Fractional Brownian Motion and Applications to Option Pricing

by

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Abstract

Fractional Brownian Motion (fBM) is a Gaussian process which generalizes the standard Brownian motion by relaxing its independent increments property. Motivated by empirical observations in financial markets, numerous authors studied the use of FBMs in modelling market dynamics. However, since FBMs are not semimartingales (unless Brownian motion), classical Itô calculus no longer applies in deriving Black Scholes formulas for option pricing. Therefore, various other mathematical techniques are used to derive the fractional Black Scholes formulas.

This dissertation provides a review on mathematical techniques that are used to derive the fractional Black Scholes formulas and conducted work-out exercises in option pricing. We have mentioned the fundamental mathematical theories with regard to fractional Brownian Motion, fractional Itô integrals as well as their conditional distributions. Moreover, we investigated and compared the derivation of several pricing models including Wick fractional Black Scholes model, mixed fractional Brownian model as well as the risk preference one. Finally, we estimated the H parameter and implied volatility of UK and Chinese stock markets, and compared the performance of fractional pricing models with classical Black Scholes model. We came to a conclusion that, in terms of the estimation efficiency, the risk preference model outperforms the other two in UK stock market, whereas the Wick fractional model gives the smallest error in Chinese stock market. Another finding is that the estimation efficiency does depend on moneyness of the option.

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Contents

Chapter 1

INTRODUCTION

Since Black and Scholes first introduced classic option pricing model in [1], numerous studies have been carried upon this model and admitted the assumption that stock prices are log-normally distributed and the increments are independent. Despite its wide implementation in derivative pricing and risk management, more and more empirical researches have revealed some incompatibilities with respect to real market data.

On one hand, a major limitation of the Black Scholes model is due to the non-normal distribution of asset returns. According to Fama (1965)[2], stock returns follow a distribution with fat tails. In order to capture the excess kurtosis feature, more realistic distributions were considered such as the Pareto stable distribution and the generalized Hyperbolic distribution. Furthermore, for mapping the jumps in price processes, more realistic price dynamics were suggested in vast of researches including the Lévy process and variance gamma process among others. However, independent and stationary increments are still a common hypothesis in these models.

On the other hand, another major problem causing model unaccuracy is the long range dependence of asset returns. Long range dependence is a characteristic that captures the decay speed of auto-correlated time series data. For a long range dependent process, the auto-correlation function decays in a lower rate than an exponential decay, especially a power-like decay. Thus there is persistent dependence even at long lags. Among the first to have considered the fractal behavior in asset prices and the existence of long range dependence were Mandelbrot (1963)[3] and Mandelbrot and Taylor (1967)[4].

Since then, more studies further detected the presence of long range de-

pendence in different financial markets. Greene and Fielitz (1977)[5] found that many stocks in NYSE(New York Stock Exchange) market have long range dependence based on daily stock returns. Lo and Mac Kinlay (1988)[6] also rejected the random walk model by empirical evidence from daily returns in CRSP (Center for Research in Security prices) data base. Huang and Yang (1995)[7] showed the evidence of long range dependence in UK market indices using modifies R/S (Rescaled-range Statistic), while rejected persistent statistical dependence in nine Asian Stock markets and US market. Beben and Ohowski (2001)[8] investigated that emerging market indexes showed more significant persistence behavior than established market indexes through Hurst exponent and de-trended fluctuations analysis. Jamdee and Los (2005)[9] and (2007)[10] demonstrated the impact of long range dependence on European option values via time-dependent volatility.

In order to factor long range dependence in financial markets, the standard Brownian Motion was generalized to a new process with Hurst parameter H called fractional Brownian Motion (fBM). fBM was originally proposed by Mandelbrot and van Ness (1968) in [11]. The self-similarity and long range dependence of fBM makes it a good candidate in factoring the empirical behavior of price processes.

As proved by Rogers (1997)[12] , fBM is not a semi-martingale, therefore traditional Itô calculus no more applies. In fact, the general result from Delbaen and Schachermayer (1994)[13] has suggested the existence of free lunch with vanishing risk in a market where the asset price is not a semi-martingale, given the traditional definition of arbitrage and self-financing as well as admissibility. Moreover, it has been shown by a number of studies that fBM allows outright arbitrage in many ways. See, for example, Rogers (1997)[12], Salopek (1998)[14], Shiryaev (1998)[15] and Cheridito (2003)[16].

In result, how to extend the Itô integration theory had been concerned by many scholars. One of these methods is fractional path-wise integration theory established by Lin (1995)[17]. However Rogers (1997)[12] pointed out that pricing options under this theory will allow arbitrage opportunities because of the non-zero expectation of fBM.

A milestone is the Wick product and Wick Itô Skorohod integral introduced by Duncan et al.(2000)[18], under which the fBM has a zero expectation. The parallels to Itô calculus and option pricing theory was demonstrated by this theory so that the fractional market can be set up. Yet, there is still a limitation in original version of Wick-based approach which is that Hurst parameter is larger than $\frac{1}{2}$.

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Hu and Øksendal (2003)[19] proposed that, in order to consider non semi-martingale models, one needs to modify the underlying definition of the portfolio value. A Wick self-financing condition was imposed on the portfolio and then it could be shown that the market was complete and arbitrage-free. Finally, the fractional Black Scholes formula for the initial option price was derived in this paper. Elliot and van der Hoek (2003)[20] derived similar results as Hu and Øksendal.

Necula (2002)[21] further generalized a fractional Black-Scholes formula to price option from any arbitrary time before maturity. Using the results of the quasi-conditional expectation, which is a conditional expectation with respect to the filtration generated by fractional Brownian Motion. This study gave a fractional risk-neutral valuation theorem for option pricing.

On one hand,Björkand Hult (2005)[22] criticized the work of both Hu and Øksendal (2000) and Necula(2002), stating that the self-financing strategies used in the above researches were economically meaningless.

On the other hand , although the approaches of Hu and Øksendal and Necula are perfectly correct and accurate in mathematics, when trading in continuous time, the Wick Itó integration theory still admits weak arbitrage.

Several Researchers further proposed suitable restrictions to exclude arbitrage. For instance, Cheridito(2001)[16] proposed to modify the stock price process by introducing a mixed fractional Brownian Motion in order to get a semi-martingale. Guasoni et al (2010)[23] introduced transaction costs into the fractional Black-Scholes model and showed that proportional transaction cost eliminate arbitrage opportunities. Cheridito (2003)[16] suggested that arbitrage can be excluded when restricting the model to a discontinuous trading strategy.

Rostek (2009)[24] derived a formula for pricing fractional European options using conditional expectation in a risk preference based pricing approach by assuming a minimal time between trading strategies. This model also assumed that traders are risk neutral and possess limited knowledge of the past. Rostek and Schöbel (2010)[25] derived the same model by assuming that participants have a constant relative risk aversion and trade in discrete time. Under assumed investor objectives a stochastic discount factor is introduced to satisfy an equilibrium condition.

The goal of this dissertation is to understand the mathematical application of fractional Brownian Motion in option pricing, the empirical applicability of these models and to get a deeper insight into how these models perform compared to the classical Black-Scholes one.

This dissertation is arranged in the following way. In chapter 2, we will introduce the fractional Brownian Motion (fBM) and its basic properties. Then we will present the Wick-calculus based world and the conditional distributions as basis of the pricing models. In chapter 3, we will cover 3 different fractional Black Scholes models, including the Wick fractional Black Scholes, mixed Brownian model as well as the risk preference one. Finally in chapter 4, a work-out exercise is given based on UK and Chinese stock markets. Where the H parameters are calculated and the prices are estimated and compared with the real market data.

Chapter 2

fBM and Wick-calculus World

In this chapter we will first define fractional Brownian Motion (fBM) and then briefly review the basic properties of fBM, like self-similarity, long-range dependence and non semi-martingale. Next, we will introduce the Wick-calculus based world, including the factional Itô theorem and fractional Girsanov theorem. Finally, we will look at the conditional distribution of fBM based on unlimited and limited history.

2.1   Definition and Basic Properties

Definition 2.1.1. *Let the Hurst parameter $H \in (0,1)$, a fBM with H is a continuous Gaussian Process $\left\{B_t^H, t \in \mathbb{R}\right\}$ satisfies:*

- $B_0^H = 0$;

- $\mathbb{E}[B_t^H] = 0$;

- $\mathbb{E}[B_t^H B_s^H] = \dfrac{1}{2}(\mid t \mid^{2H} + \mid s \mid^{2H} - \mid t - s \mid^{2H}), t, s \in \mathbb{R}.$

   Notice that B_t^H is the standard Brownian Motion B_t if $H = \frac{1}{2}$. This justifies that fBM is a generalization of standard Brownian Motion.

   Also there is a main difference between fBM and the standard Brownian Motion. The increments of standard Brownian Motion follow independent normal distribution. However it's not the case for fBM. More precisely, the covariance of fBM is positive when $H > \frac{1}{2}$ and negative when $H < \frac{1}{2}$.

Following the definition, we will briefly review some properties of fBM.

1. Stationary increments:

From the definition of fBM, it can be immediately deduced that the increments of fBM in any interval $[s, t]$, that is $B_t^H - B_s^H$, has the same covariance function as that of B_{t-s}^H. According to the definition, the process is a centered Gaussian, and therefore B_{t-s}^H is normally distributed with mean 0 and variance $|t - s|^{2H}$. Thus the increment over time interval $[s, t]$ only depends on $|t - s|$. For this reason it is said to have stationary increments.

2. Self-similarity:

Definition 2.1.2. *Let $H \in (0, 1)$, a stochastic process $X : [0, +\infty) \times \Omega \to \mathbb{R}$ is called H self-similar, if for all $a > 0$, the finite d-dimensional distribution of $\{X_{at}, t \in \mathbb{R}\}$ which is denoted by $(X_{at_1}, \cdots, X_{at_d})$ are identical to $(a^H X_{t_1}, \cdots, a^H X_{t_d})$.*

In other words, a process is self-similar if its quantitative properties present symmetry under finer levels of spatial and time scales. This property can be easily derived from the fact that the covariance function of fBM is homogeneous of order $2H$.

3. Long range dependence:

Definition 2.1.3. *A stationary random sequence $\{W_n\}_{n \in \mathbb{N}}$ exhibits long range dependence if the auto-covariance function $\rho_n = \mathbb{E}[W_0 W_n]$ fulfils*

$$\sum_{i=1}^{\infty} \rho_n = \infty$$

This phenomenon relates to the decay rate of statistical dependence which arises during time series analysis. The sequence is considered to have long range dependence if its dependence decays in an order lower than an exponential decay, typically a power-like one.

Definition 2.1.4. *Set $W_n^H = B_n^H - B_{n-1}^H, n \geq 1$, then $\{W_n^H\}_{n \in \mathbb{N}}$ is called the fractional Gaussian noise with Hurst parameter H.*

It is a stationary sequence normally distributed with unit variance and the covariance function is given by

$$\rho_n^H = \frac{1}{2} \left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H} \right) \quad \sim H(2H-1) n^{2H-2} \to 0, \quad as \quad n \to \infty.$$

The independent increments of the standard Brownian Motion B_t indicates there is no long range dependence for $H = \frac{1}{2}$. When $H < \frac{1}{2}$, $\rho_n^H < 0$ for n large enough and $\sum_{i=1}^{\infty} \rho_n^H < \infty$. Therefore the fractional Gaussian noise has short range dependence property and can be used to model sequences with counter-persistence which means the price is more likely to decrease in the future if it was decreasing, and vice versa.

