Computational Statistics & Data Analysis 18 (1994) 485–498
North-Holland

Counting rules for identification in linear structural models *

Paul A. Bekker
Department of Economics, Groningen University, Groningen, The Netherlands

Received March 1991
Revised June 1993

Abstract: The paper gives 24 counting rules or order conditions that are necessary for the local identification of the parameters of linear structural models.

Keywords: Linear structural models; Identification

1. Introduction

General linear structural models, such as the one given by Jöreskog and Sörbom (1979), describe the covariance matrix of an observable random vector as a function of a number of linearly restricted parameter matrices. We say that the model is locally not identifiable if for almost all parameter values there is another parameter value, arbitrarily close to the first one, that generates the same covariance matrix. In that case estimation of the parameters becomes meaningless. Unfortunately, the characterization of the identification situation is a problem that is hard to solve for the general model. Verifiable identifiability conditions, that allow for a computerized evaluation, have only been given for special cases such as factor models (e.g. Bekker, 1989) and simultaneous equations models (e.g. Fisher, 1966). For the general model the identifiability conditions become very complicated, and in practice we have only one counting rule. The number of parameters should not exceed the number of equations as provided by the parametrization of the covariance matrix. However, it is not clear how the parameters should be distributed over the different parameter matrices.

Here we give 24 counting rules that do give an indication of how the parameters should be distributed over the system. The rules are order condi-

* This research has been made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences.
Correspondence to: P.A. Bekker, Department of Economics, Groningen University, P.O. Box 800, 9700 AV Groningen, The Netherlands.

SSDI 0167-9473(93)E0035-3

tions related to the rank condition on the Jacobian matrix. They provide only necessary conditions for identifiability. On the other hand, the conditions are very easy to evaluate, and they provide a practical check-list for model construction. In case of an identified model it may tell the researcher which parameter restrictions cannot be relaxed; and in case of an underidentified model it may tell the researcher where additional restrictions should be applied.

The model is described in the next section. The counting rules are given in Tables 1 and 2 in Section 3, and the derivation is given in Sections 4 and 5.

2. The model

Consider the following model:

$$\begin{aligned} \eta' B &= \xi' \Gamma + \zeta', \\ y' &= \eta' \Lambda_y + \epsilon', \\ x' &= \xi' \Lambda_x + \delta'. \end{aligned} \tag{1}$$

The matrix B is nonsingular, so that

$$\eta' = (\zeta', \xi') \begin{bmatrix} B^{-1} \\ \Gamma B^{-1} \end{bmatrix}.$$

Consequently, the observable vector (y', x') can be written as

$$(y', x') = (\zeta', \xi') \begin{bmatrix} B^{-1} & 0 \\ \Gamma B^{-1} & I_n \end{bmatrix} \begin{bmatrix} \Lambda_y & 0 \\ 0 & \Lambda_x \end{bmatrix} + (\epsilon', \delta'). \tag{2}$$

The matrices are of order:

$$\begin{cases} B : m \times m, \\ \Gamma : n \times m, \\ \Lambda_y : m \times p \\ \Lambda_x : n \times q. \end{cases}$$

It is assumed that the random vector $(\zeta', \xi', \epsilon', \delta')$ has zero expectation and covariance matrix (with order):

$$\Phi = \begin{array}{c} \begin{bmatrix} \Phi_\zeta & 0 & 0 & 0 \\ 0 & \Phi_\xi & 0 & 0 \\ 0 & 0 & \Phi_\epsilon & 0 \\ 0 & 0 & 0 & \Phi_\delta \end{bmatrix} \end{array} \begin{array}{l} \} \, m \\ \} \, n \\ \} \, p \\ \} \, q \end{array}. \tag{3}$$

