Lecture 1

Alexander Goltsev

Plan

1. Randomness. Random numbers.
2. Probability and probability distributions.
3. The central limiting theorem.

Randomness is a lack of pedictibility in events, symbols, steps, etc.

A random sequence of events, ~~or~~ symbols, or steps has no order.

Individual random events are by definition unpredictable.

Examples:

Throwing a coin. Eagle or tail?

Eagle = 1, tail = 0

1, 0, 0, 1, 1, 1, 0, 0,

After N attempts, ~~we~~ the eagle appears N_e times and the tail appears $N_t = N - N_e$ times.

$$N_e + N_t = N$$

We introduce the probability to find the eagle (tail).

We define

$$P_e \equiv \frac{N_e}{N}, \quad P_t = \frac{N_t}{N}$$

Normalization

$$P_e + P_t = \frac{N_e}{N} + \frac{N_t}{N} = \frac{N_e + N_t}{N} = 1$$

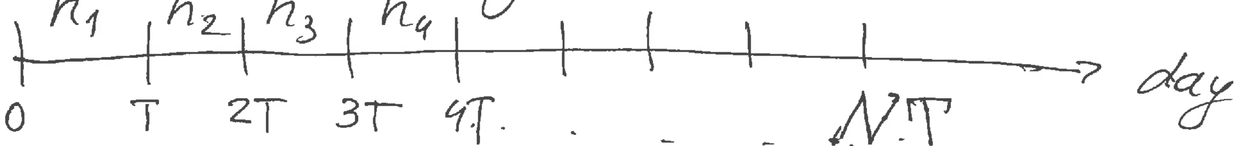
$$p_e + p_t = 1$$

Probability is defined only in the limit $N \to \infty$

$$p_e = \lim_{N\to\infty} \frac{N_e}{N}, \quad p_t = \lim \frac{N_t}{N}$$

Another example.

The number of cars that passed a street during an interval T



Sequence of random numbers

$n_1, n_2, n_3, \ldots$

$N(n)$ is the number of intervals when the number n (cars) appears

$$p(n) = \lim_{N\to\infty} \frac{N(n)}{N} = \text{the probability to find } n \text{ cars}$$

It is obvious that

$$\sum_{n=0}^{\infty} N(n) = N$$

~~the~~ <u>The normalization condition</u>

$$\sum_{n=0}^{\infty} p(n) = \sum_{n} \frac{N(n)}{N} = \frac{\sum_n N(n)}{N} = 1$$

$$\boxed{\sum_{n\geq 0} p(n) = 1}$$

<u>The mean value of random numbers</u>

$$\frac{1}{N}\sum_{i=1}^{N} n_i \equiv \langle n \rangle \quad (\text{or } \overline{n})$$

We can write this equation in another form

$$\langle n \rangle = \frac{1}{N}\sum_{i=1}^{N} n_i = \frac{1}{N}\sum_{n} N(n)\, n =$$

$$= \sum_{n} \frac{N(n)}{N} n = \sum_{n\geq 0} p(n)\, n$$

$$\langle n \rangle = \sum_{n\geq 0} p(n)\, n$$

Fluctuations. We define

$$\delta n_i \equiv n_i - \langle n \rangle$$

δn_i shows a deviation of n_i from the mean value.

It is obvious that

$$\sum_{i=1}^{N} \delta n_i = \sum_{i=1}^{N} (n_i - \langle n \rangle) =$$

$$= \sum_{i=1}^{N} n_i - \sum_{i=1}^{N} \langle n \rangle = N \frac{1}{N} \sum_i n_i - N\langle q \rangle =$$

$$= N\langle q \rangle - N\langle q \rangle = 0$$

Variance.

In order to find how strong are fluctuations we define the variance as follows

$$\sigma^2 \equiv \frac{1}{N} \sum_{i=1}^{N} \delta n_i^2$$

we have

$$\sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (n_i - \langle n \rangle)^2 = \frac{1}{N} \sum_{i=1}^{N} (n_i^2 - 2n_i \langle n \rangle +$$

$$+ \langle n \rangle^2) = \frac{1}{N} \sum_{i=1}^{N} n_i^2 - \frac{2\langle n \rangle}{N} \sum_{i=1}^{N} n_i + \langle n \rangle^2$$

$$= \langle n^2 \rangle - 2\langle n \rangle \langle n \rangle + \langle n \rangle^2$$