On the contrary, when $H > \frac{1}{2}$, $\rho_n^H > 0$ for n large enough and $\sum_{i=1}^{\infty} \rho_n^H = \infty$. In this case, the fractional Gaussian noise exhibits an aggregation behaviour which can be used to describe cluster phenomena: it is more likely to keep trend than to break it. For such H, the fBM has the property of long range dependence. Moreover, the closer H is to 1, the greater the degree of long range dependence shows.

4. Non semi-martingale:

Definition 2.1.5. *A stochastic process $X : [0, +\infty) \times \Omega \longrightarrow \mathbb{R}$ is called semi-martingale with respect to filtration $\{\mathcal{F}_t\}$, if it admits a representation that*

$$X_t = X_0 + M_t + A_t$$

where M_t is a local martingale starts from zero, A_t is a process of locally bounded variation and X_0 is $\mathcal{F}_0$- measurable.

This definition indicates the property that any semi-martingale has locally bounded quadratic variation. Moreover, if X_t is continuous then M_t and A_t are continuous.

Consider a function $f : [0, T] \to \mathbb{R}$, and a partition of the interval $[0, T]$ $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$. For $p \geq 1$ and we denote

$$v_p(f; \pi) = \sum_{k=1}^{n} | f(t_k) - f(t_{k-1})|^p$$

Definition 2.1.6. *f is said to have finite $p - variation$ if the limit*

$$v_p(f) = \lim_{|\pi| \to 0} v_p(f; \pi)$$

exists and is finite.

Rogers [12] obtained the following:

Lemma 2.1.1. *Let $\pi_n = \{\frac{j}{2^n} : j = 0, \cdots, 2^n\}$ be a partition of $[0, T]$. Then*

$$\lim_{n \to \infty} v_p(B^H; \pi_n) \xrightarrow{P} \begin{cases} \infty, & \text{if } pH < 1, \\ \mathbb{E} \mid B_1^H - B_0^H \mid^p, & \text{if } pH = 1, \\ 0, & \text{if } pH > 1. \end{cases}$$

in probability.

If $H < \frac{1}{2}$, for any $p \in \left[2, \frac{1}{H}\right)$, $lim_{n \to \infty} v_p\left(B^H; \pi_n\right) \to \infty$ in probability. Thus the quadratic variation of B^H on any fixed interval must be infinite. Therefore it can not be a semi-martingale.

If $H > \frac{1}{2}$, suppose B^H is a semi-martingale, then the quadratic variation of B^H is zero. Thus by definition of semi-martingale,

$$[M_t, M_t] = \left[B_t^H, B_t^H\right] = 0.$$

Since a continuous local martingale with zero quadratic variation is constant, $M_t = M_0 = 0$ almost surely. It follows that $B_t^H = A_t$ has locally bounded variation, which contradicts to the fact that for any $p \in \left(1, \frac{1}{H}\right)$, $lim_{n \to \infty} v_p\left(B^H; \pi_n\right) \to \infty$ in probability. Thus B^H can not be a semi-martingale.

As we just proved, fBm with Hurst parameter $H \neq \frac{1}{2}$ is not a semi-martingale, therefore traditional Itô integration theory no more applies. Several stochastic integral theories were considered to extend the concept of integration with respect to fBM, among which Wick-calculus was suggested.

2.2 Wick-calculus based world

This section we introduce the Wick-calculus universe based on the Wick product, which is denoted by the symbol $\diamond$. A Wick-Itô integral defined by the Wick product, along with some basic theories will be discussed here.

2.2.1 Wick-Itô integral

Here we briefly summarize a milestone in stochastic integration with respect to the fBM introduced by Duncan et al in [18], Wick-Itô integral. The main basis of this theory is to generalize the ordinary Riemann-Stieltjes sums by

introducing a new multiplicative concept, Wick product which is denoted by $\diamond$.

As a start, we first introduce the Wick integral with a deterministic integrand $f : \mathbb{R} \to \mathbb{R}$. Then a generalised concept of Wick product between two random variables is given subsequently. Finally based on the Wick product, the Wick integral with a stochastic integrand is defined.

Before giving definition of Wick integral with deterministic integrand, we first introduce the fractional kernel $\phi : \mathbb{R}^2 \to \mathbb{R}$ that satisfies

$$\phi(s,t) = H(2H-1) \mid s-t \mid^{2H-2} .$$

Furthermore we define the norm of deterministic function $f : \mathbb{R} \to \mathbb{R}$ as:

$$\mid f \mid_\phi^2 := \int_0^\infty \int_0^\infty f(s)f(t)\phi(s,t)dsdt.$$

Especially when $H = \frac{1}{2}$, the fractional kernel ϕ is nothing but the Dirac-Delta function, that is

$$\mid f \mid_{1/2}^2 := \int_0^\infty f(t)^2 dt.$$

Accordingly, the resulting Hilbert space $L_\phi^2 = \left\{ f : \mathbb{R} \to \mathbb{R}, \mid f \mid_\phi^2 < \infty \right\}$ is equipped with the inner product:

$$< f,g >_\phi := \int_0^\infty \int_0^\infty f(s)g(t)\phi(s,t)dsdt.$$

The definition of Wick integral of deterministic function is given in the following way.

Definition 2.2.1. *Consider a deterministic function $f \in L_\phi^2$ defined on $[0,T]$, and a partition π of the interval $[0,T]$, $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$. Function f can be approximated by a series of simple functions f_n which are constant c_i over the respective partition:*

$$f_n(t) = \sum_i^{n-1} c_i^n \mathbb{1}_{[t_i,t_{i+1})}(t) \xrightarrow{L_\phi^2} f(t)$$

Then the Wick integral of these simple functions can be calculated simply using Riemann-Stieltjes sums

$$\int_0^T f_n(t)d^\diamond B_t^H := \sum_i^{n-1} c_i^n \left(B_{t_{i+1}}^H - B_{t_i}^H \right)$$

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where $d^\diamond$ represents that the integration is conducted in terms of the Wick calculation world.

Finally, the Wick integral of deterministic function is given by a limit of these Riemann-Stieltjes sums above

$$\int_0^T f(t)d^\diamond B_t^H := \lim_{n\to\infty} \int_0^T f_n(t)d^\diamond B_t^H = \lim_{n\to\infty} \sum_i^{n-1} c_i^n \left(B_{t_{i+1}}^H - B_{t_i}^H \right)$$

After determining the Wick integral of deterministic functions, we introduce the Wick exponential $\varepsilon(f)$, which is the basis of Wick-Itô integral.

$$\varepsilon(f) := exp\left(\int_0^\infty f(t)d^\diamond B_t^H - \tfrac{1}{2} \mid f \mid_\phi^2 \right), \quad f \in L_\phi^2,$$

where f denotes a deterministic function in L_ϕ^2. Note that $\varepsilon(f)$ is a log-normally distributed random variable. As according to Gripenberg and Norros[26], $\int_0^\infty f(t)d^\diamond B_t^H$ is normally distributed with zero mean and variance $\mid f \mid_\phi^2$.

Now we shall give the definition of Wick product in terms of Wick exponentials in [18].

Definition 2.2.2. *For $f, g \in L_\phi^2$, the Wick product of $\varepsilon(f)$ and $\varepsilon(g)$ is:*

$$\varepsilon(f) \diamond \varepsilon(g) = \varepsilon(f + g)$$

As shown in [18], the random variables in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ can be approximated by linear combinations of Wick exponentials. The definition of Wick product in terms of Wick exponentials can be extended to define the Wick product of any two random variables in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Base on this, we can generalize the concept of Wick integration with respect to stochastic process, the Wick-Itô integral, by taking limit of the sum of Wick products.

Definition 2.2.3. *[18] Let for all $t \in [0, T], X_t \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, and $\pi = \{0 = t_0, \cdots, t_n = T\}$ be a partition of $[0, T]$. The fractional Wick-Itô integral is given by:*

$$\int_0^T X_s d^\diamond B_s^H = \lim_{|\pi|\to 0} \sum_{i=0}^{n-1} X_{t_i} \diamond [B_{t_{i+1}}^H - B_{t_i}^H]$$

if the limit and the Wick products exist in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

2.2.2 Wick-Itô isometry

Before introducing the Wick-Itô isometry, we first introduce the Malliavin derivative D_t^ϕ. Recall the result in [18] that the fractional Malliavin derivative of a fractional integral with deterministic integrand $f : [0, T] \rightarrow \mathbb{R}$, $\int_0^T f(s) d^\diamond B_s^H$ equals:

$$D_t^\phi \left(\int_0^T f(s) d^\diamond B_s^H \right) = \int_0^T \phi(s, t) f(s) ds.$$

For example, the fractional Malliavin derivative of B_t^H in time t is

$$D_t^\phi(B_t^H) = H \mid -t \mid^{2H-1} .$$

For the special case $H = \frac{1}{2}$, we get the classical Malliavin derivative

$$D_t^\phi(B_t^H) = \frac{1}{2}.$$

Now, we shall give the definition of the family $\mathcal{L}_\phi(0, T)$, the set of stochastic processes where Wick-Itô isometry applies.

Definition 2.2.4. *Given the stochastic process $X_t : [0, T] \times \Omega \rightarrow \mathbb{R}$, we say $X_t \in \mathcal{L}_\phi(0, T)$ if:*

(i) $\mathbb{E}(\mid X_t \mid_\phi^2) < +\infty$

(ii) X_t *is ϕ-Malliavin differentiable*

(iii) *the trace of $D_s^\phi X_t : s, t \in [0, T]$ exists*

(iv) $\mathbb{E}(\int_0^T \int_0^T |D_s^\phi X_t|^2 ds dt) < +\infty$

(v) *for each partition $|\pi| \rightarrow 0$:*

$$\sum_{i=0}^{n-1} \mathbb{E} \left(\int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |D_s^\phi X_{t_i} D_t^\phi X_{t_j} - D_s^\phi X_t D_t^\phi X_s| ds dt \right) \rightarrow 0$$

and
$$\mathbb{E} \mid X^\pi - X \mid_\phi^2 \rightarrow 0$$

where $X_t^\pi = X_{t_i}$ for $t_i \leq t < t_{i+1}$.

Finally, using the results above, the moment properties of the fractional Wick-Itô integrals are obtained in [18].

Theorem 2.2.1 (Wick-Itô isometry). *[18] For $X_t \in \mathcal{L}_\phi(0, T)$, the first two moments of Wick-Itô integral $\int_0^T X_t d^\diamond B_t^H$ are:*

$$\mathbb{E}\left[\int_0^T X_t d^\diamond B_t^H \right] = 0,$$

$$\mathbb{E}\left[\left(\int_0^T X_t d^\diamond B_t^H \right)^2 \right] = \mathbb{E}\left[\left(\int_0^T D_t^\phi X_t dt \right)^2 + \mid X \mid_\phi^2 \right],$$

Note that the latter equation is called fractional Itô isometry. Since the first moment of fractional integral is zero, there is no systemic bias over all possible price paths. This convenient feature enables Wick integrals to be widely used in financial industry.

2.2.3 Fractional Girsanov formula

The classical Girsanov theorem describes the change in dynamics of classical Itô integrals under the change of measure. Here we state the fractional version of Girsanov theorem obtained in Norros and Valkeila[27].

