Lecture 7.

First passage time and survival probability

Lévy flights.

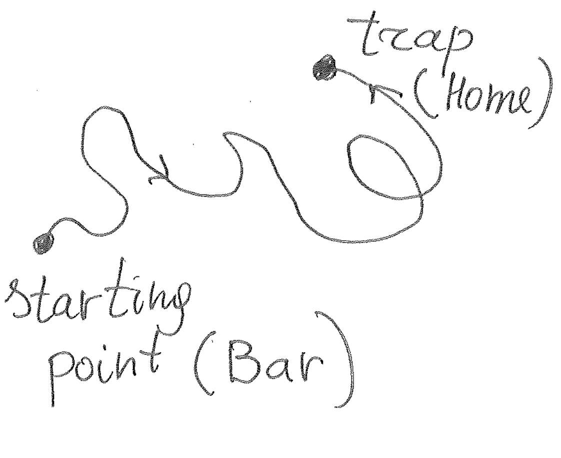
We have showed that the probability ~~this~~ to find a particle at point $(x, y, z, \dots)$ after t random jumps is

$$P(x, y, z, \dots, t) = \frac{1}{(\pi t)^{d/2}} e^{-\frac{x^2+y^2+\dots}{t}}$$

Probability to find the particle in the starting point is

$$P(0,t) = \begin{cases} \frac{1}{t^{1/2}}, & d=1 \\ t^{-1}, & d=2 \\ t^{-3/2}, & d=3 \\ t^{-d/2}, & d \end{cases}$$

First passage time



What is the probability to reach the trap after t jumps?

Definition

The first passage time probability $F(\vec{r},t)\Delta t$ is the probability that a particle hits a specific point $\vec{r}$ for the first time during time interval $[t,t+\Delta t]$

$F(\vec{r},t)$ is the probability density of the first passage time process.

$$F(\vec{r},t) = \frac{\text{the number of trajectories that ends up at point } \vec{r} \text{ for the first time in the interval } [t,t+\Delta t]}{\text{total number of trajectories}} \cdot \frac{1}{\Delta t}$$

Survival probability

The survival probability $S(t)$ is the probability that a particle do not hit the absorbing trap (or boundary) during the time interval $[0,t]$.

Calculation of $S(t)$..
We have N particles (or we make N attempts).
At time $t=0$ they stay at point $\vec{r}=0$.
Then the particles make random walks.

If a particle hits the trap, it dissappears. After time t (t jumps) we calculate the number of survival particles $N_s(t)$. The survival probability

$$S(t) = \lim_{N \to \infty} \frac{N_s(t)}{N}$$

Relationship between $F(t)$ and $S(t)$

$$S(t) = 1 - \int_0^t F(t)\,dt$$

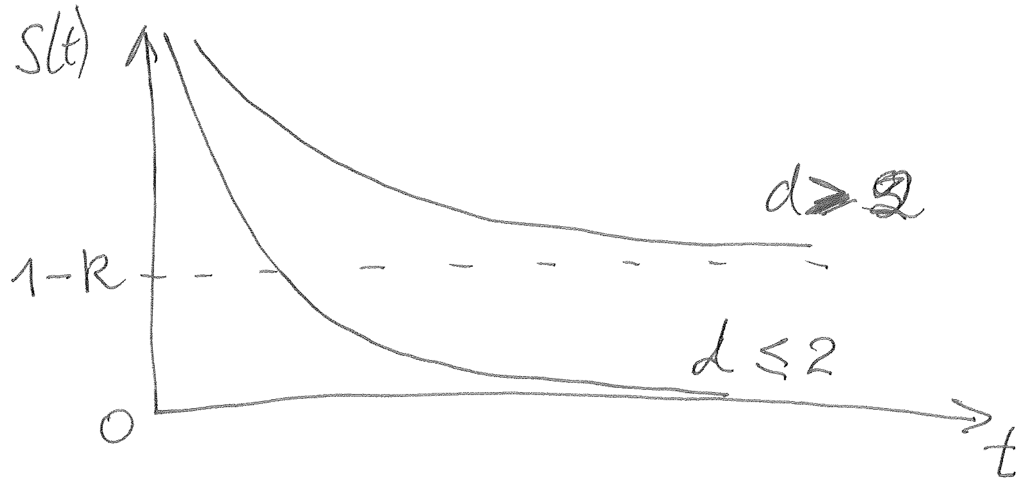
probability to hit the trap before time t.

$$\frac{dS}{dt} = -F(t)$$

Theoretical calculations show that at $t \gg 1$

$$F(t) \propto \begin{cases} t^{-3/2}, & d = 1 \\ \dfrac{1}{t \ln^2 t}, & d = 2 \\ \dfrac{1}{t^{d/2}}, & d \geq 3 \end{cases}$$

$$S(t) = \begin{cases} \dfrac{1}{t^{1/2}}, & d = 1 \\ \dfrac{1}{\ln t}, & d = 2 \\ \text{const} + A t^{1-\frac{d}{2}}, & d \geq 3 \end{cases}$$



A random walker has a finite probability to survive in a system with a trap if dimentionality of the system is larger than 2.

