Engg. Math. I

Unit-I

Differential Calculus

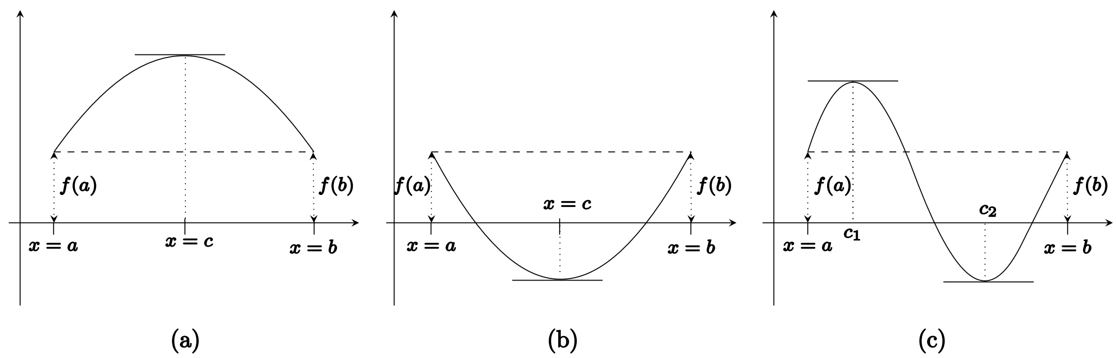
Syllabus: Limits of functions, continuous functions, uniform continuity, monotone and inverse functions. Differentiable functions, Rolle's theorem, mean value theorems and Taylor's theorem, power series. Functions of several variables, partial derivatives, chain rule, Tangent planes and normals. Maxima, minima, saddle points, Lagrange multipliers, exact differentials.

ROLLE'S THEOREM. Let f be a function which is continuous everywhere on the interval $[a, b]$ and has a derivative at each point of the open interval (a, b). Also, assume that

$$f(a) = f(b).$$

Then there is at least one point c in the interval (a, b) such that $f'(c) = 0$.

Proof. We prove the Rolle's theorem geometrically.



Geometric Interpretation of Roll's Theorem

Since $f(a) = f(b)$ and function f is continuous in $[a, b]$ we have the following three cases:

Case (a): Suppose that the function increases after point $x = a$. Since $f(a) = f(b)$ and function f is continuous, there must exists a point c such that $a < c < b$ and f has its maximum value at c. Therefore, we have $f'(c) = 0$.

Case (b): Suppose that the function decreases after point $x = a$. Since $f(a) = f(b)$ and function f is continuous, there must exists a point c such that $a < c < b$ and f has its minimum value at c. Therefore, we have $f'(c) = 0$.

Case (c): Suppose that the function increases after point $x = a$ and then attains its maximum values and then decreases and attains its minimum value, i.e., function oscillates. Since $f(a) = f(b)$ and function f is continuous, it finally returns to its initial value. Thus, we have more than one point $c_1, c_2, \dots$ such that $a < c_1, c_2, \dots < b$ and f has its maximum and minimum values at $c_1, c_2, \dots$. Therefore, we have $f'(c_1) = f'(c_2) = \dots = 0$.

Thus, in each case we obtain the desired point c. $\square$

Example 1. Verify the Rolle's theorem for $f(x) = |x|$ in $[-1, 1]$.

Sol: Here $a = -1, b = 1$. Given function $f(x)$ is continuous in $[-1, 1]$ and $f(a) = f(1) = |1| = 1$, $f(b) = f(-1) = |-1| = 1$, but we know that the function $f(x) = |x|$ is not differentiable at point $x = 0$, and $0 \in [-1, 1]$, therefore the Rolle's theorem cannot be verified. $\square$

Example 2. Verify the Rolle's theorem for $f(x) = e^x \sin x$ in $[0, \pi]$.

Sol: Here $a = 0, b = \pi$. Given function $f(x)$ is continuous in $[0, \pi]$ and $f(a) = f(0) = e^0 \sin 0 = 0$, $f(b) = f(\pi) = e^\pi \sin \pi = 0$, so $f(a) = f(b)$. Also, the function $f(x) = e^x \sin x$ is differentiable at every point of the interval $(0, \pi)$. Therefore, all the conditions of the Rolle's theorem are satisfied and by Rolle's theorem, there exists $0 < c < \pi$ such that $f'(c) = 0$. Then

$$f'(x) = \frac{d}{dx}(e^x \sin x) = e^x \sin x + e^x \cos x.$$

Therefore,

$$\begin{aligned} f'(c) = 0 \quad &\Longrightarrow \quad e^c \sin c + e^c \cos c = 0 \Longrightarrow e^c[\sin c + \cos c] = 0 \\ &\Longrightarrow \quad \sin c + \cos c = 0 \Longrightarrow (\sin c + \cos c)^2 = 0 \\ &\Longrightarrow \quad \sin 2c = -1 \Longrightarrow 2c = \frac{3\pi}{2} \\ &\Longrightarrow \quad c = \frac{3\pi}{4}. \end{aligned}$$

Since $c = \dfrac{3\pi}{4} \in (0, \pi)$ the Rolle's theorem is verified. $\square$

Example 3. Verify the Rolle's theorem for $f(x) = \sin 3x$ in $\left[0, \dfrac{\pi}{3}\right]$.

Sol: Here $a = 0, b = \dfrac{\pi}{3}$. Given function $f(x)$ is continuous in $\left[0, \dfrac{\pi}{3}\right]$ and $f(a) = f(0) = \sin 0 = 0$, $f(b) = f\left(\dfrac{\pi}{3}\right) = \sin\left(\dfrac{3\pi}{3}\right) = 0$, so $f(a) = f(b)$. Also, the function $f(x) = \sin 3x$ is differentiable at every point of the interval $\left(0, \dfrac{\pi}{3}\right)$. Therefore, all the conditions of the Rolle's theorem are satisfied and by Rolle's theorem, there exists $c \in \left(0, \dfrac{\pi}{3}\right)$ such that $f'(c) = 0$. Then

$$f'(x) = \frac{d}{dx}(\sin 3x) = 3\cos 3x.$$

Therefore,

$$\begin{aligned} f'(c) = 0 \quad &\Longrightarrow \quad 3\cos 3c = 0 \Longrightarrow \cos 3c = 0 \\ &\Longrightarrow \quad 3c = \frac{\pi}{2} \\ &\Longrightarrow \quad c = \frac{\pi}{6}. \end{aligned}$$

Since $c = \dfrac{\pi}{6} \in \left(0, \dfrac{\pi}{3}\right)$ the Rolle's theorem is verified. $\square$

Example 4. Verify the Rolle's theorem for $f(x) = \cos 2x$ in $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$.

Sol: Here $a = -\frac{\pi}{4}, b = \frac{\pi}{4}$. Given function $f(x)$ is continuous in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ and $f(a) = f\left(-\frac{\pi}{4}\right) = \cos\left(-\frac{2\pi}{4}\right) = 0$, $f(b) = f\left(\frac{\pi}{4}\right) = \cos\left(\frac{2\pi}{4}\right) = 0$, so $f(a) = f(b)$. Also, the function $f(x) = \cos 2x$ is differentiable at every point of the interval $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$. Therefore, all the conditions of the Rolle's theorem are satisfied and by Rolle's theorem, there exists $c \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ such that $f'(c) = 0$. Then

$$f'(x) = \frac{d}{dx}(\cos 3x) = -2\sin 2x.$$

Therefore,

$$\begin{aligned} f'(c) = 0 \quad &\implies \quad -2\sin 2c = 0 \implies \sin 2c = 0 \\ &\implies \quad 2c = 0 \\ &\implies \quad c = 0. \end{aligned}$$

Since $c = 0 \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ the Rolle's theorem is verified. □

Example 5. Verify the Rolle's theorem for $f(x) = 2 + (x-1)^{2/3}$ in $[0, 2]$.

Sol: Here $a = 0, b = 2$. Given function $f(x)$ is continuous in $[0, 2]$ and $f(a) = f(0) = 2 + (0-1)^{2/3} = 3$, $f(b) = f(2) = 2 + (2-1)^{2/3} = 3$, so $f(a) = f(b)$. Note that f is not differentiable in the interval $(0, 2)$. Indeed:

$$f'(x) = \frac{d}{dx}\left(2 + (x-1)^{2/3}\right) = \frac{2}{3}(x-1)^{-1/3}.$$

Therefore, $f'(1)$ does not exist and since $1 \in (0, 2)$, therefore all the conditions of the Rolle's theorem are not satisfied, and so, it cannot be verified. □

Example 6. Verify the Rolle's theorem for $f(x) = x^3 - 4x$.

Sol: Here the interval where the theorem is to be verified is not given. To find the interval put $f(x) = 0$, i.e.,

$$x^3 - 4x \implies x(x^2 - 4) = 0 \implies x = 0, \pm 2.$$

So we obtain the intervals $[-2, 0]$, $[0, 2]$ and $[-2, 2]$. Given function $f(x)$ is a polynomial in x, so, continuous and differentiable everywhere and $f(-2) = f(0) = f(2) = 0$. Therefore, all the conditions of the Rolle's theorem are satisfied and by Rolle's theorem, there exists $c \in (0, 2)$ such that $f'(c) = 0$. Then

$$f'(x) = \frac{d}{dx}\left(x^3 - 4x\right) = 3x^2 - 4.$$

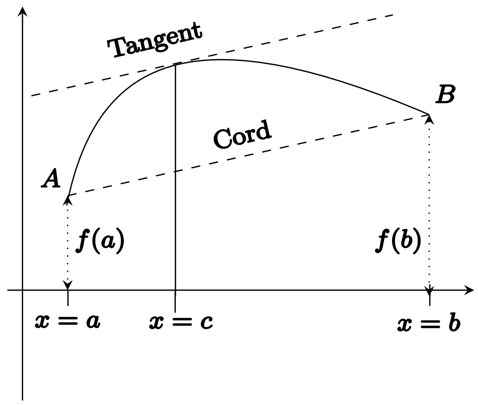
Therefore,

$$f'(c) = 0 \quad \implies \quad 3c^2 - 4 = 0 \implies c = \pm\frac{2}{\sqrt{3}}.$$

Since $c = -\frac{2}{\sqrt{3}} \in (-2, 0)$ and $c = \frac{2}{\sqrt{3}} \in (0, 2)$ the Rolle's theorem is verified. □

MEAN VALUE THEOREM OR LAGRANGE'S MEAN VALUE THEOREM. Let f be a function which is continuous everywhere on the interval $[a, b]$ and has a derivative at each point of the open interval (a, b). Then there is at least one point c in the interval (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Proof. We prove the this theorem with the help of Rolle,s theorem.



Geometric Interpretation of Mean Value Theorem

Define a function $F(x)$ by

$$F(x) = f(x) + \alpha x \tag{1}$$

where α is an arbitrary constant. Then, we shall show that F satisfies all the conditions of Rolle's theorem. Then:

(I) Since f is continuous in $[a, b]$ and αx is a polynomial so it is continuous everywhere, and so, their sum $F(x) = f(x) + \alpha x$ is also continuous in $[a, b]$.

(II) Since f is differentiable in (a, b) and αx is a polynomial so it is differentiable everywhere, and so, their sum $F(x) = f(x) + \alpha x$ is also differentiable in (a, b).

(III) Finally, since α was an arbitrary constant, choose α such that:

$$\begin{aligned} F(a) = F(b) \quad &\implies \quad f(a) + \alpha a = f(b) + \alpha b \\ &\implies \quad \alpha = -\frac{f(b) - f(a)}{b - a)}. \end{aligned}$$

Thus, F satisfies all the conditions of Rolle's theorem. Therefore, by Rolle's theorem there exists $c \in (a, b)$ such that

$$\begin{aligned} F'(c) = 0 \quad &\implies \quad f'(c) + \alpha = 0 \\ &\implies \quad f'(c) = -\alpha \\ &\implies \quad f'(c) = \frac{f(b) - f(a)}{b - a)}. \end{aligned}$$

Hence the proof is completes. □

Example 7. Find the c of mean value theorem for the function $f(x) = (x - 1)(x - 2)(x - 3)$ in the interval $[0, 4]$.

Sol: Here $a = 0, b = 4$ and the function f is a polynomial, and so, it is continuous and differentiable everywhere. Therefore, all the conditions of mean value theorem are satisfied. By mean value theorem there exists a point $c \in (0,4)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. Now

$$f'(x) = (x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2) = 3x^2 - 12x + 11.$$

Therefore,

$$\begin{aligned} f'(c) = \frac{f(b)-f(a)}{b-a} \quad &\implies \quad 3c^2 - 12c + 11 = \frac{f(4)-f(0)}{4-0} \\ &\implies \quad 3c^2 - 12c + 11 = \frac{6-(-6)}{4} \\ &\implies \quad 3c^2 - 12c + 11 = 3 \\ &\implies \quad 3c^2 - 12c + 8 = 0 \\ &\implies \quad c = 2 \pm \frac{2\sqrt{3}}{3}. \end{aligned}$$

Since $c = \dfrac{2\sqrt{3}}{3} \in (0,4)$, hence the mean value theorem is verified. □

Example 8. Verify mean value theorem for the function $f(x) = \ln x$ in the interval $\dfrac{1}{e} \leq x \leq e$.

Sol: Here $a = \dfrac{1}{e}, b = e$ and the function f is logarithmic, and so, it is continuous in the interval $\left[\frac{1}{e}, e\right]$ and differentiable in the interval $\left(\frac{1}{e}, e\right)$. Therefore, all the conditions of mean value theorem are satisfied. By mean value theorem there exists a point $c \in \left(\frac{1}{e}, e\right)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. Now

$$f'(x) = \frac{1}{x}.$$

Therefore,

$$\begin{aligned} f'(c) = \frac{f(b)-f(a)}{b-a} \quad &\implies \quad \frac{1}{c} = \frac{f(e) - f\left(\frac{1}{e}\right)}{e - 1/e} \\ &\implies \quad \frac{1}{c} = \frac{\ln(e) - \ln\left(\dfrac{1}{e}\right)}{e - 1/e} \\ &\implies \quad \frac{1}{c} = \frac{e(1-(-1))}{e^2 - 1} \\ &\implies \quad c = \frac{e^2 - 1}{e}. \end{aligned}$$

Since $c = \dfrac{e^2 - 1}{e} \in \left(\frac{1}{e}, e\right)$, hence the mean value theorem is verified. □

Example 9. Verify mean value theorem for the function $f(x) = \ln x$ in the interval $[1, e]$.

Sol: Here $a = 1, b = e$ and the function f is logarithmic, and so, it is continuous in the interval $[1, e]$ and differentiable in the interval $(1, e)$. Therefore, all the conditions of mean value theorem are satisfied. By mean value theorem there exists a point $c \in (1, e)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. Now

$$f'(x) = \frac{1}{x}.$$

Therefore,

$$\begin{aligned} f'(c) = \frac{f(b) - f(a)}{b - a} \quad & \implies \quad \frac{1}{c} = \frac{f(e) - f(1)}{e - 1} \\ & \implies \quad \frac{1}{c} = \frac{\ln(e) - \ln(1)}{e - 1} \\ & \implies \quad \frac{1}{c} = \frac{1}{e - 1} \\ & \implies \quad c = e - 1. \end{aligned}$$

Since $c = e - 1 \in (1, e)$, hence the mean value theorem is verified. □

Example 10. Show that on the graph of any quadratic polynomial the chord joining the points for which $x = a, x = b$ is parallel to the tangent line at the midpoint $x = \dfrac{a + b}{2}$.

OR

If $f(x) = \alpha x^2 + \beta x + \gamma$, where α, β, γ are constants and $\alpha \neq 0$, then find the value of c in Lagrange's mean value theorem in the interval $[a, b]$.

Sol: The function f is polynomial, and so, it is continuous in the interval $[a, b]$ and differentiable in the interval (a, b). Therefore, all the conditions of mean value theorem are satisfied. By mean value theorem there exists a point $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. Now

$$f'(x) = 2\alpha x + \beta.$$

Therefore,

$$\begin{aligned} f'(c) = \frac{f(b) - f(a)}{b - a} \quad & \implies \quad 2\alpha c + \beta = \frac{f(b) - f(a)}{b - a} \\ & \implies \quad 2\alpha c + \beta = \frac{\alpha b^2 + \beta b + \gamma - (\alpha a^2 + \beta a + \gamma)}{b - a} \\ & \implies \quad 2\alpha c + \beta = \frac{\alpha(b^2 - a^2) + \beta(b - a)}{b - a} \\ & \implies \quad 2\alpha c + \beta = \alpha(b + a) + \beta \\ & \implies \quad c = \frac{b + a}{2} \text{ midpoint of } a, b. \end{aligned}$$

Since $f'(c) = \dfrac{f(b) - f(a)}{b - a}$, hence the slope of tangent at midpoint $c = \dfrac{b + a}{2}$ (i.e., $f'(c)$) is equal to the slope of chord at the end points a, b. Therefore, the tangent and cord are parallel. □

Exercise (Assignment)

(Q.1) Discuss the conditions of Rolle's theorem for the function $f(x) = \tan x$ in the interval $0 \leq x \leq \pi$.

Ans. Since $\tan x$ is not continuous at $x = \frac{\pi}{2}$, the Rolle's theorem is not applicable.

(Q.2) Verify the Rolle's theorem for the function $f(x) = x^2$ in the interval $[-1, 1]$.

Ans. $c = 0$.

(Q.3) Can Rolle's theorem be applied for the function $f(x) = 1 - (x - 3)^{2/3}$.

Hint. For the interval, put $f(x) = 0$, it gives the interval $[2, 4]$. Then, since f is not differentiable at $x = 3 \in (2, 4)$, so, the Rolle's theorem cannot be verified.

