Introduction to Bayesian Data Analysis
Computational Bayesian Analysis (Chapter 4 of book)

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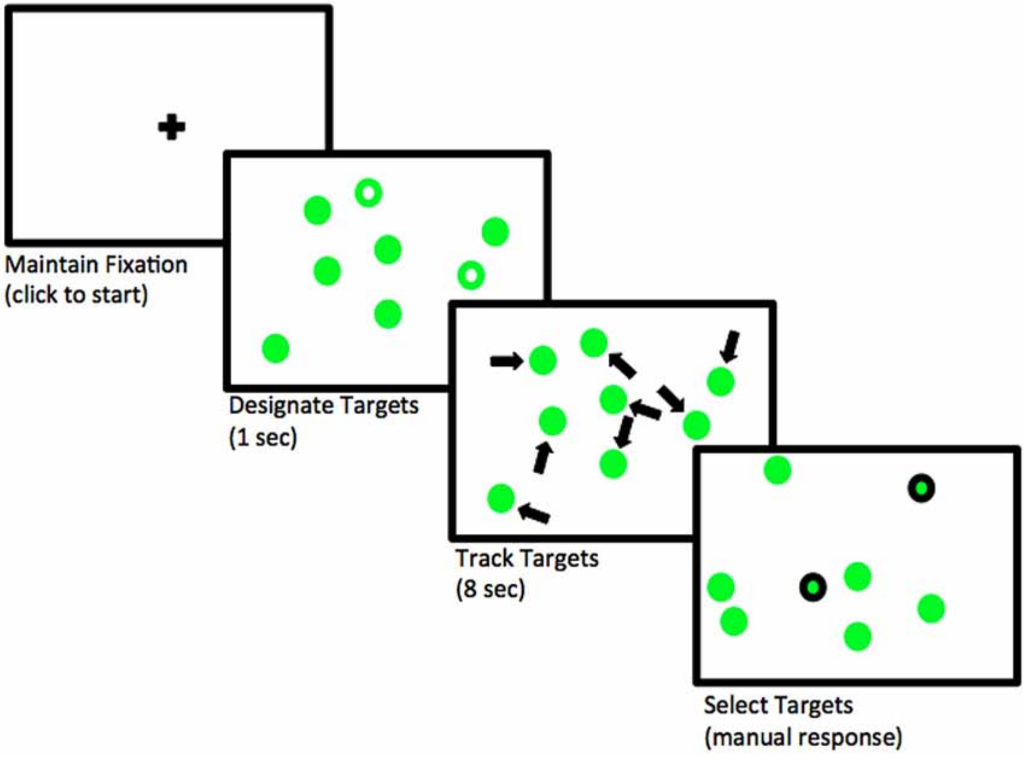
Example: Multiple object tracking

- The subject covertly tracks between zero and five objects among several randomly moving objects on a computer screen.
- First, several objects appear on the screen, and a subset of them are indicated as "targets" at the beginning.
- Then, the objects start moving randomly across the screen and become indistinguishable.
- After several seconds, the objects stop moving and the subject need to indicate which objects were the targets.

Our research goal is to examine **how the attentional load affects pupil size**.

Example: Multiple object tracking



Example: Multiple object tracking

A model for this experiment design:

$$p\_size_n \sim Normal(\alpha + c\_load_n \cdot \beta, \sigma) \tag{1}$$

- n indicates the observation number with $n = 1, \ldots, N$
- c_load refers to centered load.
- Every data point is assumed to be independent (in frequentist terms: iid).

Example: Multiple object tracking: Prior specification

Some pilot data helps us work out priors:

```
data("df_pupil_pilot")
df_pupil_pilot$p_size %>% summary()

##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##     852     856     862     861     866     868
```

This suggests we can use the following regularizing prior for α:

$$\alpha \sim Normal(1000, 500) \tag{2}$$

What we are expressing with this prior:

```
qnorm(c(.025, .975), mean = 1000, sd = 500)

## [1]   20 1980
```

Example: Multiple object tracking: Prior specification

For σ, we use an uninformative prior:

$$\sigma \sim Normal_{+}(0, 1000) \tag{3}$$

```
extraDistr::qtnorm(c(.025,0.975), mean = 0, sd = 1000, a = 0)

## [1]   31 2241
```

Example: Multiple object tracking: Prior specification

$$\beta \sim Normal(0, 100) \tag{4}$$

```r
qnorm(c(.025, .975), mean = 0, sd = 100)

## [1] -196  196
```

Example: Multiple object tracking: Fit model

First, center the predictor:

```
data("df_pupil")
(df_pupil <- df_pupil %>%
  mutate(c_load = load - mean(load)))
```

```
## # A tibble: 41 x 5
##      subj trial  load p_size c_load
##     <int> <int> <int>  <dbl>  <dbl>
##  1   701     1     2  1021. -0.439
##  2   701     2     1   951. -1.44
##  3   701     3     5  1064.  2.56
##  4   701     4     4   913.  1.56
##  5   701     5     0   603. -2.44
##  6   701     6     3   826.  0.561
##  7   701     7     0   464. -2.44
```

Example: Multiple object tracking: Fit model

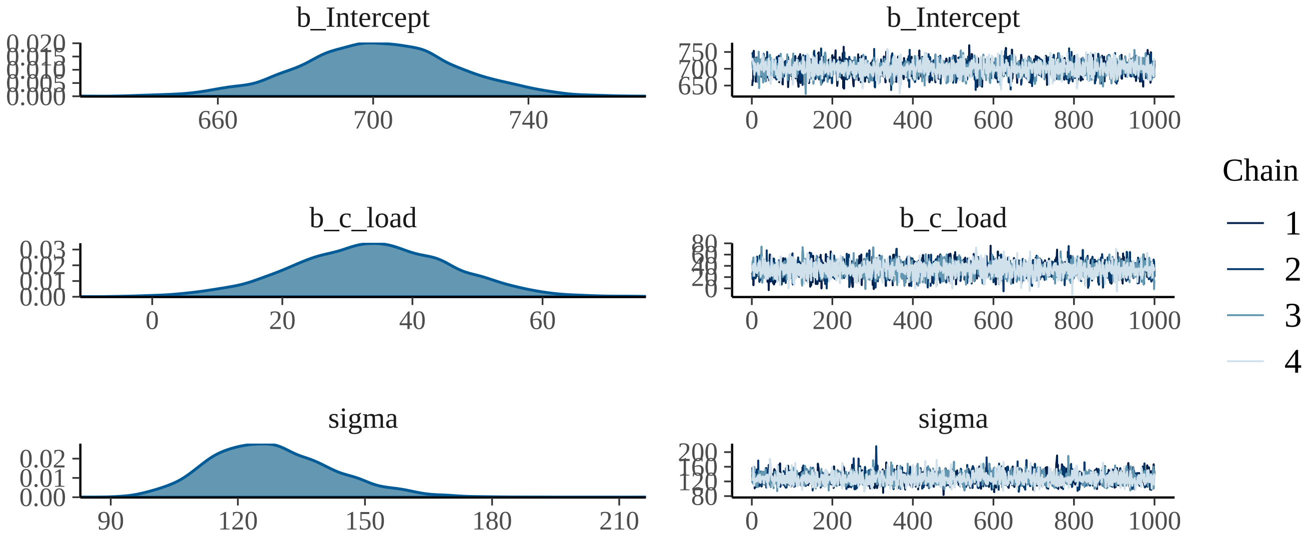
```
fit_pupil <- brm(p_size ~ 1 + c_load,
  data = df_pupil,
  family = gaussian(),
  prior = c(
    prior(normal(1000, 500), class = Intercept),
    prior(normal(0, 1000), class = sigma),
    prior(normal(0, 100), class = b, coef = c_load)
  )
)
```

The next steps

Next, we will plot the posterior distributions of the parameters, and the posterior predictive distributions for the different load levels.