$$= \langle n^2 \rangle - \langle n \rangle^2$$

where

$$\langle n^2 \rangle = \frac{1}{N} \sum_{i=1}^{N} n_i^2$$

So, we get

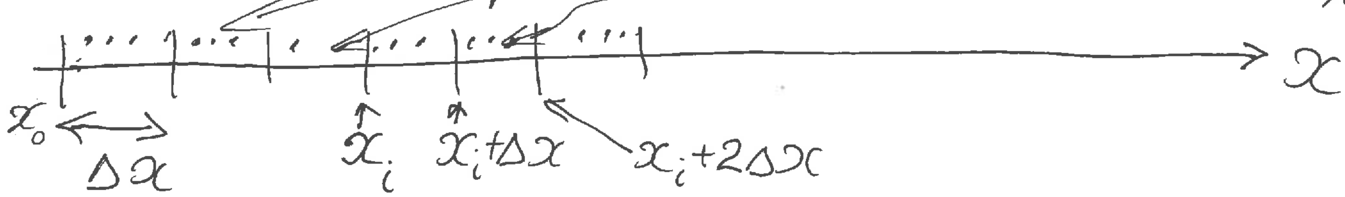
$$\sigma^2 = \langle (n - \langle n \rangle)^2 \rangle$$

$$= \langle n^2 \rangle - \langle n \rangle^2$$

Probability density distributions

In a general case, random numbers α_i can be arbitrary real numbers

We have a sequence, $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots \alpha_N$



We devide the axes into the intervals of the width Δx.

We define $\Delta N(x_i, x_i + \Delta x)$ as the number of random numbers α_i in the interval $[x_i, x_i + \Delta x]$, so

$$x_i < \alpha_i \leq x_i + \Delta x$$

Probability density

$$P(x_i) \equiv \lim_{\substack{\Delta x \to 0 \\ N \to \infty}} \frac{\Delta N(x_i, x_i + \Delta x)}{N \Delta x}$$

Normalization

$$\sum_{i \text{ (all intervals)}} P(x_i)\Delta x = \sum_i \frac{\Delta N(x_i, x_i+\Delta x)}{N\cancel{\Delta x}}\cancel{\Delta x} =$$

$$= \frac{\sum_i \Delta N(x_i, x_i+\Delta x)}{N} = \frac{N}{N} = 1$$

<u>Integral representation.</u> $\Delta N(x_i, x_i+\Delta) \gg 1$

In the limit $\Delta x \to 0$, $N \to \infty$, can write

$$\sum_i P(x_i)\Delta x = \int_{-\infty}^{\infty} P(x)\,dx$$

Thus, the normalization condition is

$$\int_{-\infty}^{\infty} P(x)\,dx = 1$$

The central limiting theorem. (The law of large numbers)

Hystory: Moivre (1733), Laplace (1812), Lyapunov (1901).

We have a sequence of random numbers

$$\underbrace{a_1, a_2, a_3 \dots a_N}_{\alpha=1} \Big| \underbrace{a_{N+1}, a_{N+2}, a_{N+3} \dots a_{2N}}_{\alpha=2} \Big| \dots$$

$$\dots \Big| a_{\alpha N+1} \dots a_{\alpha N} \Big|$$

We devide this sequence into groups of N numbers. The group are numerated by the index $\alpha = 1, 2 \dots M$. The total number of numbers is NM.

We introduce a new variable

$$X_\alpha \equiv \frac{1}{N} \sum_{i \in [N(\alpha-1)+1, N\alpha]} a_i$$

So we have a sequence: $X_1, X_2, X_3, \dots X_M$

The mean value of a

$$\langle a \rangle \equiv \frac{1}{NM} \sum_{i=1}^{NM} a_i$$

The variance

$$\sigma^2 = \langle (a - \langle a \rangle)^2 \rangle$$

Let us find the variance of random numbers X_α and the mean value of X_α

X

$$\langle X \rangle = \frac{1}{M}\sum_{\alpha=1}^{M} X_\alpha =$$

$$= \frac{1}{M}\sum_{\alpha=1}^{M} \frac{1}{N} \sum_{i\in[N(\alpha-1)+1,\, N\alpha]} a_i =$$