(Q.4) Explain Rolle's theorem for the function $f(x) = (x - a)^m (x - b)^n$ in the interval $[a, b]$.

Ans. $c = \frac{mb+na}{m+n} \in (a, b)$.

(Q.5) Find the c of mean value theorem for the function $f(x) = x^3$ in the interval $[-2, 2]$.

Ans. $c = \pm\frac{2}{\sqrt{3}}$.

(Q.6) Verifiy mean value theorem for the function $f(x) = x^3 - 3x - 1$ in the interval $[0, 1]$.

Ans. $c = \frac{1}{\sqrt{3}}$.

Taylor's Theorem. Suppose that the $(n-1)$th derivative $f^{(n-1)}$ of f is continuous on the interval $[a, b]$ and the nth derivative $f^{(n)}$ of f exists in the open interval (a, b). Then for each $x \neq a$ in I there is a value c such that $a < c < x$ and:

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c).$$

The last term $R_n = \dfrac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$ is called the remainder therm (Lagrange's form) after n terms.

Taylor's Series. Suppose $R_n \to 0$ as $n \to \infty$, then the expression for $f(x)$ in the Taylor's theorem reduces into an infinite series and this series is called the Taylor's series or Taylor's series expansion of $f(x)$ about the point $x = a$; and it is given by:

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \cdots.$$

Various forms of Taylor's series:

Maclaurin's series Put $a = 0$ in Taylor's series obtain:

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots.$$

Expansion of $f(x+h)$ in powers of x Replace x by $x+h$ and a by h in Taylor's series obtain:

$$f(x+h) = f(h) + \frac{x}{1!}f'(h) + \frac{x^2}{2!}f''(h) + \cdots + \frac{x^n}{n!}f^{(n)}(h) + \cdots.$$

Expansion of $f(x+h)$ in powers of h Replace x, h by h, x respectively, in the previous series:

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + \cdots.$$

Example 11. Expand $\ln\left(\frac{1+x}{1-x}\right)$ using Maclaurin's theorem.

Sol: Here $f(x) = \ln\left(\frac{1+x}{1-x}\right)$. By Maclaurin's theorem we know that

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots.$$

Putting $y = f(x)$, $(y)_0 = f(0), (y_1)_0 = f'(0), (y_2)_0 = f''(0)$ etc., in the above we obtain:

$$y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \cdots + \frac{x^n}{n!}(y_n)_0 + \cdots. \tag{2}$$

Now differentiating successively and putting $x = 0$ we obtain:

$$\begin{aligned}
y = f(x) = \ln(1+x) - \ln(1-x) &\implies (y)_0 = 0\\
y_1 = \frac{1}{1+x} + \frac{1}{1-x} &\implies (y_1)_0 = 2\\
y_2 = -\frac{1}{(1+x)^2} + \frac{1}{(1-x)^2} &\implies (y_2)_0 = 0\\
y_3 = \frac{2}{(1+x)^3} + \frac{2}{(1-x)^3} &\implies (y_3)_0 = 4\\
y_4 = -\frac{6}{(1+x)^4} + \frac{6}{(1-x)^4} &\implies (y_4)_0 = 0\\
y_5 = \frac{24}{(1+x)^5} + \frac{24}{(1-x)^5} &\implies (y_5)_0 = 48 \text{ and so on.}
\end{aligned}$$

Putting these values in (2) we obtain:

$$\begin{aligned}
\ln\left(\frac{1+x}{1-x}\right) &= 0 + \frac{x}{1!}(2) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(4) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(48) + \cdots\\
&= 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots.
\end{aligned}$$

It is the required series. □

Example 12. If $\ln\sec x = \frac{1}{2}x^2 + Ax^4 + Bx^6 + \cdots$, then find the values of A and B.

Sol: Since the given value of $\ln \sec x$ is a series in powers of x, we will expand $\ln \sec x$ by Maclaurin's series. Then, here $f(x) = \ln \sec x$ and by Maclaurin's theorem we know that

$$y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \cdots + \frac{x^n}{n!}(y_n)_0 + \cdots. \tag{3}$$

Now differentiating successively and putting $x = 0$ we obtain:

$$\begin{aligned}
y = f(x) = \ln \sec x &\implies (y)_0 = 0\\
y_1 = \frac{\sec x \tan x}{\sec x} = \tan x &\implies (y_1)_0 = 0\\
y_2 = \sec^2 x = 1 + \tan^2 x = 1 + y_1^2 &\implies (y_2)_0 = 1\\
y_3 = 2y_1y_2 &\implies (y_3)_0 = 2(y_1)_0(y_2)_0 = 0\\
y_4 = 2y_1y_3 + 2y_2y_2 = 2y_1y_3 + 2y_2^2 &\implies (y_4)_0 = 2(y_1)_0(y_3)_0 + 2(y_2)_0^2 = 2
\end{aligned}$$

$$y_5 = 2y_1y_4 + 2y_2y_3 + 4y_2y_3 = 2y_1y_4 + 6y_2y_3$$

$$\implies (y_5)_0 = 2(y_1)_0(y_4)_0 + 6(y_2)_0(y_3)_0 = 0$$

$$y_6 = 2y_1y_5 + 2y_2y_4 + 6y_2y_4 + 6y_3y_3 = 2y_1y_5 + 8y_2y_4 + 6y_3^2$$

$$\implies (y_6)_0 = 2(y_1)_0(y_5)_0 + 8(y_2)_0(y_4)_0 + 6y_3^2(0) = 16 \text{ and so on.}$$

Putting these values in (3) we obtain:

$$\begin{aligned}
\ln \sec x &= 0 + \frac{x}{1!}(0) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(2) + \frac{x^5}{5!}(0) + \frac{x^6}{6!}(16) + \cdots\\
&= \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \cdots.
\end{aligned}$$

On comparing the coefficients of various powers of x in the above and given series we obtain

$$A = \frac{1}{12},\ B = \frac{1}{45}.$$

□

Example 13. Find the first five terms in the expansion of $e^{\sin x}$ by Maclaurin's series.

Sol: Here $f(x) = e^{\sin x}$. By Maclaurin's theorem we know that

$$y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \cdots + \frac{x^n}{n!}(y_n)_0 + \cdots. \tag{4}$$

Now differentiating successively and putting $x = 0$ we obtain:

$$\begin{aligned}
y = f(x) = e^{\sin x} &\implies (y)_0 = 1\\
y_1 = \cos x e^{\sin x} = y \cos x &\implies (y_1)_0 = 1\\
y_2 = y_1 \cos x - y \sin x &\implies (y_2)_0 = 1\\
y_3 = y_2 \cos x - 2y_1 \sin x - y \cos x = y_2 \cos x - 2y_1 \sin x - y_1 &\implies (y_3)_0 = 0\\
y_4 = y_3 \cos x - 3y_2 \sin x - 2y_1 \cos x - y_2 &\implies (y_4)_0 = -3\\
y_5 = y_4 \cos x - 4y_3 \sin x - 5y_2 \cos x + 2y_1 \sin x - y_3 &\implies (y_5)_0 = -8
\end{aligned}$$

and so on. Putting these values in (4) we obtain:

$$\begin{aligned}
e^{\sin x} &= 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(-3) + \frac{x^5}{5!}(-8) + \cdots\\
&= 1 + x + \frac{x^3}{2} - \frac{x^4}{8} - \frac{x^5}{15} + \cdots.
\end{aligned}$$

It is the required series. □

Example 14. Expand $e^{ax}\cos(bx)$ by Maclaurin's theorem.

Sol: Here $f(x) = e^{ax}\cos(bx)$. By Maclaurin's theorem we know that

$$y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \cdots + \frac{x^n}{n!}(y_n)_0 + \cdots . \tag{5}$$

Now differentiating successively and putting $x = 0$ we obtain:

$$\begin{aligned} y = f(x) = e^{ax}\cos(bx) &\implies (y)_0 = 1 \\ y_1 = ae^{ax}\cos(bx) - be^{ax}\sin(bx) = ay - be^{ax}\sin(bx) &\implies (y_1)_0 = a \\ y_2 = ay_1 - b^2e^{ax}\cos(bx) - abe^{ax}\sin(bx) = ay_1 - b^2y + a(y_1 - ay) &= 2ay_1 - (a^2+b^2)y \\ &\implies (y_2)_0 = a^2 - b^2 \\ y_3 = 2ay_2 - (a^2+b^2)y_1 &\implies (y_3)_0 = a(a^2 - 3b^2) \end{aligned}$$

and so on. Putting these values in (5) we obtain:

$$\begin{aligned} e^{\sin x} &= 1 + \frac{x}{1!}(a) + \frac{x^2}{2!}(a^2 - b^2) + \frac{x^3}{3!}a(a^2 - 3b^2) + \cdots \\ &= 1 + ax + (a^2 - b^2)\frac{x^2}{2!} + a(a^2 - 3b^2)\frac{x^3}{3!} + \cdots . \end{aligned}$$

It is the required series. □

Example 15. Expand $e^{a\sin^{-1}x}$ by Maclaurin's theorem. Hence show that

$$e^{\theta} = 1 + \sin\theta + \frac{1}{2!}\sin^2\theta + \frac{2}{3!}\sin^3\theta + \cdots$$

where $\theta = \sin^{-1}x$.

Sol: Here $f(x) = e^{a\sin^{-1}x}$. By Maclaurin's theorem we know that

$$y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \cdots + \frac{x^n}{n!}(y_n)_0 + \cdots . \tag{6}$$

Since $y = f(x) = e^{a\sin^{-1}x}$ we have $\boxed{(y)_0 = 1}$.
Differentiating we get

$$\begin{aligned} y_1 &= e^{a\sin^{-1}x} \times \frac{a}{\sqrt{1-x^2}} \\ \implies \quad y_1 &= \frac{ay}{\sqrt{1-x^2}} \\ \implies (1-x^2)y_1^2 &= a^2y^2. \end{aligned} \tag{7}$$

Therefore, $\boxed{(y_1)_0 = a}$. Again differentiating (7) we get:

$$\begin{aligned} (1-x^2)y_2 - 2xy_1 &= 2a^2yy_1 \\ \implies (1-x^2)y_2 - xy_1 - a^2y &= 0. \end{aligned} \tag{8}$$

Therefore, $(y_2)_0 - a^2(y)_0 = 0$, i.e., $\boxed{(y_2)_0 = a^2}$. Again differentiating (8) we get:

$$(1-x^2)y_3 - 3xy_2 - (1+a^2)y_1 = 0.$$

Therefore, $(y_3)_0 - (1+a^2)(y_1)_0 = 0$, i.e., $\boxed{(y_3)_0 = a(1+a^2)}$ and so on. Putting these values in (6) we obtain:

$$\begin{aligned} e^{a\sin^{-1}x} &= 1 + \frac{x}{1!}(a) + \frac{x^2}{2!}(a^2) + \frac{x^3}{3!}(1+a^2) + \cdots \\ &= 1 + ax + \frac{a^2x^2}{2!} + \frac{a(1+a^2)x^3}{3!} + \cdots . \end{aligned}$$

Putting $a = 1$ and $\sin^{-1} x = \theta$ we get

$$e^\theta = 1 + \sin\theta + \frac{\sin^2\theta}{2!} + \frac{2\sin^3\theta}{3!} + \cdots .$$

It is the required series. □

Example 16. Find the first five terms in the expansion of $\ln(1+\sin x)$ by Maclaurin's series.

Sol: Here $f(x) = \ln(1+\sin x)$. By Maclaurin's series we know that

$$y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \cdots + \frac{x^n}{n!}(y_n)_0 + \cdots . \tag{9}$$

Since $y = f(x) = \ln(1+\sin x)$ we have $\boxed{(y)_0 = 0}$.
Differentiating we get

$$\begin{aligned} y_1 &= \frac{\cos x}{1+\sin x} \\ \implies \quad (1+\sin x)y_1 &= \cos x. \end{aligned} \tag{10}$$

Therefore, $\boxed{(y_1)_0 = 1}$. Again differentiating (10) we get:

$$(1+\sin x)y_2 + y_1\cos x = -\sin x. \tag{11}$$

Therefore, $(1+0)(y_2)_0 + (y_1)_0 = 0$, i.e., $\boxed{(y_2)_0 = -1}$. Again differentiating (11) we get:

$$(1+\sin x)y_3 + 2y_2\cos x - y_1\sin x = -\cos x. \tag{12}$$

Therefore, $(1+0)(y_3)_0 + 2(y_2)_0 - 0 = -1$, i.e., $\boxed{(y_3)_0 = 1}$. Again differentiating (12) we get:

$$(1+\sin x)y_4 + 3y_3\cos x - 3y_2\sin x - y_1\cos x = \sin x. \tag{13}$$

Therefore, $(1+0)(y_4)_0 + 3(y_3)_0 - 0 - (y_1)_0 = 0$, i.e., $\boxed{(y_4)_0 = -2}$. Again differentiating (13) we get:

$$(1+\sin x)y_5 + 4y_4\cos x - 6y_3\sin x - 4y_2\cos x + y_1\sin x = \cos x. \tag{14}$$

Therefore, $(1+0)(y_5)_0+4(y_4)_0-0-4(y_2)_0+0=1$, i.e., $\boxed{(y_5)_0=5}$. Putting these values in (9) we obtain:

$$\begin{aligned}\ln(1+\sin x) &= 0+\frac{x}{1!}(1)+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(1)+\frac{x^4}{4!}(-2)+\frac{x^4}{5!}(5)+\cdots\\ &= x-\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{2x^4}{4!}+\frac{5x^5}{5!}+\cdots.\end{aligned}$$

It is the required series. □

Example 17. Expand $\tan^{-1}x$ in the ascending powers of $x-1$.

Sol: By Taylor's series we know that

$$f(x)=f(a)+\frac{x-a}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots.$$

Here $f(x)=\tan^{-1}x$ and $a=1$, therefore:

$$f(x)=f(1)+\frac{x-1}{1!}f'(1)+\frac{(x-1)^2}{2!}f''(1)+\cdots. \tag{15}$$

Since $f(x)=\tan^{-1}x$ we have $\boxed{f(1)=\frac{\pi}{4}}$. Differentiating we get:

$$f'(x)=\frac{1}{1+x^2} \implies \boxed{f'(1)=\frac{1}{2}}.$$

Rearranging the terms in the above we get:

$$(1+x^2)f'(x)=1.$$

Again differentiating we get:

$$(1+x^2)f''(x)+2xf'(x)=0 \implies \boxed{f''(1)=-1}.$$

On putting these values in (15) we get

$$\tan^{-1}x=\frac{\pi}{4}+\frac{x-1}{2\cdot 1!}-\frac{(x-1)^2}{2!}+\cdots.$$ □

Example 18. Expand $\sin x$ in powers of $x-\frac{\pi}{2}$ and hence evaluate $\sin 91°$ correct to four places of decimals.

Sol: By Taylor's series we know that

$$f(x)=f(a)+\frac{x-a}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots.$$

Here $f(x)=\sin x$ and $a=\frac{\pi}{2}$, therefore:

$$f(x)=f\left(\frac{\pi}{2}\right)+\frac{x-\frac{\pi}{2}}{1!}f'\left(\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}f''\left(\frac{\pi}{2}\right)+\cdots. \tag{16}$$

Since $f(x) = \sin x$ we have $\boxed{f\left(\frac{\pi}{2}\right) = 1}$. Differentiating we get:

$$f'(x) = \cos x \implies \boxed{f'\left(\frac{\pi}{2}\right) = 0}.$$

Again differentiating we get:

$$f''(x) = -\sin x \implies \boxed{f''\left(\frac{\pi}{2}\right) = -1}.$$

Again differentiating we get:

$$f'''(x) = -\cos x \implies \boxed{f'''\left(\frac{\pi}{2}\right) = 0}.$$

Again differentiating we get:

$$f^{(\text{iv})}(x) = \sin x \implies \boxed{f^{(\text{iv})}\left(\frac{\pi}{2}\right) = 1}.$$

On putting these values in (16) we get

$$\sin x = 1 - \frac{\left(x - \frac{\pi}{2}\right)^2}{2!} + \frac{\left(x - \frac{\pi}{2}\right)^4}{4!} - \cdots . + \cdots .$$

Let $x = 91°$, so that,

$$x - \frac{\pi}{2} = 91° - 90° = 1° = \frac{\pi}{180} \text{ radians} = 0.0174 \text{ radians.}$$

Putting the value of $x - \frac{\pi}{2}$ in the above series we obtain:

$$\begin{aligned} \sin 91° &= 1 - \frac{(0.0174)^2}{2!} + \frac{(0.0174)^4}{4!} \\ &= 0.9999 \end{aligned}$$

correct up to four places of decimals. $\square$

Example 19. Expand $\ln x$ in powers of $x - 1$ and hence evaluate $\ln(1.1)$ correct to four decimal places.

Sol: By Taylor's series we know that

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^4}{4!} f^{(\text{iv})}(a) + \cdots .$$

Here $f(x) = \ln x$ and $a = 1$, therefore:

$$f(x) = f(1) + \frac{x-1}{1!} f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \frac{(x-1)^4}{4!} f^{(\text{iv})}(1) + \cdots . \quad (17)$$

Since $f(x) = \ln x$ we have $\boxed{f(1) = 0}$. Differentiating we get:

$$f'(x) = \frac{1}{x} \implies \boxed{f'(1) = 1}.$$

Again differentiating we get:

$$f''(x) = -\frac{1}{x^2} \implies \boxed{f''(1) = -1}.$$

Again differentiating we get:

$$f'''(x) = \frac{2}{x^3} \implies \boxed{f'''(1) = 2}.$$

Again differentiating we get:

$$f^{(\text{iv})}(x) = -\frac{6}{x^4} \implies \boxed{f^{(\text{iv})}\left(\frac{\pi}{2}\right) = -6}.$$

On putting these values in (17) we get

$$\ln x = x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots.$$

Putting $x = 1.1$ in the above we get:

$$\begin{aligned}
\ln(1.1) &= 1.1 - 1 - \frac{1}{2}(1.1-1)^2 + \frac{1}{3}(1.1-1)^3 - \frac{1}{4}(1.1-1)^4 + \cdots \\
&= 0.1 - 0.005 + 0.0003 - 0.00002 \\
&= 0.0953
\end{aligned}$$

correct up to four places of decimals. □

Example 20. Expand $2x^3 + 7x^2 + x - 1$ in powers of $x - 2$.