Example: Multiple object tracking: Summarize posteriors

```r
plot(fit_pupil)
```

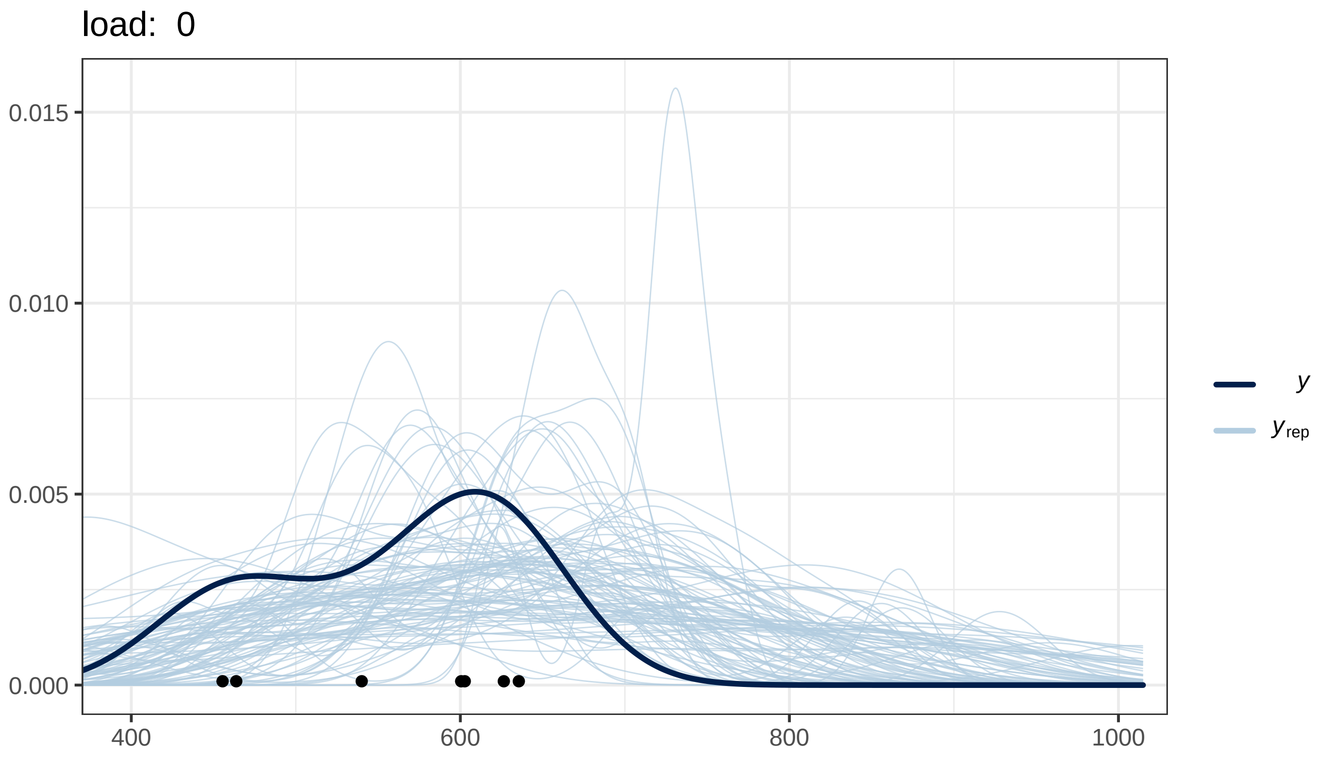


Example: Multiple object tracking: Summarize posteriors

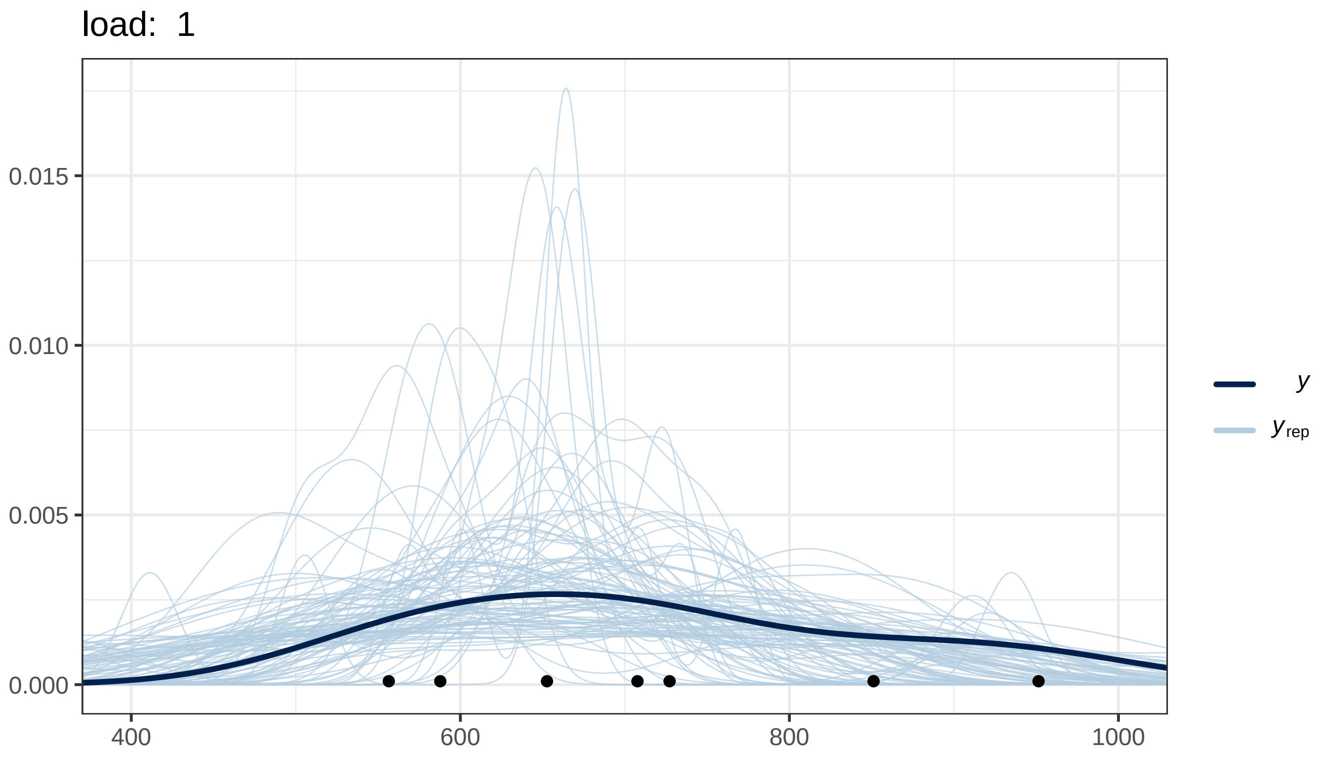
```
## Note: short_summary is a function we wrote
short_summary(fit_pupil)

## ...
## Population-Level Effects:
##           Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
## Intercept   701.06     20.00   660.47   739.65 1.00     3761     2844
## c_load       33.85     12.06     9.78    57.08 1.00     3639     2859
##
## Family Specific Parameters:
##       Estimate Est.Error l-95% CI u-95% CI Rhat Bulk_ESS Tail_ESS
## sigma   128.19     14.64   102.94   159.66 1.00     3751     2937
##
## ...
```

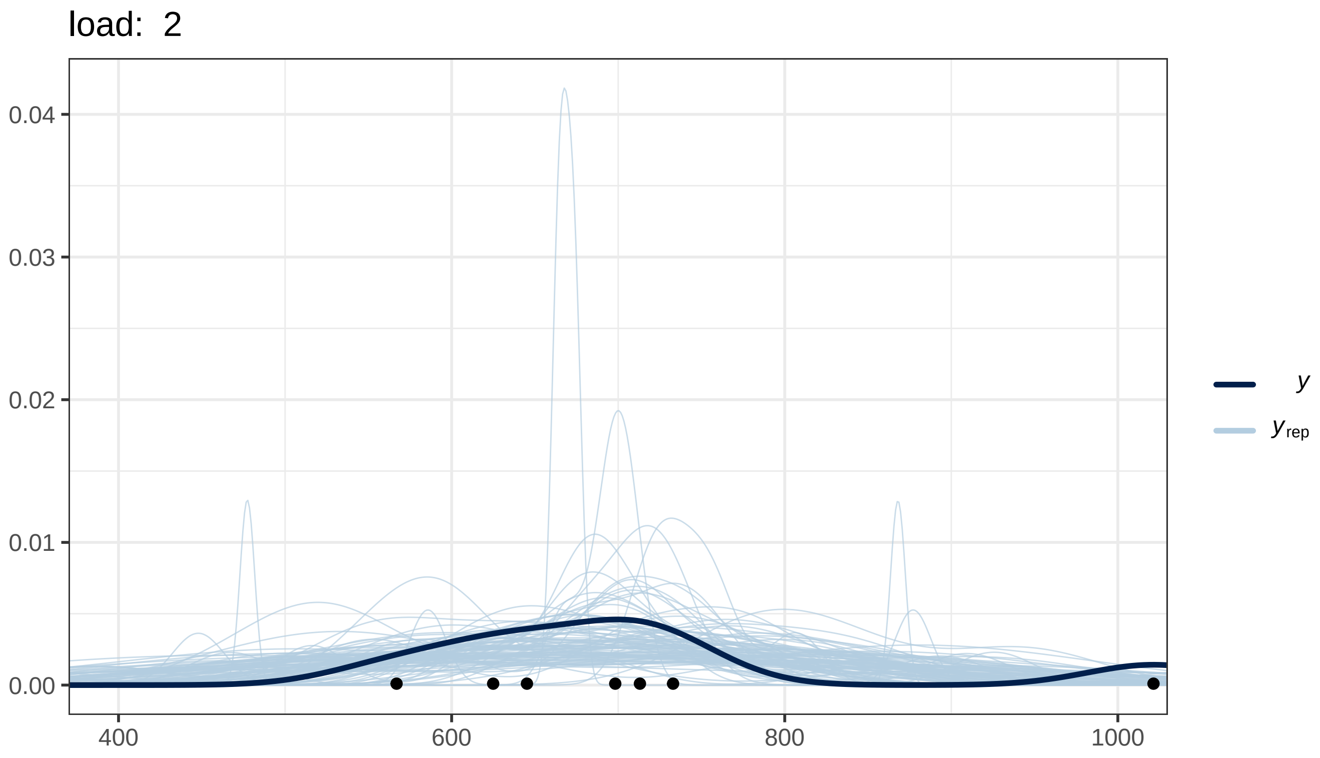
Example: Multiple object tracking: Posterior predictive check



