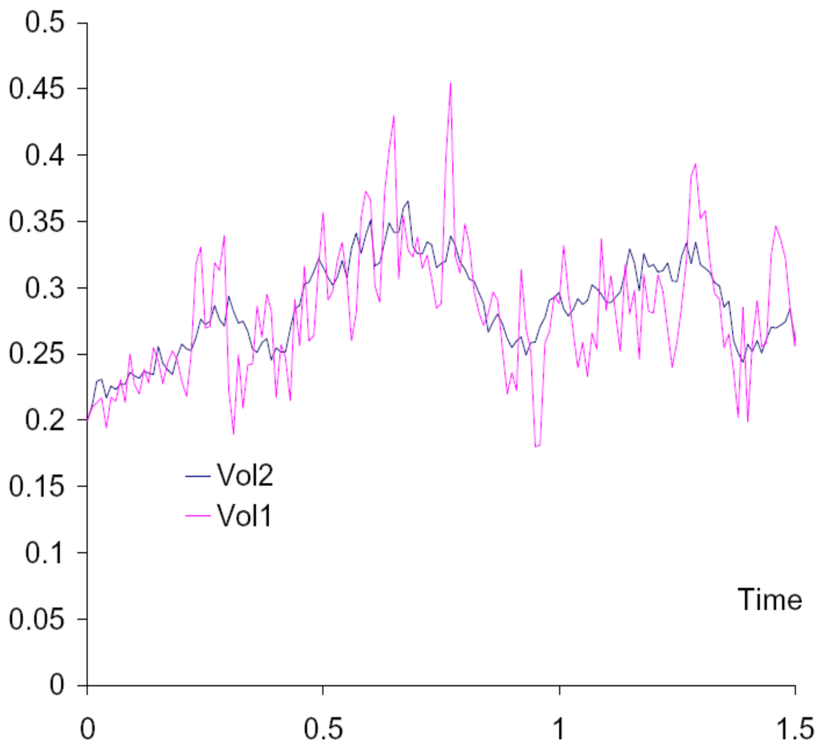
The λ terms represent the market prices of risk.

The a and b coefficients and the correlations between the three sources of randomness give this system seven parameters.

These parameters are related to the parameters of the original REGARCH model and can be estimated from asset data.

Example: let's look at some parameters.

$a_1 = 56.6$, $b_1 = 1.138$, $a_2 = 2.82$, $b_2 = 0.388$, $c_2 = -1.25$. ($\lambda_1 = \lambda_2 = 0$.)



σ_1 is very rapidly mean reverting to the level of σ_2. This is a 'short-term' volatility. The time scale for mean reversion is about one week.

σ_2, the 'long-term volatility, reverts more slowly, over a period of about six months.

a_1 is the speed for $\log \sigma_1$. The bigger this is, the faster the reversion to $\log \sigma_2$.

$a_1 dt$ is non-dimensional therefore a_1 has dimensions of 1/time $\Longrightarrow 1/a_1$ has dimensions of time.

So a time scale of approximately 1 week, for $\log \sigma_1$ to mean revert.

$b_1 \gg b_2$, volatility of $d(\log \sigma_1)$ much greater than $d(\log \sigma_2)-$ which it is chasing.

$1/a_2$ is approximately 0.5 years, so it takes $\log \sigma_2$ 6 months to revert back to its (long term) mean.

How do you solve these equations?

- Monte Carlo: The solutions of the two-factor partial differential equations you get with stochastic volatility models can still be interpreted as 'the present value of the expected payoff.' So all you have to do is to simulate the relevant random walks for the underlying and volatility (risk neutral) many times, calculate the average payoff and then present value it.

- Finite differences: The partial differential equations can still be solved by finite differences but you will need to work with a three-dimensional grid.

Pros and cons of stochastic volatility models

Pros:

- Evidence (and common sense) suggests that volatility changes, possibly randomly

- More parameters means that calibration can be ‘better’

Cons:

- As with any incomplete-market model hedging is only possible if you believe in the market price of (volatility) risk

Jump Diffusion Models

Introduction

Some of the ideal assumptions of the classic Black-Scholes framework continue to be addressed in this chapter. Brownian motion has been the canonical random process driving asset price models. A basic property of Brownian motion is that it has continuous sample paths. It follows a Gaussian distribution whose thin exponentially decaying tails make large changes in the underlying less probable than actually observed in the market. This fact that it often fits financial data very poorly is widely acknowledged.