Sol: By Taylor's series we know that

$$f(x) = f(a) + \frac{x-a}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{(\text{iv})}(a) + \cdots.$$

Here $f(x) = 2x^3 + 7x^2 + x - 1$ and $a = 2$, therefore:

$$f(x) = f(2) + \frac{x-2}{1!}f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \frac{(x-1)^4}{4!}f^{(\text{iv})}(1) + \cdots. \quad (18)$$

Since $f(x) = 2x^3 + 7x^2 + x - 1$ we have $\boxed{f(2) = 45}$. Differentiating we get:

$$f'(x) = 6x^2 + 14x + 1 \implies \boxed{f'(2) = 53}.$$

Again differentiating we get:

$$f''(x) = 12x + 14 \implies \boxed{f''(2) = 38}.$$

Again differentiating we get:

$$f'''(x) = 12 \implies \boxed{f'''(2) = 12}.$$

All other higher order derivatives are zero. On putting these values in (18) we get

$$2x^3 + 7x^2 + x - 1 = 45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3.$$

It is the required expansion. □

Example 21. Use Taylor's theorem to prove that

$$\begin{aligned}\tan^{-1}(x+h) &= \tan^{-1}x + h\sin\theta\cdot\frac{\sin\theta}{1} - (h\sin\theta)^2\cdot\frac{\sin 2\theta}{2} + (h\sin\theta)^3\cdot\frac{\sin 3\theta}{3}\\ &\quad - \cdots + (-1)^{n-1}(h\sin\theta)^n\cdot\frac{\sin n\theta}{n} + \cdots\end{aligned}$$

where $\theta = \cot^{-1}x$.

Sol: By Taylor's series we know that

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots. \tag{19}$$

Here $f(x+h) = \tan^{-1}(x+h)$, and so, $f(x) = \tan^{-1}x$ therefore differentiating we get:

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1+\cot^2\theta} = \sin^2\theta.$$

Again differentiating (w.r.t. x) we get:

$$\begin{aligned}f''(x) &= 2\sin\theta\cos\theta\cdot\frac{d\theta}{dx} = \sin 2\theta\cdot\frac{d}{dx}(\cot^{-1}x)\\ &= -\sin 2\theta\cdot\frac{1}{1+x^2}\\ &= -\sin 2\theta\sin^2\theta \qquad (\text{since } x = \cot\theta).\end{aligned}$$

Again differentiating we get:

$$\begin{aligned}f'''(x) &= \left(-2\cos 2\theta\sin^2\theta - 2\sin\theta\sin 2\theta\cos\theta\right)\cdot\frac{d\theta}{dx}\\ &= 2\sin\theta\,(\cos 2\theta\sin\theta + \sin 2\theta\cos\theta)\cdot\frac{1}{1+x^2}\\ &= 2\sin^3\theta\sin 3\theta \qquad (\text{since } x = \cot\theta).\end{aligned}$$

On putting these values in (19) we get

$$\begin{aligned}\tan^{-1}(x+h) &= \tan^{-1}x + \frac{h}{1!}(\sin^2\theta) + \frac{h^2}{2!}(-\sin 2\theta\sin^2\theta) + \frac{h^3}{3!}(2\sin^3\theta\sin 3\theta) + \cdots\\ &= \tan^{-1}x + h\sin\theta\cdot\frac{\sin\theta}{1} - (h\sin\theta)^2\cdot\frac{\sin 2\theta}{2} + (h\sin\theta)^3\cdot\frac{\sin 3\theta}{3}\\ &\quad - \cdots + (-1)^{n-1}(h\sin\theta)^n\cdot\frac{\sin n\theta}{n} + \cdots\end{aligned}$$

□

Example 22. Expand $\tan\left(x+\frac{\pi}{4}\right)$ as far as the term x^4 and evaluate $\tan 46.5°$ to four places of decimals.

OR

Approximate the value of $\tan(46°30')$ using Taylor's theorem.$(1° = 60')$

Sol: By Taylor's series we know that the expansion of $f(x+h)$ in powers of x is:

$$f(x+h) = f(h) + \frac{x}{1!}f'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \cdots. \tag{20}$$

Here $f(x+h) = \tan\left(x + \frac{\pi}{4}\right)$, $f(x) = \tan x$, $h = \frac{\pi}{4}$, and so, $f\left(\frac{\pi}{4}\right) = 1$. Differentiating $f(x)$ we get:

$$f'(x) = \sec^2 x = 1 + \tan^2 x = 1 + [f(x)]^2 \implies f'\left(\frac{\pi}{4}\right) = 2.$$

Again differentiating we get:

$$f''(x) = 2f(x)f'(x) \implies f''\left(\frac{\pi}{4}\right) = 4.$$

Again differentiating we get:

$$\begin{aligned} f'''(x) &= 2f(x)f''(x) + 2f'(x)f'(x) = 2f(x)f''(x) + 2[f'(x)]^2 \\ &\implies f'''\left(\frac{\pi}{4}\right) = 16. \end{aligned}$$

Again differentiating we get:

$$\begin{aligned} f^{(\text{iv})}(x) &= 2f(x)f'''(x) + 2f'(x)f''(x) + 4f'(x)f''(x) = 2f(x)f'''(x) + 6f'(x)f''(x) \\ &\implies f^{(\text{iv})}\left(\frac{\pi}{4}\right) = 80. \end{aligned}$$

On putting these values in (20) we get

$$\begin{aligned} \tan\left(x + \frac{\pi}{4}\right) &= 1 + \frac{x}{1!}(2) + \frac{x^2}{2!}(4) + \frac{x^3}{3!}(16) + \frac{x^4}{4!}(80) + \cdots \\ &= 1 + 2x + 2x^2 + \frac{8x^3}{3} + \frac{10x^4}{3} + \cdots. \end{aligned}$$

On putting $x = 1.5^\circ = 1.5 \times \frac{\pi}{180}$radians $= 0.0262$ (approximately) in the above equation we get:

$$\begin{aligned} \tan(46.5^\circ) &= 1 + 2(0.0262) + 2(0.0262)^2 + \frac{8(0.0262)^3}{3} + \frac{10(0.0262)^4}{3} + \cdots \\ &= 1.0538. \end{aligned}$$

Thus, $\tan(46^\circ 30') = \tan(46.5^\circ) = 1.0538$ (correct to four places of decimals). □

Example 23. Find the value of $\sqrt{10}$.

Sol: Let $f(x+h) = \sqrt{x+h}$. By Taylor's series we know that the expansion of $f(x+h)$ in powers of h is:

$$f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f''(x) + \cdots. \tag{21}$$

Here $f(x+h) = \sqrt{x+h}$, and so, $f(x) = \sqrt{x}$. Differentiating $f(x)$ we get:

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

Again differentiating we get:

$$f''(x) = -\frac{1}{4x^{3/2}}.$$

Again differentiating we get:

$$f'''(x) = \frac{3}{8x^{5/2}}.$$

On putting these values in (21) we get

$$\sqrt{x+h} = \sqrt{x} + \frac{h}{2\sqrt{x}} - \frac{h^2}{8x^{3/2}} + \frac{h^3}{16x^{5/2}} + \cdots.$$

On putting $x = 9, h = 1$ in the above equation we get:

$$\begin{aligned}\sqrt{10} &= \sqrt{9} + \frac{1}{2\sqrt{9}} - \frac{1}{8 \cdot 9^{3/2}} + \frac{1}{16 \cdot 9^{5/2}} + \cdots \\ &= 3 + 0.16667 - 0.00463 + 0.00025 \\ &= 3.16229.\end{aligned}$$

Thus, $\sqrt{10} = 3.1623$ (correct to four places of decimals). □

Exercise (Assignment)

(Q.1) Expand $\dfrac{e^x}{1+e^x}$ in Maclaurin's series as far as the terms x^3.

Ans. $\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{8x^3}{3!} + \cdots$.

(Q.2) Expand $e^{x\cos x}$ in Maclaurin's series.

Ans. $e^{x\cos x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \cdots$.

(Q.3) Prove that: $\left(\sin^{-1} x\right)^2 = \dfrac{2}{2!}x^2 + \dfrac{2 \cdot 2^2}{4!}x^4 + \cdots$.

Hint. Use Maclaurin's series for $y = \left(\sin^{-1} x\right)^2$.

(Q.4) Prove that: $\ln(1+e^x) = \ln(2) + \dfrac{1}{2}x + \dfrac{1}{8}x^2 - \dfrac{1}{192}x^4 + \cdots$.

Hint. Use Maclaurin's series for $y = \ln(1+e^x)$.

(Q.5) Prove that: $e^x \sin x = x + x^2 + \dfrac{2}{3!}x^3 - \dfrac{2^2}{5!}x^5 + \cdots$.

Hint. Use Maclaurin's series for $y = e^x \sin x$.

(Q.6) Find the Maclaurin's series for $y = \sin\left(m \sin^{-1} x\right)$.

Ans. $y = mx + \dfrac{m(m^2-1)}{3!}x^3 + \cdots$.

(Q.7) Expand $\tan x$ in powers of $x - \dfrac{\pi}{4}$.

Ans. $\tan x = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \cdots$.

(Q.8) Expand $7x^6 - 3x^5 + x^2 + 2$ in powers of $x - 1$.

Ans. $7x^6 - 3x^5 + x^2 + 2 = 7 + 29(x-1) + 76(x-1)^2 + 110(x-1)^3 + 90(x-1)^4 + 39(x-1)^5 + 7(x-1)^6$.

(Q.9) Find the Taylor's series expansion of $\ln(\cos x)$ about the point $\dfrac{\pi}{3}$.

Ans. $\ln(\cos x) = \ln\frac{1}{2} - \sqrt{3}\left(x - \frac{\pi}{3}\right) - \frac{4}{2!}\left(x - \frac{\pi}{3}\right)^2 - \cdots$.

(Q.10) Prove that $\ln(x+h) = \ln x + \dfrac{h}{x} - \dfrac{h^2}{2x^2} + \dfrac{h^3}{3x^3} - \cdots$.

Hint. Use Taylor's series and expand $f(x+h)$ in powers of h.

(Q.11) Calculate the value of $\sqrt{5}$ correct to four places of decimals by taking first four terms in Taylor's series.

Hint. Use Taylor's series and expand $f(x+h) = \sqrt{x+h}$ in powers of h, and put $x = 4, h = 1$.

(Q.12) Approximate the value of $\sin(61°30')$ using Taylor's theorem.

Ans. $\sin(61°30') = 0.87881711$(approximate).

L'Hospital's Rule $\left(\dfrac{0}{0}\text{ form}\right)$. Let $f'(t)$ and $g'(t)$ exist and $g'(t) \neq 0$ for all $t \in (a,b)$. If $f(c) = g(c) = 0$ for some $c \in (a,b)$, then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

Proof. Suppose $a < c < x < b$ (or $a < x < c < b$). Define a new function $h\colon [c,x] \to \mathbb{R}$ by:

$$h(t) = f(t) - \frac{f(x)}{g(x)} \cdot g(t).$$

Then $h(c) = h(x) = 0$ and the function h is continuous in the interval $[c,x]$ and differentiable in the interval (c,x). Therefore, by the Rolle's theorem there exists ζ such that $c < \zeta < x$ and $h'(\zeta) = 0$, i.e.,

$$\begin{aligned} f'(\zeta) - \frac{f(x)}{g(x)} \cdot g'(\zeta) &= 0 \\ \implies \quad \frac{f(x)}{g(x)} &= \frac{f'(\zeta)}{g'(\zeta)}. \end{aligned}$$

Since $c < \zeta < x$ (or $x < \zeta < c$), as $x \to c$ we have $\zeta \to c$, and so, the above inequality yields:

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

which proves the required result. □

Remark 1. If $f(c) = g(c) = \infty$, then the L'Hospital's rule remains true ($\frac{\infty}{\infty}$ form). Also, the forms $0 \cdot \infty, 0^0, \infty^0, 1^\infty$ etc., can be converted into either $\frac{\infty}{\infty}$ or $\frac{0}{0}$ form, and then can be solved by L'Hospital's rule.

Example 24. Evaluate the following limits:

(i) $\lim\limits_{x \to 0} \dfrac{x}{\tan x}$ (ii) $\lim\limits_{x \to 0} (1+x)^{1/x}$ (iii) $\lim\limits_{x \to \pi/2} \dfrac{2x - \pi}{\cos x}$

(iv) $\lim\limits_{x \to \infty} \left(\dfrac{x+2}{x+1}\right)^{x+3}$ (v) $\lim\limits_{x \to 0} [\tan(x + \pi/4)]^{1/x}$ (vi) $\lim\limits_{x \to a} \left(\dfrac{\int_a^x f(t)dt}{x - a}\right)$.

Sol: (i) Given limits is:
$$\begin{aligned} L &= \lim_{x \to 0} \frac{x}{\tan x} && \frac{0}{0} \text{ form} \\ &= \lim_{x \to 0} \frac{1}{\sec^2 x} \\ &= 1. \end{aligned}$$

(ii) Given limits is:
$$\begin{aligned} L &= \lim_{x \to 0} (1+x)^{1/x} && 1^\infty \text{ form} \\ \implies \ln(L) &= \lim_{x \to 0} \frac{\ln(1+x)}{x} && \frac{0}{0} \text{ form} \\ &= \lim_{x \to 0} \frac{1}{1+x} \\ &= 1. \end{aligned}$$

Thus, $\ln(L) = 1$, and so, $L = e$.

(iii) Given limits is:
$$\begin{aligned} L &= \lim_{x \to \pi/2} \frac{2x - \pi}{\cos x} && \frac{0}{0} \text{ form} \\ &= \lim_{x \to \pi/2} \frac{2}{-\sin x} \\ &= -2. \end{aligned}$$

(iv) Given limits is:
$$\begin{aligned} L &= \lim_{x \to \infty} \left(\frac{x+2}{x+1}\right)^{x+3} = \lim_{x \to \infty} \left(\frac{1+2/x}{1+1/x}\right)^x \lim_{x \to \infty} \left(\frac{1+2/x}{1+1/x}\right)^3 \\ &= \lim_{x \to \infty} \left(\frac{1+2/x}{1+1/x}\right)^x \cdot 1 \\ &= \lim_{x \to \infty} \left(\frac{1+2/x}{1+1/x}\right)^x && 1^\infty \text{ form} \end{aligned}$$

Therefore:

$$\begin{aligned}
\ln(L) &= \lim_{x\to\infty} \frac{\ln\left(\dfrac{1+2/x}{1+1/x}\right)}{1/x} && \frac{0}{0} \text{ form}\\
&= \lim_{x\to\infty} \left[-\frac{(1+1/x)x^2}{1+2/x} \times \frac{-(1+1/x)(2/x^2)+(1+2/x)(1/x^2)}{(1+1/x)^2}\right]\\
&= \lim_{x\to\infty} \left[-\frac{-2(1+1/x)+1+2/x}{(1+2/x)(1+1/x)}\right]\\
&= \lim_{x\to\infty} \left[-\frac{-2+1}{1\cdot 1}\right]\\
&= 1.
\end{aligned}$$

Thus, $\ln(L) = 1$, and so, $L = e$.

(v) Given limits is:

$$\begin{aligned}
L &= \lim_{x\to 0} \left[\tan\left(x+\pi/4\right)\right]^{1/x} && 1^\infty \text{ form}\\
\implies \ln(L) &= \lim_{x\to 0} \frac{\ln\left[\tan(x+\pi/4)\right]}{x} && \frac{0}{0} \text{ form}\\
&= \lim_{x\to 0} \frac{\sec^2(x+\pi/4)}{1\cdot\tan(x+\pi/4)}\\
&= 2.
\end{aligned}$$

Thus, $\ln(L) = 2$, and so, $L = e^2$.

(vi) Given limits is:

$$\begin{aligned}
L &= \lim_{x\to a} \left(\frac{\int_a^x f(t)dt}{x-a}\right) && \frac{0}{0} \text{ form}\\
&= \lim_{x\to a} \left(\frac{\dfrac{d}{dx}\left[\int_a^x f(t)dt\right]}{\dfrac{d}{dx}(x-a)}\right)\\
&= \lim_{x\to a} \left(\frac{f(x)}{1}\right)\\
&= f(a).
\end{aligned}$$

Here we assumed that the function is continuous so that the integral $\int_a^x f(t)dt$ exists and $\lim\limits_{x\to a} f(x) = f(a)$. □

Example 25. *Evaluate:* $\lim\limits_{x\to 0} \dfrac{a^x - b^x}{x}$.

Sol: Given limits is:

$$\begin{aligned}
L &= \lim_{x\to 0} \frac{a^x - b^x}{x} && \frac{0}{0} \text{ form}\\
&= \lim_{x\to a} \left(\frac{a^x \ln a - b^x \ln b}{1}\right)\\
&= \ln a - \ln b.
\end{aligned}$$

□

Example 26. *Evaluate:* $\lim\limits_{x\to 1} \dfrac{x^5 - 2x^3 - 4x^2 + 9x - 4}{x^4 - 2x^3 + 2x - 1}$.