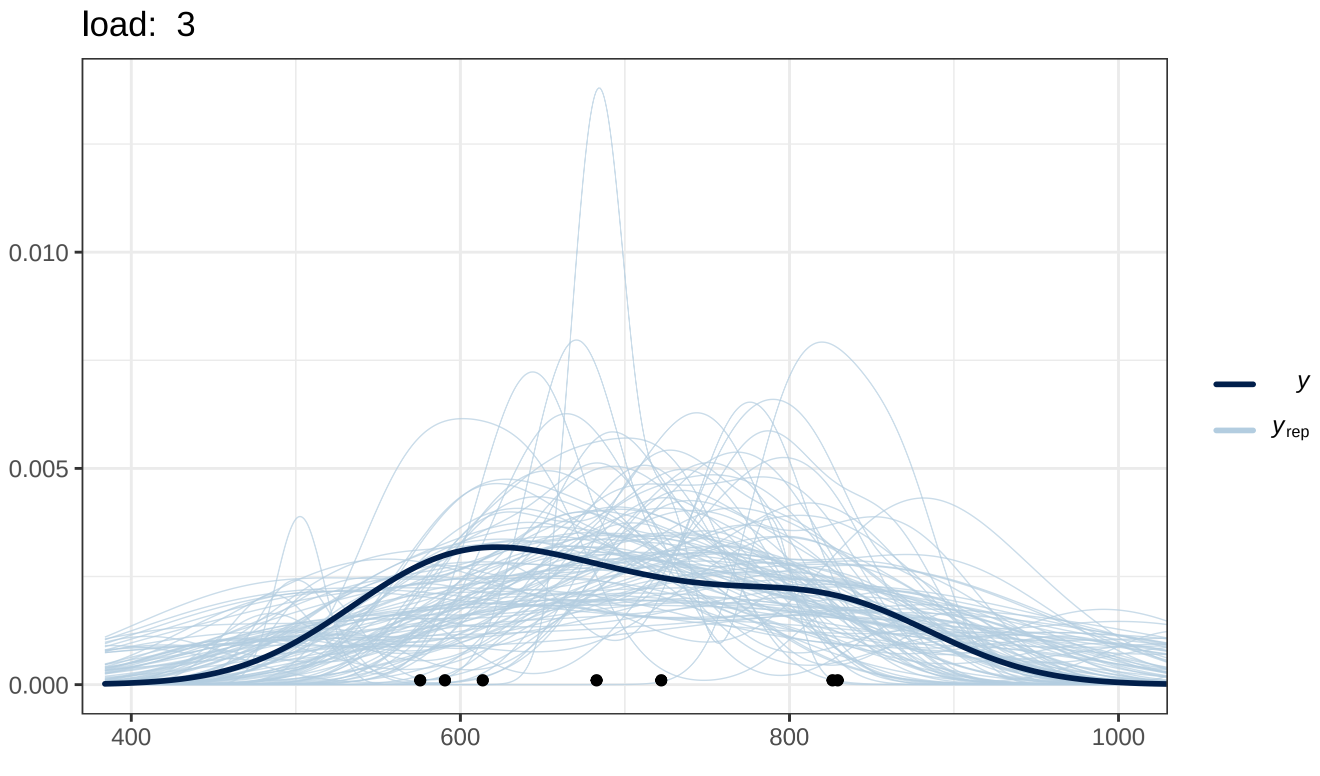
Example: Multiple object tracking: Posterior predictive check



Example: Multiple object tracking: Posterior predictive check



Example: Multiple object tracking: Posterior predictive check



Bayesian Data Analysis

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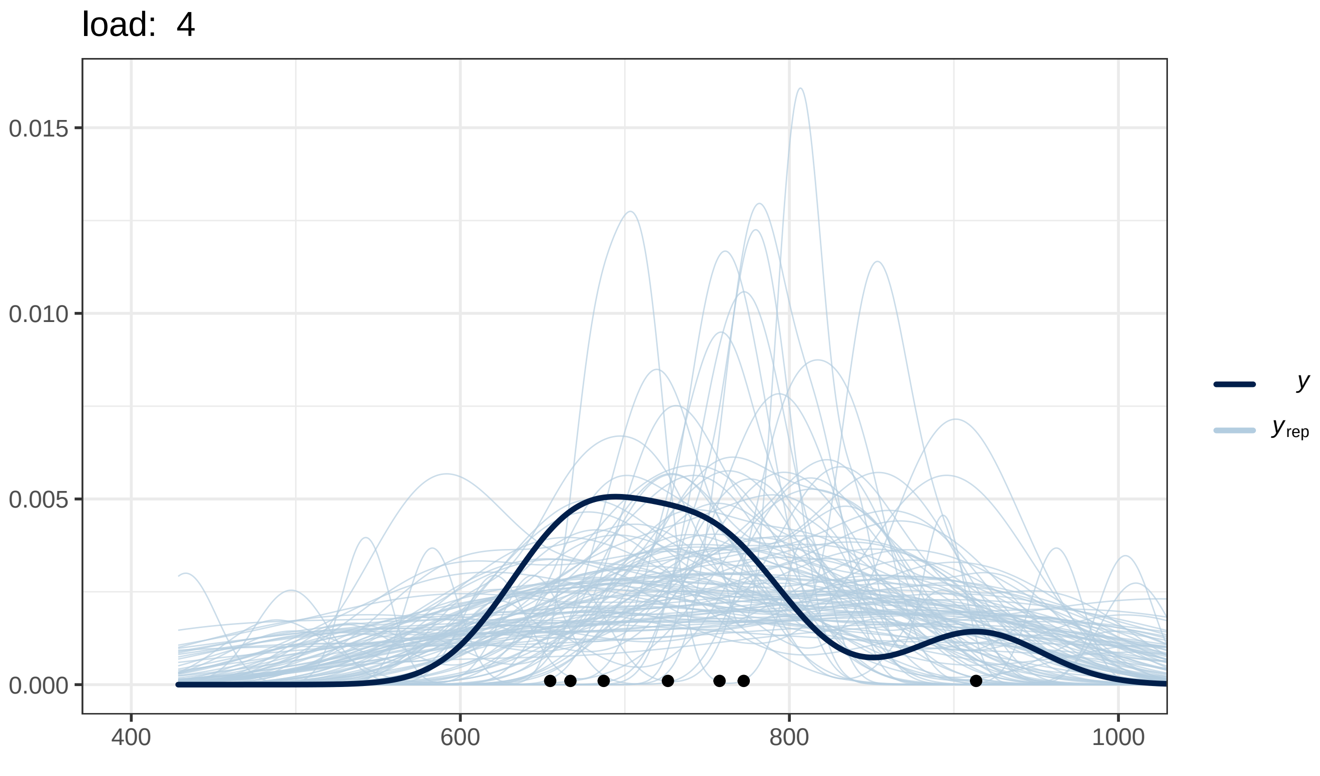
Example: Multiple object tracking: Posterior predictive check



The next steps

Next, we will look at another example: the effect of trial id on button-pressing times.
This time, we will use the log-normal likelihood.

