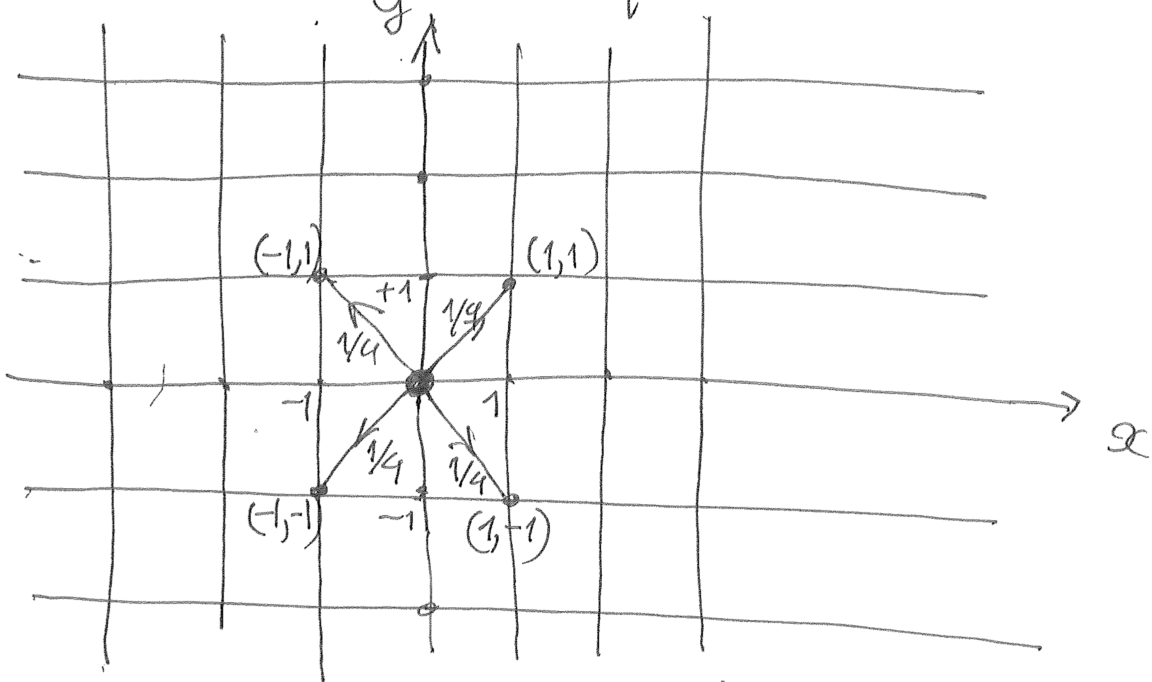
Lecture 6

2D-random walks

Let us study random walks of a particle in a 2D-square lattice.



We consider the case when the particle jumps along diagonals. Every of 4 possible jumps has the probability 1/4. After a jump, the particle will be in a site with coordinate

$$(x,y) = (\pm 1, \pm 1).$$

or

$$= (S_x, S_y)$$

wher $S_x = \pm 1$, and $S_y = \pm 1$

We can consider the jump, as simultaneous jumps along x and y axes with steps $S_x = \pm 1$ and $S_y = \pm 1$.
These jumps along x and y axes are assumed to be uncorrelated and have the probabilites

$$p_x(S_x) = \frac{1}{2}, \quad p_y(S_y) = \frac{1}{2}$$

i.e., $p_x(1) = p_x(-1) = p_y(+1) = p_y(-1)$

The probability to reach the site $(\pm 1, \pm 1)$ is

$$P(S_x, S_y) = P_x(S_x)\, P_y(S_y) = \frac{1}{4}$$

Let us find the probability that after t jumps our particle will be in a point (x, y).