Sol: Given limits is:

$$\begin{aligned} L &= \lim_{x\to 1} \frac{x^5-2x^3-4x^2+9x-4}{x^4-2x^3+2x-1} && \frac{0}{0} \text{ form} \\ &= \lim_{x\to 1} \frac{5x^4-6x^2-8x+9}{4x^3-6x^2+2} && \frac{0}{0} \text{ form} \\ &= \lim_{x\to 1} \frac{20x^3-12x-8}{12x^2-12x} && \frac{0}{0} \text{ form} \\ &= \lim_{x\to 1} \frac{60x^2-12}{24x-12} \\ &= 4. \end{aligned}$$

□

Example 27. *Evaluate:* $\lim_{x\to 0} \frac{e^x-e^{-x}-2x}{\tan x - x}$.

Sol: Given limits is:

$$\begin{aligned} L &= \lim_{x\to 0} \frac{e^x-e^{-x}-2x}{\tan x - x} && \frac{0}{0} \text{ form} \\ &= \lim_{x\to 0} \frac{e^x+e^{-x}-2}{\sec^2 x - 1} && \frac{0}{0} \text{ form} \\ &= \lim_{x\to 0} \frac{e^x-e^{-x}}{2\sec x(\sec x \tan x)} && \frac{0}{0} \text{ form} \\ &= \lim_{x\to 0} \frac{e^x-e^{-x}}{2(1+\tan^2 x)\tan x} \\ &= \lim_{x\to 0} \frac{e^x-e^{-x}}{2\tan x + 2\tan^3 x} && \frac{0}{0} \text{ form} \\ &= \lim_{x\to 0} \frac{e^x+e^{-x}}{2\sec^2 x + 6\tan^2 x \sec^2 x} \\ &= \frac{1+1}{2+0} \\ &= 1. \end{aligned}$$

□

Example 28. *Evaluate:* $\lim_{x\to 0} \frac{\tan x - x}{x^2 \tan x}$.

Sol: We have,

$$\begin{aligned} L &= \lim_{x\to 0} \frac{\tan x - x}{x^2 \tan x} \\ &= \lim_{x\to 0} \left(\frac{\frac{\sin x}{\cos x} - x}{x^2 \frac{\sin x}{\cos x}} \right) \\ &= \lim_{x\to 0} \frac{\sin x - x\cos x}{x^2 \sin x} && \frac{0}{0} \text{ form} \\ &= \lim_{x\to 0} \frac{\cos x - \cos x + x \sin x}{2x\sin x + x^2 \cos x} \\ &= \lim_{x\to 0} \frac{\sin x}{2\sin x + x\cos x} && \frac{0}{0} \text{ form} \\ &= \lim_{x\to 0} \frac{\cos x}{2\cos x + \cos x - x\sin x} \\ &= \lim_{x\to 0} \frac{\cos x}{3\cos x - x\sin x} \\ &= \frac{1}{3}. \end{aligned}$$

□

Example 29. *Evaluate:* $\lim\limits_{x\to\frac{\pi}{2}} (\cos x)^{\cos x}$.

Sol: We have,

$$\begin{aligned} L &= \lim_{x\to\frac{\pi}{2}} (\cos x)^{\cos x} \\ \log L &= \lim_{x\to\frac{\pi}{2}} \log (\cos x)^{\cos x} \\ &= \lim_{x\to\frac{\pi}{2}} \cos x \log (\cos x) \\ &= \lim_{x\to\frac{\pi}{2}} \frac{\log (\cos x)}{\sec x} \\ &= \lim_{x\to\frac{\pi}{2}} \frac{-\dfrac{\sin x}{\cos x}}{\sec x \tan x} \\ &= \lim_{x\to\frac{\pi}{2}} \cos x \\ &= 0 \\ L &= e^0 \\ &= 1. \end{aligned}$$

□

Exercise (Assignment)

(Q.1) If $\lim\limits_{x\to 0} \dfrac{\sin 2x + a\sin x}{x^3}$ be finite, find the value of a and also the limit.

Ans. $a = -2$ and the value of limit is -1.

(Q.2) Determine the value of a, b, c so that $\lim\limits_{x\to 0} \dfrac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2$.

Ans. $a = 1, b = 2, c = 1$.

(Q.3) Evaluate: $\lim\limits_{x\to 0} \dfrac{e^x - e^{\sin x}}{x - \sin x}$.

Ans. 1.

(Q.4) Evaluate: $\lim\limits_{x\to 0} \dfrac{\sin^2 \pi x}{2e^{\frac{x}{2}} - xe}$.

Ans. $4\dfrac{\pi^2}{e}$.

(Q.5) Evaluate: $\lim\limits_{x\to 0} \dfrac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}$.

Ans. $-\dfrac{2}{3}$.

(Q.6) Evaluate: $\lim\limits_{x\to 0} \dfrac{1+\sin x - \cos x + \log(1-x)}{x\tan^2 x}$.

Ans. $-\dfrac{1}{3}$.

(Q.7) Evaluate: $\lim\limits_{x\to 0} \dfrac{\tan x - \sin x}{x^3}$.

Ans. $\dfrac{1}{2}$.

(Q.8) Evaluate: $\lim\limits_{x\to 0} \dfrac{1 - x + \frac{1}{2}x^2 - e^{-x}}{x^3}$.

Ans. $\dfrac{1}{6}$.

(Q.9) Evaluate: $\lim\limits_{t\to 0} \dfrac{1 - \cos t - \frac{1}{2}t^2}{t^4}$.

Ans. $-\dfrac{1}{24}$.

(Q.10) Evaluate: $\lim\limits_{x\to 0} \dfrac{\sin x^2 - \sin^2 x}{x^4}$.

Ans. $\dfrac{1}{3}$.

(Q.11) Evaluate: $\lim\limits_{x\to a} \dfrac{xe^{-x} - ae^{-a}}{x-a}$.

Ans. $(1-a)e^{-a}$.

(Q.12) Evaluate: $\lim\limits_{x\to 1}(2-x)^{\tan\frac{\pi x}{2}}$.

Ans. $e^{\frac{2}{\pi}}$.

(Q.13) Evaluate: $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$.

Ans. $e^{-\frac{1}{6}}$.

(Q.14) Evaluate: $\lim\limits_{x\to 0}(1+\tan x)^{\cot x}$.

Ans. e.

(Q.15) Evaluate: $\lim\limits_{x\to 0}\left(\dfrac{1}{x}\right)^{2\sin x}$.

Ans. 0.

Functions of Several Variables

Suppose, a particle is moving parallel to the earth surface, then at any instant its energy depends only upon its velocity (surely, we neglect the effect of other celestial and terrestrial bodies on the energy of particle). Precisely, the energy of particle

$$E(v) = \frac{1}{2}mv^2 + K_0$$

where v is the velocity of particle and K_0 is its potential energy (which is constant). Thus, the $E(v)$ depends only on the velocity v. We say that the energy E of particle is an output, while its velocity is the input for this output function, and for various values of input we obtain the different outputs.

Now consider same particle but with a different situation. Suppose, the particle is moving in such a way that its hight from the earth surface changes continuously. Then, at any instant its energy depends upon its velocity v, as well as, its hight h from the earth surface. Precisely, the energy of particle

$$\mathcal{E}(v,h) = \frac{1}{2}mv^2 + mgh.$$

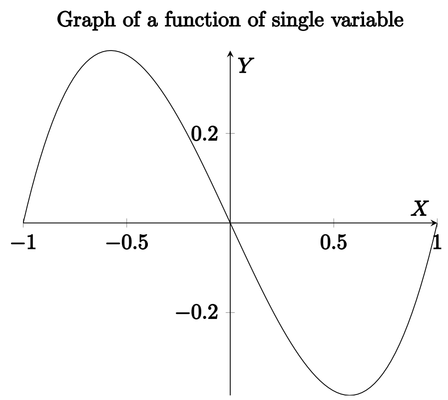
What we see? We now see that the output function $\mathcal{E}$ depends on the two inputs, namely, the velocity v and the hight h of the particle.

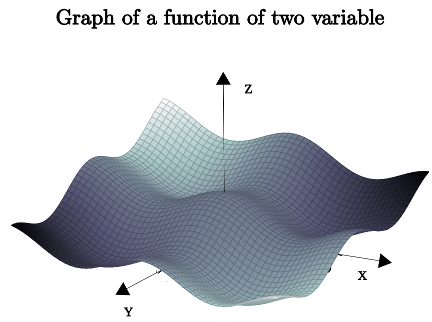
We say that the energy function E is a function of a single variable v, while the energy function $\mathcal{E}$ is a function of two variables v, h.

In general, we say that a quantity y is a function f of n variables if its value depends on n variables $x_1, x_2, \ldots, x_n$. Mathematically, we represent this fact by:

$$y = f(x_1, x_2, \ldots, x_n).$$

Partial Derivatives. Suppose, $y = f(x)$ is a function of single variable. If we draw a graph of this function by taking the values of x on X-axis and of y on Y-axis, then we get a two-dimensional curve. Clearly, the input x can change only along the X-axis (either towards left or towards right), and so, we can find the rate of change of y only along X-axis. This rate is called the derivative (total derivative) of y with respect to x and denoted by $\dfrac{dy}{dx}$.





We call the inputs as *independent variable* and the output as *dependent variable.* Now consider a different case, when the dependent variable z is a function of two independent variables (x, y). We write $z = f(x, y)$. Now if we draw the graph of this function by taking the values of x, y and z on three mutually perpendicular axes, we obtain a three dimensional surface. Then, apart from the previous case the independent variables (x, y) (the inputs) now can change in the XY-plane in any direction (right or left, up or down; or in any direction different from these two), and so, we can find the rate of change of z along any such direction. Such rate of change is called the directional derivative of f. In particular, we are interested in finding the rate of change (directional derivative) of f in two directions (i) along X-axis; and (ii) along Y-axis, and so, we get two directional derivatives along these two axes. The rate of change of f (or z) along X-axis is called the partial derivative of f (or z) with respect to x and it is denoted by $\dfrac{\partial f}{\partial x}$. Similarly, the rate of change of f (or z) along Y-axis is called the partial derivative of f (or z) with respect to y and it is denoted by $\frac{\partial f}{\partial y}$.

Because, in moving along X-axis y remains constant, and $\frac{\partial f}{\partial x}$ is the rate of change of f along X axis, we have:

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

Similarly, in moving along Y-axis x remains constant, and $\frac{\partial f}{\partial y}$ is the rate of change of f along Y axis, we have:

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}.$$

In similar way, we can define the partial derivatives of higher orders.

To find the partial derivative of $z = f(x, y)$ with respect to x we differentiate z by usual rules of differentiation with respect to x, but treat the variable y as constant. Similarly, when we find the partial derivative of $z = f(x, y)$ with respect to y we differentiate z by usual rules of differentiation with respect to y, but treat the variable x as constant. If $u = f(x, y, z)$ is a function of three variables, then find the partial derivative of $u = f(x, y, z)$ with respect to x we differentiate u by usual rules of differentiation with respect to x, but treat all other variables y and z as constant, and so on..

Example 30. Find the first and second partial derivatives of the function $z = x^3 + y^3 - 3axy$.

Sol: Given function is

$$z = x^3 + y^3 - 3axy. \tag{22}$$

Differentiating (22) partially with respect to x we get:

$$\frac{\partial z}{\partial x} = 3x^2 - 3ay. \tag{23}$$

Differentiating (22) partially with respect to y we get:

$$\frac{\partial z}{\partial y} = 3y^2 - 3ax. \tag{24}$$

Differentiating (23) partially with respect to x and y we get:

$$\frac{\partial^2 z}{\partial x^2} = 6x; \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = -3a.$$

Differentiating (24) partially with respect to y we get:

$$\frac{\partial^2 z}{\partial y^2} = 6y.$$

□

Example 31. If $z(x + y) = x^2 + y^2$, then show that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$

Sol: Given function is

$$z = \frac{x^2+y^2}{x+y}. \tag{25}$$

Differentiating (25) partially with respect to x we get:

$$\frac{\partial z}{\partial x} = \frac{(x+y)2x - (x^2+y^2)}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2}. \tag{26}$$

Differentiating (25) partially with respect to y we get:

$$\frac{\partial z}{\partial x} = \frac{(x+y)2y - (x^2+y^2)}{(x+y)^2} = \frac{y^2+2xy-x^2}{(x+y)^2}. \tag{27}$$

From (26) and (27) we obtain:

$$\begin{aligned}
\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 &= \left[\frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2}\right]^2 \\
&= \left[\frac{2x^2-2y^2}{(x+y)^2}\right]^2 \\
&= \left[\frac{2(x-y)(x+y)}{(x+y)^2}\right]^2 \\
&= \frac{4(x-y)^2}{(x+y)^2}
\end{aligned}$$

and

$$\begin{aligned}
4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) &= 4\left[1 - \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2}\right] \\
&= 4\left[\frac{(x+y)^2 - (x^2+2xy-y^2) - (y^2+2xy-x^2)}{(x+y)^2}\right] \\
&= 4\left[\frac{(x^2+y^2+2xy) - (x^2+2xy-y^2) - (y^2+2xy-x^2)}{(x+y)^2}\right] \\
&= 4\left[\frac{x^2+y^2-2xy}{(x+y)^2}\right] \\
&= \frac{4(x-y)^2}{(x+y)^2}.
\end{aligned}$$

Therefore: $$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$ □

Example 32. If $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$, then show that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}$ and $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Sol: Differentiating given function partially with respect to y we get:

$$\begin{aligned}
\frac{\partial u}{\partial y} &= x^2 \times \frac{1}{1+(y/x)^2} \times \frac{1}{x} - \left[2y\tan^{-1}\left(\frac{x}{y}\right) + y^2 \times \frac{1}{1+(x/y)^2} \times \left(-\frac{x}{y^2}\right)\right] \\
&= \frac{x^3}{x^2+y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right) + \frac{xy^2}{x^2+y^2} \\
&= x - 2y\tan^{-1}\left(\frac{x}{y}\right).
\end{aligned}$$

Differentiating the above equation partially with respect to x we get:

$$\begin{aligned}\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x}\left[x - 2y\tan^{-1}\left(\frac{x}{y}\right)\right]\\ &= 1 - 2y \times \frac{1}{1+(x/y)^2} \times \frac{1}{y}\\ &= \frac{x^2-y^2}{x^2+y^2}.\end{aligned}$$

Similarly, $\dfrac{\partial u}{\partial x} = 2x\tan^{-1}\left(\dfrac{y}{x}\right) - y$. Differentiating with respect to y we get:

$$\frac{\partial^2 u}{\partial y \partial x} = 2x \times \frac{1}{1+(y/x)^2} \times \frac{1}{x} - 1 = \frac{x^2-y^2}{x^2+y^2}.$$

Therefore:

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{x^2-y^2}{x^2+y^2}.$$

□

Example 33. If $v = (x^2+y^2+z^2)^{-1/2}$, then prove that $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2} = 0$.

Sol: Differentiating given function partially with respect to x we get:

$$\begin{aligned}\frac{\partial v}{\partial x} &= -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2} \times 2x\\ &= -x\left(x^2+y^2+z^2\right)^{-3/2}.\end{aligned}$$

Again differentiating with respect to x we obtain:

$$\begin{aligned}\frac{\partial^2 v}{\partial x^2} &= -\left(x^2+y^2+z^2\right)^{-3/2} - x\left(-\frac{3}{2}\right)\left(x^2+y^2+z^2\right)^{-5/2} \times 2x\\ &= \left(x^2+y^2+z^2\right)^{-5/2}\left[3x^2 - \left(x^2+y^2+z^2\right)\right]\\ &= \left(x^2+y^2+z^2\right)^{-5/2}\left(2x^2-y^2-z^2\right).\end{aligned}$$

Using symmetry of v in x, y and z we obtain:

$$\frac{\partial^2 v}{\partial y^2} = \left(x^2+y^2+z^2\right)^{-5/2}\left(2y^2-x^2-z^2\right)$$

and

$$\frac{\partial^2 v}{\partial z^2} = \left(x^2+y^2+z^2\right)^{-5/2}\left(2z^2-x^2-z^2\right).$$

Adding the above three we get:

$$\begin{aligned}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} &= \left(x^2+y^2+z^2\right)^{-5/2}\left(2x^2-y^2-z^2\right) + \left(x^2+y^2+z^2\right)^{-5/2}\left(2y^2-x^2-z^2\right)\\ &\quad + \left(x^2+y^2+z^2\right)^{-5/2}\left(2z^2-x^2-z^2\right)\\ &= \left(x^2+y^2+z^2\right)^{-5/2}\left(2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-z^2\right)\\ &= \left(x^2+y^2+z^2\right)^{-5/2}\cdot 0\\ &= 0.\end{aligned}$$

□

Example 34. If $u = \ln\left(x^3+y^3+z^3-3xyz\right)$, then show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}.$$

Sol: Differentiating given function partially with respect to x we get:

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}.$$

Using symmetry of u in x, y and z we obtain:

$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$$

and

$$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}.$$

Adding the above three we get:

$$\begin{aligned}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} &= \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}+\frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}+\frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}\\ &= \frac{3x^2-3yz+3y^2-3xz+3z^2-3xy}{x^3+y^3+z^3-3xyz}\\ &= \frac{3(x^2+y^2+z^2-yz-xz-xy)}{x^3+y^3+z^3-3xyz}\\ &= \frac{3(x^2+y^2+z^2-yz-xz-xy)}{(x+y+z)(x^2+y^2+z^2-yz-xz-xy)}\\ &= \frac{3}{x+y+z}.\end{aligned}$$

Thus:

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3}{x+y+z}.$$

Therefore:

$$\begin{aligned}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u &= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)\\ &= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)\\ &= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)\\ &= -\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}\\ &= -\frac{9}{(x+y+z)^2}.\end{aligned}$$

□

Example 35. If $u = e^{xyz}$, then show that $\dfrac{\partial^3 u}{\partial x\partial y\partial z} = (1+3xyz+x^2y^2z^2)e^{xyz}$.