Example: Button-pressing time and the effect of trial

```
df_spacebar <- df_spacebar %>%
  mutate(c_trial = trial - mean(trial))
```

Example: Button-pressing time and the effect of trial

If we assume that response times are log-normally distributed, we could proceed as follows:

$$rt_n \sim LogNormal(\alpha + c\_trial_n \cdot \beta, \sigma) \tag{5}$$

where

- N is the total number of (independent!) data points
- $n = 1, \ldots, N$, and
- rt is the dependent variable (response times in milliseconds).

Example: Button-pressing time and the effect of trial

The priors have to be defined on the log scale:

$$
\begin{aligned}
\alpha &\sim Normal(6, 1.5) \\
\sigma &\sim Normal_{+}(0, 1)
\end{aligned}
\tag{6}
$$

A new parameter, β, needs a prior specification:

$$
\beta \sim Normal(0, 1) \tag{7}
$$

This prior on β is very uninformative.

Example: Button-pressing time and the effect of trial

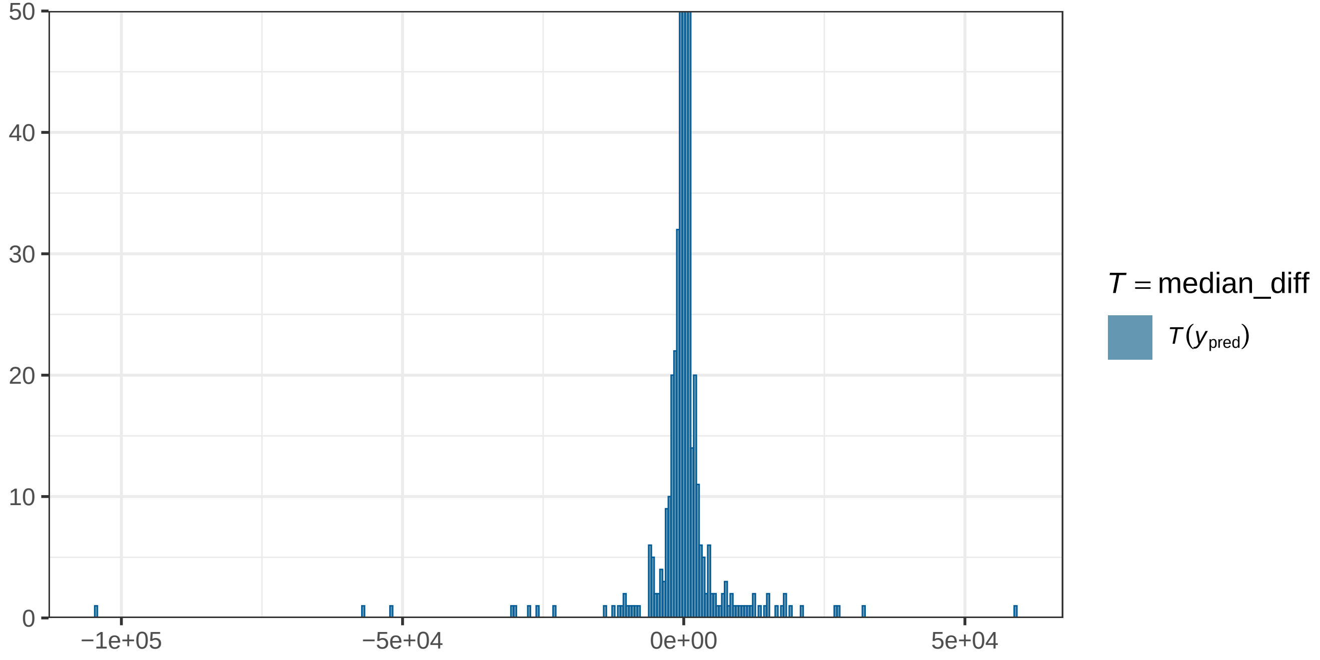
Prior predictive distribution:

```
df_spacebar_ref <- df_spacebar %>%
  mutate(rt = rep(1, n()))
fit_prior_press_trial <- brm(t ~ 1 + c_trial,
  data = df_spacebar_ref,
  family = lognormal(),
  prior = c(
    prior(normal(6, 1.5), class = Intercept),
    prior(normal(0, 1), class = sigma),
    prior(normal(0, 1), class = b, coef = c_trial)
  ),
  sample_prior = "only",
  control = list(adapt_delta = .9)
)
```

Example: Button-pressing time and the effect of trial

```
median_diff <- function(x) {
  median(x - lag(x), na.rm = TRUE)
}
pp_check(fit_prior_press_trial,
         type = "stat",
         stat = "median_diff",
  # show only prior predictive distributions
         prefix = "ppd",
  # each bin has a width of 500ms
         binwidth = 500) +
  # cut the top of the plot to improve its scale
  coord_cartesian(ylim = c(0, 50))+theme_bw()
```

Example: Button-pressing time and the effect of trial



Example: Button-pressing time and the effect of trial

What would the prior predictive distribution look like if we set the following more informative prior on β?

$$\beta \sim Normal(0, 0.01) \tag{8}$$

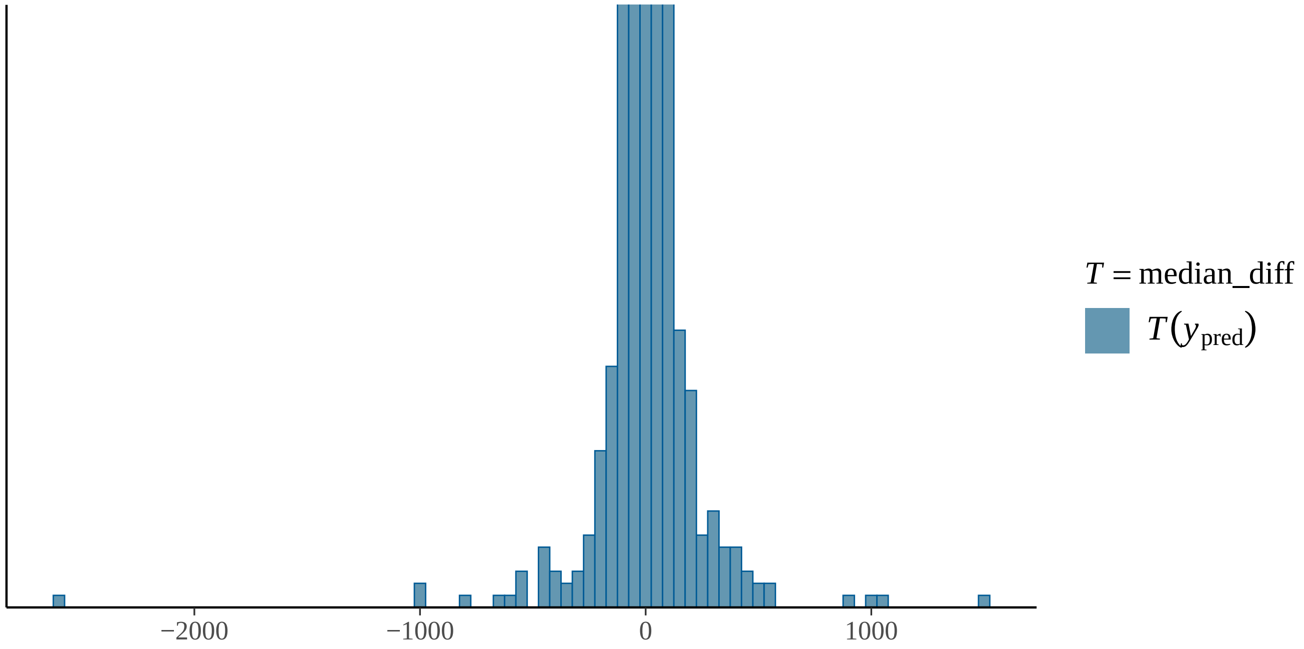
Example: Button-pressing time and the effect of trial

```r
fit_prior_press_trial <- brm(t ~ 1 + c_trial,
  data = df_spacebar_ref,
  family = lognormal(),
  prior = c(
    prior(normal(6, 1.5), class = Intercept),
    prior(normal(0, 1), class = sigma),
    prior(normal(0, .01), class = b, coef = c_trial)
  ),
  sample_prior = "only",
  control = list(adapt_delta = .9)
)
```

Example: Button-pressing time and the effect of trial

```
fit_prior_press_trial <- brm(t ~ 1 + c_trial,
  data = df_spacebar_ref,
  family = lognormal(),
  prior = c(
    prior(normal(6, 1.5), class = Intercept),
    prior(normal(0, 1), class = sigma),
    prior(normal(0, .01), class = b, coef = c_trial)
  ),
  sample_prior = "only",
  control = list(adapt_delta = .9)
)
```

Example: Button-pressing time and the effect of trial



The next steps

Now that we have decided on our priors, we fit the model.

