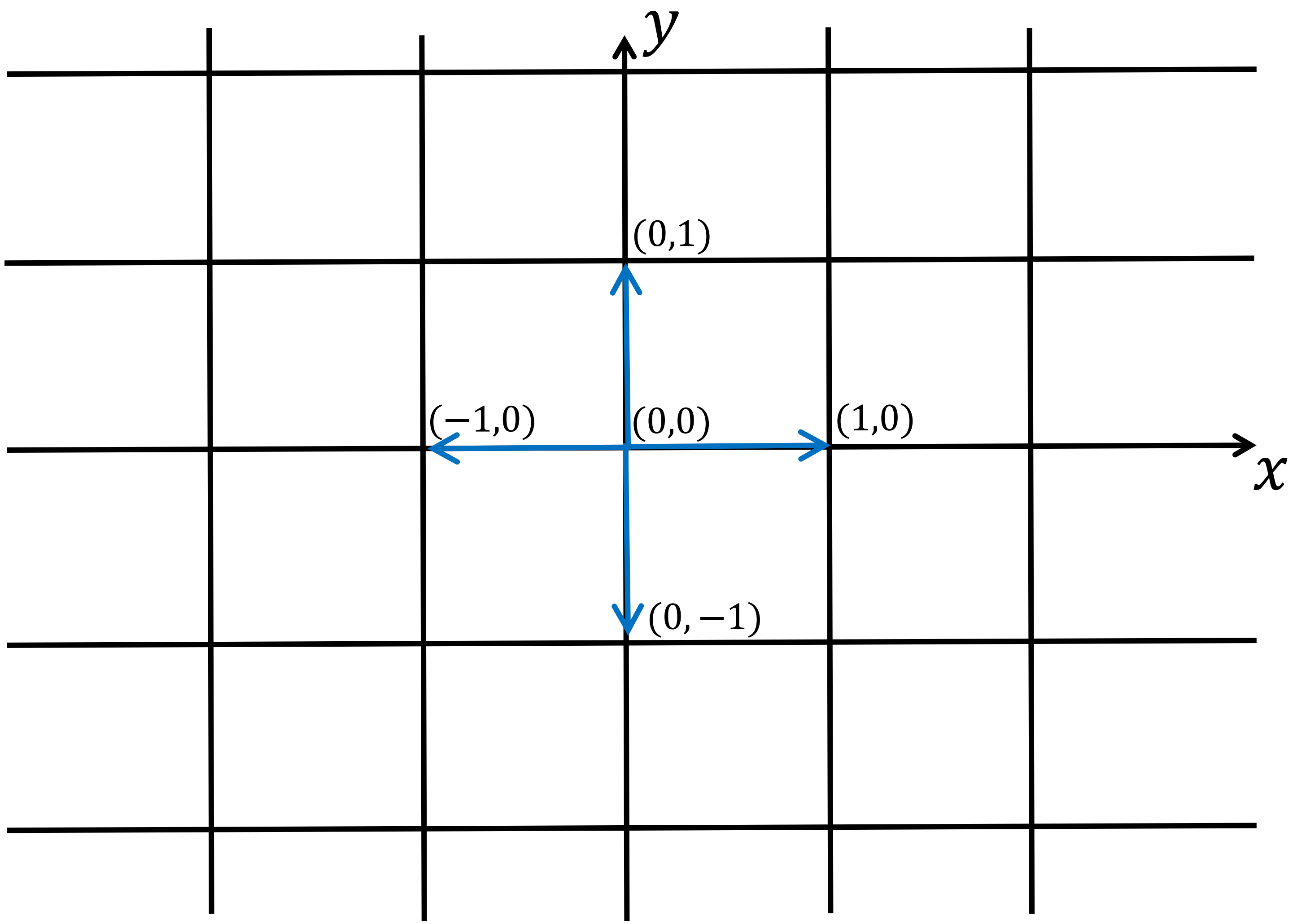
Modelling of Complex Systems

Two-dimensional Random Walks

Random walks on a two dimensional regular a square lattice:

- Explicit solution
- Master equation

Two-dimensional Random Walks



The particle has four possible jumps with equal probabilities of $1/4$.

Two-dimensional Random Walks

The variables x and y are not independent, i.e., $P(x, y; t) \neq P(x; t)P(y; t)$.