For the covariance matrix, $\Sigma_{(y', x')}$, of (y', x') we thus find:

$$\begin{aligned} \Sigma_{(y',x')} = {} & \begin{bmatrix} \Lambda_y & 0 \\ 0 & \Lambda_x \end{bmatrix}' \begin{bmatrix} B & 0 \\ -\Gamma & I_n \end{bmatrix}'^{-1} \begin{bmatrix} \Phi_\zeta & 0 \\ 0 & \Phi_\xi \end{bmatrix} \begin{bmatrix} B & 0 \\ -\Gamma & I_n \end{bmatrix}^{-1} \begin{bmatrix} \Lambda_y & 0 \\ 0 & \Lambda_x \end{bmatrix} \\ & + \begin{bmatrix} \Phi_\epsilon & 0 \\ 0 & \Phi_\delta \end{bmatrix}. \end{aligned} \tag{4}$$

The parameter matrices are subject to a set of a priori linear restrictions:

$$R'\begin{bmatrix} \mathrm{vec}(B) \\ \mathrm{vec}(\Gamma) \\ \mathrm{vec}(\Lambda_y) \\ \mathrm{vec}(\Lambda_x) \\ \mathrm{vec}(\Phi_\zeta) \\ \mathrm{vec}(\Phi_\xi) \\ \mathrm{vec}(\Phi_\epsilon) \\ \mathrm{vec}(\Phi_\delta) \end{bmatrix} = R'\psi = c; \qquad \begin{cases} \Phi_\zeta = \Phi'_\zeta, \\ \Phi_\xi = \Phi'_\xi, \\ \Phi_\epsilon = \Phi'_\epsilon, \\ \Phi_\delta = \Phi'_\delta, \end{cases} \tag{5}$$

where the parameter vector ψ is a vectorization of the parameter matrices; the vector c is fixed, just as the matrix R', which is assumed to be of full row rank. This rank corresponds to the number of restriction on ψ. The restrictions are assumed to be operating on the diagonal and sub-diagonal elements of the symmetrical matrices, so, for example, Φ_ζ is subject to at most $\frac{1}{2}m(m+1)$ restrictions in addition to the symmetry restrictions $\Phi_\zeta = \Phi'_\zeta$.

In practice the restrictions on ψ can often be formulated in terms of restrictions on subvectors of ψ. If, after a suitable rearrangement, the restrictions on $\psi = (\psi'_1, \psi'_2)'$ can be formulated as

$$R'\psi = \begin{bmatrix} R'_1 & 0 \\ 0 & R'_2 \end{bmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} R'_1\psi_1 \\ R'_2\psi_2 \end{pmatrix} = c, \tag{6}$$

then we say that the restrictions on ψ can be *separated* into restrictions on ψ_1 and ψ_2. If separate elements of ψ are fixed, the restrictions on ψ can be separated into restrictions on separate elements of ψ.

3. Order conditions

We say that a parameter vector ψ^0, satisfying the restrictions (5), is *locally no identifiable* it there are other parameter vectors arbitrarily close to ψ^0, that satisfy (5) and generate the same covariance matrix $\Sigma_{(y',x')}$. Here we give necessary conditions for the local uniqueness of parameter vectors ψ. That is, if one of the conditions given below is not satisfied, then almost all points satisfying (5) will not be locally identifiable. The conditions are necessary; they are not sufficient for the identifiability of almost all parameter points.

The conditions are given in Tables 1 and 2. For each condition separately it is assumed that the restrictions on the parameters considered in the condition are separable from the remaining restrictions on ψ. With respect to the conditions in Table 2 we make some additional separability-assumptions, as indicated at the bottom of the table. If the restrictions fix separate elements of the parameter matrices, then all separability-assumptions are satisfied.

