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Systemas Complexos e Desordenados

Lecture 3

Random walks

In 1785, Jan Ingenhousz (phisiologist, biologist) observed coal dust particles moving on the surface of alcohol.

In 1827, Robert Brown (botanist) observed pollen particles floating in water

Einstein (1905) found a relationship between difision coefficient D and the temperature T

$$D = \mu K_B T$$

μ is the mobility.

Smoluchowski (1906) derived the smoluchowski' equations.

Applications:

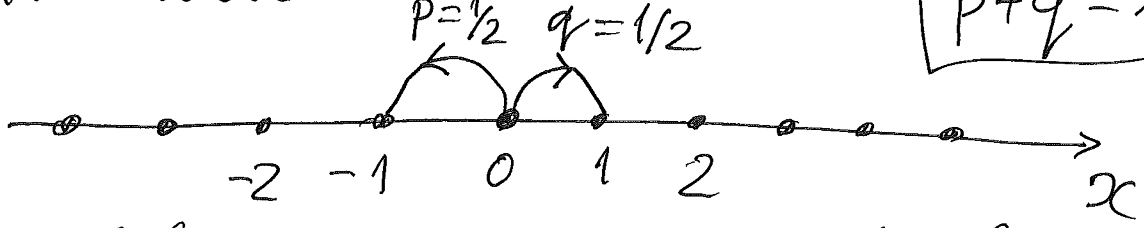
- diffusion of particles
- folding of polymer molecules
- economics (fluctuations of stockmarket)
- ecology and biology (animals searching a food

- computer science
- mathematics (spectrum of random matrices)
- neuroscience (activation of neurons)

One-dimensional random walks

A particle jumps randomly on 1D lattice

$\boxed{p+q=1}$



The Probabilities to jump on the left is p, q is the probability to jump on the right.

At first we consider the case

$$p = q = \frac{1}{2}$$

Each time step the particle jump either on the right or on the left.

Where is the particle after t jumps?

$$x(t) - ?$$

We introduce a random variable

$$s = \pm 1 = \begin{cases} +1, & \text{jump on the right} \\ -1, & \text{jump on the left} \end{cases}$$

The probability distribution

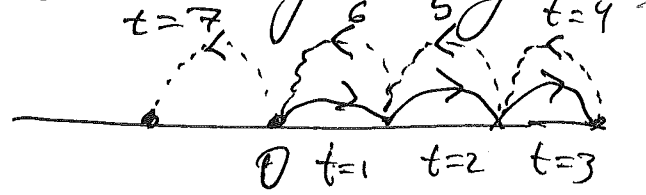
$$P(s) = \underbrace{\frac{1}{2}\delta(s-1)}_{\text{right}} + \underbrace{\frac{1}{2}\delta(s+1)}_{\text{left}}$$

After t jumps the particle will be at the point

$$x(t) = \sum_{i=1}^{t} s_i \equiv x(s_1, s_2 \ldots s_t)$$

where $s_i = \pm 1$ for the jump i.

Every random walk is characterized by its trajectory → trajectory is a list $[1, 1, 1, -1, -1, -1, -1]$



The total number of trajectories is

$$2^t \qquad [\underbrace{\pm 1, \pm 1, \ldots, \pm 1}_{t}] \rightarrow \underbrace{2 \cdot 2 \cdots 2}_{t} = 2^t$$

The mean value of $x(t)$

$$\langle x(t) \rangle = \frac{1}{2^t} \sum_{\text{trajectories}} x(s_1, s_2, \ldots s_t)$$

$$\sum_{\text{trajectories}} = \sum_{s_1=\pm1}\sum_{s_2=\pm1}\dots\sum_{s_t=\pm1}$$ (4)

$$\langle x(t)\rangle = \frac{1}{2^t}\sum_{\{s_i=\pm1\}}\left(\sum_{i=1}^{t} s_i\right) = \sum_{i=1}^{t}\langle s_i\rangle_{\text{traj}}$$

Summation over trajectories

$$\frac{1}{2^t}\sum_{\{s_i=\pm1\}} = \prod_{i=1}^{t} P(s_i) = \frac{1}{2^t}\prod_{i=1}^{t}\left(\sum_{s_i=\pm1}\right)$$

the mean value of a single jump

$$\langle x(t)\rangle = \frac{1}{2}\sum_{i=1}^{t}\left(\sum_{s_i=\pm1} s_i\right)$$

$$\sum_{s_i=\pm1} s_i = +1-1 = 0$$

We find

$$\langle x(t)\rangle = 0$$

The variance

$$\langle (x(t)-\langle x(t)\rangle)^2\rangle = \langle x^2(t)\rangle =$$

$$= \left\langle\left(\sum_{i=1}^{t} s_i\right)^2\right\rangle =$$

$$= \sum_{i=1}^{t}\sum_{j=1}^{t}\langle s_i s_j\rangle =$$

$$= \sum_{i=j=1}^{t} \langle s_i^2 \rangle + \sum_{i \neq j}^{t} \langle s_i s_j \rangle$$

We have

$$s_i^2 = (\pm 1)^2 = 1$$

Assuming that the jump are independent (the Markov process), i.e. there is no correlation between jump, or, in other words there is no memory, we

we obtain

$$\langle s_i s_j \rangle = \langle s_i \rangle \langle s_j \rangle = 0$$

($s_i = \pm 1$, $s_i = \pm 1$) with equal probability

Therefore the variance is

$$\langle x^2(t) \rangle = \sum_{i=1}^{t} \langle s_i^2 \rangle = \sum_{i=1}^{t} 1 = t$$

$$\boxed{\langle x^2(t) \rangle = t}$$

$$\sqrt{\langle x^2(t) \rangle} = \sqrt{t}$$

The mean distance from the starting point increases as $\sqrt{t}$.

