①

$$\begin{cases} u_{tt} - 9u_{xx} = 0 & 0 < x < 3 \\ u(x,0) = 0 \;;\; u_t(x,0) = \cos\dfrac{\pi x}{6} \\ u(0,t) = u(3,t) = 0 \end{cases}$$

$u = XT \quad \rightarrow \quad XT'' - 9X''T = 0$

$\dfrac{X''}{X} = \dfrac{T''}{9T} = \lambda = \text{cost}$

(cn) (x <0) $\quad X'' - \lambda X = 0 \quad \begin{cases} X(0) = 0 \\ X(3) = 0 \end{cases}$

$X(x) = C_1 \cos\sqrt{-\lambda}\, x + C_2 \operatorname{sen}\sqrt{-\lambda}\, x$

$X(0) = 0 \rightarrow C_1 = 0$

$X(3) = 0 \rightarrow \sqrt{-\lambda}\cdot 3 = k\pi$

$\lambda = -\left(\dfrac{k\pi}{3}\right)^2$

$X_k(x) = \operatorname{sen}\dfrac{k\pi}{3} x$

$T_k(t) = A_k \cos\dfrac{3k\pi}{3} t + B_k \operatorname{sen}\dfrac{3k\pi}{3} t$

$A_k = \dfrac{2}{3}\int_0^3 0 \operatorname{sen}\dfrac{k\pi}{3} x\, dx = 0$

$B_k = \dfrac{2}{3}\int_0^3 \cos\dfrac{\pi x}{6} \cdot \operatorname{sen}\dfrac{k\pi}{3}\, dx$

$B_k = -\dfrac{4(-2k + \operatorname{sen} k\pi)}{\pi(4k^2-1)} = -\dfrac{8k}{\pi(4k^2-1)}$

② Δε σερεση συστημα

$$\begin{cases} \dot{x} = -4x + 3y \\ \dot{y} = -2x + 3y \end{cases} \rightarrow y = \frac{1}{3}(\dot{x} + 4x)$$

$$\dot{y} = \frac{1}{3}(\ddot{x} + 4\dot{x})$$

$$\frac{1}{3}(\ddot{x} + 4\dot{x}) = -2x + 3 \cdot \frac{1}{3}(\dot{x} + 4x)$$

$$\ddot{x} + 4\dot{x} = -6x + 3\dot{x} + 12x$$

$$\ddot{x} + \dot{x} - 6x = 0 \rightarrow \lambda^2 + \lambda - 6 = 0$$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1+24}}{2} = \frac{-1 \pm 5}{2} \rightarrow \begin{array}{l} \lambda_1 = -3 \rightarrow e^{-3t} \\ \lambda_2 = 2 \rightarrow e^{2t} \end{array}$$

$$\therefore \boxed{x = c_1 e^{-3t} + c_2 e^{2t}}$$

$$y = \frac{1}{3}(\dot{x} + 4x) = \frac{1}{3}\left(\overbrace{(c_1 e^{-3t} + c_2 e^{2t})}^{\cdot} + 4(c_1 e^{-3t} + c_2 e^{2t})\right)$$

$$= \frac{1}{3}\left(\underline{-3c_1 e^{-3t}} + 2c_2 e^{2t} + \underline{4c_1 e^{-3t}} + 4c_2 e^{2t}\right)$$

$$\Rightarrow y = \frac{1}{3}(c_1 e^{-3t} + 6c_2 e^{2t})$$

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③ $u_{tt} - 5u_{xx} = 0 \qquad x \in \mathbb{R},\ t > 0$

$u(x,0) = \cos x \ , \ u_t(x,0) = \frac{1}{2}\sin 3\pi x$

Δελεμπερ

$$u(x,t) = \frac{1}{2}\left(\cos(x - \sqrt{5}t) + \cos(x + \sqrt{5}t)\right) + \frac{1}{2\sqrt{5}} \int_{x-\sqrt{5}t}^{x+\sqrt{5}t} \frac{1}{2}\sin 3\pi x \, dx$$

$$= \frac{1}{2}\cos(x - \sqrt{5}t) + \frac{1}{2}\cos(x + \sqrt{5}t) + \frac{\sqrt{5}}{60\pi}\left(\cos 3\pi(\sqrt{5}t - x) - \cos 3\pi(\sqrt{5}x + t)\right)$$