Sol: Differentiating given function partially with respect to x we get:

$$\frac{\partial u}{\partial z} = e^{xyz}\cdot xy.$$

Again differentiating with respect to y we get:

$$\begin{aligned}\frac{\partial^2 u}{\partial y\partial z} &= (e^{xyz}\cdot xz)\cdot xy + e^{xyz}\cdot x\\ &= e^{xyz}\left(x+x^2yz\right).\end{aligned}$$

Again differentiating with respect to x we get:

$$\begin{aligned}\frac{\partial^3 u}{\partial x \partial y \partial z} &= e^{xyz} \cdot yz \cdot \left(x + x^2yz\right) + e^{xyz} \cdot (1 + 2xyz).\\ &= (1 + 3xyz + x^2y^2z^2)e^{xyz}.\end{aligned}$$

□

Example 36. If $x^x y^y z^z = c$, then show that $\frac{\partial^2 z}{\partial x \partial y} = -(x \ln ex)^{-1}$ at point $x = y = z$.

Sol: Given that: $x^x y^y z^z = c$. Taking logarithm we obtain:

$$x \ln x + y \ln y + z \ln z = \ln c.$$

Differentiating given function partially with respect to x (note that, z is a function of x and y both, so, it will not be treated as constant) we get:

$$\begin{aligned}x \cdot \frac{1}{x} + \ln x + z \cdot \frac{1}{z}\frac{\partial z}{\partial x} + \ln z \frac{\partial z}{\partial x} &= 0\\ \implies 1 + \ln x + (1 + \ln z)\frac{\partial z}{\partial x} &= 0\\ \implies \frac{\partial z}{\partial x} &= -\frac{1 + \ln x}{1 + \ln z}.\end{aligned}$$

By symmetry of the function z in the variables x and y we obtain:

$$\frac{\partial z}{\partial y} = -\frac{1 + \ln y}{1 + \ln z}.$$

Differentiating the above equation partially with respect to x we obtain:

$$\begin{aligned}\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left[-\frac{1 + \ln y}{1 + \ln z}\right]\\ &= (1 + \ln y) \cdot \frac{1}{(1 + \ln z)^2} \cdot \frac{1}{z} \cdot \frac{\partial z}{\partial x}\\ &= (1 + \ln y) \cdot \frac{1}{(1 + \ln z)^2} \cdot \frac{1}{z} \cdot \left[-\frac{1 + \ln x}{1 + \ln z}\right]\\ &= -\frac{(1 + \ln x)(1 + \ln y)}{z(1 + \ln z)^3}.\end{aligned}$$

Putting $x = y = z$ in the above equation we get:

$$\begin{aligned}\frac{\partial^2 z}{\partial x \partial y} &= -\frac{(1 + \ln x)(1 + \ln x)}{z(1 + \ln x)^3}\\ &= -\frac{1}{z(1 + \ln x)}\\ &= -\frac{1}{z(\ln e + \ln x)}\\ &= -(x \ln ex)^{-1}.\end{aligned}$$

□

Example 37. If $v = r^n$, where $r^2 = x^2 + y^2 + z^2$, then show that

$$v_{xx} + v_{yy} + v_{zz} = n(n + 1)r^{n-2}.$$

Sol: Given that: $r^2 = x^2 + y^2 + z^2$. Differentiating partially with respect to x we get:

$$\begin{aligned} 2r\frac{\partial r}{\partial x} &= 2x \\ \implies \frac{\partial r}{\partial x} &= \frac{x}{r}. \end{aligned}$$

By symmetry of the function r in the variables x and y we obtain:

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \ \frac{\partial r}{\partial z} = \frac{z}{r}.$$

Now, given that $v = r^n$. Differentiating partially with respect to x we obtain:

$$\begin{aligned} \frac{\partial v}{\partial x} &= nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-1}\cdot\frac{x}{r} \\ &= nxr^{n-2}. \end{aligned}$$

Again differentiating with respect to x we get:

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} &= \frac{\partial}{\partial x}(nxr^{n-2}) = nr^{n-2} + n(n-2)xr^{n-3}\frac{\partial r}{\partial x} \\ &= nr^{n-2} + n(n-2)xr^{n-3}\cdot\frac{x}{r} \\ &= nr^{n-4}\left[r^2 + (n-2)x^2\right] \end{aligned}$$

Again by symmetry we obtain:

$$\frac{\partial^2 v}{\partial y^2} = nr^{n-4}\left[r^2 + (n-2)y^2\right]$$

and

$$\frac{\partial^2 v}{\partial z^2} = nr^{n-4}\left[r^2 + (n-2)z^2\right].$$

Adding the above three equalities we obtain:

$$\begin{aligned} v_{xx} + v_{yy} + v_{zz} &= \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \\ &= nr^{n-4}\left[r^2 + (n-2)x^2\right] + nr^{n-4}\left[r^2 + (n-2)y^2\right] \\ &\quad + nr^{n-4}\left[r^2 + (n-2)z^2\right] \\ &= nr^{n-4}\left[3r^2 + (n-2)\left(x^2 + y^2 + z^2\right)\right] \\ &= nr^{n-4}\left[3r^2 + (n-2)r^2\right] \\ &= nr^{n-4}(n+1)r^2 \\ &= n(n+1)r^{n-2}. \end{aligned}$$

□

Example 38. *If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$, then prove that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r).$$

Sol: Given that $x = r\cos\theta$ and $y = r\sin\theta$. On squaring and adding these two we obtain

$$r^2 = x^2 + y^2.$$

Differentiating the above equation with respect to x partially we obtain: $2r\dfrac{\partial r}{\partial x} = 2x$, i.e.

$$\frac{\partial r}{\partial x} = \frac{x}{r}.$$

Similarly we obtain:

$$\frac{\partial r}{\partial y} = \frac{y}{r}.$$

Given that $u = f(r)$. Differentiating u with respect to x partially we get:

$$\begin{aligned} \frac{\partial u}{\partial x} &= f'(r)\frac{\partial r}{\partial x} \\ \implies \frac{\partial u}{\partial x} &= \frac{x}{r}f'(r). \end{aligned}$$

Again differentiating the above equation with respect to x partially and using the formula $\dfrac{d}{dx}(f_1f_2f_3) = f_1'f_2f_3 + f_1f_2'f_3 + f_1f_2f_3'$ we get:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{x}{r}f'(r)\right] \\ \implies \frac{\partial^2 u}{\partial x^2} &= \frac{x}{r}f''(r)\frac{\partial r}{\partial x} + \frac{1}{r}f'(r) - \frac{x}{r^2}\frac{\partial r}{\partial x}f'(r) \\ \implies \frac{\partial^2 u}{\partial x^2} &= \frac{x^2}{r^2}f''(r) + \frac{1}{r}f'(r) - \frac{x^2}{r^3}f'(r). \end{aligned} \tag{28}$$

Since the given functions are symmetric in x and y we obtain:

$$\frac{\partial^2 u}{\partial y^2} = \frac{y^2}{r^2}f''(r) + \frac{1}{r}f'(r) - \frac{y^2}{r^3}f'(r). \tag{29}$$

Adding equations (28) and (29) we obtain

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{1}{r^2}f''(r)\left(x^2+y^2\right) + \frac{2}{r}f'(r) - \frac{1}{r^3}f'(r)\left(x^2+y^2\right) \\ &= \frac{1}{r^2}f''(r)\cdot r^2 + \frac{2}{r}f'(r) - \frac{1}{r^3}f'(r)\cdot r^2 \\ &= f''(r) + \frac{2}{r}f'(r) - \frac{1}{r}f'(r) \\ &= f''(r) + \frac{1}{r}f'(r). \end{aligned}$$

□

Example 39. *If $x = r\cos\theta$, $y = r\sin\theta$, then prove that $\dfrac{\partial r}{\partial x} = \dfrac{\partial x}{\partial r}$ and $\dfrac{1}{r}\dfrac{\partial x}{\partial \theta} = r\dfrac{\partial \theta}{\partial x}$.*

Sol: Given that

$$x = r\cos\theta \tag{30}$$

$$y = r\sin\theta. \tag{31}$$

Squaring and adding these two we obtain: $r^2 = x^2 + y^2$. So, as we found in the previous example: $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$. Again dividing (31) by (30) we get

$$\begin{aligned} \frac{y}{x} &= \frac{\sin\theta}{\cos\theta} = \tan\theta \\ \implies \theta &= \tan^{-1}\left(\frac{y}{x}\right). \end{aligned}$$

Differentiating the above equation partially with respect to θ we obtain:

$$\begin{aligned} \frac{\partial\theta}{\partial x} &= \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \left(-\frac{y}{x^2}\right) \\ &= -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2} \\ &= -\frac{\sin\theta}{r}. \end{aligned}$$

On differentiating equation (30) partially with respect to r we get:

$$\begin{aligned} \frac{\partial x}{\partial r} &= \cos\theta = \frac{x}{r} \\ \implies \frac{\partial x}{\partial r} &= \frac{\partial r}{\partial x}. \end{aligned}$$

Again differentiating equation (30) partially with respect to θ we get:

$$\begin{aligned} \frac{\partial x}{\partial\theta} &= -r\sin\theta = -r\left(-r\frac{\partial\theta}{\partial x}\right) \\ \implies \frac{1}{r}\frac{\partial x}{\partial\theta} &= r\frac{\partial\theta}{\partial x}. \end{aligned}$$

□

Chain Rule for Partial Differentiation. Suppose $z = f(x, y)$ be a function of two variable, where $x = x(t), y = y(t)$ are functions of another variable t. Suppose there is a small change δt in the variable t, due to which there are small changes δx and δy in the variables x and y respectively. Because of these changes in x and y, suppose there is a small change δz in the function $z = f(x, y)$. Then, the rate of change of z in X direction will be $\dfrac{\partial z}{\partial x}$, and so the change in z along the X direction will be $\dfrac{\partial z}{\partial x}\delta x$. Similarly, the change in z in Y direction will be $\dfrac{\partial z}{\partial y}\delta y$. Since the changes are very small the total approximate change in z will be:

$$\delta z \approx \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y.$$

Therefore, the rate of change of z with respect to t:

$$\frac{\delta z}{\delta t} \approx \frac{\partial z}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial z}{\partial y}\frac{\delta y}{\delta t}.$$

For instantaneous rate of change, letting $\delta \to 0$, and so, $\delta x, \delta y \to 0$ in the above inequality we obtain:

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.}$$

In a general case, if $z = f(x, y)$, $x = x(r, s)$ and $y = y(r, s)$, then we have:

$$\begin{aligned} \frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \\ \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}. \end{aligned}$$

The above results can be generalized for a function of n variables.

Example 40. *If $u = f(x, y)$ and $x = r\cos\theta, y = r\sin\theta$, then prove that*

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2.$$

Sol: Since $x = r\cos\theta, y = r\sin\theta$ we have:

$$\frac{\partial x}{\partial r} = \cos\theta,\ \frac{\partial x}{\partial \theta} = -r\sin\theta$$

and

$$\frac{\partial y}{\partial r} = \sin\theta,\ \frac{\partial y}{\partial \theta} = r\cos\theta.$$

Now by chain rule we have:

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\ &= \cos\theta\,\frac{\partial u}{\partial x} + \sin\theta\,\frac{\partial u}{\partial y} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \\ &= -r\sin\theta\,\frac{\partial u}{\partial x} + r\cos\theta\,\frac{\partial u}{\partial y}. \end{aligned}$$

Therefore:

$$\begin{aligned} \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 &= \left(\cos\theta\,\frac{\partial u}{\partial x} + \sin\theta\,\frac{\partial u}{\partial y}\right)^2 + \frac{1}{r^2}\left(-r\sin\theta\,\frac{\partial u}{\partial x} + r\cos\theta\,\frac{\partial u}{\partial y}\right)^2 \\ &= \left(\cos^2\theta + \sin^2\theta\right)\left(\frac{\partial u}{\partial x}\right)^2 + \left(\sin^2\theta + \cos^2\theta\right)\left(\frac{\partial u}{\partial y}\right)^2 \\ &= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2. \end{aligned}$$

□

Example 41. *If $u = x\log(xy)$ where $x^3 + y^3 + 3xy = 1$, find $\frac{du}{dx}$.*

Sol: Since

$$u = x\log(xy) \tag{32}$$

we have:

$$\frac{\partial u}{\partial x} = x\left\{\frac{1}{xy}.y\right\} + \log(xy) = 1 + \log(xy),$$

and
$$\frac{\partial u}{\partial y} = x\left\{\frac{1}{xy}.x\right\} = \frac{x}{y}.$$

also,

$$x^3 + y^3 + 3xy = 1 \tag{33}$$

Differentiating (33), w.r.t. x, we get

$$3x^2 + 3y^2\frac{dy}{dx} + 3\left(x\frac{dy}{dx} + y\right) = 0$$

$$\frac{dy}{dx} = -\left(\frac{x^2+y}{x+y^2}\right) \tag{34}$$

we know that

$$\begin{aligned}\frac{du}{dx} &= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} \\ &= 1 + \log(xy) + \frac{x}{y}\left\{-\left(\frac{x^2+y}{x+y^2}\right)\right\} \\ &= 1 + \log(xy) - \frac{x(x^2+y)}{y(x+y^2)}.\end{aligned}$$

□

Example 42. *If $u = f(x-y, y-z, z-x)$, then prove that* $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

Sol: Let

$$X = x - y \tag{35}$$
$$Y = y - z \tag{36}$$
$$Z = z - x. \tag{37}$$

Then we have $u = f(X, Y, Z)$, i.e., u is a function of X, Y, Z. Differentiating (35), (36) and (37) with respect to x, y, z we get:

$$\begin{aligned}&\frac{\partial X}{\partial x} = 1,\ \frac{\partial X}{\partial y} = -1,\ \frac{\partial X}{\partial z} = 0 \\ &\frac{\partial Y}{\partial x} = 0,\ \frac{\partial Y}{\partial y} = 1,\ \frac{\partial Y}{\partial z} = -1 \\ &\frac{\partial Z}{\partial x} = -1,\ \frac{\partial Z}{\partial y} = 0,\ \frac{\partial Z}{\partial z} = 1.\end{aligned}$$

Now by chain rule of partial differentiation we get:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}\frac{\partial Z}{\partial x} \\ &= \frac{\partial u}{\partial X}\cdot 1 + \frac{\partial u}{\partial Y}\cdot 0 + \frac{\partial u}{\partial Z}(-1) \\ &= \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z},\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}\frac{\partial Z}{\partial y}\\ &= \frac{\partial u}{\partial X}(-1) + \frac{\partial u}{\partial Y}\cdot 1 + \frac{\partial u}{\partial Z}\cdot 0\\ &= -\frac{\partial u}{\partial X} + \frac{\partial u}{\partial Y}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial u}{\partial z} &= \frac{\partial u}{\partial X}\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}\frac{\partial Z}{\partial z}\\ &= \frac{\partial u}{\partial X}\cdot 0 + \frac{\partial u}{\partial Y}(-1) + \frac{\partial u}{\partial Z}\cdot 1\\ &= -\frac{\partial u}{\partial Y} + \frac{\partial u}{\partial Z}.\end{aligned}$$

Adding the above three equalities we get

$$\begin{aligned}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z} - \frac{\partial u}{\partial X} + \frac{\partial u}{\partial Y} - \frac{\partial u}{\partial Y} + \frac{\partial u}{\partial Z}\\ &= 0.\end{aligned}$$

□

Example 43. *Transform the Laplace equation* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ *into the polar coordinates.*

Sol: We know that the relation between cartesian coordinates (x, y) and the polar coordinates (r, θ) are given by $x = r\cos\theta, y = r\sin\theta$, i.e., $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. Therefore:

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta,\ \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$

and

$$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r},\ \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}.$$

Now by chain rule we have:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\\ &= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)u.\end{aligned}$$

The above relation is true for all functions u, and so:

$$\frac{\partial}{\partial x} \equiv \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}.$$

Similarly, we have

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\\ &= \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)u.\end{aligned}$$

The above relation is true for all functions u, and so:

$$\frac{\partial}{\partial y} \equiv \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}.$$

Therefore:

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \\
&= \left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial \theta}\right)\left(\cos\theta\,\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial u}{\partial \theta}\right) \\
&= \cos\theta\,\frac{\partial}{\partial r}\left(\cos\theta\,\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}\,\frac{\partial}{\partial \theta}\left(\cos\theta\,\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial u}{\partial \theta}\right) \\
&= \cos^2\theta\,\frac{\partial^2 u}{\partial r^2} - \sin\theta\cos\theta\left(-\frac{1}{r^2}\,\frac{\partial u}{\partial \theta} + \frac{1}{r}\,\frac{\partial^2 u}{\partial r\partial \theta}\right) - \frac{\sin\theta}{r}\left(-\sin\theta\,\frac{\partial u}{\partial r} + \cos\theta\frac{\partial^2 u}{\partial r\partial \theta}\right) \\
&\quad + \frac{\sin\theta}{r^2}\left(\cos\theta\,\frac{\partial u}{\partial \theta} + \sin\theta\frac{\partial^2 u}{\partial \theta^2}\right)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \\
&= \left(\sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial}{\partial \theta}\right)\left(\sin\theta\,\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial u}{\partial \theta}\right) \\
&= \sin^2\theta\,\frac{\partial^2 u}{\partial r^2} + \sin\theta\cos\theta\left(-\frac{1}{r^2}\,\frac{\partial u}{\partial \theta} + \frac{1}{r}\,\frac{\partial^2 u}{\partial r\partial \theta}\right) + \frac{\cos\theta}{r}\left(\cos\theta\,\frac{\partial u}{\partial r} + \sin\theta\frac{\partial^2 u}{\partial r\partial \theta}\right) \\
&\quad + \frac{\cos\theta}{r^2}\left(-\sin\theta\,\frac{\partial u}{\partial \theta} + \cos\theta\,\frac{\partial^2 u}{\partial \theta^2}\right).
\end{aligned}$$

Therefore the Laplace equation will be:

$$\begin{aligned}
& \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\
\implies & \left(\cos^2\theta + \sin^2\theta\right)\frac{\partial^2 u}{\partial r^2} + \frac{\sin^2\theta + \cos^2\theta}{r}\,\frac{\partial u}{\partial r} + \frac{\sin^2\theta + \cos^2\theta}{r^2}\,\frac{\partial^2 u}{\partial \theta^2} = 0 \\
\implies & \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\,\frac{\partial u}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 u}{\partial \theta^2} = 0.
\end{aligned}$$

□

Exercise (Assignment)

(Q.1) If $z = f(x, y)$, $x = e^u + e^{-v}, y = e^{-u} + e^v$, then prove that:

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

Hint: Use the chain rule of partial differentiation.