An observation when pricing derivatives is the fact that the underlying ocassionally jumps. The use of processes with jumps have become increasingly popular. Their detailed practice has already been seen in modelling credit events (jumps to default) although given the extreme market moves of 2008, they may well become more common in other asset classes as well. It is important to note how ever that large moves are very rare occurrences.

Jump processes have discontinuous sample paths and, therefore, they allow for large sudden moves in the underlying price process. They can also capture skewness and excess kurtosis in price returns.

So far, our model for asset prices has been

$$dS = A(S,t)\,dt + B(S,t)\,dW$$

with the usual properties $\mathbb{E}[dW] = 0$ and $\mathbb{V}[dW] = dt$. As $dt \longrightarrow 0$, it gives a continuous realisation of the random walk for S.

We cannot always rely on *complete markets*.

In complete markets we can hedge derivatives with the underlying in such a way as to eliminate risk.

Most Quant Finance books deal with the Black Scholes model or Binomial Model which are examples of Complete Markets Models.

If markets are complete then derivatives are redundant because we can replicate them using the underlying

$$\Pi = V - \Delta S \Rightarrow V = \Pi + \Delta S$$

The whole purpose of derivatives is that markets are incomplete!

In fixed income we model the spot rate r which is random, but we can't trade it so can't use it to get rid off risk.

This is the idea underpinning derivatives theory, i.e. dynamic/delta hedging.

The presence of jumps means we cannot hedge continuously because we need a continuous process with which to hedge. We will use the Poisson Process for modelling jumps.

Discontinuous in practice often refers to a move which is significantly large so that we can't hedge our way through it.

The foundations of Mathematical Finance are based upon the idea of continuous hedging - so if stock is not continuous - then we cannot hedge.

Equally if we can't hedge quickly, the asset path may as well be discontinuous.

To model a discontinuous realization we need a *Poisson process* or *jump process*.

This gives the building block for the *jump-diffusion model* for an asset price.

One simple way to represent jumps is using this Poisson process.

This is an example of a *counting process*.

A random process $\{q(t)\}_{t\geq 0}$ is called a counting process if $q(t)$ represents the total number of occurrences that have taken place in the interval $[0, t]$. $q(t)$ is an integer value quantity with $q(0) = 0$.

We will start to build up the theory using this process in conjunction with Brownian motion (Itô calculus).

The Poisson process is also used in Credit Modelling.

Usual notation to use is $dq(t)$ with the following definition

$$dq = \begin{cases} 1 & \text{with } \mathbb{P} = \lambda dt \\ 0 & \text{with } \mathbb{P} = 1 - \lambda dt \end{cases}$$

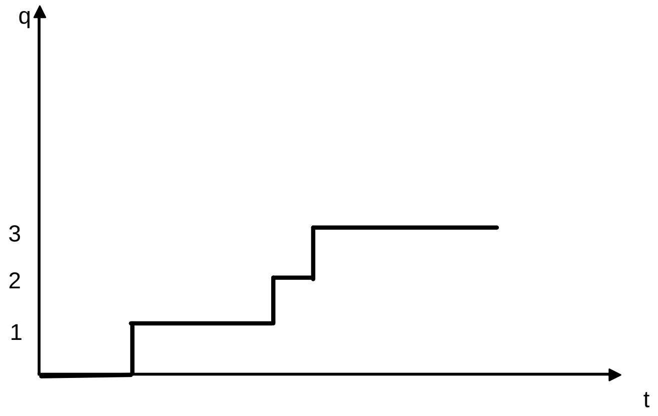
Thus in each interval either $dq(t)$ stays fixed, or it increases by 1.

So we think of dq as a Poisson counter.

The parameter λ is called the *intensity* of the Poisson process. The larger it is, the greater likelihood there is of a jump.

The scaling of the probability of a jump with the size of the time step dt is crucial in making the resulting process 'meaningful,' i.e. there being a finite chance of a jump occurring in a finite time, with q remaining finite. This is a classic Poisson process.

q is the integral of dq



This is a typical representation of a counting process.