Задача $\dot{x} = -y + y^2$
$\dot{y} = x^2 - y^2$ - да се изследват на т. на равновесие

$f(x,y) = -y + y^2 \rightarrow f'_x = 0$, $f'_y = -1 + 2y$

$g(x,y) = x^2 - y^2 \rightarrow g'_x = 2x$ $g'_y = -2y$

Точки на равновесие $\left| \begin{array}{l} -y + y^2 = 0 \\ x^2 - y^2 = 0 \end{array} \right.$ $\rightarrow y = 0, y = 1$

$y = 0 \rightarrow x^2 = 0 \rightarrow (x,y) = (0,0)$

$y = 1 \quad x^2 = 1 \quad (x,y) = (0,1) ; (-1,0)$

$A(0,0) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \Big|_{(0,0)} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$

$\lambda^2 = 0$ $\lambda_{1,2} = 0$ $\rightarrow$ Няма отговор не върши за работа

може да е устойчиво или не!

$A(1,0) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ $\lambda^2 - 2 = 0$ неустойчиво

$A(-1,0) = \begin{pmatrix} 0 & -3 \\ -2 & 0 \end{pmatrix}$ $\rightarrow$ седло!

Задача Да се определи типа на особената точка $(0,0)$ за системата $\left| \begin{array}{l} \dot{x} = x - 4y \\ \dot{y} = x - 9y \end{array} \right.$

в зависимост от параметъра $a \in \mathbb{R}$.

Реш.

Хар.характеристичен полином $\lambda^2 - (1-q)\lambda + 4 - q = 0$

1 сп $\Delta = (1-q)^2 - 4(4-q) = q^2 - 2q + 1 - 16 + 4q$

$= q^2 + 2q - 15$ $\quad q_{1,2} = -1 \pm \sqrt{16} = -1 \pm 4 \rightarrow \begin{matrix} q_1 = 3 \\ \rightarrow q_2 = -5 \end{matrix}$

1 сп (реален) $\Delta \geq 0$ $\quad q \in (-\infty; -5] \cup [3; +\infty)$

2 сп (комплексни) $\Delta < 0$ $\quad q \in (-5, 3)$

(3 сп) $\lambda_1 < 0 < \lambda_2 \rightarrow \lambda_1 \lambda_2 < 0$

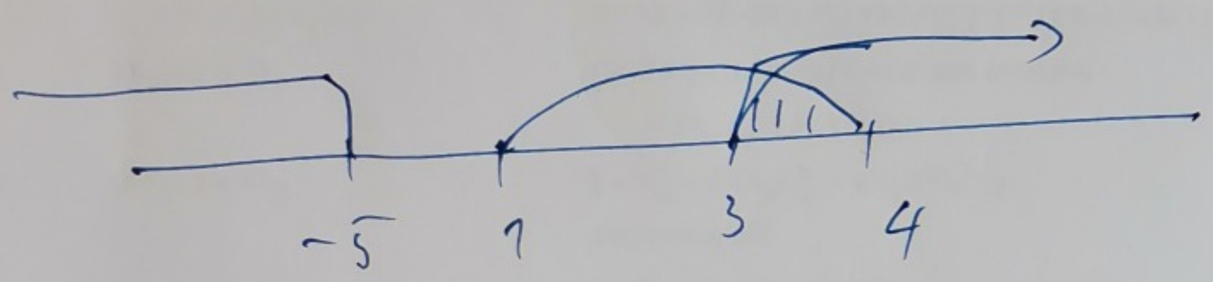
A. $4 - q < 0 \rightarrow q \in (4; +\infty)$

