Уравнения с разделени променливи

① $\left.\begin{aligned} y' &= x \cdot y \\ y(1) &= 1 \end{aligned}\right\}$ З. Коши

Реш $y=0$ не е решение (не удовлетворява началното у.е)

Делим на y

$$\frac{y'}{y} = x \quad \to \quad \int \frac{y'\,dx}{y} = \int x\,dx + K$$

$$\int \frac{dy}{y} = \frac{x^2}{2} + K \quad \to \quad \ln y = \frac{x^2}{2} + K$$

$$y = e^{\frac{x^2}{2}+K} = e^K \cdot e^{\frac{x^2}{2}} \Rightarrow C e^{\frac{x^2}{2}}$$

$$y(1) = C \cdot e^{\frac{1^2}{2}} = C e^{1/2} = 1 \quad \Rightarrow C = e^{-1/2}$$

$$\boxed{y(x) = e^{-1/2 + \frac{x^2}{2}}}$$

② $y' = e^{x+y}$ пол $x+y = z \to y = z - x$

$$y' = z' - 1 \quad \to \quad z' - 1 = e^z \quad \to \quad z' = e^z + 1$$

$$\int \frac{z'}{e^z+1}\,dx = \int dx + K \quad \to \quad \int \frac{dz}{e^z+1} = x + K$$

$$\int \frac{e^z\,dz}{e^z(e^z+1)} = x + K \quad \to \quad \int \frac{d\,e^z}{e^z(e^z+1)} = x + K$$

$$\int \frac{du}{u(u+1)} = \int \frac{1}{u}\,du - \int \frac{1}{u+1}\,du = \ln \frac{u}{u+1}$$

$$\Rightarrow \ln \frac{e^z}{e^z+1} = x + K \quad \to \quad \boxed{\frac{e^z}{e^z+1} = C e^x}$$

Втори начин $\quad y' = e^x \cdot e^y \quad | : e^y$

$\Rightarrow \quad y' e^{-y} = e^x \quad \to \int e^{-y} dy = \int e^x dx$

$\Rightarrow -e^{-y} = e^x + C$

<u>Пример</u> $\quad (y-x)y' = 2x \to y' = \frac{2x}{y-x} = \frac{2}{y/x - 1}$

$z = \frac{y}{x} \quad \to y = z \cdot x \quad \to y' = z' \cdot x + z$

$z' \cdot x + z = \frac{2}{z-1} \quad \to \quad z' \cdot x = \frac{2}{z-1} - \overset{z-1}{z}$

$z' \cdot x = \frac{2 - z^2 + z}{z-1} = -\frac{(z-2)(z+1)}{z-1}$ $\qquad z = -1, \; z = 2$ стац.

$\int \frac{z-1}{(z-2)(z+1)} dz = -\int \frac{dx}{x} + C$

$\frac{z-1}{(z-2)(z+1)} = \frac{A}{z-2} + \frac{B}{z+1} \quad \to A = -1/3 \; ; B = -2/3$

$\Rightarrow -\frac{1}{3} \int \frac{dz}{z-2} - \frac{2}{3} \int \frac{dz}{z+1} = \ln(x \cdot C)$

$\to -\frac{1}{3} \ln|z-2| - 2/3 \ln|z+1| = \ln(xC)$

$\ln (z+1)^{-2/3} (z-2)^{-1/3} = \ln xC$

$(z+1)^{-2/3} (z-2)^{-1/3} = x \cdot C$

$\left(\frac{y}{x} + 1\right)^{-2/3} \left(\frac{y}{x} - 2\right)^{-1/3} = x + C$

$\begin{cases} z = -1 \to y = -x \\ z = 2 \quad\; y = 2x \end{cases}$

особени.

Линейни уравнения

$$y' = a(x)y + b(x) \longrightarrow y(x) = e^{\int a(x)dx}\left[c + \int b(x) e^{-\int a(x)dx} dx\right]$$

$$y' = \frac{2xy}{x^2+1} + 2x(x^2+1)$$

$$a(x) = \frac{2x}{x^2+1} \qquad b(x) = 2x(x^2+1)$$

$$y(x) = e^{\int \frac{2x}{x^2+1}dx}\left[c + \int 2x(x^2+1) e^{-\int \frac{2x}{x^2+1}dx} dx\right]$$

$$y(x) = e^{\int \frac{d(x^2+1)}{x^2+1}}\left[c + \int 2x(x^2+1) e^{-\int \frac{d(x^2+1)}{x^2+1}} dx\right]$$

$$\Rightarrow y(x) = e^{\ln(x^2+1)}\left[c + \int 2x\, e^{-\ln(x^2+1)} \cdot (x^2+1)\, dx\right]$$

$$y(x) = (x^2+1)\left[c + \int 2x\,(x^2+1)(x^2+1)^{-1} dx\right]$$

$$y(x) = (x^2+1)\left[c + \int 2x\, dx\right] = (x^2+1)(c + x^2)$$

$$\left.\begin{aligned} y' &= \frac{2}{x}y + x^3 \\ y(1) &= 1 \end{aligned}\right\}\ \text{З. К.} \qquad a(x) = \frac{2}{x} \qquad b(x) = x^3$$

$$y(x) = e^{\int \frac{2}{x}dx}\left[c + \int x^3 e^{-\int \frac{2}{x}dx} dx\right]$$

$$y(x) = e^{2\ln x}\left[c + \int x^3 e^{-2\ln x} dx\right]$$

$$y(x) = e^{\ln x^2}\left[c + \int x^3 e^{\ln x^{-2}} dx\right] = x^2\left[c + \int x^3 x^{-2} dx\right]$$

$$y(x) = x^2\left[c + \int x\, dx\right] = x^2\left[c + \frac{x^2}{2}\right]$$

$$y(1) = 1 \rightarrow \qquad 1 = 1^2\left[c + 1/2\right] \rightarrow c + 1/2 = 1$$

$$c = 1/2$$

$$y(x) = x^2\left[\frac{1}{2} + \frac{x^2}{2}\right]$$