Table 1
Necessary conditions for identifiability

Cond	Parameters	Min. # restrictions	Max. # free pars.
1	l rows of Γ	$l(m-p)$	lp
2	l rows of (B, Λ_y)	lm	lp
3	l rows of $(B', \Gamma', \Phi_\zeta)$	lm	$l(m+n-\frac{1}{2}l+\frac{1}{2})$
4	l rows of $(\Lambda'_y, \Phi_\epsilon)$	$l\{m-\min(n, q)\}$	$l\{p+\min(n, q)-\frac{1}{2}l+\frac{1}{2}\}$
5	l rows of $(\Lambda'_x, \Phi_\delta)$	$l\{n-\min(m, p)\}$	$l\{q+\min(m, p)-\frac{1}{2}l+\frac{1}{2}\}$
6	$\Phi_\zeta, \Phi_\epsilon$	$\frac{1}{2}m(m+1)$	$\frac{1}{2}p(p+1)$
7	B, Γ, Φ_ζ	$m^2+mn-\min(mn, pn, mq, pq)$	$\frac{1}{2}m(m+1)+\min(mn, mq, pn, pq)$
8	$B, \Gamma, \Phi_\zeta, \Phi_\epsilon$	$\frac{1}{2}m(m+1)+m^2+mn-\min(mn, mq)$	$\frac{1}{2}p(p+1)+\min(mn, mq)$
9	$B, \Gamma, \Lambda_y, \Phi_\zeta, \Phi_\epsilon$	$\frac{1}{2}m(m+1)+m^2+mn+mp-\min(pn, pq)$	$\frac{1}{2}p(p+1)+\min(pn, pq)$
10	$B, \Gamma, \Phi_\zeta, \Phi_\xi, \Phi_\delta$	$\frac{1}{2}n(n+1)+m^2+mn-\min(mn, pn)$	$\frac{1}{2}m(m+1)+\frac{1}{2}q(q+1)+\min(mn, pn)$
11	$B, \Gamma, \Lambda_x, \Phi_\zeta, \Phi_\xi, \Phi_\delta$	$\frac{1}{2}n(n+1)+m^2+mn+nq-\min(mq, pq)$	$\frac{1}{2}m(m+1)+\frac{1}{2}q(q+1)+\min(mq, pq)$
12	$B, \Gamma, \Phi_\zeta, \Phi_\xi, \Phi_\epsilon, \Phi_\delta$	$\frac{1}{2}m(m+1)+\frac{1}{2}n(n+1)+m^2$	$\frac{1}{2}p(p+1)+\frac{1}{2}q(q+1)+mn$
13	$B, \Gamma, \Lambda_y, \Phi_\zeta, \Phi_\xi, \Phi_\epsilon, \Phi_\delta$	$\frac{1}{2}m(m+1)+\frac{1}{2}n(n+1)+m^2+mn+mp-pn$	$\frac{1}{2}p(p+1)+\frac{1}{2}q(q+1)+pn$
14	$B, \Gamma, \Lambda_x, \Phi_\zeta, \Phi_\xi, \Phi_\epsilon, \Phi_\delta$	$\frac{1}{2}m(m+1)+\frac{1}{2}n(n+1)+m^2+mn+nq-mq$	$\frac{1}{2}p(p+1)+\frac{1}{2}q(q+1)+mq$
15	$B, \Gamma, \Lambda_y, \Lambda_x, \Phi_\zeta, \Phi_\xi, \Phi_\epsilon, \Phi_\delta$	$\frac{1}{2}m(m+1)+\frac{1}{2}n(n+1)+m^2+mn+mp+nq-pq$	$\frac{1}{2}p(p+1)+\frac{1}{2}q(q+1)+pq$

It is assumed that the restrictions on the parameters considered, in each condition separately, are separable from the remaining restrictions.

The conditions give the minimal number of restrictions that should be operating on the elements of the parameter matrices, or vectors, under consideration. They also give the difference between the number of elements in the matrices, or vectors (omitting the elements above the diagonal of symmetrical matrices) and this minimal number of restrictions. The latter quantity can be considered as a maximum for the number of free parameters that can be present in the matrices, or vectors. The method to derive the conditions, which is described in the next section, can also be used to formulate additional conditions. However, these additional conditions are implied by the conditions 1–24, on the assumption that the restrictions can be separated into restrictions on the separate parameter matrices. For example, it is possible to derive the necessary condition for identifiability that says that the parameter matrices B, Γ, Φ_ζ and Φ_ϵ should be subject to at least $\frac{1}{2}m(m+1)+m^2+mn-pn$ restrictions. This means that there can be at most $m^2+mn+\frac{1}{2}m(m+1)+\frac{1}{2}p(p+1)-\{\frac{1}{2}m(m+1)+m^2+mn-pn\}=\frac{1}{2}p(p+1)+pn$ free parameters in these matrices. This condition is clearly implied by condition 9 in Table 1.