Consider fBM X_t with drift μ under the measure $\mathbb{P}$, that is

$$X_t = B_t^H + \mu t \quad \text{under } \mathbb{P}.$$

Then there exists a measure $\mathbb{P}^\mu$ so that

$$X_t = B_t^H \quad \text{under } \mathbb{P}^\mu.$$

The measure $\mathbb{P}^\mu$ is given by the Radon-Nikodym derivative

$$\frac{d\mathbb{P}^\mu}{d\mathbb{P}} = exp(-\mu q_t - \frac{1}{2}\mu^2 < q, q >_t),$$

where

$$q_t = \int_0^t c_1 s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H} d^\diamond B_s^H,$$

and

$$c_1 = \left[2H\Gamma(\tfrac{2}{3} - H)\Gamma(H + \tfrac{1}{2}) \right]^{-1}.$$

The process q_t is a martingale with independent increments, zero mean and variance function

$$E(q_t^2) = c_2^2 t^{2-2H}$$

where

$$c_2 = \frac{c_H}{2H\sqrt{2-2H}}.$$

$$c_H = \left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}\right)^{1/2}.$$

Based on the results above, we can then rewrite the Radon-Nikodym derivative by

$$\frac{d\mathbb{P}^\mu}{d\mathbb{P}} = exp(-\mu q_t - \frac{1}{2}\mu^2 c_2^2 t^{2-2H}).$$

We can conclude directly from this expression that when $H = \frac{1}{2}$, the representation is consistent with the classical Radon-Nikodym derivative.

2.2.4 Wick-Itô formula

Now we extend Itô formula to the case of Wick-Itô integral.[18]

Theorem 2.2.2 (Wick-Itô formula). *For $X_t \in \mathcal{L}_\phi(0,T)$ and $Y_t = \int_0^t X_s d^\diamond B_s^H$. If:*

(i) there exists $\alpha > 1 - H$ such that

$$\mathbb{E}[|X_u - X_v|^2] \leq C|u-v|^{2\alpha}$$

where $|u-v| \leq \delta$ for some $\delta > 0$ and

$$\lim_{0 \leq u,v \leq t; |u-v| \to 0} \mathbb{E}[|D_u^\phi(X_u - X_v)|^2] = 0;$$

(ii) function $F(t,x) \in C^{1,2}([0,T] \times \mathbb{R} \to \mathbb{R})$ has bounded derivatives;

(iii) $\mathbb{E}[\int_0^T |X_s D_s^\phi Y_s| ds] < +\infty$ and $F'(s, Y_s)X_s \in \mathcal{L}_\phi(0,T)$.

Then

$$\begin{aligned} F(t, Y_t) = & F(0,0) + \int_0^t \frac{\partial F}{\partial s}(s, Y_s)ds + \int_0^t \frac{\partial F}{\partial x}(s, Y_s)X_s d^\diamond B_s^H \\ & + \int_0^t \frac{\partial^2 F}{\partial x^2}(s, Y_s)X_s D_s^\phi Y_s ds \qquad a.s. \end{aligned} \tag{2.1}$$

where

$$D_s^\phi Y_s = \int_0^s D_s^\phi X_u d^\diamond B_u^H + \int_0^s X_u \phi(s,u)du \quad a.s.$$

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2.3 Conditional distribution of fBM

In order to investigate the conditional equilibrium distribution of asset prices in the market driven by fBM, we first look at the the conditional fBM framework, which is the theoretical basis of the risk preference model in next chapter. Here we first introduce the prediction formula based on infinite and finite knowledge of the past, then give the conditional Wick-Itô formula.

2.3.1 Prediction formula based on infinite knowledge of the past

First, we consider the distribution of fBM based on all the historical information. In this section we regard the conditional distribution of $\hat{B}_{T,t}^{H} :=$ $\mathbb{E}[B_{T}^{H}|\mathcal{F}_{t}^{H}]$, where $T > t$ and $\mathcal{F}_{t}^{H} = \sigma\left(\{B_{s}^{H}\} : s \in (-\infty, t)\right)$ is the σ-field generated by $\left(\{B_{s}^{H}\} : s \in (-\infty, t)\right)$.

At first step, we introduce the following theorem given by Gripenberg and Norros [26] which give a formula for conditional expectation of fBM.

Theorem 2.3.1. *Let $H \in (\frac{1}{2}, 1)$. For each $T > t > 0$, the conditional expectation of B_T^H can be represented by:*

$$\hat{B}_{T,t}^{H} = B_t^H + \int_{-\infty}^{t} g(T - t, s - t) d^{\diamond} B_s^H$$

where

$$
\begin{aligned}
g(v, \omega) &= \frac{sin(\pi(H - \frac{1}{2}))}{\pi}(-\omega)^{-H+\frac{1}{2}} \int_{0}^{v} \frac{x^{H-\frac{1}{2}}}{x - \omega} dx \\
&= \frac{sin(\pi(H - \frac{1}{2}))}{\pi}\left[\frac{1}{H - \frac{1}{2}}\left(\frac{-\omega}{v}\right)^{-H+\frac{1}{2}} - \beta\frac{v}{v - \omega}(H - \frac{1}{2}, \frac{3}{2} - H)\right]
\end{aligned}
$$

(2.2)

and β denotes the incomplete Beta function.

Note that the conditional projection on the future value can be interpreted as adjusting the current value B_t^H with an additional integration accounting for the observed path of the process. Consequently, when $H = \frac{1}{2}$,

the correction part vanishes, and the best estimation of future value becomes the current value, which is consistent with the martingale property of classical Brownian Motion.

For the prediction purpose, we take a further look at the expected conditional distribution of B_T^H.

Theorem 2.3.2. *[28] The conditional distribution of B_T^H based on the infinite observation on the realized path is $N(B_t^H + \hat{\mu}_{T,t}\hat{\sigma}_{T,t}^2)$, where:*

$$\hat{\mu}_{T,t} := \mathbb{E}[B_T^H|\mathcal{F}_t^H] = \int_{-\infty}^t g(T-t, s-t)d^\diamond B_s^H$$

$$\hat{\sigma}_{T,t}^2 := Var[B_T^H|\mathcal{F}_t^H] = \mathbb{E}[(B_T^H - \hat{B}_{T,t}^H)^2|\mathcal{F}_t^H] = \rho_H(T-r)^{2H}$$

with

$$\rho_H = \frac{sin(\pi(H-\frac{1}{2}))\Gamma(\frac{3}{2}-H)^2}{\pi(H-\frac{1}{2})\Gamma(2-2H)}$$

One thing worth mentioning is that, as H tends to $\dfrac{1}{2}$, $g(T-t, s-t)$ and ρ_H both tends to 1, therefore $\hat{\mu}_{T,t}$ equals zero, consequently the conditional distribution becomes $N(B_t^H, T-t)$. Hence, again this result coincides with the classical Brownian Motion case.

2.3.2 Prediction formula based on Part of knowledge of the past

For practical purposes it is desirable to make predictions that are based on only part of the history and to go back only to a finite point in time $t-a$. That is, we restrict ourselves to a finite observation interval of length a. Thus in the following theorem, we consider this kind of conditional expectation and give the prediction formula.

Similar with the preceding subsection, we regard the conditional distribution of $\hat{B}_{T,t,a}^H := \mathbb{E}[B_T^H|\mathcal{F}_{t,a}^H]$, where $T > t$ and $\mathcal{F}_{t,a}^H = \sigma(\{B_s^H\} : s \in (t-a, t))$ is the σ-field generated by $(\{B_s^H\} : s \in (t-a, t))$.

Theorem 2.3.3. *Let $H \in (\frac{1}{2}, 1)$. For all $T, t, a > 0$, the conditional expectation of B_T^H can be represented by:*

$$\hat{B}_{T,t,a}^H := \int_{t-a}^t g_a(T-t, s-t)d^\diamond B_s^H$$

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where

$$g_a(v, \omega) = \frac{sin(\pi(H - \frac{1}{2}))}{\pi}(-\omega)^{-H+\frac{1}{2}}(a + \omega)^{-H+\frac{1}{2}}\int_0^v \frac{x^{H-\frac{1}{2}}(x + a)^{H-\frac{1}{2}}}{x - \omega}dx$$

Again, we can obtain the expected conditional distribution of B_T^H, this time based on limited knowledge about the past, which is expressed in the restriction on the filtration.

Theorem 2.3.4. *For all $t - a \le s \le t$. The conditional distribution of B_T^H based on the observation on the realized path back to $t - a$ in the past until t is normal, where:*

$$\hat{\mu}_{T,t,a} := \mathbb{E}[B_T^H|\mathcal{F}_{t,a}^H] = \int_{t-a}^t g_a(T - t, s - t)d^\diamond B_s^H$$

$$\hat{\sigma}_{T,t,a}^2 := Var[B_T^H|\mathcal{F}_{t,a}^H] = \mathbb{E}[(B_T^H - \hat{B}_{T,t,a}^H)^2|\mathcal{F}_{t,a}^H] = \rho_{H,a}(T - r)^{2H}$$

with

$$\rho_{H,a} = 1 - H\int_0^{\frac{a}{T-t}} g_{\frac{a}{T-t}}(1, -s)((1 + s)^{2H-1} - s^{2H-1})ds$$

Note that according to [26], when the observation interval extend to as large as the projection interval, $\rho_{H,a}$ and $\hat{\sigma}_{T,t,a}^2$ tend to ρ_H and $\hat{\sigma}_{T,t}^2$ respectively. Thus in terms of estimating the conditional variance, limited observation on the historical value seems sufficient. Whereas the influence of additionally historical information on the conditional mean still counts, yet is decreasing. Hence, [24] concluded that it is reasonable to evaluate the expected conditional distribution based on limited historical observation of the past.

2.3.3 Conditional Wick-Itô formula

Base on the preceding results, we can obtain the conditional Wick-Itô formula by extending the Wick-Itô formula.

Theorem 2.3.5. *Let $F(t, x) \in C^{1,2}([0, T] \times \mathbb{R} \to \mathbb{R})$ be a function with bounded derivatives, then under some regularity conditions in [28],*

$$\begin{aligned} F(t, \hat{B}_t^H) =& F(0, 0) + \int_0^t \frac{\partial F}{\partial s}(s, \hat{B}_s^H)ds + \int_0^t \frac{\partial F}{\partial x}(s, \hat{B}_s^H)\hat{B}_s^H d^\diamond \hat{B}_s^H \\ &+ \rho_H H\int_0^t s^{2H-1}\frac{\partial^2 F}{\partial x^2}(s, \hat{B}_s^H)(\hat{B}_s^H)^2 ds \quad a.s. \end{aligned}$$

(2.3)

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with

$$\rho_H = \frac{sin(\pi(H - \frac{1}{2}))\Gamma(\frac{3}{2} - H)^2}{\pi(H - \frac{1}{2})\Gamma(2 - 2H)}$$

2.3.4 Quasi-Conditional Expectation

As shown above, the conditional expectation of the fBM process correlates with the past. This property imposes restriction on applying the traditional pricing theory to the fractional market, as the assets driven by fBM are not martingales in the traditional market setting. For this reason a system of quasi-conditional expectation $\tilde{E}[\cdot]$ was introduced in [29], under which the Wick-Itô integrals are quasi-martingales. This theory forms the fundamental part of the Wick fractional Black Scholes model in next chapter, as it allows us to price the asset values using quasi-conditional expectations instead of the original one.

Before defining the quasi-conditional expectation, we introduce the iterated integral defined in [18] as follows:

$$I_n(f_n) := \int_{\mathbb{R}^n} f_n d^\diamond B_s^{H \otimes n} := n! \int_{s_1 < \cdots < s_n} f_n(s_1, \cdots, s_n) d^\diamond B_{s_1}^H \cdots d^\diamond B_{s_n}^H.$$

where deterministic function $f_n \in L_\phi^2(\mathbb{R}^n)$. For $n = 0$ and $f_n = f_0$ constant we set $I_0(f_0) = f_0$.

