Digital Assignment - 1 (MAT3005)

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Topic: Classification of Second Order Linear partial Differential Equations

Basic overview:

1. Elliptic: The eigenvalues are all positive or all negative.
2. Parabolic: The eigenvalues are all positive or all negative, save one which is zero. i.e $b^2 - 4ac = 0$
3. Hyperbolic: There is only one negative eigenvalue and all rest are positive or vice-versa.

Problem 1: $\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$

It can also be written as: $u_{xx} + y\,u_{yy} + u_x + u_y = 0$

Comparing this with

$$Au_{xx} + Bu_{xy} + Cu_{yy} + H(u_x, u_y, u, G) = 0$$

$A = 1$, $B = 0$ and $C = y$

The discriminant is:

$$\Delta = B^2 - 4AC = 0 - 4y = -4y$$

So, the given equation is Parabolic when $y = 0$

Hyperbolic when $y < 0$

elliptic when $y > 0$

Problem (2)

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

$$\longrightarrow \quad x u_{xx} + u_{xy} + u_{yy} + u_x + u_y = 0$$

Comparing with:

$$A u_{xx} + B u_{xy} + C u_{yy} + H(u_x, u_y, u, G) = 0$$

$$A = x \ , \ B = 1 \ , C = 1$$

Discriminant

$$\Delta = B^2 - 4AC = 1 - 4x$$

The equation given is Hyperbolic $\rightarrow x < 1/4$

Parabolic $x = 1/4$

Elliptic $x > 1/4$

Problem (3) Determine if the given equation is parabolic, elliptic or Hyperbolic.

$$(1 - M_\infty^2)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Also write its canonical form.

Comparing the given equation with the general we get.

$$A = 1 - M_\infty^2 \quad B = 0 \quad C = 1$$

Discriminant $\Delta = B^2 - 4AC = -4(1 - M_\infty^2)$

So the given eqn is Hyperbolic for $M > 1$

Elliptic for $M < 1$

Parabolic for $M = 1$

The roots of characteristic polynomial is given by

$$\lambda_1 = \frac{B + \sqrt{\Delta}}{2A} = \frac{\sqrt{4(M_\infty^2 - 1)}}{2(1 - M_\infty^2)} = -\frac{1}{\sqrt{M_\infty^2 - 1}}$$

$$\lambda_2 = \frac{B - \sqrt{\Delta}}{2A} = \frac{\sqrt{4(M_\infty^2 - 1)}}{2(1 - M_\infty^2)} = \frac{1}{\sqrt{M_\infty^2 - 1}}$$

Hence

$$\frac{dy}{dx} = \frac{1}{\sqrt{M_\infty^2 - 1}}, \quad -\frac{1}{\sqrt{M_\infty^2 - 1}}$$

On integrating

$$y = \frac{x}{\sqrt{M_\infty^2 - 1}} + C_1, \quad y = -\frac{x}{\sqrt{M_\infty^2 - 1}} + C_2$$

So $\xi = y - \dfrac{x}{\sqrt{M_\infty^2 - 1}}$ $\quad \eta = y + \dfrac{x}{\sqrt{M_\infty^2 - 1}}$

Reducing it to canonical form,

$$\phi_{xx} = \frac{1}{M_\infty^2 - 1} w_{\xi\xi} - \frac{2}{M_\infty^2 - 1} w_{\xi\eta} + \frac{1}{M^2 - 1} w_{\eta\eta}$$

$$\phi_{yy} = w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}$$

We get $w_{\xi\eta} = 0$

This is the canonical form of given PDE

Here, $\xi =$ const and $\eta =$ const

$$\text{Slope} = \frac{\pm 1}{\sqrt{M_\infty^2 - 1}}$$

Problem (4) A simple two-dimension heat equation

$$u_t - \alpha(u_{xx} + u_{yy}) = 0$$

Creating coefficient matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & -\alpha \end{bmatrix}$$

As the matrix is diagonalized we know it has zero eignvalues. Hence we can say it is parabolic..

Problem (5) $$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

We can write it as:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}\frac{\partial^2 u}{\partial x \partial y} - \frac{1}{2}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

Making coefficient matrix of it

$$A = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}$$

Solving the matrix.

$$\begin{bmatrix} 1-\lambda & -1/2 \\ -1/2 & 1-\lambda \end{bmatrix} = 0$$

Expanding $(1-\lambda)^2 - \frac{1}{4} = 0 \Rightarrow \lambda^2 - 2\lambda + \frac{3}{4} = 0$

We get two eigenvalues $\lambda_1 = 1/2$ $\lambda_2 = 3/2$

Both are positive, so it is elliptical.