MODELAÇÃO DE SISTEMAS COMPLEXOS

PROJECT 2

One dimensional random walks

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Part 1: One-dimensional random walks with symmetric jumps

The intent of this part is to simulate random walks of a particle on a one-dimensional chain, where the probabilities to jump on the left and on the right are equal to 0.5, and analyse the results.

1. A particle starts at t=0 at x=0 and makes t=50 steps. Plot three trajectories of random walks, i.e., plot the dependence of the particle position x(t) on time t:

$$x(t) = \sum_{i=1}^{t} S_i$$

where $S_i = \pm 1$.

We generated (in each iteration of the algorithm) with the *MATLAB* script, pseudo-random uniformly distributed numbers, between 0 and 1. And if there was a number bellow 0.5, then S=-1 (the particle would move to the left), and vice-versa for the right. The simulation was done for t=50 steps. As we see in figure 1, the positions eventually diverge between each other as time progresses. The observed trajectories in this example were very similiar untill the sixth second, and then the divergence is clearly visible. This results show the chaotic nature of the random walks, after merely some dozen steps.

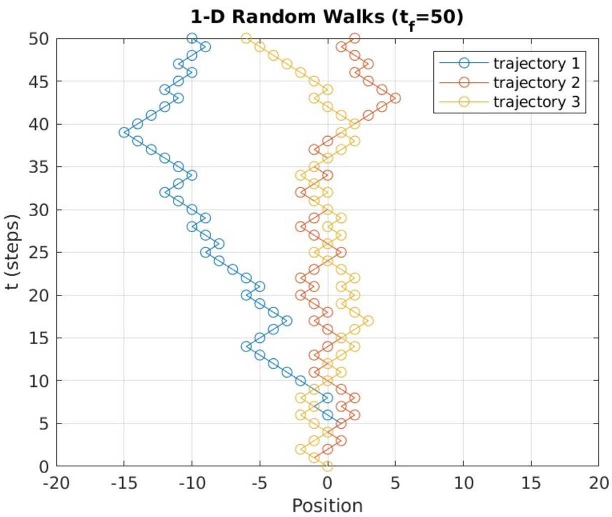


Figure 1: Three different trajectories of symmetric random walks, for t=50 steps and starting at x=0.

2. Compute the probability P(t,x) to find the particle at time t (i.e., after t jumps) at site x. At t=0 the particle places at site x=0. Use (i) t=40, 41; (ii) t= 400, 401; (iii) t= 4000, 4001, then average over even and odd t and find the averaged probability <P(x,t)>. Compare between your simulations and the theoretical distribution function:

$$P(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \tag{1}$$

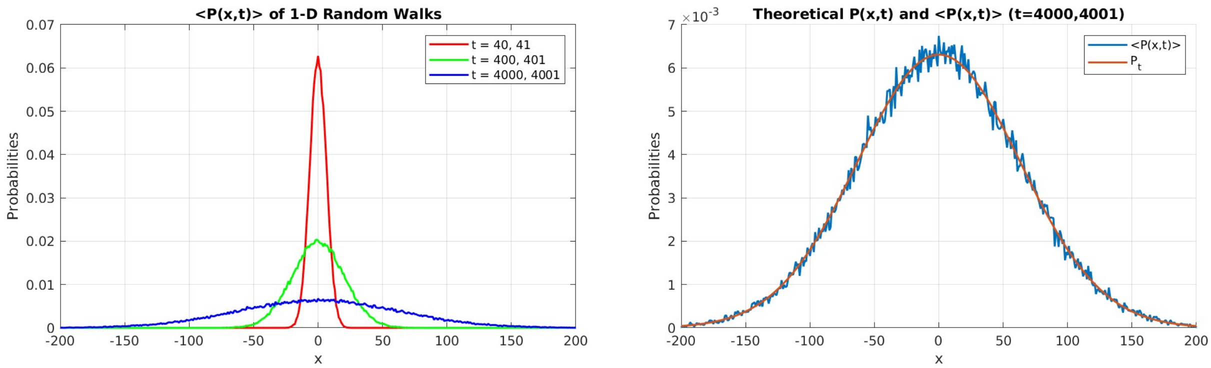
By following the algorithm presented in the work sheet, the averaged probability in function of the position is computed for the three test cases presented (3 pairs of t steps). From the resulting plots in figure 2a we see that as the final step increases, the curve stretches more, i.e., is less probable to end up near the starting position, as expected.

The Gaussian distribution presented in (1) is the solution to the 1-D diffusion equation,

$$\frac{\partial P(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 P(x,t)}{\partial^2 x^2}$$

which is our theoretical prediction.

Comparing the Gaussian function with the third case of the probability distributions, where t= 4000, 4001 (figure 2b), we see lots of similarities in the behavior, of course with some deviations and noise due to the randomness of the problem.



(a) Averaged probabilities of symmetric random walks, for 3 sets of times.

(b) Comparison of averaged probabilities with the Gaussian probability distribution.

Figure 2: Results of probability distribution for 1-D Random Walks and comparison with theoretical prediction.

The reason why we do the averaged probability for two consecutive final steps is because if the particle has an even number of steps in its trajectory, then the resulting final 1-D position will allways be even (because the initial position 0 is even), and the reverse for an odd number of steps. So for a propper smoothing of the probability distribution plot, it is better to compute an averaged probability between two consecutive t's, and the Gaussian function can be presented and be more representative of 1-D random walks.