Example: Button-pressing time and the effect of trial

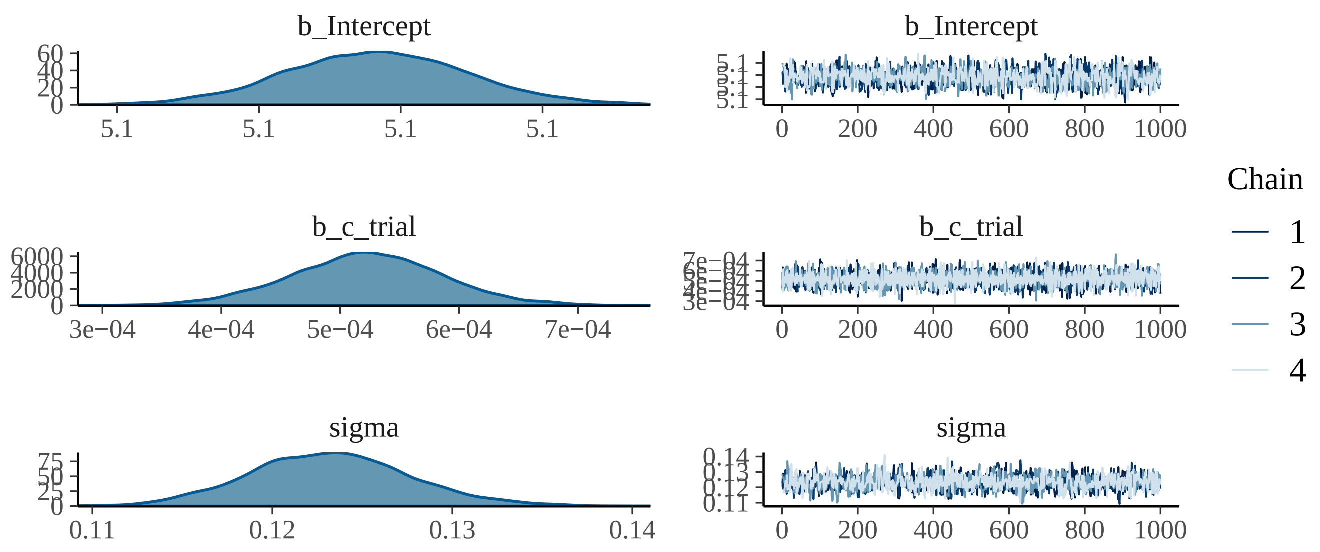
Fit the model:

```
fit_press_trial <- brm(t ~ 1 + c_trial,
  data = df_spacebar,
  family = lognormal(),
  prior = c(
    prior(normal(6, 1.5), class = Intercept),
    prior(normal(0, 1), class = sigma),
    prior(normal(0, .01), class = b, coef = c_trial)
  )
)
```

Example: Button-pressing time and the effect of trial

Summarize posteriors (graphically or in a table, or both):

```r
plot(fit_press_trial)
```



Example: Button-pressing time and the effect of trial

Summarize results on the ms scale (the effect estimate from the middle of the expt to the preceding trial):

```
alpha_samples <- as_draws_df(fit_press_trial)$b_Intercept
beta_samples <- as_draws_df(fit_press_trial)$b_c_trial

beta_ms <- exp(alpha_samples) - exp(alpha_samples - beta_samples)

beta_msmean <- round(mean(beta_ms), 5)
beta_mslow <- round(quantile(beta_ms, prob = 0.025), 5)
beta_mshigh <- round(quantile(beta_ms, prob = 0.975), 5)
c(beta_msmean , beta_mslow, beta_mshigh)
```

```
##        2.5% 97.5%
## 0.087 0.066 0.108
```

Example: Button-pressing time and the effect of trial

The effect estimate at the first vs second trial:

```
first_trial <- min(df_spacebar$c_trial)
second_trial <- min(df_spacebar$c_trial) + 1
effect_beginning_ms <-
  exp(alpha_samples + second_trial * beta_samples) -
  exp(alpha_samples + first_trial * beta_samples)
## ms effect from first to second trial:
c(mean = mean(effect_beginning_ms),
  quantile(effect_beginning_ms, c(0.025, 0.975)))
```

```
##  mean  2.5% 97.5%
## 0.079 0.062 0.096
```

Example: Button-pressing time and the effect of trial

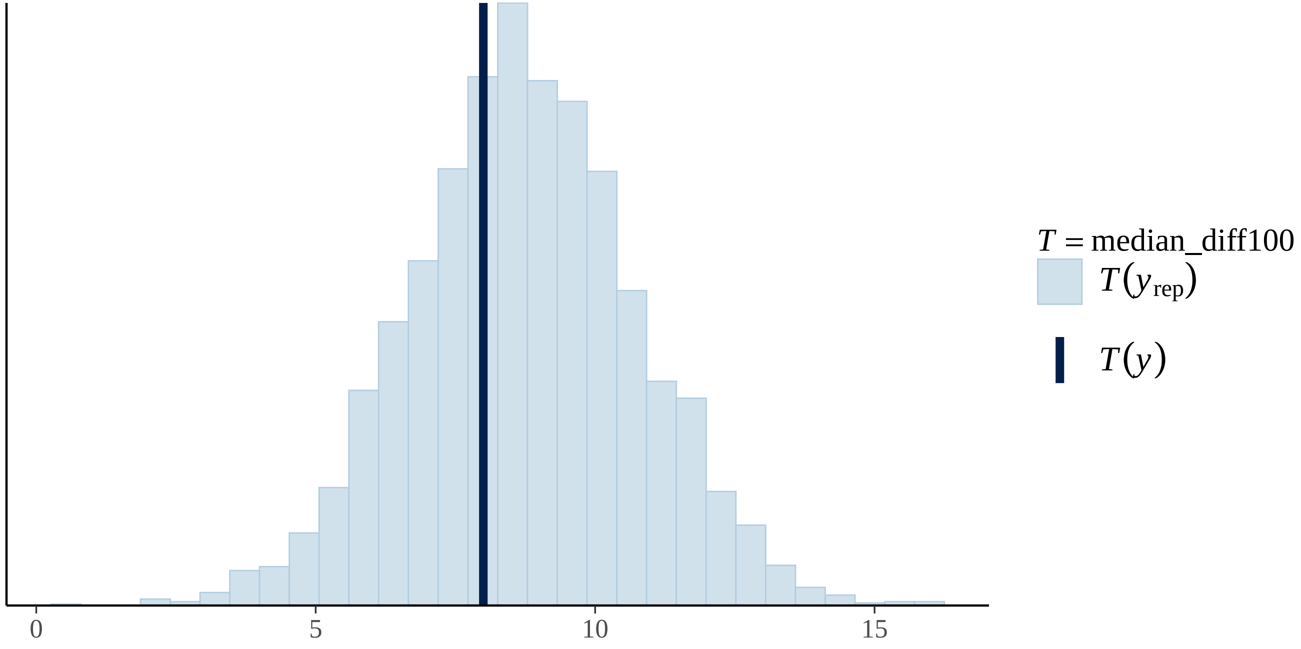
Slowdown after 100 trials from the middle of the expt:

```
effect_100 <-
  exp(alpha_samples + 100 * beta_samples) -
  exp(alpha_samples)
c(mean = mean(effect_100),
  quantile(effect_100, c(0.025, 0.975)))
```

```
##  mean  2.5% 97.5%
##   9.0   6.7  11.2
```