Вбсен

B. $\lambda_1 \leq \lambda_2 < 0$ – устойчив възел

$\lambda_1 + \lambda_2 < 0$ | $1 - q < 0$ $\quad q > 1$

$\lambda_1 \lambda_2 > 0$ | $4 - q > 0$ $\quad q < 4$



$q \in [3, 4)$

устойчив възел

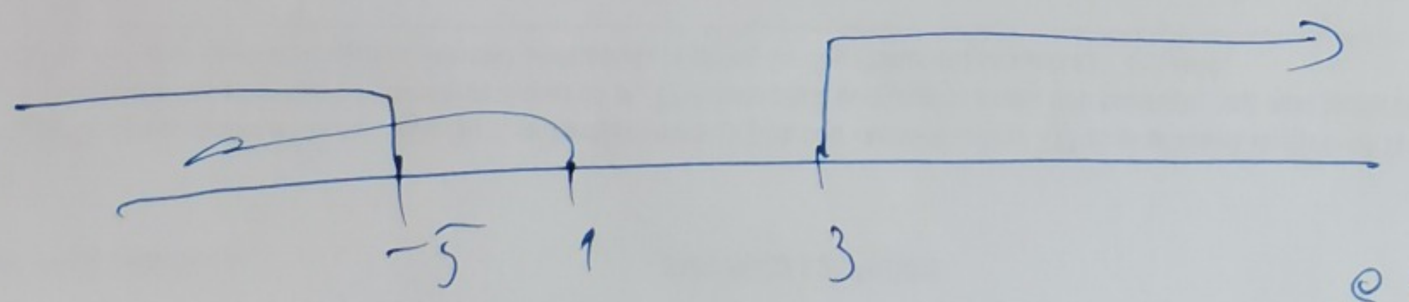
$q = 3$ (изроден)

C. $\lambda_1 \geq \lambda_2 > 0$

$\lambda_1 + \lambda_2 > 0 \rightsquigarrow 1 - q > 0$

$q < 1$

$\lambda_1 \lambda_2 > 0 \rightsquigarrow q < 4$



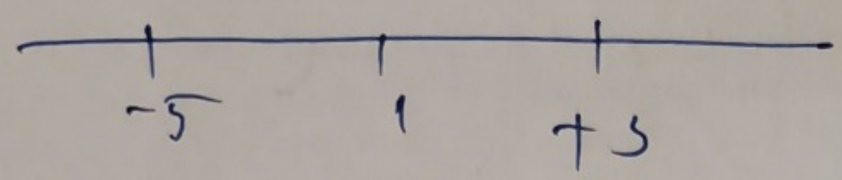
$q \in (-\infty; -5)$

Неустойчив възел $\quad q = -5$ (изроден)

2сл - комплексни корени

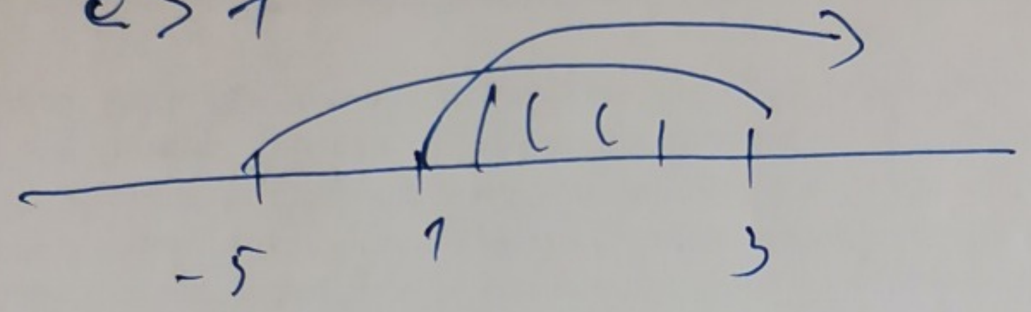
$\mathrm{Re}\,\lambda_i = 1-q$

Ако $1-q = 0 \quad q = 1$ - двоен



Ако $1-q < 0$ - двойна точка

$q > 1$ $\quad q \in (1;3)$



Ако $q < 1$ - неустойчива точка

Ако $q \in (-5, 1)$