The mean-squared error (MSE) measures the average of the squares of the errors—that is, the average squared difference between the estimated values and the actual value. By calculating the MSE between the Gaussian distribution and each of the cases of the averaged probabilities (i, ii, and iii), we get a good grasp on the precision of the experiments, so that we can somehow validate the theoretical probability function. The resulting errors from MSE are as follows:

- $MSE_{t=40,41} = 3.6105 \cdot 10^{-5}$
- $MSE_{t=400,401} = 0.6536 \cdot 10^{-5}$
- $MSE_{t=4000,4001} = 0.0010 \cdot 10^{-5}$

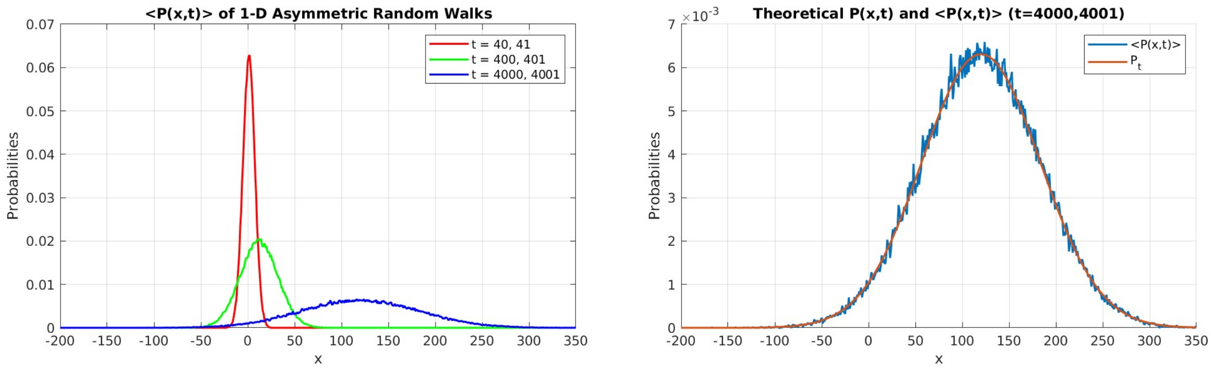
These results also prove that, as the time steps increase, the resulting probabilities converge more to the theoretical predictions. As referenced before aas t increases the chaos increases. We see that for t=4000,4001 there's a broader range of possible final positions of the particle, in figure 2a. For such case, the continuous theoretical function should present more similarities to the experimental function than for the other cases. This is because there are more broad probabilities presented and therefore, as the Gaussian function is continuous, the results of MSE should converge more. That's what happens here.

Part 2: Random walks with a drift

Simulate random walks of a particle on a one-dimensional chain with asymmetric probabilities. The probabilities to jump on the left and on the right equal to $p = 0.5 - \delta$ *and* $q = 0.5 + \delta$*, respectively. Take* $\delta = 0.015$*. Compute the probability* $P(x, t)$ *to find this particle after time t (i.e., after t jumps) at site x. At t=0 the particle places at site x =0. Use (i) t =40, 41; (ii) t = 400, 401; (iii) t = 4000, 4001, then average over even and odd t and find the averaged probabilities <P(x,t)>. Compare between your simulations and a theoretical distribution function,*

$$P(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-(x-2t\delta)^2/2t} \tag{2}$$

If we compare figures 2a and 3a, the outputs of this problem are very similar to the last one's (symmetric random walk). The differences in the probability distributions of the assymetric random walks lie in the x-axis displacement, or drift. The probabilities are displaced to higher positions of x. And this occurs as predicted, because any jump to the right (towards higher values of x) has higher probability ($q = 0.5015 (> p)$) than a jump to the left.



(a) Averaged probabilities of asymmetric random walks, for 3 sets of times.

(b) Comparison of averaged probabilities with the Gaussian probability distribution.

Figure 3: Results of probability distribution for 1-D Random Walks and comparison with theoretical prediction.

The mean-squared error between the theory (eq. (2)) and the experiments done was calculated, and the results we as follows:

- $MSE_{t=40,41} = 4.6425 \cdot 10^{-5}$
- $MSE_{t=400,401} = 1.5375 \cdot 10^{-5}$
- $MSE_{t=4000,4001} = 0.0009 \cdot 10^{-5}$

In formula (2), the term 2δ is the *drift* velocity of the maximum of $P(x,t)$. Hence, $x_{max} = v_d \cdot t = 2\delta t$ and so $max\ P(x,t)|_{x=2\delta t} = \frac{1}{\sqrt{2\pi t}}$. With this knowledge, we compare each three given maximum values of the averaged probabilities and then compare (in %) to the theoretical ones, by calculating the relative error (table 1). Judging from the low errors in the following table and from the previous results of the MSE, the theoretical function predicts well the 1-D assymetric random jumps.

N	$t = 40, 41$	$t = 400, 401$	$t = 4000, 4001$
$\max[P_{exp}(x,t)]$	0.0628	0.0205	0.0066
$\max[P_{theory}(x,t)]$	0.0627	0.0199	0.0063
Rel. Error (%)	0.2110 %	3.0366%	4.4798%

Table 1: Maximum values of the probabilities for each pair of t's, with theoretical and experimental values. Relative error calculation.