Tangent Plane and Normal (Normal Line)

Definition 1. Given a point P on a surface S, the tangent plane of S at point P, is the plane passing through P which contains the tangent lines of all the curves on S passing through P.

Formula for tangent plane. Let $f(x, y, z) = 0$ be the equation of surface S, and $P(x_0, y_0, z_0)$ be a point on S. Then the equation of any plane passing through P is given by:

$$\hat{n}(\overrightarrow{r} - \overrightarrow{r_0}) = 0 \tag{38}$$

where $\overrightarrow{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\overrightarrow{r_0} = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$ and $\hat{n}$ is the unit normal vector to the surface S at point P. Since $P(x_0, y_0, z_0)$ is given, the vector $\overrightarrow{r_0}$ is known and we have to find only $\hat{n}$. Then, it is sufficient to obtain the direction cosines/ratios of $\hat{n}$. For this, we know that the normal vector to the surface S at point P is given by:

$$\text{grad} f = \nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z},$$

where all the partial derivatives are calculated at point P. Therefore, it follows from (38) that the equation of tangent plane:

$$\left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right) \cdot \left[\hat{i}(x - x_0) + \hat{j}(y - y_0) + \hat{k}(z - z_0)\right] = 0$$

or

$$\frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) + \frac{\partial f}{\partial z}(z - z_0) = 0. \tag{39}$$

Definition 2. *Given a point P on a surface S, the normal line or normal to S at point P, is the line passing through P which is perpendicular to the tangent plane at point P.*

Formula for Normal. Let $f(x, y, z) = 0$ be the equation of surface S, and $P(x_0, y_0, z_0)$ be a point on S. Then the equation normal to the surface S passing through P is given by:

$$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z} \tag{40}$$

where $f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}, f_z = \frac{\partial f}{\partial z}$ are calculated at point P. In parametric form:

$$\begin{aligned} x &= x_0 + f_x \cdot t \\ y &= y_0 + f_y \cdot t \\ z &= z_0 + f_z \cdot t. \end{aligned}$$

Example 44. Find the equation of the tangent plane and the normal to the surface $z^2 = 4(1 + x^2 + y^2)$ at point $(2, 2, 6)$.

Sol. The equation of given surface can be written as $f(x, y, z) = z^2 - 4(1 + x^2 + y^2) = 0$. Therefore, $\frac{\partial f}{\partial x} = -8x, \frac{\partial f}{\partial y} = -8y, \frac{\partial f}{\partial z} = 2z$. At point $(2, 2, 6)$ we have $\frac{\partial f}{\partial x} = -16, \frac{\partial f}{\partial y} = -16, \frac{\partial f}{\partial z} = 12$. Also, here $x_0 = 2, y_0 = 2, z_0 = 6$, therefore, by (39) the equation of tangent plane of the surface S will be:

$$\begin{aligned}
& \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) + \frac{\partial f}{\partial z}(z - z_0) = 0 \\
\Longrightarrow \quad & -16(x - 2) - 16(y - 2) + 12(z - 6) = 0 \\
\Longrightarrow \quad & 4x + 4y - 3z + 2 = 0.
\end{aligned}$$

By (40) the equation of normal at point P will be

$$\begin{aligned}
& \frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z} \\
\Longrightarrow \quad & \frac{x - 2}{-4} = \frac{y - 2}{-4} = \frac{z - 6}{3}.
\end{aligned}$$

Example 45. Find the equation of tangent plane and the normal to each of the following surfaces at the given points:

(A) $2x^2 + y^2 = 3 - 2z$ at $(2, 1, -3)$;

(B) $x^3 + y^3 + 3xyz = 3$ at $(1, -1, 2)$.

Sol. (A). The equation of given surface is $f(x, y, z) = 2x^2 + y^2 - 3 + 2z = 0$, and the point is $x_0 = 2, y_0 = 1, z_0 = -3$. Therefore, $\frac{\partial f}{\partial x} = 4x, \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = 2$ and at point $(2, 1, -3)$, $\frac{\partial f}{\partial x} = 8, \frac{\partial f}{\partial y} = 2, \frac{\partial f}{\partial z} = 2$. Therefore, the equation of tangent plane at point $(2, 1, -3)$ is

$$\begin{aligned}
& \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) + \frac{\partial f}{\partial z}(z - z_0) = 0 \\
\Longrightarrow \quad & 8(x - 2) + 2(y - 1) + 2(z + 3) = 0 \\
\Longrightarrow \quad & 4x + y + z - 6 = 0.
\end{aligned}$$

By (40) the equation of normal at point P will be

$$\begin{aligned}
& \frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z} \\
\Longrightarrow \quad & \frac{x - 2}{4} = \frac{y - 1}{2} = \frac{z + 3}{2}.
\end{aligned}$$

Sol. (B). The equation of given surface is $f(x, y, z) = x^3 + y^3 + 3xyz - 3 = 0$, and the point is $x_0 = 1, y_0 = 2, z_0 = -1$. Therefore, $\frac{\partial f}{\partial x} = 3x^2 + 3yz, \frac{\partial f}{\partial y} = 3y^2 + 3xz, \frac{\partial f}{\partial z} = 3xy$ and at point $(1, 2, -1)$, $\frac{\partial f}{\partial x} = -3, \frac{\partial f}{\partial y} = 9, \frac{\partial f}{\partial z} = 6$. Therefore, the equation of tangent plane at point $(1, 2, -1)$ is

$$\begin{aligned} & \frac{\partial f}{\partial x}(x-x_0)+\frac{\partial f}{\partial y}(y-y_0)+\frac{\partial f}{\partial z}(z-z_0)=0 \\ \implies & -3(x-1)+9(y-2)+6(z+1)=0 \\ \implies & x-3y-2z+3=0. \end{aligned}$$

By (40) the equation of normal at point P will be

$$\begin{aligned} & \frac{x-x_0}{f_x}=\frac{y-y_0}{f_y}=\frac{z-z_0}{f_z} \\ \implies & \frac{x-1}{-3}=\frac{y-2}{9}=\frac{z+1}{6}. \end{aligned}$$

Example 46. Show that the plane $ax + by + cz + d = 0$ touches the surface $px^2 + qy^2 + 2z = 0$, if $\dfrac{a^2}{p} + \dfrac{b^2}{q} + 2cd = 0$.

Sol. Given equations of surface and plane are:

$$f(x, y, z) = px^2 + qy^2 + 2z = 0 \tag{41}$$

$$ax + by + cz + d = 0. \tag{42}$$

Suppose, plane (42) is a tangent plane of surface (41) at point $P(x_0, y_0, z_0)$. Then, we have $\dfrac{\partial f}{\partial x} = 2px, \dfrac{\partial f}{\partial y} = 2qy, \dfrac{\partial f}{\partial z} = 2$ and at point $P(x_0, y_0, z_0)$, $\dfrac{\partial f}{\partial x} = 2px_0, \dfrac{\partial f}{\partial y} = 2qy_0, \dfrac{\partial f}{\partial z} = 2$. Therefore, the equation of tangent plane to the surface (41) at point P will be:

$$\begin{aligned} & \frac{\partial f}{\partial x}(x-x_0)+\frac{\partial f}{\partial y}(y-y_0)+\frac{\partial f}{\partial z}(z-z_0)=0 \\ \implies & 2px_0(x-x_0)+2qy_0(y-y_0)+2(z-z_0)=0 \\ \implies & 2px_0x+2qy_0y+2z-(2px_0^2+2qy_0^2+2z_0)=0. \end{aligned} \tag{43}$$

But, by assumption, the tangent plane of surface (41) at point P is the plane (42), so, comparing the equations (42) and (43) we obtain:

$$\left.\begin{aligned} a &= 2px_0 \\ b &= 2qy_0 \\ c &= 2 \\ d &= -(2px_0^2+2qy_0^2+2z_0). \end{aligned}\right\} \tag{44}$$

Since point $P(x_0, y_0, z_0)$ is situated on the surface (41) it will satisfy (41), therefore we have

$$px_0^2 + qy_0^2 + 2z_0 = 0. \tag{45}$$

Now,

$$\begin{aligned} \text{L.H.S.} &= \frac{a^2}{p}+\frac{b^2}{q}+2cd \\ &= \frac{(2px_0)^2}{p}+\frac{(2qy_0)^2}{q}-2\cdot 2(2px_0^2+2qy_0^2+2z_0) && \text{(using (44))} \\ &= 4px_0^2+4qy_0^2-4(2px_0^2+2qy_0^2+2z_0) \\ &= -4(px_0^2+qy_0^2+2z_0) \\ &= 0 && \text{(using (45))} \\ &= \text{R.H.S.} \end{aligned}$$

Exercise (Assignment)

(Q.1) Find the equation of the normal to the surface $x^2 + y^2 + z^2 = a^2$ at any point $P(x_0, y_0, z_0)$.

(Q.2) Show that the plane $3x+12y-6z-17 = 0$ touches the conicoid $3x^2-6y^2+9z^2+17 = 0$. Find also the point of contact.

(Q.3) Find the tangent plane and normal to the surface $2xz^2 - 3xy - 4x = 7$ at point $P(1, -1, 2)$.

(Q.4) Find the tangent plane and normal to the surface $xyz = a^2$ at point $P(x_1, y_1, z_1)$.

(Q.5) Derive the formula for a tangent plane and normal to a given surface.

Maxima and Minima of function of two variables

Necessary condition for maxima or minima of a function of two variables: Suppose $z = f(x, y)$ is a function of two variables x and y. We say that there is a maxima of function f at point (a, b) if $f(a+h, b+k) - f(a, b) < 0$ for all h, k (positive or negative). Similarly, say that there is a minima of function f at point (a, b) if $f(a + h, b + k) - f(a, b) > 0$ for all h, k (positive or negative). We discuss the necessary condition for maxima or minima of f analytically and geometrically.

The Taylor's series for the function f about the point (a, b) is given by:

$$f(a + h, b + k) = f(a, b) + \left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right] f(a, b) + \frac{1}{2!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f(a, b) + \cdots$$

Neglecting the higher order terms we get:

$$f(a+h, b+k) = f(a, b)+hf_x(a, b)+kf_y(a, b)+\frac{1}{2!}\left[h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)\right]$$

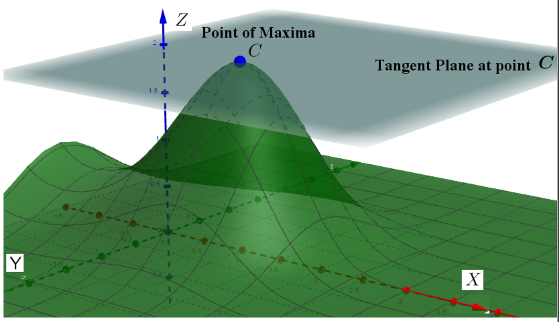
or

$$f(a + h, b + k) - f(a, b) = hf_x(a, b) + kf_y(a, b) + \frac{1}{2!}\left[h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)\right]. \quad (46)$$

If there is a maxima (or minima) at (a, b) the LHS of the above equation is negative (or positive). Therefore the RHS must be negative (or positive) for all values of h and k. Note that, the first two terms of the RHS changes their sign with change in the signs of h and k (as h and k becomes positive and negative), and so, LHS will be negative (or positive) for all h and k if the first two terms becomes zero, i.e.,

$$\boxed{f_x(a, b) = f_y(a, b) = 0.}$$

Geometrically, since at maxima or minima, the tangent plane to the surface $z = f(x, y)$ becomes parallel to the XY-plane, its normal at point (a, b) must be in Z-direction. Since the direction ratios of normal are $f_x(a, b), f_y(a, b)$ and $f_z(a, b)$, at maxima or minima we must have $f_x(a, b) = f_y(a, b) = 0$.



Second Derivative Test. Therefore putting $f_x(a, b) = f_y(a, b) = 0$ in equation (46) we obtain:

$$f(a + h, b + k) - f(a, b) = \frac{1}{2!}\left[h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)\right]. \quad (47)$$

Let $r = f_{xx}(a,b), s = f_{xy}(a,b), t = f_{yy}(a,b)$, then:

$$\begin{aligned} h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b) &= h^2 r + 2hks + k^2 t \\ &= \frac{1}{r}(hr+ks)^2 + k^2\left(t - \frac{s^2}{r}\right). \end{aligned}$$

On putting this value in (47) we get:

$$f(a+h, b+k) - f(a,b) = \frac{1}{2!}\left[\frac{1}{r}(hr+ks)^2 + k^2\left(t - \frac{s^2}{r}\right)\right].$$

For maxima the LHS, and so the RHS should be negative and it is possible if $r < 0$ and $t - \dfrac{s^2}{r} < 0$, i.e., $rt - s^2 > 0$.

For minima the LHS, and so the RHS should be positive and it is possible if $r > 0$ and $t - \dfrac{s^2}{r} > 0$, i.e., $rt - s^2 > 0$.

For saddle point the LHS, and so the RHS should be positive as well as negative (should change the sign) and it is possible in the following two ways: (i) if $r > 0$ and $t - \dfrac{s^2}{r} < 0$, i.e., $rt - s^2 < 0$. (ii) if $r < 0$ and $t - \dfrac{s^2}{r} > 0$, i.e., $rt - s^2 < 0$. Thus, for saddle point we must have $rt - s^2 < 0$.

Finally, if $rt - s^2 = 0$, then the neglected terms in the series becomes effective and we need the further investigation.

Working Rules for finding the Maxima and Minima. Let

$$D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - \left[f_{xy}(x,y)\right]^2.$$

We follow the following steps:

(1) Find the first derivatives $f_x(x,y)$ and $f_y(x,y)$ and solve the equations:

$$\begin{aligned} f_x(x,y) &= 0 \\ f_y(x,y) &= 0. \end{aligned}$$

Solution(s) of the above system is (are) the *critical* point(s). Suppose, a critical point is (a,b);

(2) if $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then $f(x,y)$ has a local minimum at (a,b);

(3) if $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then $f(x,y)$ has a local maximum at (a,b);

(4) if $D(a,b) < 0$, then $f(x,y)$ has a saddle point at (a,b);

(5) if $D(a,b) = 0$, then we cannot draw any conclusions and further investigations are required.

Example 47. Discuss the maxima and minima of $f(x,y) = x^3 + y^3 - 3axy$.

Sol: Given function is $f(x,y) = x^3 + y^3 - 3axy$. Differentiating partially with respect to x and y we get:

$$f_x(x,y) = 3x^2 - 3ay, \quad f_y(x,y) = 3y^2 - 3ax.$$

First we find the critical point. Then:

$$f_x(x,y)=0,\ \ f_y(x,y)=0 \quad\Longrightarrow\quad 3x^2-3ay=0,\ \ 3y^2-3ax=0.$$

Since f is symmetric in x and y, a solution of the above system is $x=y$. Putting $x=y$ in the above equation we get:

$$\begin{aligned} 3x^2-3ax=0 &\quad\Longrightarrow\quad 3x(x-a)=0\\ &\quad\Longrightarrow\quad x=0,a. \end{aligned}$$

Since $x=y$, we get two critical points $(0,0)$ and (a,a). Now, by differentiating $f_x(x,y)$ and $f_y(x,y)$ again with respect to x and y we get:

$$f_{xx}(x,y)=6x,\ \ f_{xy}(x,y)=-3a,\ \ f_{yy}(x,y)=6y.$$

Now we find D at each critical point. Then:
(i).

$$\begin{aligned} D(0,0) &= f_{xx}(0,0)f_{yy}(0,0)-\big[f_{xy}(0,0)\big]^2\\ &= 0\cdot 0-\big[-3a\big]^2\\ &= -9a^2\\ &< 0. \end{aligned}$$

Since $D(0,0)<0$, the critical point $(0,0)$ is a saddle point.
(ii).

$$\begin{aligned} D(a,a) &= f_{xx}(a,a)f_{yy}(a,a)-\big[f_{xy}(a,a)\big]^2\\ &= 6a\cdot 6a-\big[-3a\big]^2\\ &= 36a^2-9a^2=27a^2\\ &> 0. \end{aligned}$$

Since $D(0,0)>0$, there is a maxima or minima at the critical point (a,a). We consider two cases:

If $a<0$, then $f_{xx}(a,a)=6a<0$ and so there is a maxima of function f and its maximum value is

$$f_{\max}=f(a,a)=a^3+a^3-3a\cdot a\cdot a=-a^3.$$

If $a>0$, then $f_{xx}(a,a)=6a>0$ and so there is a minima of function f and its minimum value is

$$f_{\min}=f(a,a)=a^3+a^3-3a\cdot a\cdot a=-a^3.$$

□

Example 48. Discuss the maxima and minima of $u=xy+\dfrac{a^3}{x}+\dfrac{a^3}{y}$.