$$= \frac{1}{MN}\sum_{i=1}^{N} a_i = \langle a \rangle$$

Thus

$$\langle X \rangle = \langle a \rangle$$

Then, the varience

$$\Lambda^2 \equiv \langle (X - \langle X \rangle) \rangle =$$

$$= \frac{1}{M}\sum_{\alpha=1}^{N} (X_\alpha - \langle x \rangle)^2 = \frac{1}{M}\sum_{\alpha=1}^{N}\left[\frac{1}{N}\sum_{i\in[N(\alpha-1)+1,\, N\alpha]} (a_i - \langle a \rangle)\right]^2$$

$$= \frac{1}{MN}\sum_{\alpha=1}^{N}\left[\sum_{i_\alpha} (a_{i_\alpha} - \langle a \rangle)\right]^2$$

where

$$\sum_{i_\alpha} \equiv \sum_{i\in[N(\alpha-1)+1,\, N\alpha]}$$

Thus

$$\Lambda^2 = \frac{1}{MN}\sum_{\alpha=1}^{N}\sum_{i_\alpha}\sum_{j_\alpha} (a_{i_\alpha} - \langle a \rangle)(a_{j_\alpha} - \langle a \rangle)$$

Assuming that fluctuations $\delta a = a - \langle a \rangle$ are uncorrelated we find that

$$\sum_{i_\alpha} \sum_{j_\alpha} (a_{i_\alpha} - \langle a \rangle)(a_{j_\alpha} - \langle a \rangle) =$$

$$= \sum_{i_\alpha = j_\alpha} (a_{i_\alpha} - \langle a \rangle)^2 + \cancelto{0}{\sum_{i_\alpha \neq j_\alpha} \delta a_{i_\alpha} \delta a_{j_\alpha}} =$$

$$= \sum_{i_\alpha} \delta a_{i_\alpha}^2$$

Therefore

$$\Lambda^2 = \frac{1}{MN^2} \sum_{\alpha=1}^{M} \sum_{i_\alpha} \delta a_{i_\alpha}^2 =$$

$$= \frac{1}{MN} \sum_{\alpha=1}^{M} \left(\frac{1}{N} \sum_{i_\alpha} \delta a_{i_\alpha}^2 \right)$$

In the limit $M, N \to \infty$

$$\frac{1}{N} \sum_{i_\alpha} \delta a_{i_\alpha}^2 = \sigma^2$$

We get

$$\Lambda^2 = \frac{1}{MN} \sum_{\alpha=1}^{M} \sigma^2 = \frac{\sigma^2}{N}$$

The central limiting theorem states that

$$\boxed{\Lambda^2 = \frac{\sigma^2}{N}}$$

Applications of the central limiting theorem.

Kinetic energy ε_i of molecules in gases.
The mean energy of a molecule

$$\langle \varepsilon \rangle = \frac{1}{N}\sum_i \varepsilon_i$$

The variance

$$\sigma^2 = \langle (\varepsilon - \langle \varepsilon \rangle)^2 \rangle$$

The total energy of molecules per molecule

$$E(N) = \frac{1}{N}\sum_{i=1}^{N} \varepsilon_i$$

$$\langle E(N) \rangle = \langle \varepsilon \rangle$$

The variance

$$\Lambda^2 = \langle (E(N) - \langle E(N) \rangle)^2 \rangle = \left\langle \left(\frac{1}{N}\sum_i (\varepsilon_i - \langle \varepsilon \rangle) \right)^2 \right\rangle$$

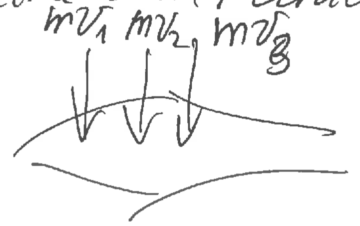
$$= \frac{\langle \sigma^2 \rangle}{N}$$

In the limit $N \to \infty$ ($N_A = 10^{23}$)
The variance of the total energy is very small

$$\Lambda^2 = \frac{\langle \sigma^2 \rangle}{N} \ll \langle \sigma^2 \rangle$$

The total energy is defined by the temperature T. The total energy defines all thermodynamic properties. A small varience means <u>stability of thermodynamic properties with respect to fluctuations.</u>

The pressure of air on our skin is determined by the total momentum of molecules that hit our skin per unit time and unit square



$$P = \frac{1}{N_\Delta} \sum_i m v_i$$

$$\langle (P - \langle P \rangle)^2 \rangle = \frac{\langle m v \rangle}{N_\Delta} \ll \langle m v \rangle$$