Let us introduce two auxiliary variables:

$$u = x + y,$$
$$v = x - y.$$

Conveniently, the new coordinates are independent random variables:

$$P(u, v; t) = P(u; t)P(v; t).$$

To show this, we notice that:

u and v are the sum of random displacements $\Delta u, \Delta v = \pm 1$,

the probabilities $P(\Delta u = \pm 1) = P(\Delta v = \pm 1) = 1/2$,

and, the displacements Δu and Δv are independent, i.e.,

$$P(\Delta u = \pm 1, \Delta v = \pm 1) = P(\Delta u = \pm 1)P(\Delta v = \pm 1) = 1/4.$$

Two-dimensional Random Walks

Therefore, the variables u and v make **independent random walks on a one-dimensional lattice**.

Let us find the probability that after t jumps in a square lattice, the particle will be at coordinates (u, v), and then convert our results back to the original coordinate system (x, y).

Previously we showed that in a **one-dimensional lattice**:

$$P(x;t) = \binom{t}{(t+x)/2} 2^{-t} \approx \frac{2}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

Thus, on a two-dimensional **square lattice** we have

$$P(u,v;t) = P(u;t)P(v;t) \approx \frac{2}{\pi t} e^{-\frac{u^2+v^2}{2t}}.$$

Two-dimensional Random Walks

Returning to the original variables, the probability of finding the particle at position (x, y):

$$P(x, y; t) = P(u = x + y, v = x - y; t)$$
$$\approx \frac{2}{\pi t} e^{-\frac{(x+y)^2 + (x-y)^2}{2t}} = \frac{2}{\pi t} e^{-\frac{x^2 + y^2}{t}}.$$

Again, if t is even/odd only even/odd sites can be reached (i.e., sites with even/odd $x + y$).
So we must divide by 2 to obtain the **probability density distribution function**

$$P(x, y; t) = \frac{1}{\pi t} e^{-\frac{x^2 + y^2}{t}}$$

Two-dimensional Random Walks

$$P(x, y; t) = \frac{1}{\pi t} e^{-\frac{x^2+y^2}{t}}$$

- $P(0,0; t) = \frac{1}{\pi t}$
- Normalization: $\int_{-\infty}^{\infty} P(x, y, t) dx dy = 1.$
- Mean values: $\langle x \rangle = \langle y \rangle = \int_{-\infty}^{\infty} x P(x\, y, t) dx dy = 0$
- Variances: $\langle x^2 \rangle = \langle y^2 \rangle = \frac{t}{2},$

$$\langle x^2 \rangle + \langle y^2 \rangle = \langle r^2 \rangle = t.$$

d-dimensions

Random walks in *d*-dimensional square lattice with jumps along the diagonals

$$(\Delta x, \Delta y, \Delta z, \dots) = (\pm 1, \pm 1, \pm 1, \dots)$$

$$P(x, y, z, \dots, t) = \left(\frac{2}{\pi t}\right)^{d/2} e^{-\frac{x^2+y^2+z^2+\cdots}{2t}}$$

The probability to find the particle in the initial point $(x, y, z, \dots) = 0$ is

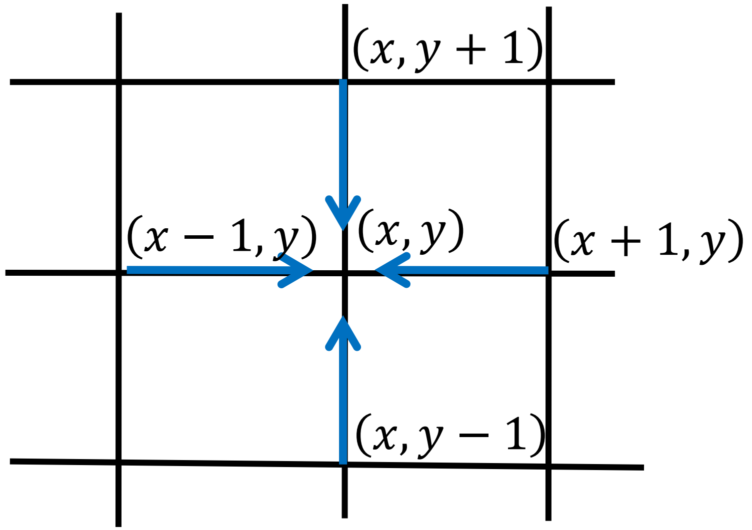
$$P(0, t) = \left(\frac{2}{\pi t}\right)^{d/2} \propto \begin{cases} \frac{1}{\sqrt{t}}, & d = 1 \\ t^{-1}, & d = 2 \\ t^{-3/2}, & d = 3 \end{cases}$$

Therefore,

$$\ln P(0, t) \propto \frac{d}{2} \ln t + \text{constant}$$

Measuring $\ln P(0, t)$ as function of $\ln t$, we can find dimensionality d of the space we are living in.