Definition 2.3.1. *[19] Let $X = \sum_{n=0}^{\infty} I_n(f_n)$, the quasi-conditional expectation of F with respect to $\mathcal{F}_t^H = \sigma(\{B_s^H\} : s \in (0, t))$ is defined as*

$$\tilde{E}_t\left[X \mid \mathcal{F}_t^H\right] = \sum_{n=0}^{\infty} \int_{[0,t]^n} f_n d^\diamond (B_s^H)^{\otimes n}.$$

Then quasi-martingale can be defined accordingly.

Definition 2.3.2. *For a stochastic process M_t adapted to filtration $\mathcal{F}_t^H = \sigma(\{B_s^H\} : s \in (0, t))$, we call M_t a quasi-martingale if for $0 \leqslant s < t$,*

$$\tilde{E}_s\left[M_t \mid \mathcal{F}_s^H\right] = M_s$$

Based on the definition above, [21] concluded that fBM B_t^H, Wick-Itô integrals including the Wick exponentials are quasi-martingales.

Finally, we give the fractional Clark-Ocone theorem.

Theorem 2.3.6. *Consider a stochastic process $X \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_T^H$-measurable, then $\tilde{E}_t[D_t^\phi X \mid \mathcal{F}_t^H] \in L_\phi^{1,2}$ and*

$$X = \mathbb{E}[X] + \int_0^T \tilde{E}_t[D_t^\phi X \mid \mathcal{F}_t^H] d^\diamond B_t^H, \qquad (2.4)$$

where D_t^ϕ donates the Malliavin derivatives.

Chapter 3

fBM Black-Scholes Models

From the preceding chapter we know that stochastic processes driven by fBM are not semi-martingales. In fact, this directly leads to the existence of free lunch in fractional Black-Scholes market, since the serial correlation enables prediction of future based on information up to date. However, several studies have excluded arbitrage by imposing different restrictions to trading strategies.

In this chapter, we will first introduce the Wick fractional Black Scholes model which determined a risk-neutral measure and give pricing formula based on martingale approach. In addition, we will mention the mixed fBM model in order to exclude arbitrage opportunities. Finally, the risk preference model will be presented as an alternative approach.

3.1   Wick fractional Black Scholes model

We first introduce the fractional Black Scholes market under continuous time, which is parallel to the traditional Black-Scholes market set up. However, explicit arbitrage strategy has been constructed in [16].

Consider two tradable assets in a fractional Black Scholes market with respect to probability space $(\Omega, \mathcal{F}_t^H, \mathbb{P})$, where $\mathcal{F}_t^H = \sigma(\{B_s^H\} : s \in (0, t))$.

(i) a money market account denoted as A such that:

$$dA_t = rA_t dt, \quad A_0 = 1, \quad 0 \le t \le T$$

where constant r represents the risk-free interest rate.

(ii) a stock S which follows the SDE:

$$dS_t = \mu S_t dt + \sigma S_t d^\diamond B_t^H, \quad S_0 = S > 0, \quad 0 \le t \le T$$

where μ, σ are constants and $\delta, \sigma \ne 0$

From the SDE above, it follows immediately that the explicit solutions of the basic market assets are

$$A_t = e^{rt}$$

and

$$S_t = S_0 e^{\sigma B_t^H + \mu t - \frac{\sigma^2}{2} t^{2H}}$$

Definition 3.1.1. *A portfolio (trading strategy) is a pair of progressively-measurable processes $\phi = (\beta_t, \gamma_t)_{t \in [0,T]}$ where β_t and γ_t separately denote the amount invested in the bank account and the shares of stocks. The value process of such a portfolio is:*

$$V_t(\phi) = \beta_t A_t + \gamma_t S_t$$

Since [16] showed that this market has arbitrage opportunities, Hu and Oksendal [19] incorporated the Wick product into the definition of a self-financing portfolio to exclude arbitrage.

Definition 3.1.2 (Wick-self-financing). *The value process is assumed to follow*

$$V_t(\phi) = \beta_t A_t + \gamma_t \diamond S_t$$

And the portfolio $\phi = (\beta_t, \gamma_t)$ is called Wick-self-financing if for all $t \in [0, T]$

$$\begin{aligned} dV_t(\phi) &= \beta_t dA_t + \gamma_t d^\diamond S_t \\ &= \beta_t r A_t dt + \mu \gamma_t S_t dt + \sigma \gamma_t S_t d^\diamond B_t^H \end{aligned} \tag{3.1}$$

Applying the fractional Girsanov formula, Hu and Oksendal [19] obtained the stochastic process of portfolio value under the risk-neutral probability measure $\mathbb{Q}_H$ defined on $\mathcal{F}_T^H$

$$dV_t(\phi) = rV_t(\phi)dt + \sigma \gamma_t S_t d^\diamond \tilde{B}_t^H$$

where

$$\frac{d\mathbb{Q}_H}{d\mathbb{P}} = exp(-\int_0^T q_t d^\diamond B_t^H - \frac{1}{2} \mid q \mid_\phi^2)$$

and

$$\int_0^T q_t \phi(s,t)dt = \frac{\mu - r}{\sigma} \quad \text{holds for all } s \in [0,T].$$

Note that $\tilde{B}_t^H = B_t^H + \frac{\mu - r}{\sigma}$ is a fBM under the measure $\mathbb{Q}_H$. In addition, the stock price under $\mathbb{Q}_H$ follows $S_t = S_0 e^{\sigma \tilde{B}_t^H + rt - \frac{\sigma^2}{2}t^{2H}}$.

Hence by discounting the portfolio value with risk-free rate and applying fractional Clark-Ocone theorem under $(\Omega, \tilde{\mathcal{F}}_t^H, \mathbb{Q}_{\mathbb{H}})$, where $\tilde{\mathcal{F}}_t^H = \sigma(\{\tilde{B}_s^H\}: s \in (0,t))$, they obtained the price of European call option at initial time 0:

$$V_0 = e^{-rT} \mathbb{E}^{\mathbb{Q}_H}[V_T(\phi)]$$

As they further proved that the Wick fractional Black Scholes model is complete and arbitrage-free with the class of Wick-self-financing portfolios, the risk-neutral measure $\mathbb{Q}_H$ mentioned above is unique and the initial price of the European call option with maturity T and strike K is determined by the value of the replicating portfolio.

$$C(0, S_0) = e^{-rT} \mathbb{E}^{\mathbb{Q}_H}[max(S_T - K, 0)]$$

Finally they gave the price of the European call option at initial time 0:

Theorem 3.1.1. *[19] [Hu and Oksendal fractional Black Scholes formula] Given an European call option with maturity T and strike price K. The price of such an option at initial time 0 is*

$$C(0, S_0) = S_0 N(d_+^H) - Ke^{-rT} N(d_-^H)$$

where

$$d_{\pm}^H = \frac{ln\left(\frac{S_0}{K}\right) + rT \pm \frac{\sigma^2}{2}T^{2H}}{\sigma\sqrt{T^{2H}}}$$

and $N(\cdot)$ denotes the standard normal cumulative function.

Based on Wick framework, Necula[21] derived a plausible evaluation theorem using results regarding quasi-conditional expectation with respect to the risk-neutral measure $\mathbb{Q}_H$.

Theorem 3.1.2 (fractional risk-neutral evaluation). *Consider a bounded $\tilde{\mathcal{F}}_T^H$-measurable claim V in $L^2(\Omega, \tilde{\mathcal{F}}_t^H, \mathbb{Q}_H)$, its price at $t \in [0,T]$ follows*

$$V_t = e^{-r(T-t)} \tilde{E}_t^{\mathbb{Q}_H}[V_T \mid \tilde{\mathcal{F}}_t^H]$$

Note that the major difference lies in this pricing formula is that while Hu and Oksendal [19] are using the traditional concept of expectation under the risk-neutral measure $\mathbb{Q}_H$, Necula[21] gave the formula based on the quasi-conditional expectation, which is the conditional expectation with respect to $\tilde{\mathcal{F}}_t^H$ generated by fBM $\tilde{B}_s^H$ under $\mathbb{Q}_H$.

Then the price of the European call option with maturity T and strike K at an arbitrary current time t is given by

$$C(t, S_t) = e^{-r(T-t)} \tilde{E}_t^{\mathbb{Q}_H}[max(S_T - K, 0) \mid \tilde{\mathcal{F}}_t^H].$$

Thus the fractional Black Scholes formula is extended to an arbitrary current time t .

Theorem 3.1.3 (Necula fractional Black Scholes formula). *Given an European call option with maturity T and strike price K. The price of such an option at time $t \in [0, T]$ is*

$$C(t, S_t) = S_t N(d_+^H) - Ke^{-r(T-t)} N(d_-^H)$$

where

$$d_\pm^H = \frac{ln\left(\dfrac{S_t}{K}\right) + r(T - t) \pm \dfrac{\sigma^2}{2}(T^{2H} - t^{2H})}{\sigma\sqrt{T^{2H} - t^{2H}}}$$

and $N(\cdot)$ denotes the standard normal cumulative function.

Corollary 3.1.4. *Using similar arguments as above, the price at every $t \in [0, T]$ of a European put option with strike price K and maturity T is given by*

$$P(t, S_t) = Ke^{-r(T-t)} N(-d_-^H) - S_t N(-d_+^H)$$

Since the law of one price holds in the fractional market, the put-call parity still holds.

Corollary 3.1.5 (Put-Call parity). *The price of a European call option and European put option in the fractional Black Scholes market satisfies the relationship:*

$$C(t, S_t) - P(t, S_t) = S_t - Ke^{-r(T-t)}$$

There is a lot of controversy about this model. Although the result by Hu, Oksendal and Necula is mathematically a good analogue of the no-arbitrage result for classical Black-Scholes model, severe critique arose concerning the economic meaning of Wick products that are used beyond pure

integration. In particular, the definition of the value process used in the model contradicts to economic intuition. Due to the reason that Wick product is an operator of random variables, the current price and positions are not sufficient to calculate current value of the portfolio. Thus one needs to calculate the value based on knowledge of all possible prices and corresponding positions, including the states of nature not realized.

The second surprising feature appears when one takes a deeper look at the Wick fractional Black Scholes formula itself. The option value does not depend on time to maturity $T - t$, but is up to the current point in time t. This makes it necessary to determine the line of time absolutely and not only relatively and causes arbitrage opportunity. If we have $t_1 \leq t_2 \leq t \leq T$ and two options with the same maturity T written in date t_1 and t_2 respectively the Wick fractional Black Scholes prices (at time t) for these two options will be different.It leads to an arbitrage in this market. It is caused by the long range dependent property, the price of the first option is influenced by the evolution of the stock price in the period $[t_1, t_2]$.

In order to justify the definition of the Wick self-financing portfolios Oksendal[30] introduced the concept of a market observer. All formulas containing Wick products are interpreted as abstract quantities that become real prices by an observation. For example, the underlying firm value and trading strategy are stochastic process whereas the corresponding observable stock price and position are assumed to be the outcomes of statistic test functions.

3.2 Mixed fractional Brownian Model

In order to exclude arbitrage in the fractional Black Scholes market, Cheridito [31] first proposed to modify the stochastic process of the stock price in order to get a semi-martingale. The idea is to define the mixed fBM as a linear combination of a fBM and a standard Brownian Motion. Under $(\Omega, \mathcal{F}_t^H, \mathbb{P})$, we can define the mixed fBM as

$$Z_t = B_t^H + \epsilon W_t$$

where W_t is a standard Brownian Motion. B_t^H and W_t are independent.