Example: Button-pressing time and the effect of trial

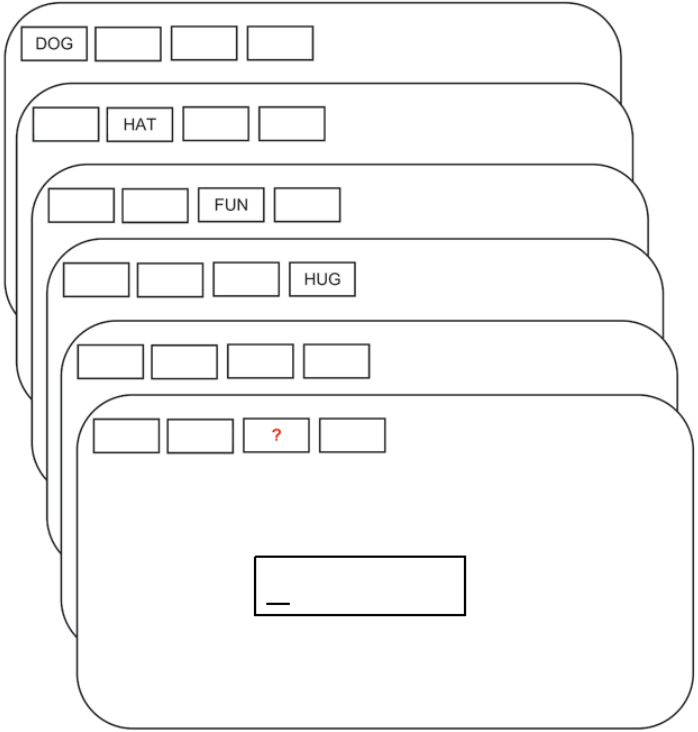
The posterior predictive distribution (distribution of predicted median differences between the n and n-100th trial):



The next steps

Next: logistic regression.

Example of logistic regression: Does set size affect free recall?



Example of logistic regression: Does set size affect free recall?

```
data("df_recall")
head(df_recall)

## # A tibble: 6 x 7
##   subj  set_size correct trial session block tested
##   <chr>    <int>   <int> <int>   <int> <int>  <int>
## 1 10           4       1     1       1     1      2
## 2 10           8       0     4       1     1      8
## 3 10           2       1     9       1     1      2
## 4 10           6       1    23       1     1      2
## 5 10           4       1     5       1     2      3
## 6 10           8       0     7       1     2      5

df_recall <- df_recall %>%
  mutate(c_set_size = set_size - mean(set_size))
```

Example of logistic regression: Does set size affect free recall?

```r
# Set sizes in the data set:
df_recall$set_size %>%
  unique() %>% sort()
```

```
## [1] 2 4 6 8
```

Example of logistic regression: Does set size affect free recall?

```
# Trials by set size
df_recall %>%
  group_by(set_size) %>%
  count()
```

```
## # A tibble: 4 x 2
## # Groups:   set_size [4]
##   set_size     n
##      <int> <int>
## 1        2    23
## 2        4    23
## 3        6    23
## 4        8    23
```

Example of logistic regression: Does set size affect free recall?

$$correct_n \sim Bernoulli(\theta_n) \tag{9}$$

$$\eta_n = g(\theta_n) = \log\left(\frac{\theta_n}{1-\theta_n}\right) \tag{10}$$

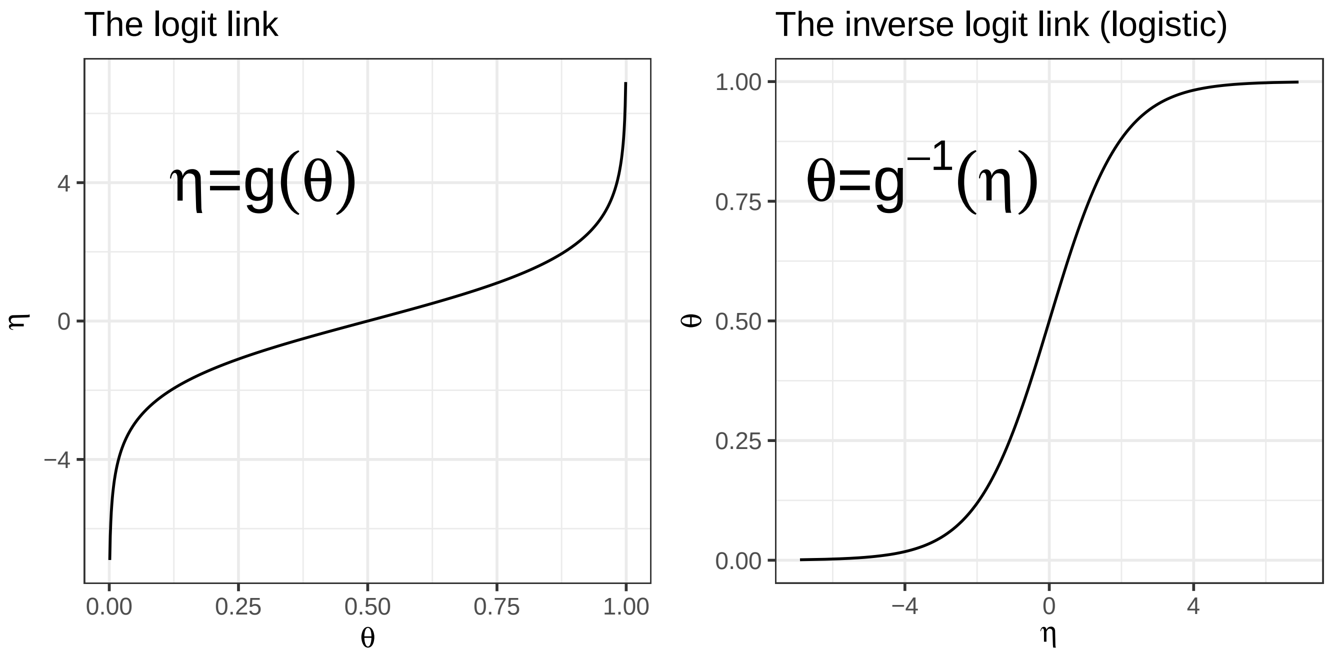
Example of logistic regression: Does set size affect free recall?

```
x <- seq(0.001, 0.999, by = 0.001)
y <- log(x / (1 - x))
logistic_dat <- data.frame(theta = x, eta = y)

p1 <- qplot(logistic_dat$theta, logistic_dat$eta, geom = "line") +
  xlab(expression(theta)) +
  ylab(expression(eta)) +
  ggtitle("The logit link") +
  annotate("text",
    x = 0.3, y = 4,
    label = expression(paste(eta, "=",
                             g(theta))), parse = TRUE,
    size = 8
  )
```

Example of logistic regression: Does set size affect free recall?

```
p2 <- qplot(logistic_dat$eta, logistic_dat$theta,
            geom = "line") + xlab(expression(eta)) +
  ylab(expression(theta)) +
  ggtitle("The inverse logit link (logistic)") +
  annotate("text",
  x = -3.5, y = 0.80,
  label = expression(paste(theta, "=", g^-1,
                           (eta))), parse = TRUE, size = 8
)

gridExtra::grid.arrange(p1, p2, ncol = 2)
```

Example of logistic regression: Does set size affect free recall?



The next steps

Next step: deciding on priors.

Example of logistic regression: Does set size affect free recall?

```
data("df_recall")
head(df_recall)
```

```
## # A tibble: 6 x 7
##   subj  set_size correct trial session block tested
##   <chr>    <int>   <int> <int>   <int> <int>  <int>
## 1 10           4       1     1       1     1      2
## 2 10           8       0     4       1     1      8
## 3 10           2       1     9       1     1      2
## 4 10           6       1    23       1     1      2
## 5 10           4       1     5       1     2      3
## 6 10           8       0     7       1     2      5
```

```
df_recall <- df_recall %>%
  mutate(c_set_size = set_size - mean(set_size))
```

Example of logistic regression: Does set size affect free recall?

The linear model is now fit not to the 0,1 responses as the dependent variable, but to η_n, i.e., log-odds, as the dependent variable:

$$\eta_n = \log\left(\frac{\theta_n}{1-\theta_n}\right) = \alpha + \beta \cdot c\_set\_size \tag{11}$$

Example of logistic regression: Does set size affect free recall?

- Unlike the linear models, the model is defined so that there is no residual error term (ε) in this model.
- Once η_n is estimated, one can solve the above equation for θ_n (in other words, we compute the inverse of the logit function and obtain the estimates on the probability scale).

This gives the above-mentioned logistic regression function:

$$\theta_n = g^{-1}(\eta_n) = \frac{\exp(\eta_n)}{1+\exp(\eta_n)} = \frac{1}{1+exp(-\eta_n)} \tag{12}$$

Example of logistic regression: Does set size affect free recall?