Properties of the Poisson process

A counting process $q(t)$ is called a **Poisson process** with non-negative *intensity* (or mean arrival rate) λ of an event in a time interval dt if

$$q(0) = 0,$$

$q(t)$ has independent increments.

The number of jumps in a finite time horizon t has a Poisson distribution with parameter λt. Then

$$\mathbb{P}[q(t) = n] = \exp(-\lambda t)\frac{(\lambda t)^n}{n!}; \quad n = 0, 1, 2, ...$$

$$\begin{aligned} \mathbb{E}[q(t)] &= \lambda t \\ \mathbb{V}[q(t)] &= \lambda t \end{aligned}$$

We can propose the following model for S

$$dS = c(S,t)\, dq$$

where $c(S,t)$ itself can be unpredictable so that both the size and timing of the jumps is random.

However a more sensible and realistic model is to use a jump diffusion version of Geometric Brownian Motion, i.e.

$$dS = a(S,t)\, dt + b(S,t)\, dW + c(S,t)\, dq.$$

So a model that follows GBM most of the time and every now and again, there is a jump. Since we are interested in the stock return it makes sense to write

$$\frac{dS}{S} = \mu dt + \sigma dW + (J-1)\, dq$$

which is the *jump diffusion* model.

The two basic building blocks of every jump-diffusion model are the Geometric Brownian motion (the diffusion part) and the Poisson process (the jump component).

We assume that the Brownian motion and Poisson process are uncorrelated.

So there are two sources of risk: $dW,\ dq$.

J is a random number with property $\mathbb{E}\left[J\right]=1$.

Most of the time $dq=0$, so we have diffusion.

Occasionally at random intervals there is a contribution from dq when it takes value one and then there is a jump because it is big.

When $dq=1$

$$S+dS\longrightarrow S+\left(J-1\right)S=JS$$

So S goes immediately goes to the value JS. Hence

$$dS = JS - S = (J - 1)\, S$$

As an example if $J = 0.9$ then $S \longrightarrow 0.9S$, i.e. a 10% fall.

So J is a factor which determines what happens to assets when there is a jump.

$$\begin{aligned} J &< 1 \Longrightarrow \text{fall in value} \\ J &> 1 \Longrightarrow \text{rise in asset} \\ J &= 0 \Longrightarrow \text{stock falls to zero} \end{aligned}$$

J can be random with its own distribution. There are a number of parameters here: μ, σ, λ, J.

J could follow any distribution with its own set of parameters, so plenty of scope for calibration/data fitting.

A convenient form for J is lognormal, so

$$\begin{aligned}\mathbb{E}[\log J] &= e^{\frac{1}{2}\sigma_J^2} \\ \mathbb{V}[\log J] &= \sigma_J^2\end{aligned}$$

The advantage is that closed form solutions are possible (see Merton's argument later). Consider the following example.

In anticipation of using Itô calculus, we need a framework for extending to Poisson.

When $dq = 0$ we know.

$$d(\log S) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW$$

If $dq = 1$, $S \longrightarrow JS$, so in addition to the expression above we have $\log S \longrightarrow \log(JS) = \log S + \log J$.

So when $dq = 1$ we have $d(\log S) =$ usual Itô terms plus $\log J$, which can be written compactly as

$$d(\log S) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW + (\log J)\, dq,$$

when $dq = 1$, we "switch on" the jumps.

Hedging options when there are jumps

Now start building up a theory of derivatives in the presence of jumps.

Usual construction of a portfolio by holding the option and $-\Delta$ of the asset (in the usual way):

$$\Pi = V(S,t) - \Delta S.$$

Across a time step dt the change is

$$\begin{aligned} d\Pi \;&=\; \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \left(\frac{\partial V}{\partial S} - \Delta\right)(\mu S dt + \sigma S dW) \\ &\quad + \left(V(JS,t) - V(S,t) - \Delta (J-1) S\right) dq. \end{aligned}$$

Again, this is a jump-diffusion version of Itô.

How do we get the second line in the expression above?

Before jump: $V(S,t) - \Delta S$.

After jump: $V(JS,t) - \Delta JS$, because S has jumped to JS.