In the mixed fractional Brownian model, the stock price follows the process

$$S_t = S_0 e^{\sigma(B_t^H + \epsilon W_t) + \mu t - \frac{\sigma^2}{2} t} \quad \mu, \sigma \in \mathbb{R}.$$

According to Cheridito's study [31], for $H \in (\frac{3}{4}, 1)$ and $\epsilon > 0$, the model is equivalent to the one driven by classical Brownian Motion. Hence it is complete and arbitrage-free. For each $\epsilon > 0$ small enough, one can price assets with respect to the unique martingale measure $\mathbb{Q}_\epsilon$, and obtain that at time $t = 0$, the mixed fBM model is equivalent to the classical Black Scholes pricing formula:

$$C(0, S_0) = e^{-rT}\mathbb{E}^{\mathbb{Q}_\epsilon}\left[max\left(S_T - K, 0\right)\right] = BS(0, S_0, \epsilon\sigma)$$

where $BS(0, S_0, \epsilon\sigma)$ is the classical Black Scholes pricing formula with initial price S_0, and volatility $\epsilon\sigma$.

Although this study obtained a pricing formula at initial time $t = 0$, it is still based on the condition that ϵ is close to zero. This is due to certain admissibility conditions it put on the trading strategy, which seems to exploit small movements of the price process over nanoseconds. One thing worth mention is that this model is only valid for $H \in (\frac{3}{4}, 1)$.

For $H \in (\frac{1}{2}, 1)$, Mishura and Valkeila [32] further proved that if the dynamics of underlying stock price satisfies

$$S_t = S_0 e^{\sigma B_t^H + \sigma_H B_t^H + \mu t} \quad \mu, \sigma, \sigma^H \in \mathbb{R},$$

then the mixed fractional Brownian model is arbitrage free[33]. The price of European call option satisfies the traditional Black Scholes pricing formula[32]:

$$C(t, S_t) = BS(t, S_t, \sigma)$$

where $BS(t, S_t, \sigma)$ is the classical Black Scholes price of a call option on a stock with price S_t, and volatility σ. Notice that Hurst parameter H has no impact on option pricing in this continuous-time trading market driven by mixed fBM.

However, as proved by [34], in a discrete-time market setting, the discrete time interval δt and the H parameter play an important role in option pricing theory.

$$C(t, S_t) = BS(t, S_t, \sqrt{\sigma^2 + \sigma_H^2 \delta t^{2H-1}})$$

Comparing this result with the pricing formula obtained in the continuous-time setting, we have

$$BS(t, S_t, \sigma) \leqslant BS(t, S_t, \sqrt{\sigma^2 + \sigma_H^2 \delta t^{2H-1}})$$

which indicates that the continuous time trading assumption will result in underestimating the value of a European call option.

3.3 Risk preference model

In this section we study the pricing approach suggested by Rostek and Schőbel [28], which they call preference based equilibrium pricing. Under this framework, a restriction is put on the minimal time between two consecutive transactions. Under this assumption, a single investor can not be as fast as the market which evolves continuously, thus under a very small time intervals all arbitrage possibilities are eliminated [16].

As we give up on continuous-time trading assumption, the preceding martingale pricing approach based on dynamic hedging under risk-neutral measure is unsuitable within this framework. The problem is solved by introducing risk preferences on equilibrium pricing approach, namely that investor should be indifferent between buying the stock and holding the amount S_t of the risk-free asset under physical measure $(\Omega, \mathcal{F}_t^H, \mathbb{P})$. Formally we demand:

$$\mathbb{E}[e^{-r(T-t)}S_T|\mathcal{F}_t^H] = S_t$$

This is less than being a martingale. [25] However, this condition is not achieved by changing the measure like in the standard Black-Scholes model, but by adjusting the drift rate of the stock process. Economic argumentation given by the authors is that in a world where all investors are risk-neutral (but possessing and using information about the past), the basic asset cannot be of an arbitrary shape, but has to be in equilibrium itself.

The price of an European call option in this model is defines to be:

$$C(t, S_t) = e^{-r(T-t)}\mathbb{E}[max(S_T - K, 0)|\mathcal{F}_t^H]$$

Again, the expectation is taken under physical measure $\mathbb{P}$.

Using the results in conditional distribution of fBM introduced in chapter 2, we have

Proposition 3.3.1. *[28] The price of a European call option with strike K and maturity T at time $t \in [0, T]$ valued by a risk-neutral investor is given by the following formula:*

$$C(t, S_t) = S_t N(d_+^H) - Ke^{-r(T-t)}N(d_-^H)$$

where

$$d_\pm^H = \frac{ln(\frac{S_t}{K}) + r(T - t) \pm \frac{1}{2}\rho_H\sigma^2(T - t)^{2H}}{\sqrt{\rho_H}\sigma(T - t)^H}$$

with

$$\rho_H = \frac{sin(\pi(H - \frac{1}{2}))\Gamma(\frac{3}{2} - H)^2}{\pi(H - \frac{1}{2})\Gamma(2 - 2H)}$$

Note that for the case $H = \frac{1}{2}$ it is the standard Black Scholes pricing formula.

Corollary 3.3.2. *The price of a fractional European put option is*

$$P(t, S_t) = Ke^{-r(T-t)}N(-d_-^H) - S_t N(-d_+^H)$$

Corollary 3.3.3 (Put-Call parity)**.** *The price of a fractional European call option and fractional European put option satisfies the relationship:*

$$C(t, S_t) - P(t, S_t) = S_t - Ke^{-r(T-t)}$$

Chapter 4

Application to Market Data

4.1 Hurst parameter estimation

The scaling character of H-self-similarity suggests that the self-affine random function $B^H(\triangle t)$ is proportional to $\mid \triangle t \mid^H$. Several methods of estimating H make use of this property.

The simplest, the oldest and the most popular is R/S (Rescaled-range Statistic) analysis. This method was introduced by Hurst and latter refined by Mandelbrot[3]. Given a physical time series of length n, $(X_i : i \in \{1, \ldots, n\})$, we divide it into k non-overlapping blocks. Next we compute two numbers

$$R(t_i, d) = max\{0, W(t_i, 1), \ldots, W(t_i, d)\} - min\{0, W(t_i, 1), \ldots, W(t_i, d)\},$$

$$W(t_i, j) = \sum_{l=1}^{j} X_{t_i+l-1} - \frac{j}{d} \sum_{l=1}^{d} X_{t_i+l-1}, \quad j = 1, \ldots, d,$$

where $t_i = \lfloor n/k \rfloor (i-1) + 1$ are the starting points of the blocks which satisfy $(t_i - 1) + d \leq n$ and $S^2((t_i, d))$ is the sample variance of $X_{t_i}, \ldots, X_{t_i+d-1}$. Here $(X_i : i \in \{1, \ldots, n\})$ is the log return of the index. For each value of d we obtain a number of R/S samples. We compute these samples for logarithmically spaced values of d. For the fractional Gaussian noise the ratio R/S follows

$$\mathbb{E}[R(d)/S(d)] \sim Cd^H,$$

where C is a positive, finite constant independent of d. Hence a plot of $log(R(d)/S(d))$ versus $logd$ should have a constant slope. Finally the slope of the regression line for the R/S samples is an estimate for the Hurst parameter. For more Hurst parameter estimation methods, see [35].

4.2 Volatility estimation

Here we fix volatility to the implied volatility at $t = 0$ so that all the price paths will start from the same point. Alternatively, Cajueiro and Barbachan presented a procedure of volatility estimation and derived estimation formula for Black Scholes model and fractional models respectively in [36].

The implied volatility is derived by substituting the market option price into the pricing formula, like Black Scholes formula, and then calculating the parameter σ.

Usually we use Newton's method to find the roots of functions. Consider we are solving an equation $f(x) = 0$, and suppose that $x = c$ is an zero of f which is unknown. If f is differentiable in an open interval that contains c, then the solution can be approximated by the algorithm:

- Make an initial approximation x_1 close to c;

- Determine a new approximation using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)};$$

- If $|x_n - x_{n+1}|$ is less than the desired accuracy, let x_{n+1} serve as the final approximation. Otherwise, return to step 2 and calculate a new approximation.

When implementing this algorithm in MATLAB, we use the built-in function $fzero$ to solve the function by Newton's method set the accuracy to the machine precision accuracy, that is 10^{-6}.

4.3 Comparison between the results of different fBM models

To be able to price index options in standard BS, Wick fractional as well as Risk preference models one needs to be able to estimate the dividend yield. Because performing this task would be very time consuming or even

impossible we instead used formula for the price of the option on futures (Black's formula). Futures contract follow the behaviour of the index itself and at the same time are sensible to the payment of dividends. The price of an option on the stock index is equivalent to the price of the option on futures.

In the standard Black-Scholes model the Black formula is:

$$C(T, t, F_t, K, r, \sigma) = e^{-r(T-t)}(F_t N(d_+) - KN(d_-)),$$

where

$$d_{\pm} = \frac{ln(\frac{F_t}{K}) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}},$$

in Wick fractional model:

$$C(T, t, F_t, K, r, \sigma, H) = e^{-r(T-t)}(F_t N(d_+^H) - KN(d_-^H)),$$

where

$$d_{\pm}^H = \frac{ln(\frac{F_t}{K}) \pm \frac{1}{2}\sigma^2(T^{2H} - t^{2H})}{\sigma\sqrt{T^{2H} - t^{2H}}},$$

and in Risk preference model:

$$C(T, t, F_t, K, r, \sigma, H) = e^{-r(T-t)}(F_t N(d_+^H) - KN(d_-^H)),$$

where

$$d_{\pm}^H = \frac{ln(\frac{F_t}{K}) \pm \frac{1}{2}\rho_H\sigma^2(T-t)^{2H}}{\sqrt{\rho_H}\sigma(T-t)^H}.$$

4.4 Numerical Result

4.4.1 UK Market Implementation

All the calculations in this section are based on the FTSE 100 option daily prices from the Bloomberg terminal. Because of liquidity, we chose the call option expired at December 2017 from the period 16/12/2015-15/12/2017.

To be closer to the practical situation, we conducted out-of-sample forecasting while estimate the Hurst parameter. In other words, we use the Hurst parameter obtained from the FTSE 100 index market data before the first trading date of the option to calculate the option prices. However, because
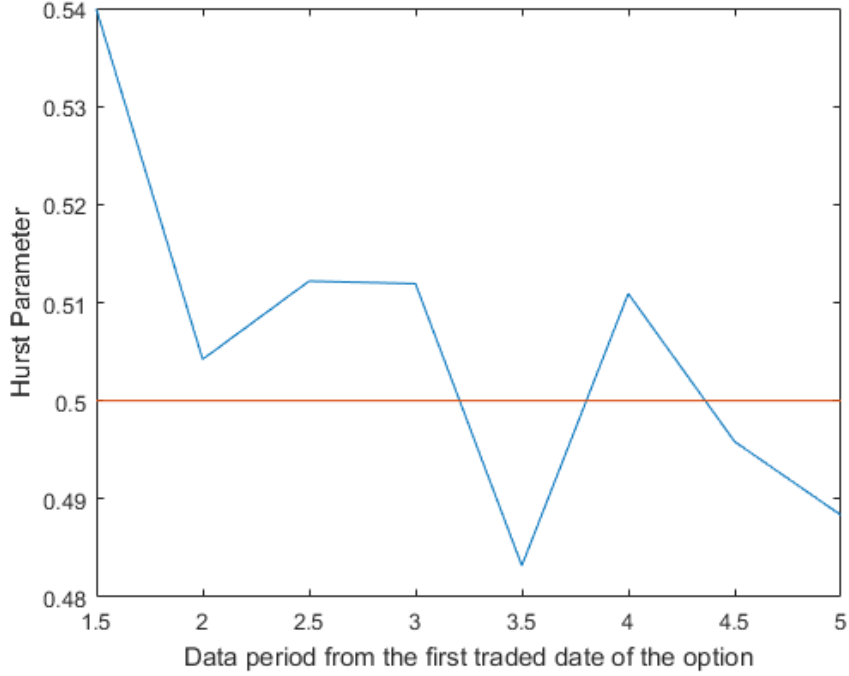
Figure 4.1: Estimated Hurst parameter with different data period

of market movements, the Hurst parameter obtained by R/S analysis fluctuated slightly near one half. The Figure 4.1 shows how Hurst parameter varies with the data period.