As for 1D – random walks, we have

$$x = S_x(1) + S_x(2) + \dots S_x(t) = \sum_{i=1}^{t} S_x(i)$$

$$y = S_y(1) + S_y(2) + \dots S_y(t) = \sum_{i=1}^{t} S_y(i)$$

If the particle made $n_+^{(x)}$ jumps with $S_x = +1$ and $n_+^{(y)}$ jumps with $S_y = +1$, then

$$x = n_+^{(x)} - n_-^{(x)}$$

$$y = n_+^{(y)} - n_-^{(y)}$$

$$t = n_+^{(x)} + n_-^{(x)} = n_+^{(y)} + n_-^{(y)}$$

Therefore

$$\begin{cases} n_+^{(x)} = \dfrac{t+x}{2} \\ n_-^{(x)} = \dfrac{t-x}{2} \end{cases} \qquad \begin{cases} n_+^{(y)} = \dfrac{t+y}{2} \\ n_-^{(y)} = \dfrac{t-y}{2} \end{cases}$$

Probability to reach ~~the~~ site (x, y) after t jumps is the product

$$P(x,y,t) = \frac{C^t_{n_+^{(x)}}}{2^t} \cdot \frac{C^t_{n_+^{(y)}}}{2^t}$$

where we used the result for 1D random walks.

Since at $t \gg 1$, $|x| \ll t$

$$\frac{C^t_{n_+^{(x)}}}{2^t} \approx \sqrt{\frac{2}{\pi t}}\, e^{-\frac{x^2}{2t}}$$

we obtain

$$P(x,y,t) \simeq \frac{2}{\pi t} e^{-\frac{x^2+y^2}{2t}}$$

Let us ~~Taking~~ take into account that at <u>even</u> t $(t=2m)$ we can reach only sites with even x and y, while for <u>odd</u> t we can reach only odd x and y. ~~m~~ this results in ~~the~~ multiplier $1/4$.

The exact result is

$$P(x,y,t) = \frac{1}{2\pi t} e^{-\frac{x^2+y^2}{2t}}$$

Normalization

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, P(x,y,t) = \left(\int_{-\infty}^{\infty} \frac{dx \, e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} \right) \left(\int_{-\infty}^{\infty} \frac{dy \, e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}} \right) = 1$$

Mean values

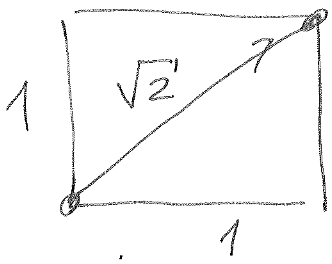
$$\langle x \rangle = \langle y \rangle = \int dx dy \, x P(x,y,t) = 0$$

Variance

$$\langle x^2 \rangle = \int dx dy \, x^2 P(x,y,t) = t$$

$$\langle x^2 + y^2 \rangle = 2t$$

In our random walks, the length of jumps is $\sqrt{2}$



If we make substitution

$$x \to \sqrt{2}\,\tilde{x}$$
$$y \to \sqrt{2}\,\tilde{y}$$

Then in terms of $\tilde{x}$ and $\tilde{y}$ the jump will have the lengh 1

$$P(\tilde{x},\tilde{y},t) = \frac{1}{\pi t} e^{-\frac{\tilde{x}^2+\tilde{y}^2}{t}}$$

Then the mean distance from the initial point is

$$\langle \tilde{x}^2+\tilde{y}^2 \rangle = \langle \tilde{r}^2 \rangle = t$$

$$\langle \tilde{r}^2 \rangle = t$$

Random walks in D-dimensional lattice with jumps along diagonals

$$(x, y, z, \ldots) = (\pm 1, \pm 1, \pm 1, \ldots)$$

We find the probability

$$P(x, y, z, \ldots, t) = \frac{1}{(2\pi t)^{D/2}} e^{-\frac{r^2}{2t}}$$

$$r^2 = x^2 + y^2 + z^2 + \ldots$$

Probabily to find the particle in the initial point $\vec{x} = 0$ is

$$P(\vec{r}=0, t) = \frac{1}{(2\pi t)^{D/2}} \propto \begin{cases} \frac{1}{t^{1/2}}, & D = 1 \\ t^{-1}, & D = 2 \\ t^{-3/2}, & D = 3 \end{cases}$$

Thus measuring $P(\vec{r}=0, t)$, we can find the dimensionality D of the lattice.