So jump in portfolio is $V(JS,t) - \Delta JS - V(S,t) + \Delta S = (V(JS,t) - V(S,t)) + \Delta S(1-J)$.

That is, the jump in the portfolio equals the jump in option price and jump in stock.

The risk sources here are dW, dq and potentially J. Yet we only have one delta term with which to hedge.

Hence Incomplete Markets.

If there is no jump at time t so that $dq = 0$, then we could have chosen $\Delta = \partial V / \partial S$ to eliminate the risk.

If there is a jump and $dq = 1$ then the portfolio changes in value by an $O(1)$ amount, that cannot be hedged away.

In that case perhaps we should choose Δ to minimize the variance of $d\Pi$.

This presents us with a dilemma.

We don't know whether to hedge the small(ish) diffusive changes in the underlying which are always present, or the large moves which happen rarely.

Let us pursue both of these possibilities.

Hedging the diffusion

If we choose

$$\Delta = \frac{\partial V}{\partial S}$$

we are following a Black-Scholes type of strategy, hedging away the diffusive movements.

The change in the portfolio value is then

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \\ & \left(V(JS,t) - V(S,t) - (J-1)\, S \frac{\partial V}{\partial S} \right) dq. \end{aligned}$$

The portfolio now evolves in a deterministic fashion, except that every so often there is a non-deterministic jump in its value.

Merton's Approach

One classic approach is Merton's 1976 model who argued that if the jump component of the asset price process is uncorrelated with the market as a whole, then the risk in the discontinuity should not be priced into the option as it is diversifiable (there is no excess reward for it).

In other words non systematic risk is **not rewarded** on average, so $\mathbb{E}[d\Pi] = r\Pi dt$.

Recall since there is uncertainty present there should be some compensation for taking risk.

Merton argued that if the dW is eliminated then there should be no compensation for the dq component.

In other words, we can take expectations of this expression and set that value equal to the risk-free return from the portfolio

$$\mathbb{E}\left[d\Pi\right] = r\Pi dt,$$

where

$$\mathbb{E}\left[()\, dq\right] = \mathbb{E}\left[()|\text{ jump occurs } dq\right].\lambda dt + \mathbb{E}\left[()|\text{ no jump } dq\right].(1 - \lambda dt)$$

$$\begin{aligned} & \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV + \\ & \lambda\mathbb{E}\left[V\left(JS,t\right) - V\left(S,t\right)\right] - \lambda\frac{\partial V}{\partial S}S\mathbb{E}\left[J-1\right] \\ = \; & 0, \end{aligned}$$

where $\mathbb{E}\left[\cdot\right]$ is the expectation taken over the jump size J, which can also be written

$$\mathbb{E}\left[X\right] = \int xp\left(J\right) dJ,$$

where $p(J)$ is the pdf for the jump size.

The equation is of the form

$$L_{BS}(V) + \lambda \int_0^\infty V(JS,t) - V(S,t)\, p(J)\, dJ = 0,$$

i.e. a PIDE (partial integro-differential equation).

If J is known then just drop the $\mathbb{E}[\cdot]$. So the original Black Scholes terms plus a new part.

As an example

$$\mathbb{E}[V(JS,t)] = \int_0^\infty V(JS,t)\, p(J)\, dJ.$$

V now depends on all stocks when there are jumps between 0 and ∞.

Aside: Are we working with real or risk-neutral expectations?

At the moment real (Merton's argument), but later we'll look at the concept of risk neutrality when there are jumps.

This is a pricing equation for an option when there are jumps in the underlying.

The important point to note about this equation that makes it different from others we have derived is its non-local nature.

• That is, the equation links together option values at distant S values, instead of just containing local derivatives.

Naturally, the value of an option here and now depends on the prices to which it can instantaneously jump.

There is a simple closed-form solution of this equation in a special case.

That special case if when J is lognormally distributed. i.e. the logarithm of J is Normally distributed.

To solve put

$$\begin{aligned} \log \frac{S}{E} &= x \\ J &= e^{-y} \end{aligned}$$

which gives

$$\text{PDE} + \int_{-\infty}^{\infty} V\left(x-y,t\right) f\left(y,t\right) dy = 0.$$

Solve this using a Fourier Transform in x.