Sol: Given function is $u=xy+\dfrac{a^3}{x}+\dfrac{a^3}{y}$. Differentiating partially with respect to x and y we get:

$$u_x(x,y)=y-\frac{a^3}{x^2},\ \ u_y(x,y)=x-\frac{a^3}{y^2}.$$

First we find the critical point. Then:

$$u_x(x,y)=0,\ \ u_y(x,y)=0 \quad\Longrightarrow\quad y-\frac{a^3}{x^2}=0,\ \ x-\frac{a^3}{y^2}=0.$$

Since f is symmetric in x and y, a solution of the above system is $x = y$. Putting $x = y$ in the above equation we get:

$$\begin{aligned} x - \frac{a^3}{x^2} = 0 \quad &\Longrightarrow \quad x^3 - a^3 = 0 \\ &\Longrightarrow \quad x = a. \end{aligned}$$

Since $x = y$, we get the critical point (a, a). Now, by differentiating $u_x(x, y)$ and $u_y(x, y)$ again with respect to x and y we get:

$$u_{xx}(x, y) = \frac{2a^3}{x^3}, \quad u_{xy}(x, y) = 1, \quad u_{yy}(x, y) = \frac{2a^3}{y^3}.$$

Now we find D at each critical point. Then:

$$\begin{aligned} D(a, a) &= u_{xx}(a, a)u_{yy}(a, a) - \left[u_{xy}(a, a)\right]^2 \\ &= 2 \cdot 2 - \left[1\right]^2 \\ &= 3 \\ &> 0. \end{aligned}$$

Since $D(a, a) > 0$, there is a maxima or minima at critical point (a, a). Then $u_{x,x}(a, a) = 2 > 0$, and so, there is a minima of function u and its minimum value is

$$u_{\min} = u(a, a) = a^2 + a^2 + a^2 = 3a^2.$$

□

Example 49. Discuss the maxima and minima of $f(x, y) = xy(a - x - y)$.

Sol: Given function is $f(x, y) = xy(a-x-y) = axy - x^2y - xy^2$. Differentiating partially with respect to x and y we get:

$$f_x(x, y) = ay - 2xy - y^2, \quad f_y(x, y) = ax - x^2 - 2xy.$$

First we find the critical point. Then:

$$f_x(x, y) = 0, \quad f_y(x, y) = 0 \quad \Longrightarrow \quad ay - 2xy - y^2 = 0, \quad ax - x^2 - 2xy = 0.$$

Since f is symmetric in x and y, a solution of the above system is $x = y$. Putting $x = y$ in the above equation we get:

$$\begin{aligned} ax - x^2 - 2x \cdot x = 0 \quad &\Longrightarrow \quad x(a - 3x) = 0 \\ &\Longrightarrow \quad x = 0, \frac{a}{3}. \end{aligned}$$

Since $x = y$, we get two critical points $\left(\frac{a}{3}, \frac{a}{3}\right)$ and (a, a). Now, by differentiating $f_x(x, y)$ and $f_y(x, y)$ again with respect to x and y we get:

$$f_{xx}(x, y) = -2y, \quad f_{xy}(x, y) = a - 2x - 2y, \quad f_{yy}(x, y) = -2x.$$

Now we find D at each critical point. Then:

(i).

$$\begin{aligned} D(0, 0) &= f_{xx}(0, 0)f_{yy}(0, 0) - \left[f_{xy}(0, 0)\right]^2 \\ &= 0 \cdot 0 - \left[a\right]^2 \\ &= -a^2 \\ &< 0. \end{aligned}$$

Since $D(0,0) < 0$, the critical point $(0,0)$ is a saddle point.

(ii).
$$\begin{aligned} D\left(\frac{a}{3},\frac{a}{3}\right) &= f_{xx}\left(\frac{a}{3},\frac{a}{3}\right) f_{yy}\left(\frac{a}{3},\frac{a}{3}\right) - \left[f_{xy}\left(\frac{a}{3},\frac{a}{3}\right)\right]^2 \\ &= -\frac{2a}{3}\cdot\left(-\frac{2a}{3}\right) - \left[-\frac{a}{3}\right]^2 \\ &= \frac{4a^2}{9} - \frac{a^2}{9} \\ &> 0. \end{aligned}$$

Since $D\left(\frac{a}{3},\frac{a}{3}\right) > 0$, there is a maxima or minima at the critical point $\left(\frac{a}{3},\frac{a}{3}\right)$. We consider two cases:

If $a > 0$, then $f_{xx}\left(\frac{a}{3},\frac{a}{3}\right) = -\frac{2a}{3} < 0$ and so there is a maxima of function f and its maximum value is
$$f_{\max} = f\left(\frac{a}{3},\frac{a}{3}\right) = \frac{a^2}{9}\left(a - \frac{a}{3} - \frac{a}{3}\right) = \frac{a^3}{27}.$$

If $a < 0$, then $f_{xx}\left(\frac{a}{3},\frac{a}{3}\right) = -\frac{2a}{3} > 0$ and so there is a minima of function f and its minimum value is
$$f_{\min} = f\left(\frac{a}{3},\frac{a}{3}\right) = \frac{a^2}{9}\left(a - \frac{a}{3} - \frac{a}{3}\right) = \frac{a^3}{27}.$$
□

Example 50. Discuss the maxima and minima of $f(x,y) = x^3y^2(1-x-y)$.

Sol: Given function is $f(x,y) = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$. Differentiating partially with respect to x and y we get:
$$f_x(x,y) = 3x^2y^2 - 4x^3y^2 - 3x^2y^3, \quad f_y(x,y) = 2x^3y - 2x^4y - 3x^3y^2.$$
First we find the critical point. Then: $f_x(x,y) = 0, \quad f_y(x,y) = 0$ implies
$$\begin{aligned} & 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0; \\ & 2x^3y - 2x^4y - 3x^3y^2 = 0 \\ \implies & x^2y^2(3 - 4x - 3y) = 0; \\ & x^3y(2 - 2x - 3y) = 0 \\ \implies & 4x + 3y = 3; \\ & 2x + 3y = 2. \end{aligned}$$
On solving the above equations we get $x = \frac{1}{2}, y = \frac{1}{3}$. Therefore, the critical point is $\left(\frac{1}{2},\frac{1}{3}\right)$. Now, by differentiating $f_x(x,y)$ and $f_y(x,y)$ again with respect to x and y we get:
$$f_{xx}(x,y) = 6xy^2 - 12x^2y^2 - 6xy^3, \quad f_{xy}(x,y) = 6x^2y - 8x^3y - 9x^2y^2,$$
$$f_{yy}(x,y) = 2x^3 - 2x^4 - 6x^3y.$$
Now at critical point $\left(\frac{1}{2},\frac{1}{3}\right)$ we have
$$\begin{aligned} D\left(\frac{1}{2},\frac{1}{3}\right) &= f_{xx}\left(\frac{1}{2},\frac{1}{3}\right) f_{yy}\left(\frac{1}{2},\frac{1}{3}\right) - \left[f_{xy}\left(\frac{1}{2},\frac{1}{3}\right)\right]^2 \\ &= \left(-\frac{1}{9}\right)\left(-\frac{1}{8}\right) - \left(-\frac{1}{12}\right)^2 = \frac{1}{72} - \frac{1}{144} \\ &> 0. \end{aligned}$$

Since $D\left(\frac{1}{2},\frac{1}{3}\right)>0$, there is a maxima or minima at the critical point $\left(\frac{1}{2},\frac{1}{3}\right)$. Since $f_{xx}\left(\frac{1}{2},\frac{1}{3}\right)=-\frac{1}{9}<0$, there is a maxima of function f and its maximum value is

$$f_{\max}=f\left(\frac{1}{2},\frac{1}{3}\right)=\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2\left(1-\frac{1}{2}-\frac{1}{3}\right)=\frac{1}{72}\cdot\frac{1}{6}=\frac{1}{432}. \qquad \square$$

Example 51. Discuss the maxima and minima of $f(x,y)=\sin x\sin y\sin(x+y)$.

Sol: Given function is $f(x,y)=\sin x\sin y\sin(x+y)$. Differentiating partially with respect to x and y we get:

$$\begin{aligned} f_x(x,y) &= \cos x\sin y\sin(x+y)+\sin x\sin y\cos(x+y)\\ &= \sin y\sin(2x+y). \end{aligned}$$

By symmetry of f in x and y we have

$$f_y(x,y)=\sin x\sin(x+2y).$$

First we find the critical point. Then:

$$f_x(x,y)=0,\ \ f_y(x,y)=0 \implies \sin y\sin(2x+y)=0,\ \ \sin x\sin(x+2y)=0.$$

Since f is symmetric in x and y, a solution of the above system is $x=y$. Putting $x=y$ in the above equation we get:

$$\begin{aligned} \sin x\sin 3x=0 &\implies \sin x=0 \text{ or } \sin 3x=0\\ &\implies x=0,\pi,\frac{\pi}{3},\frac{2\pi}{3}. \end{aligned}$$

Since $x=y$, we get four critical points $(0,0)$, (π,π), $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$ and $\left(\frac{2\pi}{3},\frac{2\pi}{3}\right)$. Now, by differentiating $f_x(x,y)$ and $f_y(x,y)$ again with respect to x and y we get:

$$f_{xx}(x,y)=2\sin y\cos(2x+y),\ \ f_{xy}(x,y)=\sin(2x+2y),\ \ f_{yy}(x,y)=2\sin x\cos(x+2y).$$

Now we find D at each critical point. Then:

(i).

$$\begin{aligned} D(0,0) &= f_{xx}(0,0)f_{yy}(0,0)-\left[f_{xy}(0,0)\right]^2\\ &= 0\cdot 0-[0]^2\\ &= 0. \end{aligned}$$

Since $D(0,0)=0$, we cannot draw any conclusions and further investigations are required.

(ii).

$$\begin{aligned} D(\pi,\pi) &= f_{xx}(\pi,\pi)f_{yy}(\pi,\pi)-\left[f_{xy}(\pi,\pi)\right]^2\\ &= 0\cdot 0-[0]^2\\ &= 0. \end{aligned}$$

Since $D(0,0)=0$, we cannot draw any conclusions and further investigations are required.

(iii).

$$\begin{aligned} D\left(\frac{\pi}{3},\frac{\pi}{3}\right) &= f_{xx}\left(\frac{\pi}{3},\frac{\pi}{3}\right)f_{yy}\left(\frac{\pi}{3},\frac{\pi}{3}\right)-\left[f_{xy}\left(\frac{\pi}{3},\frac{\pi}{3}\right)\right]^2\\ &= 2\cdot\frac{\sqrt{3}}{2}(-1)\cdot 2\cdot\frac{\sqrt{3}}{2}(-1)-\left[-\frac{\sqrt{3}}{2}\right]^2\\ &= \frac{9}{4}. \end{aligned}$$

Since $D\left(\frac{\pi}{3},\frac{\pi}{3}\right) > 0$, there is a maxima or minima at the critical point $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$. Now $f_{xx}\left(\frac{\pi}{3},\frac{\pi}{3}\right) = 2\cdot\frac{\sqrt{3}}{2}(-1) = -\sqrt{3} < 0$, and so, there is a maxima of function f and its maximum value is

$$f_{\max} = f\left(\frac{\pi}{3},\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)\cdot\sin\left(\frac{\pi}{3}\right)\cdot\sin\left(\frac{\pi}{3}+\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{8}.$$

(iv).

$$\begin{aligned} D\left(\frac{2\pi}{3},\frac{2\pi}{3}\right) &= f_{xx}\left(\frac{2\pi}{3},\frac{2\pi}{3}\right)f_{yy}\left(\frac{2\pi}{3},\frac{2\pi}{3}\right) - \left[f_{xy}\left(\frac{2\pi}{3},\frac{2\pi}{3}\right)\right]^2 \\ &= 2\cdot\frac{\sqrt{3}}{2}\cdot 1\cdot 2\cdot\frac{\sqrt{3}}{2}\cdot 1 - \left[\frac{\sqrt{3}}{2}\right]^2 \\ &= \frac{9}{4}. \end{aligned}$$

Since $D\left(\frac{2\pi}{3},\frac{2\pi}{3}\right) > 0$, there is a maxima or minima at the critical point $\left(\frac{2\pi}{3},\frac{2\pi}{3}\right)$. Now $f_{xx}\left(\frac{2\pi}{3},\frac{2\pi}{3}\right) = 2\cdot\frac{\sqrt{3}}{2}\cdot 1 = \sqrt{3} > 0$, and so, there is a maxima of function f and its minimum value is

$$f_{\min} = f\left(\frac{2\pi}{3},\frac{2\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right)\cdot\sin\left(\frac{2\pi}{3}\right)\cdot\sin\left(\frac{2\pi}{3}+\frac{2\pi}{3}\right) = -\frac{3\sqrt{3}}{8}.$$

□

Example 52. Discuss the maxima and minima of $f(x,y) = \sin x + \sin y + \sin(x+y)$.

Sol: Given function is $f(x,y) = \sin x + \sin y + \sin(x+y)$. Differentiating partially with respect to x and y we get:

$$f_x(x,y) = \cos x + \cos(x+y).$$

By symmetry of f in x and y we have

$$f_y(x,y) = \cos y + \cos(x+y).$$

First we find the critical point. Then:

$$f_x(x,y) = 0,\ \ f_y(x,y) = 0 \implies \cos x + \cos(x+y) = 0,\ \ \cos y + \cos(x+y) = 0.$$

Since f is symmetric in x and y, a solution of the above system is $x = y$. Putting $x = y$ in the above equation we get:

$$\begin{aligned} \cos x + \cos(2x) = 0 &\implies \cos x + 2\cos^2 x - 1 = 0 \\ &\implies 2\cos^2 x + \cos x - 1 = 0 \\ &\implies \cos x = \frac{-1\pm\sqrt{1+8}}{4} = -1, \frac{1}{2} \\ &\implies x = \pi, \frac{\pi}{3}. \end{aligned}$$

Since $x = y$, we two three critical points (π,π), $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)$. Now, by differentiating $f_x(x,y)$ and $f_y(x,y)$ again with respect to x and y we get:

$$f_{xx}(x,y) = -\sin x - \sin(x+y),\ \ f_{xy}(x,y) = -\sin(x+y),\ \ f_{yy}(x,y) = -\sin y - \sin(x+y).$$

Now we find D at each critical point. Then:

(i).

$$\begin{aligned} D(\pi,\pi) &= f_{xx}(\pi,\pi)f_{yy}(\pi,\pi) - \big[f_{xy}(\pi,\pi)\big]^2 \\ &= 0\cdot 0 - \big[0\big]^2 \\ &= 0. \end{aligned}$$

Since $D(0,0)=0$, we cannot draw any conclusions and further investigations are required.

(ii).

$$\begin{aligned} D\left(\frac{\pi}{3},\frac{\pi}{3}\right) &= f_{xx}\left(\frac{\pi}{3},\frac{\pi}{3}\right) f_{yy}\left(\frac{\pi}{3},\frac{\pi}{3}\right) - \left[f_{xy}\left(\frac{\pi}{3},\frac{\pi}{3}\right)\right]^2 \\ &= (-\sqrt{3})\cdot(-\sqrt{3}) - \left[-\frac{\sqrt{3}}{2}\right]^2 \\ &= \frac{9}{4} > 0. \end{aligned}$$

Since $D\left(\frac{\pi}{3},\frac{\pi}{3}\right) > 0$, there is a maxima or minima at the critical point $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$. Now $f_{xx}\left(\frac{\pi}{3},\frac{\pi}{3}\right) = -\sqrt{3} < 0$, and so, there is a maxima of function f and its maximum value is

$$f_{\text{max}} = f\left(\frac{\pi}{3},\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3}+\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}.$$

(iii).

$$\begin{aligned} D\left(\frac{5\pi}{3},\frac{5\pi}{3}\right) &= f_{xx}\left(\frac{5\pi}{3},\frac{5\pi}{3}\right) f_{yy}\left(\frac{5\pi}{3},\frac{5\pi}{3}\right) - \left[f_{xy}\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)\right]^2 \\ &= (\sqrt{3})\cdot(\sqrt{3}) - \left[\frac{\sqrt{3}}{2}\right]^2 \\ &= \frac{9}{4} > 0. \end{aligned}$$

Since $D\left(\frac{5\pi}{3},\frac{5\pi}{3}\right) > 0$, there is a maxima or minima at the critical point $\left(\frac{5\pi}{3},\frac{5\pi}{3}\right)$. Now $f_{xx}\left(\frac{5\pi}{3},\frac{5\pi}{3}\right) = \sqrt{3} > 0$, and so, there is a minima of function f and its minimum value is

$$f_{\text{min}} = f\left(\frac{5\pi}{3},\frac{5\pi}{3}\right) = \sin\left(\frac{5\pi}{3}\right) + \sin\left(\frac{5\pi}{3}\right) + \sin\left(\frac{5\pi}{3}+\frac{5\pi}{3}\right) = -\frac{3\sqrt{3}}{2}.$$

□

Example 53. Discuss the maxima or minima of $\sin x \sin y \sin z$, where x, y and z are the angles of triangle.