Condition 15 considers all parameters, and the maximum for the number of free parameters mentioned in this condition corresponds to the number of equations in (4). The difference between the number of parameters used and this maximum is equal to the degrees of freedom of the chi-square used in the evaluation of the fit of the model. It is obvious that one should not use too many parameters. The problem of how these parameters should be distributed over the system is not that clear. However, Table 1 and 2 provide a practical set of rules that should be taken care of.

4. The derivation of Table 1

Conditions 1 and 2

It follows from the equations in (4) that two parameter points, ψ, that differ only with respect to the elements in B, Γ and Λ_y, will generate the same matrix $\Sigma_{(y',x')}$, if they generate the same matrix

$$\begin{bmatrix} B & 0 \\ -\Gamma & I_n \end{bmatrix}^{-1} \begin{bmatrix} \Lambda_y \\ 0 \end{bmatrix} = \Pi, \tag{7}$$

say. Let B^0, Γ^0 and Λ_y^0 provide a solution to this equation, then any other solution can be written as

$$\begin{bmatrix} B & 0 & \Lambda_y \\ -\Gamma & I_n & 0 \end{bmatrix} = T \begin{bmatrix} B^0 & 0 & \Lambda_y^0 \\ -\Gamma^0 & I_n & 0 \end{bmatrix}; \tag{8}$$

and a necessary and sufficient condition for identifiability is that $T = I_{m+n}$. This identification problem is known from oblique factor analysis and also from classical systems of simultaneous equations in econometrics. The solution can be formulated (cf. Hsiao, 1983) in terms of a rank condition. Let the restrictions in (5), as far as they are operating on the elements of (B, Λ_y) and Γ be given by:

$$R'_{(B,\Lambda_y)'} \operatorname{vec}\begin{pmatrix} B' \\ \Lambda'_y \end{pmatrix} = c_{(B,\Lambda_y)'}, \tag{9}$$

and

$$R'_{\Gamma'} \operatorname{vec}(\Gamma') = c_{\Gamma'}. \tag{10}$$

Then a necessary and sufficient condition for $T = I_m$ is given by:

$$\operatorname{rank}\left\{ R'_{(B,\Lambda_y)'} \left(I_m \otimes \begin{pmatrix} B' \\ \Lambda'_y \end{pmatrix} \right) \right\} = m^2, \tag{11}$$

and

$$\operatorname{rank}\left\{ \begin{bmatrix} R'_{\Gamma'}(I_n \otimes B') \\ I_n \otimes \Lambda' \end{bmatrix} \right\} = mn, \tag{12}$$

where $\otimes$ is the Kronecker product.