If the logarithm of J is Normally distributed with standard deviation σ and 'mean' $k = \mathbb{E}[J-1]$ then the price of a European non-path-dependent option can be written as

$$\sum_{n=0}^{\infty} \underbrace{e^{-\lambda'(T-t)} \frac{\left(\lambda'\left(T-t\right)\right)^n}{n!}}_{=\text{ Probability of getting } n \text{ jumps}} V_{\mathsf{BS}}\left(S,t;\sigma_n,r_n\right),$$

where

$$\lambda' = \lambda\left(1+k\right), \ \sigma_n^2 = \sigma^2 + \frac{n\sigma'^2}{T-t} \text{ and } r_n = r - \lambda k + \frac{n\log\left(1+k\right)}{T-t},$$

and V_{BS} is the Black-Scholes formula for the option value in the absence of jumps.

So it is a Black-Scholes pricing formula for $0, 1, 2, \ldots\ldots$ jumps.

This formula can be interpreted as the sum of individual Black–Scholes values each of which assumes that there have been n jumps, and they are weighted according to the probability that there will have been n jumps before expiry.

There are 3 parameters we could calibrate.

Method 2: Hedging the jumps

In the above we hedged the diffusive element of the random walk for the underlying.

Another possibility is to hedge both the diffusion and jumps 'together.'

For example, we could choose Δ to minimize the variance of the hedged portfolio, after all, this is ultimately what hedging is about.

So let's return to the $d\Pi$ equation.

The change in the value of the portfolio with an arbitrary Δ is, to leading order (ignoring higher order terms),

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial S} - \Delta\right)(\mu S dt + \sigma S dW) \\ &\quad + \left(-\Delta (J-1) S + V(JS, t) - V(S, t)\right) dq + \ldots \end{aligned}$$

Square this term and take expectations, then subtract off the square of $\mathbb{E}[d\Pi]$.

The variance in this change, which is a measure of the risk in the portfolio, is

$$\begin{aligned}\mathbb{V}[d\Pi] &= \left(\frac{\partial V}{\partial S}-\Delta\right)^2\sigma^2 S^2 dt + \\ &\quad +\lambda\mathbb{E}\left[\left(-\Delta\left(J-1\right)S+V\left(JS,t\right)-V\left(S,t\right)\right)^2\right]dt+...\end{aligned}$$

which is to leading order (2 terms) - a diffusive part and a jump component.

Putting $\Delta=\frac{\partial V}{\partial S}$ only eliminates the diffusive part, not the jumps.

This is minimized by the choice

$$\Delta=\frac{\lambda\mathbb{E}\left[\left(J-1\right)\left(V\left(JS,t\right)-V\left(S,t\right)\right)\right]+\sigma^2 S\frac{\partial V}{\partial S}}{\lambda S\mathbb{E}\left[\left(J-1\right)^2\right]+\sigma^2 S}.$$

This is obtained as follows:

$$\frac{\partial}{\partial\Delta}\left(\mathbb{V}[d\Pi]\right)=0$$

which gives Δ for the minimum. This choice of Δ gives the least variance.

When $\lambda = 0$, the expression collapses to $\Delta = \frac{\partial V}{\partial S}$, the usual Black-Scholes hedge.

If we value the options as a pure discounted real expectation under this best-hedge strategy then we find that

$$\begin{aligned} & \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S\frac{\partial V}{\partial S}\left(\mu - \frac{\sigma^2}{d}(\mu + \lambda k - r)\right) - rV \\ & + \lambda \mathbb{E}\left[V(JS,t) - V(S,t)\left(1 - \frac{J-1}{d}(\mu + \lambda k - r)\right)\right] \\ = \; & 0 \end{aligned}$$

where

$$d = \lambda \mathbb{E}\left[(J-1)^2\right] + \sigma^2.$$

Note how this choice brings in μ.

Here we are not getting rid of dW, but minimizing risk so we are still left with $\mu-$ so need to measure this term.

Often happens when moving away from Complete Markets.

What about risk neutrality?

Does the concept of risk neutrality have any role when there are jumps?

N.B. The above uses ‘real’ expectations.

