<u>Project 3.</u>
Two-dimensional random walks.
First passage time and the survival probability.
Lévy flights.

1. Two-dimensional random walk.

Simulate symmetric random walks on the square lattice. Probabilities of jumps on the left, on the right, up and down are equal, 1/4.

Task 1.1

Compute and plot the probability $P(x,y,t)$ to find this particle after time t (i.e., after t jumps) at site (x,y). At $t=0$ the particle is at site $(x,y)=0$. Use $t = 5000$, 5001, then average over even and odd t and find the averaged probabilities $\overline{P}(x,y,t)$. Compare between your simulations and a theoretical distribution function,

$$P(x,y,t)=\frac{1}{\pi t}e^{-(x^2+y^2)/t}.$$

Task 1.2

Plot any 3 trajectories after 100 jumps.

Algorithm for random walks on a square lattice.

Start from the point (x,y)=(0,0).

1. Generate at random integer numbers A=1, 2, 3, and 4. If A=1 then the particle jumps up, if A=2 it jumps on the right, if A=3 it jumps down, and if A=4 it jumps on the left.
2. Update the point (x,y).
3. Repeat steps 1 and 2 t times. You will get a trajectory. Make N trajectories.
4. Calculate the number N(x,y,t) of times when the end point of the trajectory is at site (x,y). Calculate the probability
$$P(x,y,t)=N(x,y)/N$$
5. Then average over even and odd times: $\overline{P}(x,y,t)=\frac{1}{2}(P(x,y,t)+P(x,y,t+1))$.
6. Plot the averaged distribution function $\overline{P}(x,y,t)$ in the (x,y)-plane.

7. Check that $\sum_{x,y} \overline{P}(x,y,t) = 1$, $\sum_{x,y} \overline{P}(x,y,t)x = \sum_{x,y} \overline{P}(x,y,t)y = 0$, $\sum_{x,y} \overline{P}(x,y,t)(x^2 + y^2) = t$
8. Compare $\overline{P}(x,y,t)$ with the theoretical distribution function

$$P(x,y,t) = \frac{1}{\pi t} e^{-(x^2+y^2)/t} ..$$

2. First passage time and the survival probability

There is an absorbing boundary along the vertical line at $x_c = -30$, i.e., particles walk in the semi-plane with -30≤x. The starting point is at **R**=(0,0). If the particle hits this boundary, then the walk is stopped. Repeat these random walks. The number of attempts N=50000 times. (In other words, there are N=50000 particles.)

Task 2.1

Calculate the number N_{fpt} (t) of particles which hit the boundary for the first time during a time interval [t,t+Δt], Δt=10. Plot the first passage time probability F(t)= N_{fpt}(t)/(NΔt) to hit the boundary for the first time at time interval [t,t+Δt] (the first passage time probability). Analyse behaviour of F(t) at large t [let the maximum time be 50 000 jumps]. For this purpose plot the function ln$F$(t) versus ln t (log-log plot).

Calculate the survival probability S(t). For this purpose, find the number N_s(t) of particles which are not trapped, i.e., survive, at time t. By definition, S(t)= N_s(t)/N. Plot S(t).

Compare between simulation results and the theoretical predictions:

$$S(t) = erf\left(\frac{x_c - x_0}{2\sqrt{Dt}}\right),$$

$$F(t) = \frac{|x_c - x_0|}{\sqrt{4\pi D t^3}} exp\left(-\frac{(x_c - x_0)^2}{4Dt}\right),$$

$$F(t) \propto \frac{1}{t^{3/2}},\ S(t) \propto \frac{1}{t^{1/2}}$$

Here, $x_0 = 0$ is the starting point, $x_c = -30$ is the boundary position, and $D = 1/4$ is the diffusion coefficient.

2. Lévy flights.

This task is aimed to study 2D-random walks with variable length of jumps. The probability $P(l)$ that a jump has the length l is determined by the Lévy distribution,

$$P(l) = \frac{c}{l^{\mu}},$$

where c is the normalization constant. If the minimum and maximum lengths of jumps are $l_{min} = 1$ and $l_{max} = 1000$, respectively, then the normalization constant c is

$$c = \frac{\mu - 1}{1 - l_{max}^{1-\mu}}$$

After a jump the particle will be at a point with coordinates $(x, y) = (l \cos\varphi, l \sin\varphi)$ with respect to the initial point. Jumps are isotropic. It means that the probability to jump at an angle φ is constant, $p(\varphi) = 1/2\pi$.

Task 3.1

For three values of the exponent μ=1.6, 2, and 2.6 generate trajectories with N=1000 random jumps. Plot these three trajectories and compare with isotropic 2D-random walks having a fixed length of jumps, l=1. Analyse qualitatively the trajectories.

Task 3.2

Show that, if x is a random number generated uniformly at random in the interval [0,1], then the random numbers

$$l = \frac{l_{\max}}{\left[(l_{\max}^{\mu-1} - 1)x + 1\right]^{1/(\mu-1)}}$$

are distributed according the Levy flights distribution, $P(l) = \dfrac{c}{l^{\mu}}$.

Algorithm:

In order to generate the Lévy flights use the following method.

1. Generate a random number $x \in [0,1]$ with the uniform probability.
2. Calculate the length of jump as follows:

$$l = \frac{l_{\max}}{\left[(l_{\max}^{\mu-1} - 1)x + 1\right]^{1/(\mu-1)}}$$

3. Generate the angle of the jump φ with the uniform probability. Namely, generate a random number y in the interval [0,1]. Then, φ=2π y.
4. After the jump the particle will be in the point $(x_1, y_1) = (l \cos\varphi, l \sin\varphi)$.
5. Start a new jump from the point (x_1, y_1).
6. Repeat these jumps N times.
7. Plot the trajectory.