Master Equation



The particle jumps up, down, left and right with the same probability 1/4.
Let us write the probability to find the particle at point (x, y) at time $t+1$ in terms of the probabilities that at time t it is at points $(x \pm 1, y)$ and $(x, y \pm 1)$.

$$P(x, y, t+1)$$
$$= \frac{1}{4} P(x-1, y, t) + \frac{1}{4} P(x+1, y, t) + \frac{1}{4} P(x, y-1, t) + \frac{1}{4} P(x, y+1, t)$$

We just need an initial condition, for example $P(x, y, 0) = \delta_{x,0} \delta_{y,0}$.

We want to find the distribution function $P(x, y, t)$ over (x, y) at $t \gg 1$. Assuming that $P(x, y, t)$ varies slowly in time and space, we can use the Taylor expansions

$$P(x, y, t + \Delta t) = P(x, y, t) + \frac{\partial P(x, y, t)}{\partial t} \Delta t + \cdots,$$

$$P(x + \Delta x, y, t) = P(x, y, t) + \frac{\partial P(x, y, t)}{\partial x} \Delta x + \frac{1}{2} \frac{\partial^2 P(x, y, t)}{\partial x^2} (\Delta x)^2 + \cdots,$$

$$P(x, y + \Delta y, t) = P(x, y, t) + \frac{\partial P(x, y, t)}{\partial y} \Delta y + \frac{1}{2} \frac{\partial^2 P(x, y, t)}{\partial y^2} (\Delta y)^2 + \cdots,$$

and substitute them into the master equation.

The master equation takes a form

$$
\begin{aligned}
&P(x,y,t+1)\\
&=\frac{1}{4}P(x-1,y,t)+\frac{1}{4}P(x+1,y,t)+\frac{1}{4}P(x,y-1,t)+\frac{1}{4}P(x,y+1,t)\\
&\Leftrightarrow \cancel{P(x,y,t)}+\frac{\partial P(x,y,t)}{\partial t}=\cancel{\frac{1}{4}P(x,y,t)}-\cancel{\frac{1}{4}\frac{\partial P(x,y,t)}{\partial x}}+\frac{1}{4}\frac{1}{2}\frac{\partial^2 P(x,y,t)}{\partial x^2}\\
&\qquad\qquad\qquad\qquad\qquad +\cancel{\frac{1}{4}P(x,y,t)}+\cancel{\frac{1}{4}\frac{\partial P(x,y,t)}{\partial x}}+\frac{1}{4}\frac{1}{2}\frac{\partial^2 P(x,y,t)}{\partial x^2}\\
&\qquad\qquad\qquad\qquad\qquad +\cancel{\frac{1}{4}P(x,y,t)}-\cancel{\frac{1}{4}\frac{\partial P(x,y,t)}{\partial y}}+\frac{1}{4}\frac{1}{2}\frac{\partial^2 P(x,y,t)}{\partial y^2}\\
&\qquad\qquad\qquad\qquad\qquad +\cancel{\frac{1}{4}P(x,y,t)}+\cancel{\frac{1}{4}\frac{\partial P(x,y,t)}{\partial y}}+\frac{1}{4}\frac{1}{2}\frac{\partial^2 P(x,y,t)}{\partial y^2}
\end{aligned}
$$

Then, we get $\Rightarrow$ $$\frac{\partial P(x,y,t)}{\partial t}=\frac{1}{4}\frac{\partial^2 P(x,y,t)}{\partial x^2}+\frac{1}{4}\frac{\partial^2 P(x,y,t)}{\partial y^2}$$

Thus, we obtain the diffusion equation in a two dimensional system:

$$\frac{\partial P(x,y,t)}{\partial t} = D\Delta P(x,y,t)$$

Where the diffusion coefficient $D = {}^{1}\!/_{4}$ and the Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

For the initial condition $P(x,y,0) = \delta_{x,0}\delta_{y,0}$, the solution of this equation is

$$P(x,y,t) = \frac{1}{4\pi Dt} e^{-\frac{x^2+y^2}{4Dt}},$$

in complete agreement with our result above.

Diffusion:

Let us assume that at time $t = 0$ particles are distributed with the density $\rho(x,y,t=0)$. For this initial condition, the particle density at time t is

$$\rho(x,y,t) = \iint dx'dy'\,\rho(x',y',0)P(x-x',y-y',t)$$