Let’s see a special case, known jump size, J.

So start with

$$dS = \mu S dt + \sigma S dW + (J - 1) S dq.$$

but with J given.

There are now two sources of risk (there were three before), dW and dq. (No J risk)

Let's see if we can eliminate risk by having two hedging instruments, the stock and another option.

(You will recall this from the stochastic interest rate lecture and will see it again in stochastic volatility modelling.)

Construct a portfolio of the option and $-\Delta$ of the asset, and $-\Delta_1$ of another option, V_1 :

$$\Pi = V(S,t) - \Delta S - \Delta_1 V_1$$

Doing Itô's Lemma gives the change in the portfolio as

$$\begin{aligned} d\Pi &= \left(\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}-\Delta_1\left(\frac{\partial V_1}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V_1}{\partial S^2}\right)\right)dt \\ &\quad+\left(\frac{\partial V}{\partial S}-\Delta-\Delta_1\frac{\partial V_1}{\partial S}\right)(\mu S dt+\sigma S dW) \end{aligned}$$

$$+\left(V\left(JS,t\right)-V\left(S,t\right)-\Delta\left(J-1\right)S-\Delta_1\left(V_1\left(JS,t\right)-V_1\left(S,t\right)\right)\right)dq$$

To eliminate dW terms choose

$$\frac{\partial V}{\partial S}-\Delta-\Delta_1\frac{\partial V_1}{\partial S}=0$$

and to eliminate dq terms choose

$$V\left(JS,t\right)-V\left(S,t\right)-\Delta\left(J-1\right)S-\Delta_1\left(V_1\left(JS,t\right)-V_1\left(S,t\right)\right)=0.$$

We obtain the messy expressions

$$\Delta_1=\frac{\left(J-1\right)S\frac{\partial V}{\partial S}-V\left(JS,t\right)+V\left(S,t\right)}{\left(J-1\right)S\frac{\partial V_1}{\partial S}-V_1\left(JS,t\right)+V_1\left(S,t\right)}$$

and $\Delta =$

$$\frac{\partial V}{\partial S} - \frac{\partial V_1}{\partial S} \times \frac{(J-1)\, S\frac{\partial V}{\partial S} - V\,(JS,t) + V\,(S,t)}{(J-1)\, S\frac{\partial V_1}{\partial S} - V_1\,(JS,t) + V_1\,(S,t)}.$$

All risk is now eliminated, so set return on portfolio equal to risk-free rate.

End result:

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV}{(J-1)\, S\frac{\partial V}{\partial S} - V\,(JS,t) + V\,(S,t)}$$

$$= \quad \frac{\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + rS\frac{\partial V_1}{\partial S} - rV_1}{(J-1)\, S\frac{\partial V_1}{\partial S} - V_1\,(JS,t) + V_1\,(S,t)}.$$

Same functional form on each side.

So one equation in two unknowns . LHS is independent of V_1, RHS is independent of V.

$$\frac{\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV}{(J-1)\,S\frac{\partial V}{\partial S}-V\,(JS,t)+V\,(S,t)}$$

$$\begin{aligned} &= \text{universal quantity, independent of option type} \\ &= -\lambda'. \end{aligned}$$

Final equation is

$$\begin{aligned} &\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV \\ &+\lambda'\left(V\,(JS,t)-V\,(S,t)-(J-1)\,S\frac{\partial V}{\partial S}\right) \\ =\ & 0. \end{aligned}$$

This is the same equation as before but with risk-neutral λ' instead of real λ.

How do you solve these equations?

Monte Carlo: The solutions of the partial integro-differential equations you get with jump-diffusion models can still be interpreted as 'the present value of the expected payoff.' So all you have to do is to simulate the relevant random walk for the underlying (risk neutral) many times, calculate the average payoff and then present value it. As always!

Finite differences: The partial integro-differential equations can still be solved by finite differences but the method will no longer be 'local' since the governing equation contains integrations over all asset prices.