Not surprisingly, the result shows the developing FTSE market didn't show too strong persistent behavior which matches with the conclusion from Beben and Ohowski[8]. Although the Hurst parameter has fallen below the 0.5 level twice in this figure, the latest four observations are all beyond 0.5. That is, there do exists long range dependence in the FTSE 100 market.

Since the small difference in Hurst parameter is negligible in option pricing, we decided to use the result from the last three years which is $H = 0.5119$. For the risk-free interest rate, we chose the official bank rate from Bank of England, which was 0.5% from 16/12/2015 to 05/08/2016 and 0.25% for the rest of dates. Moreover we fixed volatility to implied volatility at $t = 0$ so that the option prices derived from three different models equal to market price at $t = 0$. The implied volatility is as shown in Figure 4.2

Since we have got all the parameters, we now move forward to price the options. As mentioned before, in order not to compute the dividend yield
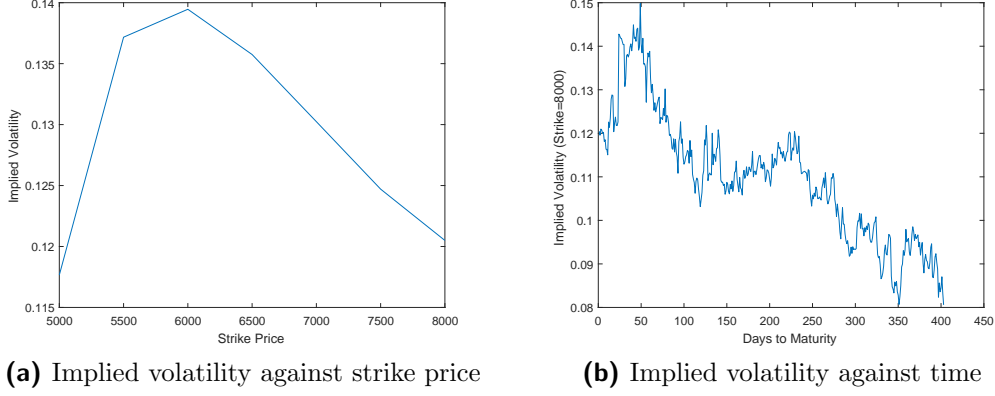
34

(a) Implied volatility against strike price

(b) Implied volatility against time

Figure 4.2: Implied volatility of FTSE 100 option

of FTSE 100 index, we used corresponding Black formulas to compute the price of FTSE 100 option and compared with the real market price. In all cases, the price curves obtained from standard Black-Scholes formula, Wick fractional formula and Risk preference formula are almost coincided with each other. This is to be expected, because the Hurst parameter is rather close to one half which means the fractional Brownian motion is almost the same as the standard Brownian motion.

To be more precise, we computed the Average Mean Square Error (AMSE) for all the methods:

$$AMSE = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{C_i^{real} - C_i^{estimated}}{C_i^{real}} \right)^2$$

where C_i^{real} is the real market price and $C_i^{estimated}$ is the option price computed by corresponding formula.

Moreover, according to Figure 4.3, the pricing formulas' performance varies among the life time of the option. For the first case where the strike price equals 5500, the pricing formulas do better in the first nine months while the other three cases are in the contrary. If we look at the FTSE 100 index spot price, we may observe that the index was in a steady up trend from June 2016. It rose from 6400 to nearly 7500. It is reasonable to believe that there might be some links between the spot price and AMSEs.

Thus, we also divided data sets into five groups depending on moneyness. We used the definition as followed:
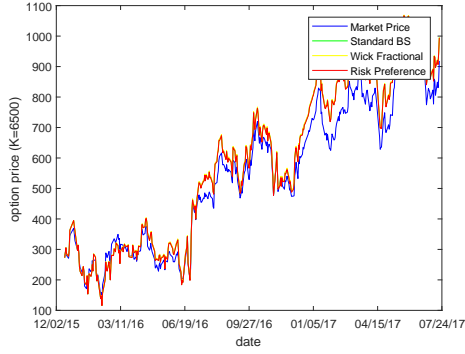
Since the option with strike price 5500 is in-the-money all the time, we
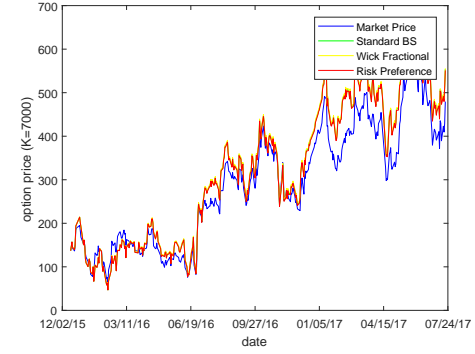
35
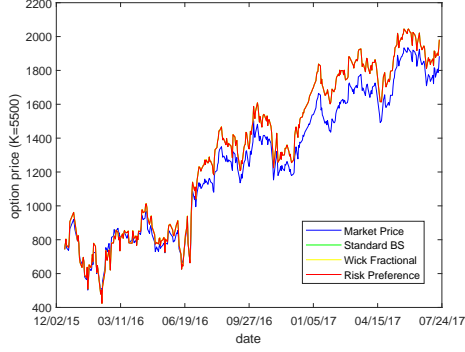
(a) Strike=5500
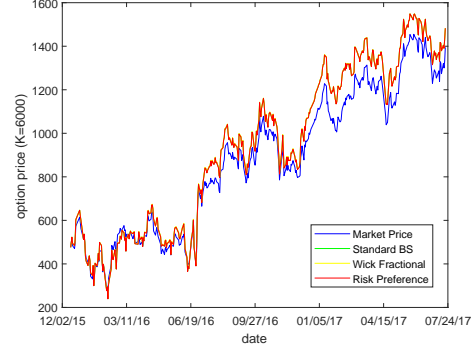
(b) Strike=6000

(c) Strike=6500

(d) Strike=7000

Figure 4.3: Price of FTSE 100 call option with different strike prices

Table 4.1: Definition of partition rule

deep out-of-the-money (deepOTM)	$S_t < 0.9K$
out-of-the-money (OTM)	$0.9K \leq S_t < 0.98K$
at-the-money (ATM)	$0.98K \leq S_t < 1.02K$
in-the-money (ITM)	$1.02K \leq S_t < 1.1K$
deep in-the-money (deepITM)	$S_t \geq 1.1K$

only listed the other three cases below.

The Risk preference formula slightly outperformed the other two formulas.

Table 4.2: Comparison of AMSEs of different pricing formula depending on moneyness in UK market

| UKX 12 C6000	deepOTM	OTM	ATM	ITM	deepITM	Total
Sample Numbers	0	17	44	80	261	402
Black Scholes	NaN	0.0025	0.0023	0.0029	0.0079	0.0060
Wick Fractional	NaN	0.0024	0.0023	0.0031	0.0081	0.0062
Risk Preference	NaN	0.0025	0.0024	0.0027	0.0077	0.0059
UKX 12 C6500	**deepOTM**	**OTM**	**ATM**	**ITM**	**deepITM**	**Total**
Sample Numbers	12	119	10	130	131	402
Black Scholes	0.0072	0.0069	0.0018	0.0071	0.0132	0.0089
Wick Fractional	0.0067	0.0073	0.0022	0.0078	0.0137	0.0094
Risk Preference	0.0074	0.0067	0.0016	0.0066	0.0127	0.0085
UKX 12 C7000	**deepOTM**	**OTM**	**ATM**	**ITM**	**deepITM**	**Total**
Sample Numbers	120	91	57	134	0	402
Black Scholes	0.9966	0.8274	0.4964	0.0600	NaN	0.0219
Wick Fractional	0.9962	0.8211	0.4899	0.0592	NaN	0.0236
Risk Preference	0.9972	0.8295	0.4971	0.0602	NaN	0.0205

4.4.2 Chinese Market Implementation

Since the Hurst parameter of UK market is near one half, the difference between Brownian motion and fractional Brownian motion is not significant. We then turn to the emerging markets like Chinese market. In this section, we study on the 50ETF option which is writing on the Shanghai Stock Exchange 50 Index. The first traded date of the option expired at 12/27/2-17 is 4/27/2017. The Hurst parameter of three years before 4/27/2017 is 0.6629. The benchmark interest rate from the People's Bank of China kept 0.35% during the trading period of the options. As for volatility, we set it to implied volatility at $t = 0$ and assumed it constant during the life time of the option. The implied volatility is as shown in Figure 4.4

By the Black formulas in the last section, we calculated the option prices and compared with the real market data.

As compared to Figure 4.3, option prices derived from three models are significantly different from each other in Figure 4.5. However the differences between real market price and the estimated prices are also relatively big. It might be because of three reasons. First, it might caused by the transaction fee charged by the exchange which is 1.3 RMB each contract. Second, the ask-bid spread in Chinese option market is too big to ignore which contradicts the assumptions for pricing models. At last, different trading strategies like delta hedge and gamma hedge might help explaining the observed option prices.

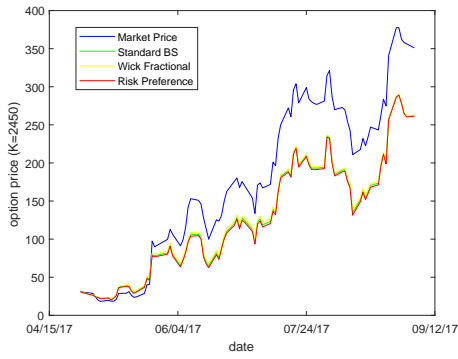Similarly, we also divided data sets into five groups depending on mon-
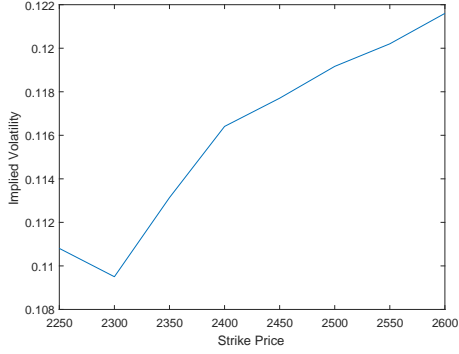
(a) Implied volatility against strike price
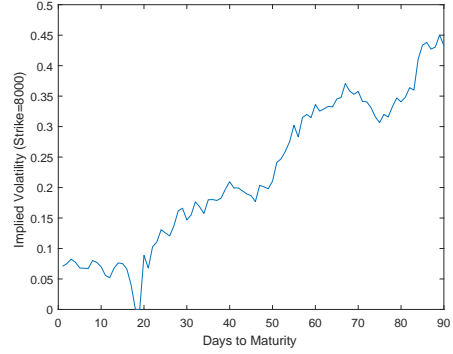
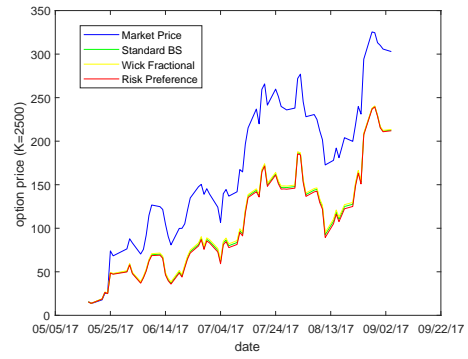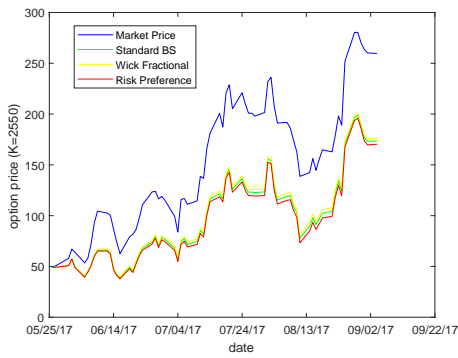**(b)** Implied volatility against time