Randamly Walking in a town you will reach your home sooner or later.

Lévy flights.

Lévy flights are random walks whose step lengths are not constant but rather are chosen from a probability distribution with a power-law tail.

The aim is to explain the cases when $\langle R^2 \rangle \propto t^{\gamma}, \gamma \neq 1$

Realization of Lévy flights:

Physics:

fluid dynamics,

turbulent diffusion,

chaotic phase diffusion in Josephson junctions

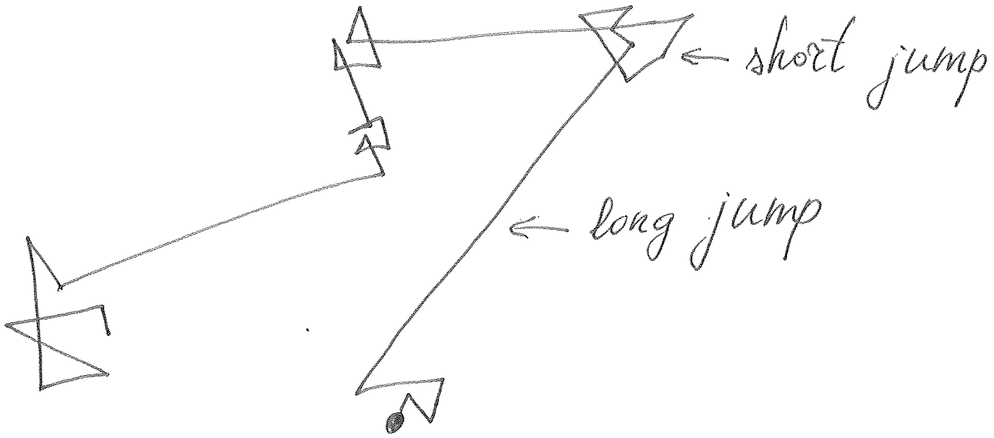
slow relaxation in glassy materials

Biology

The Lévy flights can represent an optimal search strategy.

Predators may use the search strategy where prey is sparse and distributed unpredictably.

Predators: sharks, tuna fishes, wandering albatrosses, An example of a Lévy flight trajectory.



Generations of Lévy flights

We aim to find a simple way to generate jumps of length l with probability

$$P(l) = \frac{C}{l^{\mu}}, \quad l_{min} \le l \le l_{max}$$

Normalization constant

$$\int_{1}^{l_{max}} P(l)\,dl = \int_{1}^{l_{max}} \frac{C}{l^{\mu}}\,dl = \left.\frac{-C}{(\mu-1)\,l^{\mu-1}}\right|_{1}^{l_{max}}$$

$l_{min} = 1$

$$= \frac{C}{\mu-1}\left[1 - \frac{1}{l_{max}^{\mu-1}}\right]$$

We find

$$\Rightarrow C = \frac{\mu-1}{1 - l_{max}^{1-\mu}}$$

Let us find a new random variable x distributed according $G(x)$

$$\int_1^{l_{max}} P(l)\, dl = \int_0^1 G(x)\, dx$$

Relation

$$P(l)\, dl = G(x)\, dx ,$$

We consider the case when x is a ~~random~~ random variable that distributed uniformly in the interval $[0, 1]$, i.e., $G(x) = 1$

we obtain equation

$$P(l) dl = dx \implies \frac{dx}{dl} = P(l)$$

In the case of Lévy flights

$$\frac{dx}{dl} = \frac{c}{l^{\mu}}$$

Solution of this equation

$$x = \frac{c\, l^{1-\mu}}{1-\mu} + A$$

Therefore

$$l = \left[(1-\mu)(x-A)\right]^{\frac{1}{1-\mu}}$$

In order to find A, we impose conditions

$$l(0) = l_{max} , \quad l(x=1) = 1$$

We find

$$l(x) = \frac{l_{max}}{\left[(l_{max}^{\mu-1} - 1)x + 1\right]^{1/(\mu-1)}}$$

If we generate random numbers

$$x_1, x_2, x_3, \ldots x_N$$

then we have random numbers

$$l(x_1), l(x_2), l(x_3), \ldots l(x_N)$$

Let us prove that the random numbers $l(x)$ are distributed according to the Lévy distribution.

Distribution function of $l(x)$ is

$$P(l) = \frac{[\text{the number of points in the interval } l < l(x) \leq l + \Delta l]}{N \Delta l}$$

In the limit $N \to \infty$, $\Delta l \to 0$ we have

$$P(l) = \int_0^1 dx\, G(x)\, \delta(l - l(x))$$

$$= \int_0^1 \delta(l - l(x))\, dx = \frac{1}{\left|\frac{dl}{dx}\right|_{l=l(x)}} = \frac{C}{l^\mu}$$

where we used that $\frac{dx}{dl} = \frac{C}{l^\mu}$

Thus, random numbers $l(x)$ are distributed according to the Lévy distribution.