Exact solution of 1D random walks

We want to find the probability $P(x,t)$ to observe the particle after t jumps at the point x.

$$P(x,t) = \frac{\text{the number of trajectories that end up at the point } x \text{ after } t \text{ jump}}{\text{the total number of trajectories}} = \frac{N(x,t)}{2^t}$$

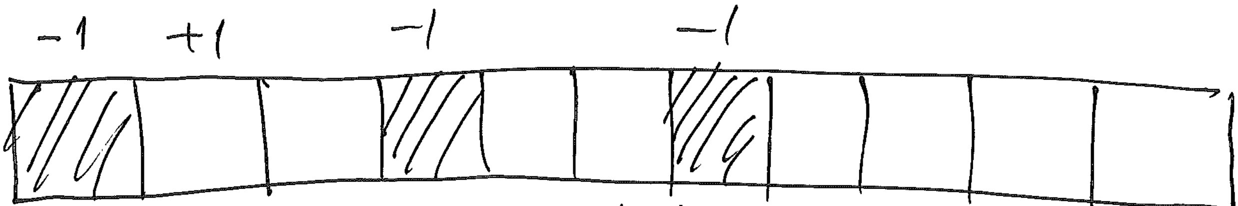
~~mm~~ Let us after t jumps we have n_+ jumps on the right, and n_- jumps on the left

$$\begin{cases} t = n_+ + n_- \\ x = n_+ - n_- \end{cases} \Rightarrow \begin{cases} n_+ = \frac{t+x}{2} \\ n_- = \frac{t-x}{2} \end{cases}$$

Obviously

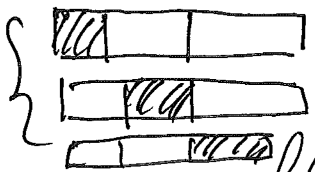
$$-t \leq x \leq t$$

Representation of a trajectory



black box jump on the left

white box jump on the right



$t=3$
$n_+=2$
$n_-=1$

We have n_+ white boxes and n_- black boxes. The number of ways we can distribute the boxes gives us the number of trajectories

The number of permutations is

$$C^{t}_{n_+} = \frac{t!}{(t-n_+)!\,n_+!} = \frac{t!}{n_+!\,n_-!} = \frac{t(t-1)\ldots(t-n_++1)}{n_+!}$$

$$= \frac{t!}{\left(\frac{t+x}{2}\right)!\left(\frac{t-x}{2}\right)!}$$

The probability distribution function

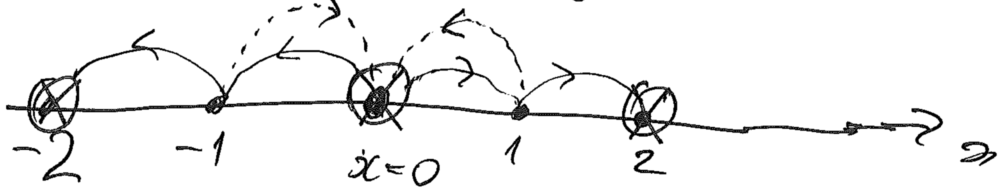
$$\boxed{P(x,t) = \frac{C^{t}_{n_+}}{2^t} = \frac{1}{2^t}\frac{t!}{\left(\frac{t+x}{2}\right)!\left(\frac{t-x}{2}\right)!}}$$

This is ~~the~~ exact result.

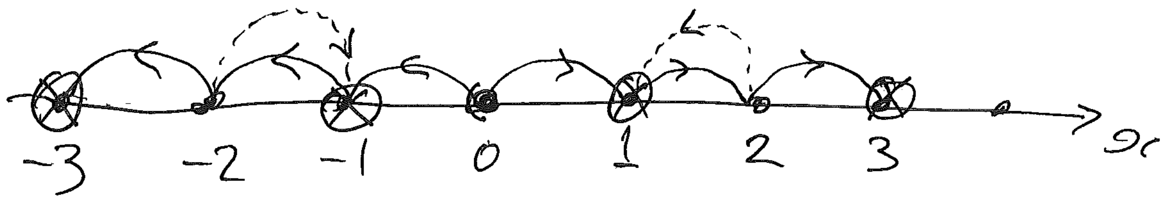
Note that after even number of jumps, i.e. $t = 2m$, we can reach a point with an even coordinate, $x = 2l$, $l = 0, \pm 1, \pm 2, \dots$

After an odd jumps, $t = 2m+1$, we can reach only points with an odd coordinate $x = 2l+1$

$t = 2$



$t = 3$



Particular cases:

$x = 0$

$$P(0,t) = \frac{1}{2^t} \frac{t!}{\left(\left(\frac{t}{2}\right)!\right)^2}$$

$x = t$

$$P(t,t) = \frac{1}{2^t} \frac{\cancel{t!}}{\cancel{t!}\, 0!} = e^{-t \ln 2}$$

Aproximate formular for $P(x,t)$.