Figure 4.4: Implied volatility of 50 ETF option
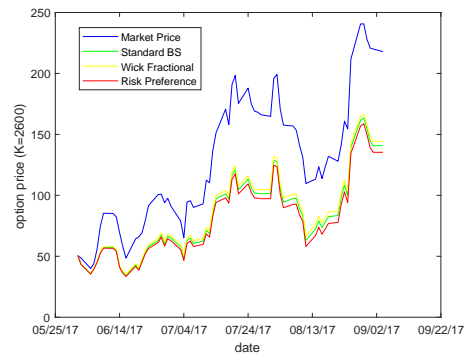

(a) Strike=2450

(b) Strike=2500

(c) Strike=2550

(d) Strike=2600

Figure 4.5: Price of 50ETF call option with different strike prices

eyness by Table 4.1 and listed the AMSEs as below. Note that the total sample numbers decreased as the strike price went up because there was no trades in the first few days since the stock price is too far away from the strike price. From Table 4.3, the Wick fractional model outperformed the other two pricing models in the last three cases while the Risk preference model had the lowest AMSE when the strike price equals 2450.

Table 4.3: Comparison of AMSEs of different pricing formula depending on moneyness in Chinese market

50ETF 12 C2450	deepOTM	OTM	ATM	ITM	deepITM	Total
Sample Numbers	0	17	16	50	7	90
Black Scholes	NaN	0.0479	0.0795	0.0870	0.0612	0.0763
Wick Fractional	NaN	0.0677	0.0700	0.0797	0.0602	0.0742
Risk Preference	NaN	0.0386	0.0874	0.0932	0.0620	0.0794
50ETF 12 C2500	deepOTM	OTM	ATM	ITM	deepITM	Total
Sample Numbers	0	6	27	44	0	77
Black Scholes	NaN	0.0011	0.1957	0.1344	NaN	0.1455
Wick Fractional	NaN	0.0020	0.1795	0.1264	NaN	0.1353
Risk Preference	NaN	0.0009	0.2064	0.1412	NaN	0.1531
50ETF 12 C2550	deepOTM	OTM	ATM	ITM	deepITM	Total
Sample Numbers	0	13	23	34	0	70
Black Scholes	NaN	0.0994	0.1340	0.1289	NaN	0.1251
Wick Fractional	NaN	0.0862	0.1145	0.1143	NaN	0.1091
Risk Preference	NaN	0.1074	0.1490	0.1424	NaN	0.1381
50ETF 12 C2600	deepOTM	OTM	ATM	ITM	deepITM	Total
Sample Numbers	0	22	33	13	0	68
Black Scholes	NaN	0.0903	0.1347	0.1247	NaN	0.1184
Wick Fractional	NaN	0.0759	0.1092	0.1079	NaN	0.0982
Risk Preference	NaN	0.0994	0.1577	0.1408	NaN	0.1356

To be more intuitive, we plot AMSEs of Wick fractional model against moneyness type in Figure 4.6. In all cases, AMSEs reached their lowest level when the option was OTM. Especially, AMSE of the option with strike price $K = 2500$ had a sharp decrese when the option went from OTM to ATM and then went down from 0.18 to nearly 0.12. In the other three cases, AMSEs kept stable when the options were ATM and ITM.

4.5   Summary

Firstly, the Hurst parameter of FTSE market fluctuate near one half. It is not astonishing that the developing market doesn't have too strong persistent behavior, because Beben and Ohowski obtained the same result in [8]. In result, the difference between fractional Brownian motion and standard Brownian motion is rather small. It is therefore hard to tell which model is
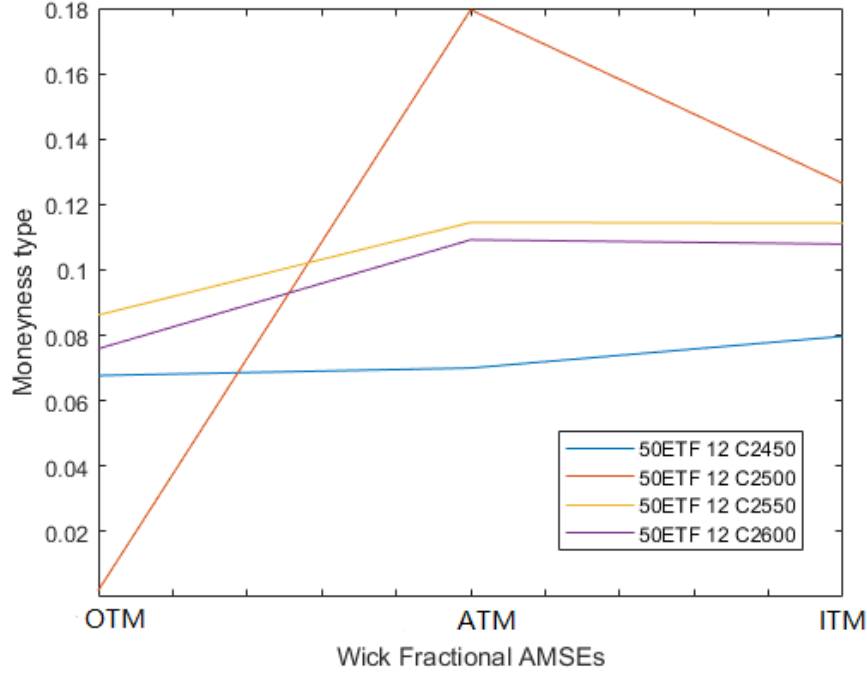
Figure 4.6: AMSEs of Wick fractional formula depending on moneyness with different strikes in Chinese market

the best to be used in practice. By contrast, Chinese market shows higher Hurst parameter where Wick factional model significantly outperformed the other two models. However, the option prices derived from all the models are still far from market prices. It should be accentuated that the emerging market may have a higher Hurst parameter but the lack of liquidity in a newly-built market will lead to unfair market prices.

In addition, the accuracy may be influenced by the moneyness type of the option. As shown in the example of Chinese market, when the option was OTM, all the pricing models reached their lowest AMSEs. But it may differ in different markets.

Finally, the pricing models performed unsatisfactory in Chinese market. It may be caused by lack of liquidity and non-negligible transaction fee. To improve the model accuracy, we may replace the constant volatility and interest rate by stochastic processes and consider jump process.

Chapter 5

CONCLUSION

The goal of this dissertation is to study the mathematical techniques that are used to derive the fractional Black Scholes formulas and the empirical implications of the corresponding models. We have mentioned the fundamental mathematical theories with regard to fractional Brownian Motion, fractional Itô integrals as well as the conditional expectation. Moreover, we investigated and compared the terminology of several pricing models including Wick fractional Black Scholes model, mixed fractional Brownian model as well as the risk preference one. Finally, we estimated the H parameter and implied volatility of UK and Chinese stock markets, and compared the performance of fractional pricing models with classical Black Scholes model. We come to a conclusion that, in terms of the estimation efficiency, in the UK market, the risk preference model outperforms the other two, whereas in the Chinese market, the Wick fractional model gives the smallest error. Another finding is that the estimation efficiency does depend on moneyness of the option.

As the fBM is not a semi-martingale, the whole theory differs from the classical framework from the start of integration. The main difficulty lies in understanding the new framework and comparing it with what I have learnt in the lectures. Especially with regards to the theoretical part of Wick-Itô integrals, conditional distributions as well as the fractional Black Scholes models.

From the empirical study, we found the estimation performed unsatisfactory in emerging market. To improve the model accuracy, further research can be conducted to replace the constant volatility and interest rate by stochastic processes driven by fBM and consider jump processes.

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APPENDIX

main.m

```matlab
1  clear all
2
3  load('C:\Users\sheep_000\Desktop\Dissertation\code\
       optionMarketData.mat');
4  load('C:\Users\sheep_000\Desktop\Dissertation\code\
       indexMarketData.mat');
5  load('C:\Users\sheep_000\Desktop\Dissertation\code\
       interestRate.mat');
6  market = 'CHN';
7
8  %% estimate Hurst exponent and volatility
9  indexDate = eval(sprintf('indexMarketData.%s(:,1)',
       market));
10 indexData = eval(sprintf('indexMarketData.%s(:,2)',
       market));
11 t0 = eval(sprintf('optionMarketData.%s.startDate',
       market));
12 tn = eval(sprintf('optionMarketData.%s.endDate',market)
       );
13 tH = t0-365*3; % use the past three years' data
14 id = logical((tH<=indexDate).*(indexDate<=t0));
15 S = indexData(id);
16 diffLogS = diff(log(S));
17 H = hurstEstimate(diffLogS);
18
19 %% Option pricing
20 strike = cell2mat(eval(sprintf('optionMarketData.%s.
       call(:,1)',market)));
```

```matlab
21  dateIR = eval(sprintf('interestRate.%s(:,1)',market));
22  rate = eval(sprintf('interestRate.%s(:,2)',market));
23  for i = 1:length(strike)
24      dateOption = eval(sprintf('optionMarketData.%s.call
            {i,2}(:,1)',market));
25      optionPrice = eval(sprintf('optionMarketData.%s.
            call{i,2}(:,2)',market));
26      dataPricing = combineData(dateOption,optionPrice,
            dateIR,rate,indexDate,indexData,t0,tn);
27      tradeDate = dataPricing(:,1);
28      S = dataPricing(:,4);
29      priceMarket = dataPricing(:,2);
30      K = strike(i);
31      r = dataPricing(:,3);
32      for j=1:length(dataPricing)
33          [sigma1(j,i),sigma2(j,i),sigma3(j,i)] =
                impliedVol( S(j), K, r(j),t0,tn, tradeDate(j
                ), priceMarket(j),H );
34      end
35      [priceBS,priceWick,priceRostek] = optionPricing(t0,
            tn,tradeDate,S,K,r,sigma1(1,i),sigma2(1,i),
            sigma3(1,i),H);
36      plotOptionPrice(tradeDate,K,priceMarket,priceBS,
            priceWick,priceRostek);
37      % impliedVolatility(i,:) = [ sigma1,sigma2,sigma3
            ];
38  end
39  %% Statistic
40  % Average Mean Square Error
41  error(1,1) = AMSE(priceMarket,priceBS);
42  error(2,1) = AMSE(priceMarket,priceWick);
43  error(3,1) = AMSE(priceMarket,priceRostek);
44
45  % AMSE regarding to Moneyness
46  [X1,X2,X3,X4,X5] = divideByMoneyness(K,S,priceMarket,
        priceBS);
47  [errorMoneyness(1,:),sampleNum] =
        computeAMSEbyMoneyness(X1,X2,X3,X4,X5);
48  [Y1,Y2,Y3,Y4,Y5] = divideByMoneyness(K,S,priceMarket,
        priceWick);
49  [errorMoneyness(2,:),~] = computeAMSEbyMoneyness(Y1,Y2,
```

```
        Y3,Y4,Y5);
50  [Z1,Z2,Z3,Z4,Z5] = divideByMoneyness(K,S,priceMarket,
        priceRostek);
51  [errorMoneyness(3,:),~] = computeAMSEbyMoneyness(Z1,Z2,
        Z3,Z4,Z5);
52
53  %% implied volatility
54  % for i=1:length(strike)
55  %      K = strike(i);
56  %      for j=1:length(S)
57  %          [sigma1(j,i),sigma2(j,i),sigma3(j,i)] =
        impliedVol( S(1), K, r(j),t0,tn, tradeDate(j),
        priceMarket(j),H );
58  %      end
59  % end
```