Pros and cons of jump-diffusion models

Pros:

- Evidence (and common sense) suggests that assets can jump in value
- Jump models can capture extreme implied volatility skews (such as seen close to expiration)
- More parameters means that calibration can be ‘better’

Cons:

- The foundations are a bit shaky (can’t hedge, hedge diffusion or minimize risk, real versus risk neutral)

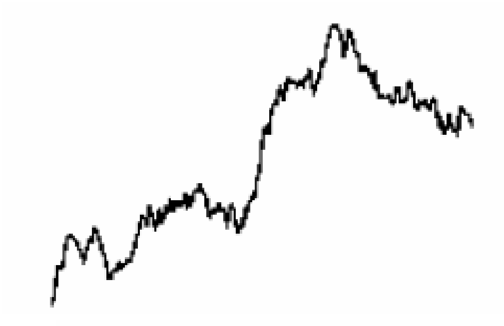
Fractional Brownian Motion

Fractional Brownian motion written fBm with Hurst exponent $H \in (0, 1)$ is a centred Gaussian process $(X_t)_{t \geq 0}$ such that

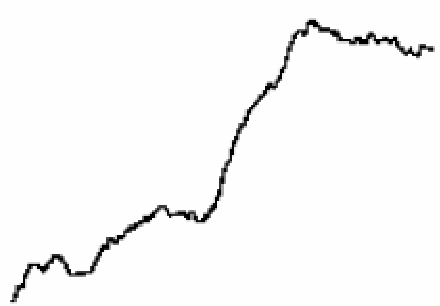
$$\forall s, t \geq 0 : \mathbb{E}\left[X_s X_t\right] = \frac{1}{2}\left[s^{2H} + t^{2H} - |s-t|^{2H}\right]$$

The fBm is a generalisation of a standard Brownian motion that allows its increments to be correlated with

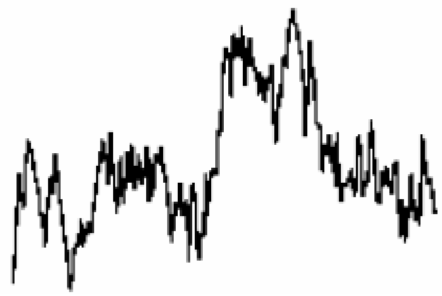
- $H = \frac{1}{2}$ Brownian motion ($H = 0.5$ below)