We use the Stirling formular

$$t! \cong \sqrt{2\pi t}\, t^{t} e^{-t}$$

We consider the case

$$t \gg 1, \qquad |x| \ll t$$

then $\quad t \pm x \gg 1$

$$P(x,t) = \frac{1}{2^t} \frac{t!}{\left(\frac{t+x}{2}\right)! \left(\frac{t-x}{2}\right)!}$$

$$\left(\frac{t\pm x}{2}\right)! = \sqrt{2\pi\left(\frac{t\pm x}{2}\right)} \left(\frac{t\pm x}{2}\right)^{(t\pm x)/2} e^{-\frac{t\pm x}{2}}$$

$$= \left(\frac{t}{2}\right)^{\frac{1}{2}+\frac{t+x}{2}} \sqrt{2\pi\left(1+\frac{x}{t}\right)} \left(1+\frac{x}{t}\right)^{\frac{t+x}{2}} e^{-\frac{t+x}{2}}$$

$$= \left(\frac{t}{2}\right)^{\frac{1}{2}+\frac{t+x}{2}} \sqrt{2\pi}\, \exp\left[-\frac{t+x}{2} + \left(\frac{1}{2}+\frac{t+x}{2}\right) \ln\left(1+\frac{x}{t}\right)\right]$$

We use the Teylor expansion

$$\ln(1+a) = a - \frac{a^2}{2}$$

$$\ln\left(1+\frac{x}{t}\right) = \frac{x}{t} - \frac{x^2}{2t^2}$$

$$\exp\left[-\frac{t+x}{2}+\left(\frac{1}{2}+\frac{t+x}{2}\right)\ln\left(1+\frac{x}{t}\right)\right]=$$

$$\cong\exp\left[-\frac{t+x}{2}+\left(\frac{1}{2}+\frac{t+x}{2}\right)\left(\frac{x}{t}-\frac{x^2}{2t^2}\right)\right]$$

$$=\exp\left[-\frac{t}{2}\cancel{-\frac{x}{2}}+\frac{x}{2t}+\cancel{\frac{t}{2}\cdot\frac{x}{t}}+\boxed{\frac{x^2}{2t}}-\frac{x^2}{4t^2}-\right.$$

$$\left.-\boxed{\frac{x^2}{4t}}-\frac{x^3}{4t^2}\right]$$

$$\cong\exp\left[-\frac{t}{2}+\frac{x}{2t}+\frac{x^2}{4t}\right]$$

We arrive at

$$\left(\frac{t+x}{2}\right)!\cong\left(\frac{t}{2}\right)^{\frac{t+x+1}{2}}\sqrt{2\pi}\,\exp\left[-\frac{t}{2}+\frac{x}{2t}+\frac{x^2}{4t}\right]$$

$$\left(\frac{t-x}{2}\right)!\simeq\left(\frac{t}{2}\right)^{\frac{t-x+1}{2}}\sqrt{2\pi}\,\exp\left[-\frac{t}{2}-\frac{x}{2t}+\frac{x^2}{4t}\right]$$

Product

$$\left(\frac{t+x}{2}\right)!\left(\frac{t-x}{2}\right)!\approx 2\pi\left(\frac{t}{2}\right)^{t+1}\exp\left[-t+\frac{x^2}{2t}\right]$$

Come back to $P(x,t)$

$$P(x,t) = \frac{1}{2^t} \frac{t!}{\left(\frac{t+x}{2}\right)! \left(\frac{t-x}{2}\right)!}$$

$$\approx \frac{1}{2^t} \frac{\sqrt{2\pi}\, t^{t+1/2} \cancel{e^{-t}}}{2\pi \left(\frac{t}{2}\right)^{t+1} e^{\cancel{-t} + \frac{x^2}{2t}}}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\left(\frac{1}{2}\right)} \frac{1}{t^{1/2}} e^{-\frac{x^2}{2t}}$$

$$\approx \frac{2}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

Now we must average over odd and even t. This result in deviding on 2

Among two attempts only one leads to a given x.

$$\Rightarrow \boxed{P(x,t) \approx \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}}$$

Normalization

$$\int_{-\infty}^{\infty} dx\, P(x,t) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2t}} dx =$$

$z = \frac{x}{\sqrt{t}}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz = \frac{\sqrt{2\pi}}{\sqrt{2\pi}} = \underline{1}$$

The Probability to observe the particle at the starting point $x = 0$

$$P(0,t) = \frac{1}{\sqrt{2\pi t}}$$