In summary, the generalized linear model with the logit link fits the following Bernoulli likelihood:

$$correct_n \sim Bernoulli(\theta_n) \tag{13}$$

- The model is fit on the log-odds scale, $\eta_n = \alpha + c\_set\_size_n \cdot \beta$.
- Once η_n has been estimated, the inverse logit or the logistic function is used to compute the probability estimates $\theta_n = \frac{\exp(\eta_n)}{1+\exp(\eta_n)}$.

Example of logistic regression: Does set size affect free recall?

There are two functions in R that implement the logit and inverse logit functions:

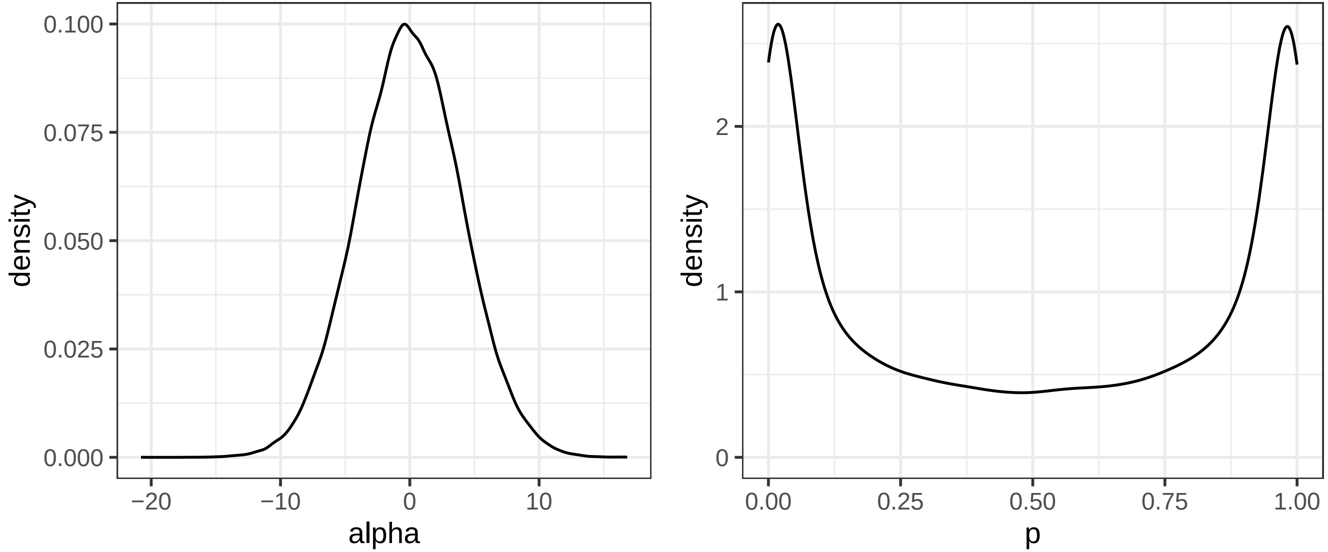
- `qlogis(p)` for the logit function and
- `plogis(x)` for the inverse logit or logistic function.

$$\alpha \sim Normal(0, 4) \tag{14}$$

Let's plot this prior in log-odds and in probability scale by drawing random samples.

Example of logistic regression: Does set size affect free recall?

Prior for $\alpha \sim Normal(0, 4)$ in log-odds and in probability space.

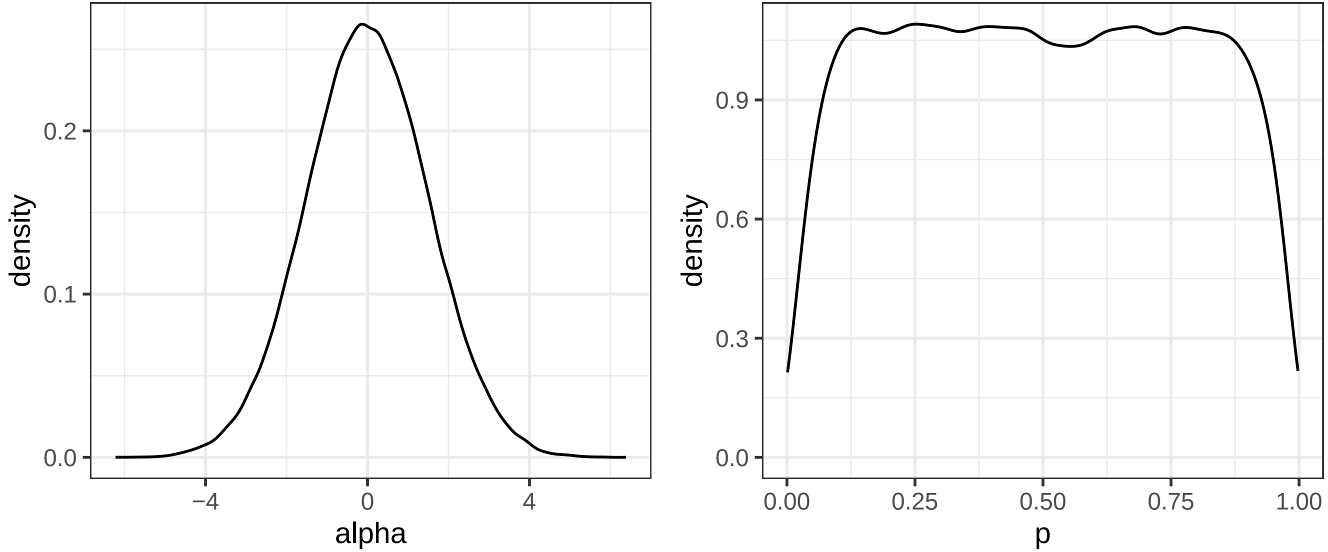


Example of logistic regression: Does set size affect free recall?

$$\alpha \sim Normal(0, 1.5) \tag{15}$$

Example of logistic regression: Does set size affect free recall?

Prior for $\alpha \sim Normal(0, 1.5)$ in log-odds and in probability space.



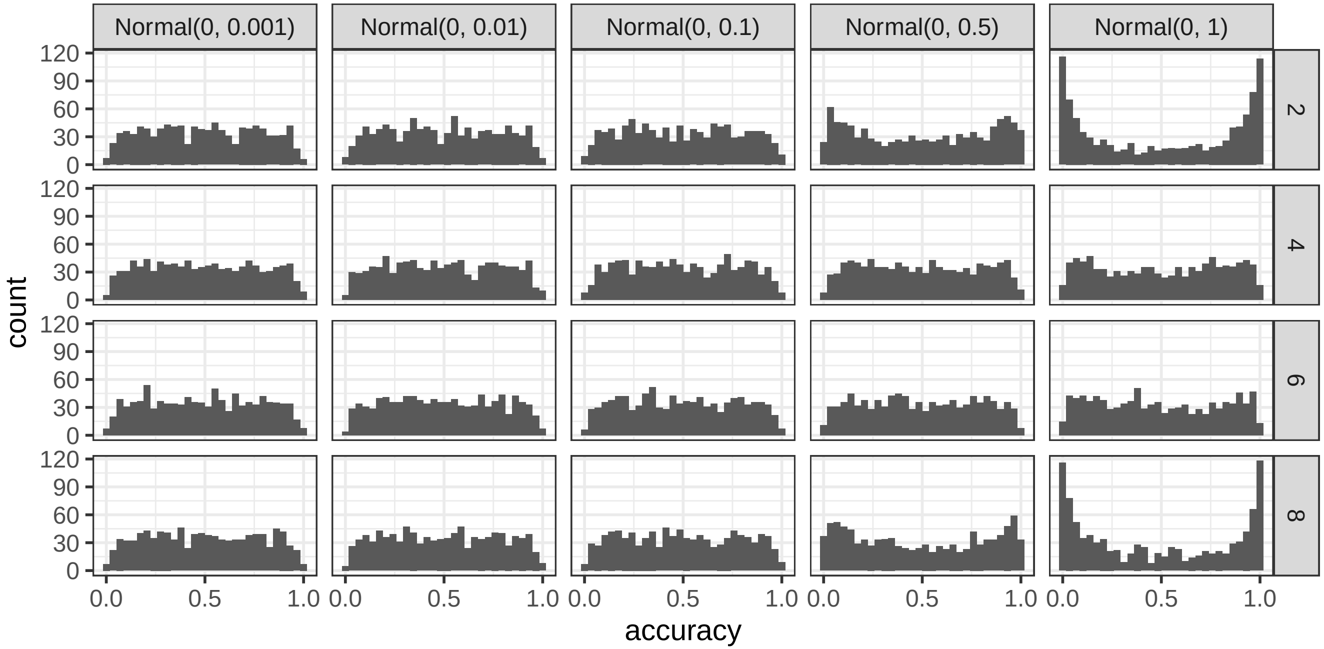
Example of logistic regression: Does set size affect free recall?