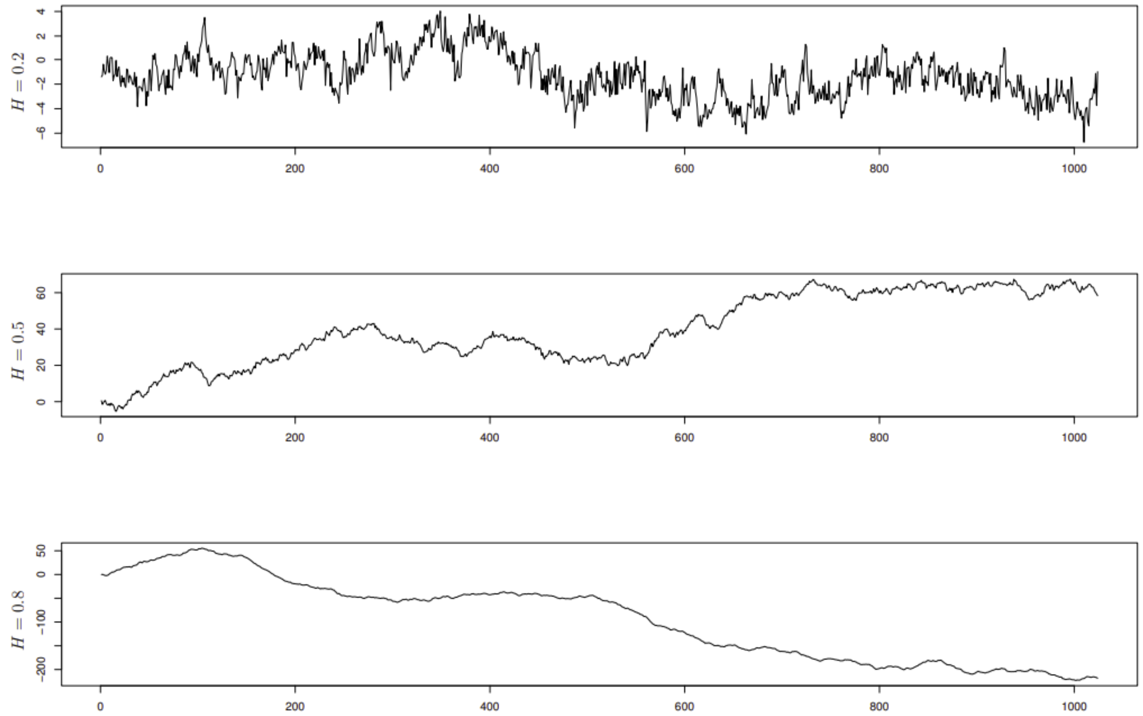


- $H > \frac{1}{2}$ increments are positively correlated so nice smooth paths ($H = 0.8$ below)



- $H < \frac{1}{2}$ increments are negatively correlated giving something very rough ($H = 0.2$ below)





Cheridito (2001) and Dieker (2004) respectively.

Motivated by empirical studies, several authors have studies financial models driven by the fBm

- Fractional stochastic vol models

- Fractional Black-Scholes model

Fractional stochastic volatility models (see Comte and Renauld (1998) or Comte, Coutin and Renault (2003)) explain better the long-time behaviour of the implied volatility. The fBm (and then the volatility) are not Markovian, and this becomes a strong difficulty to study and to put these models into practice (the usual techniques assume the Markov property).

Models driven by the fBm

Consider the fractional Black-Scholes model for a bond (B_t) and a stock (S_t) $(H > 1/2)$:

Bond dynamics: $dB_t = rB_t dt, \qquad B_0 = 1, \qquad 0 \leq t \leq T$ A stock S which follows the SDE:

$$dS_t = \mu S_t dt + \sigma S_t d^{\Diamond} X_t^H,\ S_0 = S > 0,\ 0 \leq t \leq T$$

where μ, σ are constants and $\sigma \neq 0$. Here we introduce the Wick-calculus universe based on the Wick product, which is denoted by the symbol $\Diamond$.

From the market definition we have the following explicit solutions

$$B_t = e^{rt}$$

and

$$S_t = S_0 \exp\left(\sigma X_t^H + \mu t - \frac{1}{2}\sigma^2 t^{2H}\right)$$

A portfolio (trading strategy) is a pair of progressively-measurable processes $\phi = (\beta_t, \gamma_t)$ where β_t and γ_t in turn represent the

amount invested in the bank account and the shares of stocks. The value process of such a portfolio is:

$$V_t(\phi) = \beta_t B_t + \gamma_t S_t$$

Wick-self financing: The value process is assumed to follow

$$V_t(\phi) = \beta_t B_t + \gamma_t \Diamond S_t$$

and the prtfolio $\phi = (\beta_t, \gamma_t)$ is called self-financing if for all $t \in [0, T]$

$$\begin{aligned} dV_t(\phi) &= \beta_t dB_t + \gamma_t d^{\Diamond} S_t \\ &= \beta_t r B_t dt + \mu \gamma_t S_t dt + \sigma \gamma_t S_t d^{\Diamond} X_t^H \end{aligned}$$

Applying a fractional form of Girsanov with a risk-neutral measure $\mathbb{Q}_H$

$$dV_t(\phi) = r V_t(\phi)\, dt + \sigma \gamma_t S_t d^{\Diamond} \widetilde{X}_t^H$$

where

$$\frac{d\mathbb{Q}_H}{d\mathbb{P}} = \exp\left(-\int_0^T q_t d^\diamondsuit X_t^H - \frac{1}{2}|q|_\phi^2\right)$$

and

$$\int_0^T q_t \phi(s,t)\, dt = \frac{\mu - r}{\sigma} \text{ holds for all } s \in [0, T].$$

Note: $\widetilde{X}_t^H = X_t^H + \frac{\mu - r}{\sigma}$ is a fBM under the measure $\mathbb{Q}_H$.

Under $\mathbb{Q}_H$ the stock price follows

$$S_t = S_0 \exp\left(\sigma \widetilde{X}_t^H + rt - \frac{1}{2}\sigma^2 t^{2H}\right).$$

Ultimately the price of a European call option at initial time $t = 0$ was obtained:

$$V_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}_H}[V_T(\phi)]$$