Метод на Лагранж

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = f(x)$$

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0$$

Решение на хомогенното $y_0(x) = C_1 \varphi_1(x) + \dots + C_n \varphi_n(x)$

Вариране на константите Метод на Лагранж

$$y(x) = C_1(x)\varphi_1(x) + \dots + C_n(x)\varphi_n(x)$$

$$\begin{cases} C_1' \varphi_1 + \dots + C_n' \varphi_n = 0 \\ C_1' \varphi_1' + \dots + C_n' \varphi_n' = 0 \\ \dots \\ C_1' \varphi_1^{(n-1)} + \dots + C_n' \varphi_n^{(n-1)} = f(x) \end{cases}$$

Пример $\ddot{x} + x = \frac{1}{\sin t}$

$\ddot{x} + x = 0 \rightarrow \lambda^2 + 1 = 0 \rightarrow \lambda_{1,2} = \pm i$

$x_0(t) = C_1 \cos t + C_2 \sin t$

Лагранж $\rightarrow$ $x(t) = C_1(t)\cos t + C_2(t)\sin t$

$$\begin{cases} \dot{C}_1 \cos t + \dot{C}_2 \sin t = 0 \\ \dot{C}_1(-\sin t) + \dot{C}_2(\cos t) = \frac{1}{\sin t} \end{cases}$$

ф-ла на Крамер $\dot{C}_1 = \dfrac{\begin{vmatrix} 0 & \sin t \\ \frac{1}{\sin t} & \cos t \end{vmatrix}}{\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}} = \dfrac{-1}{\cos^2 t + \sin^2 t}$

$\dot{C}_1 = -1$; $\dot{C}_2 = \begin{vmatrix} \cos t & 0 \\ -\sin t & \frac{1}{\sin t} \end{vmatrix} = \frac{\cos t}{\sin t}$

$C_1(t) = \int (-1)\,dt + K_1 \rightarrow C_1(t) = -t + K_1$

$C_2(t) = \int \frac{\cos t}{\sin t}\,dt + K_2 = \int d \ln \sin t + K_2 = \ln \sin t + K_2$

$x(t) = (-t + K_1)\cos t + (\ln \sin t + K_2)\sin t$ – общо решение

Пример $\ddot{x} + 2\dot{x} + x = 3e^{-t}\sqrt{1+t}$

$\ddot{x} + 2\dot{x} + x = 0$ $\quad \lambda^2 + 2\lambda + 1 = 0$ $\quad (\lambda + 1)^2 = 0$

$\lambda_1 = \lambda_2 = -1$

$x_0(t) = c_1 e^{-t} + c_2 t e^{-t}$

Лагранж!

$x(t) = c_1(t) e^{-t} + c_2(t) t e^{-t}$

$\dot c_1 e^{-t} + \dot c_2 t e^{-t} = 0 \qquad : e^{-t}$

$\dot c_1 (-e^{-t}) + \dot c_2 (e^{-t} - t e^{-t}) = 3 e^{-t} \sqrt{1+t} \qquad : e^{-t}$

$$\left| \begin{array}{l} \dot c_1 + \dot c_2 t = 0 \\ -\dot c_2 + \dot c_2 (1-t) = 3\sqrt{1+t} \end{array} \right.$$

$\dot c_2 = 3\sqrt{1+t} \rightarrow c_2(t) = 3 \int \sqrt{1+t}\, dt + k_2$

$\dot c_2(t) = \frac{\cancel{3}(1+t)^{3/2}}{\cancel{3}/2} + k_2 = 2(1+t)^{3/2} + k_2$

$\dot c_1 = -\dot c_2 t = -3\sqrt{1+t} \cdot t$

$c_1(t) = -3 \int \sqrt{1+t} \cdot t \, dt + k_1 = \frac{2(t+1)^{3/2}(3t-2)}{15} + k_1$

$x(t) = \left(2(1+t)^{3/2} + k_2\right) t e^{-t} + \left(2 \frac{(t+1)^{3/2}(3t-2)}{15} + k_1\right) e^{-t}$

$$\left\{ \begin{array}{l} \dot x_1 = -4x_1 - x_2 \\ \dot x_2 = 5x_1 + 2x_2 \end{array} \right.$$

$x_2 = -\dot x_1 - 4x_1$

$\dot x_2 = -\ddot x_1 - 4\dot x_1$

$-\ddot x_1 - 4\dot x_1 = 5x_1 + 2(-\dot x_1 - 4x_1)$

$-\ddot x_1 - 4\dot x_1 = 5x_1 - 2\dot x_1 - 8x_1$

$-\ddot x_1 - 2\dot x_1 + 3x_1 = 0 \longrightarrow \ddot x_1 + 2\dot x_1 - 3x_1 = 0$

$\lambda^2 + 2\lambda - 3 = 0 \qquad \lambda_{1;2} = -1 \pm \sqrt{4} = -1 \pm 2$

$\lambda_1 = 1 \qquad \lambda_2 = -3 \qquad \rightarrow \boxed{x_1(t) = c_1 e^{t} + c_2 e^{-3t}}$

$x_2(t) = -\dot x_1 - 4x_1 = -(c_1 e^{t} - 3c_2 e^{-3t}) - 4(c_1 e^{t} + c_2 e^{-3t})$

$= -5c_1 e^{t} - c_2 e^{-3t}$

-3-

$$\begin{cases} \dot{x}_1 = 2x_1 - 2x_2 \\ \dot{x}_2 = -x_1 + 3x_2 \end{cases}$$

$$x_2 = \frac{1}{2}(-\dot{x}_1 + 2x_1)$$

$$\dot{x}_2 = \frac{1}{2}(-\ddot{x}_1 + 2\dot{x}_1)$$

$$\frac{1}{2}(-\ddot{x}_1 + 2\dot{x}_1) = -x_1 + 3 \cdot \frac{1}{2}(-\dot{x}_1 + 2x_1)$$

$$-\ddot{x}_1 + 2\dot{x}_1 = -2x_1 - 3\dot{x}_1 + 6x_1$$

$$-\ddot{x}_1 + 2\dot{x}_1 + 3\dot{x}_1 - 4x_1 = 0$$

$$-\ddot{x}_1 + 5\dot{x}_1 - 4x_1 = 0$$

$$\ddot{x}_1 - 5\dot{x}_1 + 4x_1 = 0$$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$\lambda_1 = 1 \qquad \lambda_2 = 4$$

$$x_1(t) = c_1 e^{t} + c_2 e^{4t}$$

$$\dot{x}_1 = c_1 e^{t} + 4c_2 e^{4t}$$

$$x_2 = \frac{1}{2}(-c_1 e^{t} - 4c_2 e^{4t} + 2c_1 e^{t} + 2c_2 e^{4t})$$

$$= \frac{1}{2}(c_1 e^{t} - 2c_2 e^{4t})$$

Dom

① $$\begin{cases} \dot{x}_1 = 5x_1 - x_2 \\ \dot{x}_2 = 2x_1 + 2x_2 \end{cases}$$

② $$\ddot{x} - 6\dot{x} + 9x = \frac{e^{3t}}{t^2}$$

③ $$\ddot{x} + 2\dot{x} + x = te^{t} + \frac{1}{te^{t}}$$