Sol: Since x, y and z are the angles of triangle, we have $x+y+z=\pi$ or $z=\pi-(x+y)$. Now the given function is $f(x,y)=\sin x\sin y\sin z$. On putting the value of z we have

$$f(x,y) = \sin x\sin y\sin\left[\pi-(x+y)\right] = \sin x\sin y\sin(x+y).$$

Now follow the process of Example 51. □

Example 54. Find the point on the surface $z^2 = xy+1$ nearest to the origin.

Sol: Suppose the required point on the surface $z^2 = xy + 1$ is (x, y, z). Then we have to find this point such that its distance from origin, i.e.

$$d = \sqrt{x^2 + y^2 + z^2}$$

is minimum. Since d and d^2 get their minimum values together, for simplicity, we calculate the point of minima of

$$d^2 = x^2 + y^2 + z^2.$$

Since the point (x, y, z) is situated on the surface therefore $z^2 = xy + 1$. On putting this value in the above equation we get: $d^2 = f(x, y) = x^2 + y^2 + xy + 1$. Differentiating partially with respect to x and y we get:

$$f_x(x, y) = 2x + y.$$

By symmetry of f in x and y we have

$$f_y(x, y) = 2y + x.$$

First we find the critical point. Then:

$$f_x(x, y) = 0, \quad f_y(x, y) = 0 \implies 2x + y = 0, \quad 2y + x = 0.$$

Since f is symmetric in x and y, a solution of the above system is $x = y$. Putting $x = y$ in the above equation we get:

$$3x = 0 \implies x = 0.$$

Since $x = y$, the critical point is $(0, 0)$. Now, by differentiating $f_x(x, y)$ and $f_y(x, y)$ again with respect to x and y we get:

$$f_{xx}(x, y) = 2, \quad f_{xy}(x, y) = 1, \quad f_{yy}(x, y) = 2.$$

Now we find D at critical point $(0, 0)$. Then:

$$\begin{aligned} D(0,0) &= f_{xx}(0,0) f_{yy}(0,0) - \left[f_{xy}(0,0)\right]^2 \\ &= 2 \cdot 2 - [1]^2 \\ &= 3. \end{aligned}$$

Since $D(0, 0) = 3 > 0$, there is a maxima or minima at the critical point $(0, 0)$. Now $f_{xx}(0, 0) = 2 > 0$, and so, there is a minima of function f at point $x = y = 0$, and from the equation of surface $z^2 = xy + 1$, at this point we have $x = y = 0$ and so $z^2 = 0 \cdot 01$, i.e., $z = \pm 1$. Thus, the distance of point $(0, 0, \pm 1)$ of the surface will be minimum from the origin. □

Example 55. If the perimeter of a triangle is constant, prove that the area of this triangle is maximum when the triangle is equilateral.

Sol: We know that the area of triangle is given by $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where a, b, c are the sides of triangle and $2s = a+b+c$. We have to maximize the area Δ. Since Δ and Δ^2 get their maximum values together, for simplicity, we calculate the point of minima of

$$\Delta^2 = s(s-a)(s-b)(s-c).$$

Since the perimeter is constant we have $c = 2s - a - b$. On putting this value in the above equation we get: $\Delta^2 = f(a,b) = s(s-a)(s-b)(a+b-s)$. Differentiating partially with respect to a and b we get:

$$\begin{aligned} f_a(a,b) &= s(s-b)[-(a+b-s)+(s-a)] \\ &= s(s-b)(2s-2a-b). \end{aligned} \tag{48}$$

By symmetry of f in x and y we have

$$f_b(a,b) = s(s-a)(2s-2b-a).$$

First we find the critical point. Then:

$$f_a(a,b)=0,\ \ f_b(a,b)=0 \implies s(s-b)(2s-2a-b)=0,\ \ s(s-a)(2s-2b-a)=0.$$

Since f is symmetric in a and b, a solution of the above system is $a = b$. Putting $a = b$ in the above equation we get:

$$s(s-a)(2s-2a-a)=0 \implies s(s-a)(2s-3a)=0 \implies s=0, s=a, a=\frac{2s}{3}.$$

Since $s = 0$, $s = a$ are not possible, we have $a = b = \frac{2s}{3}$, and so, the critical point is $\left(\frac{2s}{3},\frac{2s}{3}\right)$. Now, by differentiating $f_a(a,b)$ and $f_b(a,b)$ again with respect to a and b we get:

$$f_{aa}(a,b) = -2s(s-b),\ \ f_{ab}(a,b) = s(2a+2b-3s),\ \ f_{bb}(a,b) = -2s(s-a).$$

Now we find D at each critical point. Then:

$$\begin{aligned} D\left(\frac{2s}{3},\frac{2s}{3}\right) &= f_{aa}\left(\frac{2s}{3},\frac{2s}{3}\right) f_{bb}\left(\frac{2s}{3},\frac{2s}{3}\right) - \left[f_{ab}\left(\frac{2s}{3},\frac{2s}{3}\right)\right]^2 \\ &= \left(-\frac{2s^2}{3}\right)\left(-\frac{2s^2}{3}\right) - \left[-\frac{s^2}{3}\right]^2 \\ &= \frac{s^4}{3}. \end{aligned}$$

Since $D\left(\frac{2s}{3},\frac{2s}{3}\right) = \frac{s^4}{3} > 0$, there is a maxima or minima at the critical point $\left(\frac{2s}{3},\frac{2s}{3}\right)$. Now $f_{aa}\left(\frac{2s}{3},\frac{2s}{3}\right) = -\frac{2s^2}{3} < 0$, and so, there is a maxima of function f at point $x = y = \frac{2s}{3}$, i.e., the area is maximum. Also, since $2s = a+b+c$, at point $\left(\frac{2s}{3},\frac{2s}{3}\right)$ we have $c = 2s - a - b = \frac{2s}{3}$. Therefore, for maximum area we have $a = b = c = \frac{2s}{3}$, i.e., the triangle is equilateral. □

Exercise (Assignment)

(Q.1) Discuss the maxima or minima of the function $f(x,y) = x^3 - 3xy^2 - 15x^2 - 15y^2 + 72x$.

Ans. Critical point $(6,0)$ (minima), with $f_{\min} = f(6,0) = 108$, $(4,0)$ (maxima), with $f_{\max} = f(4,0) = 112$, $(5,1)$ and $(5,-1)$ are saddle points.

(Q.2) Discuss the maxima or minima of the function $f(x, y) = x^3 - 4xy + 2y^2$.

Ans. Critical point $(0, 0)$ (saddle point) and $(4/3, 4/3)$ (minima), with $f_{\text{min}} = f(4/3, 4/3) = -\frac{32}{27}$.

(Q.3) Discuss the maxima or minima of the function $f(x, y) = \cos x \cos y \cos z$, where x, y and z are the angles of a triangle.

Hint. Since $x+y+z = \pi$ the given function is reduced to $f(x, y) = -\cos x \cos y \cos(x+y)$.

(Q.4) Discuss the maxima or minima of the function $f(x, y) = \cos x + \cos y + \cos z$, where x, y and z are the angles of a triangle.

Hint. Since $x + y + z = \pi$ the given function is reduced to $f(x, y) = \cos x + \cos y - \cos(x + y)$.

(Q.5) Discuss the maxima and minima of $f(x, y) = x^3 + y^3 - 3xy$.

Ans. Critical point $(0, 0)$ (saddle point) and $(1, 1)$ (minima), $f_{\text{min}} = f(a, a) = -1$.

(Q.6) Find the shortest distance from the origin to the surface $xyz^2 = 2$.

Ans. $(1, 1, \sqrt{2})$.

(Q.7) A rectangular box open at the top is with a given capacity. Find the dimensions of the box requiring least material for its construction.

Hint. Suppose the dimension of box are x (length), y (width), z (height), then given that the capacity (volume) $xyz = c$(constant). Since the top of box is open the material required for its construction is equal to the total area of the surfaces of the open box $S = xy + 2yz + 2zx = xy + \frac{2c}{x} + \frac{2c}{y}$.

Lagrange's Method of Undetermined multipliers. This method is useful when we have to find the maxima or minima of function under some given conditions. Suppose, we have to find:

$$\begin{aligned} \text{Extrema of} \quad & u = f(x, y, z) \\ \text{Subject to} \quad & g(x, y, z) = 0. \end{aligned}$$

Working Rule. Suppose:

$$F(x, y, z) = f(x, y, z) + \lambda g(x, y, z). \tag{49}$$

Now differentiate (49) with respect to x, y, z and solve the equations $F_x = 0, F_y = 0, F_z = 0$ with the constraint $g(x, y, z) = 0$ for the multiplier λ and the critical point x, y, z. The undetermined multiplier λ is called the Lagrange's multiplier.

Example 56. Find the maxima or minima of $u = x^2 + y^2 + z^2$ under the condition $ax^2 + by^2 + cz^2 = 1$.

Sol: Given function is

$$u = f(x, y, z) = x^2 + y^2 + z^2.$$

We have to find the maximum value of $u = f(x, y, z)$ under the condition $ax^2+by^2+cz^2 = 1$. Therefore the given constraint is:

$$g(x, y, z) = ax^2 + by^2 + cz^2 - 1 = 0.$$

Let

$$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z) = x^2 + y^2 + z^2 + \lambda\left[ax^2 + by^2 + cz^2 - 1\right]. \tag{50}$$

Differentiating the above equation with respect to x, y, z we get:

$$\begin{aligned} F_x &= 2x + 2a\lambda x = 0 & (51)\\ F_y &= 2y + 2b\lambda y = 0 & (52)\\ F_z &= 2z + 2a\lambda z = 0. & (53) \end{aligned}$$

Multiplying the above equations by x, y and z respectively and adding we get:

$$\begin{aligned} & 2(x^2 + y^2 + z^2) + 2\lambda\left[ax^2 + by^2 + cz^2\right] = 0\\ \implies & 2u + 2\lambda \cdot 1 = 0\\ \implies & \lambda = -u. \end{aligned}$$

On putting this value of λ in (51) we get

$$\begin{aligned} & 2x - 2aux = 0\\ \implies & u = \frac{1}{a}. \end{aligned}$$

Similarly by putting the value of λ in (52) and (53), we obtain

$$u = \frac{1}{b},\ u = \frac{1}{c}.$$

Therefore, there are three extreme values, $u = \frac{1}{a},\ u = \frac{1}{b},\ u = \frac{1}{c}$. □

Example 57. Find the maximum value of $x^m y^n z^p$ when $x + y + z = a$.

Sol: Given function is

$$u = f(x,y,z) = x^m y^n z^p.$$

We have to find the maximum value of $u = f(x,y,z)$ under the condition $x + y + z = a$. Therefore the given constraint is:

$$g(x,y,z) = x + y + z - a = 0.$$

Let

$$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z) = x^m y^n z^p + \lambda\left[x + y + z - a\right]. \tag{54}$$

Differentiating the above equation with respect to x, y, z we get:

$$\begin{aligned} F_x &= mx^{m-1}y^n z^p + \lambda = 0 & (55)\\ F_y &= nx^m y^{n-1} z^p + \lambda y = 0 & (56)\\ F_z &= px^m y^n z^{p-1} + \lambda z = 0. & (57) \end{aligned}$$

Multiplying the above equations by x, y and z respectively and adding we get:

$$\begin{aligned} & mx^m y^n z^p + nx^m y^n z^p + px^m y^n z^p + \lambda x + \lambda y + \lambda z = 0\\ & (m+n+p)x^m y^n z^p + \lambda(x + y + z) = 0\\ \implies & (m+n+p)u + a\lambda = 0\\ \implies & \lambda = -\frac{(m+n+p)u}{a}. \end{aligned}$$

On putting this value of λ in (55) we get

$$\begin{aligned} & mx^{m-1}y^nz^p - \frac{(m+n+p)u}{a} = 0 \\ \implies & \frac{mu}{x} - \frac{(m+n+p)u}{a} = 0 \\ \implies & x = \frac{ma}{m+n+p}. \end{aligned}$$

Similarly by putting the value of λ in (56) and (57), we obtain

$$y = \frac{na}{m+n+p}, \quad z = \frac{pa}{m+n+p}.$$

Therefore, the maximum value of u will be:

$$\begin{aligned} u_{\text{max}} &= f\left(\frac{ma}{m+n+p}, \frac{na}{m+n+p}, \frac{pa}{m+n+p}\right) \\ &= \left(\frac{ma}{m+n+p}\right)^m \left(\frac{na}{m+n+p}\right)^n \left(\frac{pa}{m+n+p}\right)^p \\ &= \frac{a^{m+n+p}m^m n^n p^p}{(m+n+p)^{m+n+p}}. \end{aligned}$$

□

Example 58. Show that the volume of the greatest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{8abc}{3\sqrt{3}}$.

Sol: Take the eight points $P(\pm x, \pm y, \pm z)$ as the corners of the parallelepiped on the ellipsoid, then the length of its edges will be $2x, 2y, 2z$ and its volume:

$$V = f(x, y, z) = 8xyz.$$

We have to find the maximum value of $V = f(x, y, z)$. Since the corner points are on the ellipsoid we obtain from the equation of ellipsoid that $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, therefore the given constraint is:

$$g(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0.$$

Let $$F(x, y, z) = f(x, y, z) + \lambda g(x, y, z) = 8xyz + \lambda\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right]. \tag{58}$$

Differentiating the above equation with respect to x, y, z we get:

$$F_x = 8yz + \frac{2\lambda x}{a^2} = 0 \tag{59}$$
$$F_y = 8xz + \frac{2\lambda y}{b^2} = 0 \tag{60}$$
$$F_z = 8xy + \frac{2\lambda z}{c^2} = 0. \tag{61}$$

Multiplying the above equations by x, y and z respectively and adding we get:

$$\begin{aligned} & 24xyz + 2\lambda\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right] = 0 \\ \implies & 24xyz + 2\lambda = 0 \\ \implies & \lambda = -12xyz. \end{aligned}$$

On putting this value of λ in (59) we get

$$\begin{aligned} 8yz - \frac{24x^2yz}{a^2} &= 0 \\ \implies \quad x &= \frac{a}{\sqrt{3}}. \end{aligned}$$

Similarly by putting the value of λ in (60) and (61), we obtain

$$y = \frac{b}{\sqrt{3}}, \; z = \frac{c}{\sqrt{3}}.$$

Therefore, on putting the values of x, y, z in V the maximum volume will be:

$$V_{\max} = 8 \cdot \frac{a}{\sqrt{3}} \cdot \frac{b}{\sqrt{3}} \cdot \frac{c}{\sqrt{3}} = \frac{8abc}{3\sqrt{3}}.$$

□

Example 59. Show that the rectangular parallelepiped of maximum volume that can be inscribed in sphere is a cube.

Sol: Suppose the equation of spare is $x^2+y^2+z^2 = a^2$. Take the eight points $P(\pm x, \pm y, \pm z)$ as the corners of the parallelepiped on the sphere, then the length of its edges will be $2x, 2y, 2z$ and its volume:

$$V = f(x, y, z) = 8xyz.$$

We have to show that for maximum value of V we have $x = y = z$. Since the corner points are on the sphere we obtain from the equation of sphere that $x^2 + y^2 + z^2 = a^2$, therefore the given constraint is:

$$g(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0.$$

Let
$$F(x, y, z) = f(x, y, z) + \lambda g(x, y, z) = 8xyz + \lambda \left[x^2 + y^2 + z^2 - a^2\right]. \tag{62}$$

Differentiating the above equation with respect to x, y, z we get:

$$F_x = 8yz + 2\lambda x = 0 \tag{63}$$
$$F_y = 8xz + 2\lambda y = 0 \tag{64}$$
$$F_z = 8xy + 2\lambda z = 0. \tag{65}$$

Multiplying the above equations by x, y and z respectively and adding we get:

$$\begin{aligned} & 24xyz + 2\lambda \left[x^2 + y^2 + z^2\right] = 0 \\ \implies \quad & 24xyz + 2\lambda \cdot a^2 = 0 \\ \implies \quad & \lambda = -\frac{12xyz}{a^2}. \end{aligned}$$

On putting this value of λ in (63) we get

$$\begin{aligned} 8yz - \frac{24x^2yz}{a^2} &= 0 \\ \implies \quad x &= \frac{a}{\sqrt{3}}. \end{aligned}$$

Similarly by putting the value of λ in (64) and (65), we obtain

$$y = \frac{a}{\sqrt{3}}, \; z = \frac{a}{\sqrt{3}}.$$

Therefore, for maximum volume we have $x = y = z = \dfrac{a}{\sqrt{3}}$, and so, the rectangular parallelepiped is a cube. On putting the values of x, y, z in V the maximum volume will be:

$$V_{\max} = 8 \cdot \frac{a}{\sqrt{3}} \cdot \frac{a}{\sqrt{3}} \cdot \frac{a}{\sqrt{3}} = \frac{8a^3}{3\sqrt{3}}.$$

□

Exercise (Assignment) on Continuity and Differentiability

(Q.1) Discuss continuity and differentiability of the function $f(x) = |x|$

(Q.2) Discuss continuity and differentiability of the function $f(x)=\begin{cases} 2+\sqrt{1-x^2}, & |x| \leq 1; \\ 2e^{(1-x)^2}, & |x| > 1. \end{cases}$

(Q.3) Discuss continuity and differentiability of the function $f(x) = \tan^2 x$ at $x = 0$.

(Q.4) Discuss continuity and differentiability of the function $f(x) = \begin{cases} x^3, & x^2 < 1; \\ x, & x^2 \geq 1. \end{cases}$

(Q.5) Discuss continuity and differentiability of the function $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x-1}} - 2}{e^{\frac{1}{x-1}} + 2}, & x \neq 1; \\ 1, & x = 1. \end{cases}$