Wick.m

```
1  function [C,P] = Wick(T,t,S,K,r,sigma,H)
2  k1 = log(S./K);
3  k2 = 0.5*sigma*sigma*(power(T,2*H)-power(t,2*H));
4  k3 = sigma*sqrt(power(T,2*H)-power(t,2*H));
5  d1 = (k1+k2)./k3;
6  d2 = (k1-k2)./k3;
7  C = exp(-r*(T-t))*(S*normcdf(d1)-K*normcdf(d2));
8  P = exp(-r*(T-t))*(K*normcdf(-d2)-S*normcdf(-d1));
9  end
```

Rostek.m

```
1  function [C,P] = Rostek(T,t,S,K,r,sigma,H)
2  rho = sin(pi*(H-0.5))*gamma(1.5-H)*gamma(1.5-H)/(pi*(H
        -0.5)*gamma(2-2*H));
3  k1 = log(S./K);
4  k2 = 0.5*sigma*sigma*rho*power(T-t,2*H);
5  k3 = sigma*sqrt(rho)*power(T-t,H);
6  d1 = (k1+k2)./k3;
7  d2 = (k1-k2)./k3;
8  C = exp(-r*(T-t))*(S*normcdf(d1)-K*normcdf(d2));
9  P = exp(-r*(T-t))*(K*normcdf(-d2)-S*normcdf(-d1));
```

```
10  end
```

plotOptionPrice.m

```
1  function plotOptionPrice(date,K,priceMarket,priceBS,
       priceWick,priceRostek)
2  figure;
3  plot(date,priceMarket,'b',date,priceBS,'g',date,
       priceWick,'y',date,priceRostek,'r');
4  datetick('x',2,'keepticks');
5  xlabel('date');
6  ylabel(sprintf('option price (K=%d)',K));
7  legend('Market Price','Standard BS','Wick Fractional','
       Risk Preference');
```

optionPricing.m

```
1  function [priceBS,priceWick,priceRostek] =
       optionPricing(t0,tn,date,S,K,r,sigma1,sigma2,sigma3,
       H)
2  T = (tn-t0)/365;
3  t = (date - t0)./365;
4  priceBS = zeros(length(t),1);
5  priceWick = zeros(length(t),1);
6  priceRostek = zeros(length(t),1);
7  for i = 1:length(t)
8      [priceBS(i)] = BS(T,t(i),S(i),K,r(i),sigma1);
9      [priceWick(i)] = Wick(T,t(i),S(i),K,r(i),sigma2,H);
10     [priceRostek(i)] = Rostek(T,t(i),S(i),K,r(i),sigma3
          ,H);
11 end
```

impliedVol.m

```
1  function [ sigma1,sigma2,sigma3 ] = impliedVol( S, K, r
       , t0,tn,date, value,H )
2  tol = 10e-6;
3  % limit = 10;
4  options = optimset('fzero');
```

```matlab
5  options = optimset(options, 'TolX', tol(1), 'Display',
       'off');
6  [ sigma1, ~, exitFlag ] = fzero(@objfcn, 0, options,
       ...
7      S, K, r, t0,tn,date, value,H, 1);
8  if exitFlag < 0
9      sigma1 = NaN;
10 end
11 [ sigma2, ~, exitFlag ] = fzero(@objfcn, 0, options,
       ...
12     S, K, r, t0,tn,date, value,H, 2);
13 if exitFlag < 0
14     sigma2 = NaN;
15 end
16 [ sigma3, ~, exitFlag ] = fzero(@objfcn, 0, options,
       ...
17     S, K, r, t0,tn,date, value,H, 3);
18 if exitFlag < 0
19     sigma3 = NaN;
20 end
21 end
22
23 function result = objfcn( sigma, S, K, r, t0,tn,date,
       value,H, model )
24 T = (tn-t0)./365;
25 t = (date - t0)./365;
26 if model == 1
27     call = BS(T,t,S,K,r,sigma);
28 end
29 if model == 2
30     call = Wick(T,t,S,K,r,sigma,H);
31 end
32 if model == 3
33     call = Rostek(T,t,S,K,r,sigma,H);
34 end
35 result = call - value;
36 end
```

hurstEstimate.m

```matlab
1  function H = hurstEstimate(data)
2  N = length(data);
3  dlarge = floor(N/5);
4  dsmall = max(10,log10(N)^2);
5  D = floor(logspace(log10(dsmall),log10(dlarge),50));
6  D = unique(D);
7  n = length(D);
8  x = zeros(1,n);
9  y = zeros(1,n);
10
11 R = cell(1,n);
12 S = cell(1,n);
13 for i = 1:n
14     d = D(i);
15     m = floor(N/d);
16     R{i} = zeros(1,m);
17     S{i} = zeros(1,m);
18     matrix_sequence = reshape(data(1:d*m),d,m);
19
20     Z1 = cumsum(matrix_sequence);
21     Z2 = cumsum(repmat(mean(matrix_sequence),d,1));
22     R{i} = (max(Z1-Z2)-min(Z1-Z2));
23     S{i} = std(matrix_sequence);
24
25     if min(R{i})==0 || min(S{i}) ==0
26         continue;
27     end
28
29     x(i) = log10(d);
30     y(i) = mean(log10(R{i}./S{i}));
31 end
32
33 % fit a line with middle part of sequence
34 index = x~=0;
35 x = x(index);
36 y = y(index);
37 n2 = length(x);
38 cut_min = ceil(3*n2/10);
39 cut_max = floor(9*n2/10);
40
41 X = x(cut_min:cut_max);
```

```matlab
42  Y = y(cut_min:cut_max);
43  p1 = polyfit(X,Y,1);
44  Yfit = polyval(p1,X);
45  H = (Yfit(end)−Yfit(1))/(X(end)−X(1));
46
47  % plot the figure
48  figure
49  hold on;
50  bound = ceil(log10(N));
51  axis([0 bound 0 0.75*bound]);
52
53  temp = (1:n).*index;
54  index = temp(index);
55  for i = 1:n2
56      plot(x(i),log10(R{index(i)}./S{index(i)}),'b.');
57  end
58
59  x = linspace(0,bound,10);
60  y1 = 0.5*x;
61  y2 = x;
62  h1 = plot(x,y1,'b—','LineWidth',2);
63  h2 = plot(x,y2,'b−.','LineWidth',2);
64  plot(X,Yfit,'r−','LineWidth',3);
65  legend([h1,h2],'slope 1/2','slope 1')
66  xlabel('log10(blocks of size m)'),ylabel('log10(R/S)'),
        title('R/S Method');
```

divideByMoneyness.m

```matlab
1  function [X1,X2,X3,X4,X5] = divideByMoneyness(K,S,
      priceMarket,priceEstimated)
2  % deepOTM
3  id = S<0.9*K;
4  X1 = [priceMarket(id),priceEstimated(id)];
5  % OTM
6  id = logical((S>=0.9*K).*(S<0.98*K));
7  X2 = [priceMarket(id),priceEstimated(id)];
8  % ATM
9  id = logical((S>=0.98*K).*(S<1.02*K));
10 X3 = [priceMarket(id),priceEstimated(id)];
```

```
11  % ITM
12  id = logical((S>=1.02*K).*(S<1.1*K));
13  X4 = [priceMarket(id),priceEstimated(id)];
14  % deepITM
15  id = S>=1.1*K;
16  X5 = [priceMarket(id),priceEstimated(id)];
17  end
```

computeAMSEbyMoneyness.m

```
1   function [result,sampleNum] = computeAMSEbyMoneyness(X1
        ,X2,X3,X4,X5)
2   if ~isempty(X1)
3       result(1,1) = AMSE(X1(:,1),X1(:,2));
4       sampleNum(1,1) = length(X1);
5   else
6       result(1,1) = nan;
7       sampleNum(1,1) = 0;
8   end
9   if ~isempty(X2)
10      result(1,2) = AMSE(X2(:,1),X2(:,2));
11      sampleNum(1,2) = length(X2);
12  else
13      result(1,2) = nan;
14      sampleNum(1,2) = 0;
15  end
16  if ~isempty(X3)
17      result(1,3) = AMSE(X3(:,1),X3(:,2));
18      sampleNum(1,3) = length(X3);
19  else
20      result(1,3) = nan;
21      sampleNum(1,3) = 0;
22  end
23  if ~isempty(X4)
24      result(1,4) = AMSE(X4(:,1),X4(:,2));
25      sampleNum(1,4) = length(X4);
26  else
27      result(1,4) = nan;
28      sampleNum(1,4) = 0;
29  end
```

```
30  if ˜isempty (X5)
31      result (1 ,5) = AMSE(X5 ( : ,1 ) ,X5 ( : ,2 ) ) ;
32      sampleNum ( 1 ,5 ) = length (X5) ;
33  else
34      result (1 ,5) = nan ;
35      sampleNum ( 1 ,5 ) = 0;
36  end
37  end
```

combineData.m

```
1  function [ data ] = combineData ( x1 , x2 , y1 , y2 , z1 , z2 , t0 ,
       tn )
2  % combineData returns one array by combining all the
       input
3  % vectors by their dates
4  id = logical ((x1>=t0) .*( x1<=tn ) ) ;
5  x = x1 ( id ) ;
6  id = ismember (y1 ,x) ;
7  x = y1 ( id ) ;
8  id = ismember (z1 ,x) ;
9  x = z1 ( id ) ;
10
11  data ( : ,1 ) = x;
12  id = ismember (x1 ,x) ;
13  data ( : ,2 ) = x2 ( id ) ;
14  id = ismember (y1 ,x) ;
15  data ( : ,3 ) = y2 ( id ) ;
16  id = ismember (z1 ,x) ;
17  data ( : ,4 ) = z2 ( id ) ;
18
19  [˜ , id ] = sort ( data ( : ,1 ) ) ;
20  data = data ( id , : ) ;
21  end
```

BS.m

```
1  function [C,P] = BS(T, t , S ,K, r , sigma )
2  k1 = log (S./K) ;
3  k2 = 0.5∗sigma∗sigma ∗(T−t ) ;
```

```matlab
4  k3 = sigma*sqrt((T-t));
5  d1 = (k1+k2)./k3;
6  d2 = (k1-k2)./k3;
7  C = exp(-r*(T-t))*(S*normcdf(d1)-K*normcdf(d2));
8  P = exp(-r*(T-t))*(K*normcdf(-d2)-S*normcdf(-d1));
9  end
```

AMSE.m

```matlab
1  function result = AMSE(priceMarket,priceEstimated)
2  result = sum(((priceMarket-priceEstimated)./priceMarket
       ).^2)/length(priceMarket);
3  end
```