We can examine the consequences of each of the following prior specifications:

1. $\beta \sim Normal(0, 1)$
2. $\beta \sim Normal(0, .5)$
3. $\beta \sim Normal(0, .1)$
4. $\beta \sim Normal(0, .01)$
5. $\beta \sim Normal(0, .001)$

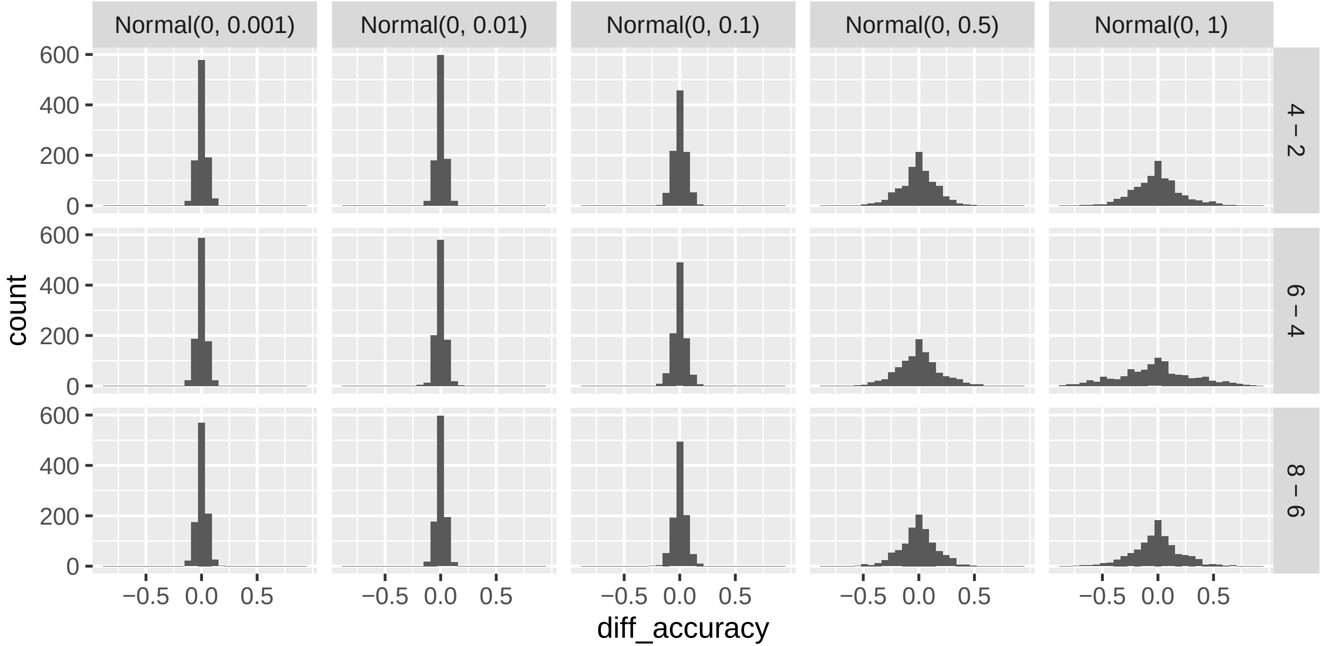
(The R code is in the textbook, chapter 4!)

Example of logistic regression: Does set size affect free recall?



Example of logistic regression: Does set size affect free recall?

It's usually more useful to look at the predicted differences in accuracy between set sizes.



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Example of logistic regression: Does set size affect free recall?

These priors seem reasonable:

$$\begin{aligned} \alpha &\sim Normal(0, 1.5) \\ \beta &\sim Normal(0, 0.1) \end{aligned} \tag{16}$$

The next steps

Next: fit the model and examine the posterior distributions of the parameters.

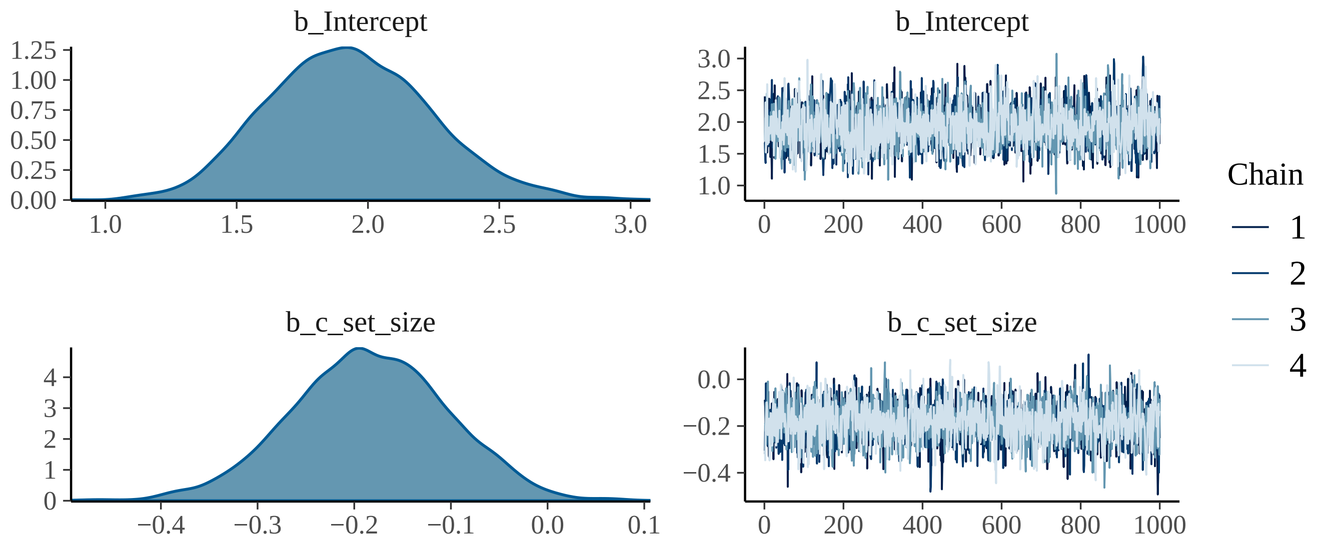
Example of logistic regression: Does set size affect free recall?

```
fit_recall <- brm(correct ~ 1 + c_set_size,
  data = df_recall,
  family = bernoulli(link = logit),
  prior = c(
    prior(normal(0, 1.5), class = Intercept),
    prior(normal(0, .1), class = b, coef = c_set_size)
  )
)

posterior_summary(fit_recall,
                  variable = c("b_Intercept", "b_c_set_size"))
```

Example of logistic regression: Does set size affect free recall?

```
plot(fit_recall)
```



Example of logistic regression: Does set size affect free recall?

```
alpha_samples <- as_draws_df(fit_recall)$b_Intercept
beta_samples <- as_draws_df(fit_recall)$b_c_set_size
beta_mean <- round(mean(beta_samples), 5)
beta_low <- round(quantile(beta_samples, prob = 0.025), 5)
beta_high <- round(quantile(beta_samples, prob = 0.975), 5)

alpha_samples <- as_draws_df(fit_recall)$b_Intercept
av_accuracy <- plogis(alpha_samples)
c(mean = mean(av_accuracy), quantile(av_accuracy, c(0.025, 0.975)))
```

```
##  mean  2.5% 97.5%
##  0.87  0.80  0.93
```

Example of logistic regression: Does set size affect free recall?

Find out the decrease in accuracy in proportions or probability scale:

```
beta_samples <- as_draws_df(fit_recall)$b_c_set_size
effect_middle <- plogis(alpha_samples) -
  plogis(alpha_samples - beta_samples)
c(mean = mean(effect_middle),
  quantile(effect_middle, c(0.025, 0.975)))
```

```
##    mean    2.5%   97.5%
## -0.0188 -0.0370 -0.0033
```

Example of logistic regression: Does set size affect free recall?

```
four <- 4 - mean(df_recall$set_size)
two <- 2 - mean(df_recall$set_size)
effect_4m2 <-
  plogis(alpha_samples + four * beta_samples) -
  plogis(alpha_samples + two * beta_samples)
c(mean = mean(effect_4m2),
  quantile(effect_4m2, c(0.025, 0.975)))
```

```
##    mean    2.5%   97.5%
## -0.0294 -0.0535 -0.0063
```

The next steps

Next: Hierarchical models.