Table 2
Necessary conditions for identifiability under additional assumptions

Cond.	Parameters	Min. # restrictions	Max. # free pars.
16	l rows of Λ'_y, if $l > p - m > 0$	$\frac{1}{2}(l-p+m)(l-p+m-1) - l\{\min(n, q)\}$	$l\{m + \min(n, q)\} - \frac{1}{2}(l-p+m)(l-p+m-1)$
17	$B, \Gamma, \Lambda_y, \Phi_\zeta$, if $p > m$	$2m^2 + mn - \min(pn, pq)$	$\frac{1}{2}m(m+1) + m(p-m) + \min(pn, pq)$
18	$B, \Gamma, \Lambda_y, \Phi_\zeta, \Phi_\xi, \Phi_\delta$, if $p > m$	$2m^2 + mn - pn + \frac{1}{2}n(n+1)$	$\frac{1}{2}m(m+1) + m(p-m) + \frac{1}{2}q(q+1) + pn$
19	$B, \Gamma, \Lambda_y, \Lambda_x, \Phi_\zeta, \Phi_\xi, \Phi_\delta$, if $p > m$	$2m^2 + mn + \frac{1}{2}n(n+1) + nq - pq$	$\frac{1}{2}m(m+1) + m(p-m) + \frac{1}{2}q(q+1) + pq$
20	l rows of Λ'_x, if $l > q - n > 0$	$\frac{1}{2}(l-q+n)(l-q+n-1) - l\{\min(m, p)\}$	$l\{n + \min(m, p)\} - \frac{1}{2}(l-q+n)(l-q+n-1)$
21	$B, \Gamma, \Lambda_x, \Phi_\zeta, \Phi_\xi$, if $q > n$	$m^2 + n^2 + mn - \min(pn, pq)$	$\frac{1}{2}n(n+1) + n(q-n) + \frac{1}{2}m(m+1) + \min(mq, pq)$
22	$B, \Gamma, \Lambda_x, \Phi_\zeta, \Phi_\xi, \Phi_\epsilon$, if $q > n$	$\frac{1}{2}m(m+1) + m^2 + n^2 + mn - mq$	$\frac{1}{2}n(n+1) + n(q-n) + \frac{1}{2}p(p+1) + mq$
23	$B, \Gamma, \Lambda_y, \Lambda_x, \Phi_\zeta, \Phi_\xi, \Phi_\epsilon$, if $q > n$	$\frac{1}{2}m(m+1) + m^2 + n^2 + mn + mp - pq$	$\frac{1}{2}n(n+1) + n(q-n) + \frac{1}{2}p(p+1) + pq$
24	$B, \Gamma, \Lambda_y, \Lambda_x, \Phi_\zeta, \Lambda_\xi$, if $p > m$ & $q > n$	$2m^2 + n^2 + mn - pq$	$\frac{1}{2}m(m+1) + m(p-m) + \frac{1}{2}n(n+1) + n(q-n) + pq$

It is assumed that the restrictions on the parameters considered, in each condition separately, are separable from the remaining restrictions. Additional assumptions apply to subsets of the conditions 16–24.

W.r.t. conditions 17–19, it is assumed that the restrictions on Φ_ϵ are separable form the remaining restrictions; for condition 16 the restrictions on the relevant l rows of Φ_ϵ should be separable.

W.r.t. conditions 21–23, it is assumed that the restrictions on Φ_δ are separable form the remaining restrictions; for condition 20 the restrictions on the relevant l rows of Φ_δ should be separable.

W.r.t. conditions 24, the restrictions on Φ_ϵ and Φ_δ, should both be separable form the remaining restrictions.

If the restrictions (9) can be separated into restrictions on separate rows of (B, Λ_y), i.e. $e_i'(B, \Lambda_y)$, $i = 1, \ldots, m$:

$$R'_{(B,\Lambda_y)'e_i}\begin{pmatrix} B' \\ \Lambda'_y \end{pmatrix} e_i = c_{(B,\Lambda_y)'e_i}, \qquad i = 1, \ldots, m, \tag{13}$$

then rank condition (11) can be separated into m rank conditions:

$$\operatorname{rank}\left\{ R'_{(B,\Lambda_y)'e_i}\begin{pmatrix} B' \\ \Lambda'_y \end{pmatrix} \right\} = m, \qquad i = 1, \ldots, m. \tag{14}$$

The corresponding order conditions say that each row of (B, Λ_y) should be subject to at least m restrictions, which provides condition 2.

Similarly, if the restrictions (10) can be separated into restrictions on separate rows of Γ, i.e. $e_j'\Gamma$, $j = 1, \ldots, n$:

$$R'_{\Gamma' e_j}(\Gamma' e_j) = c_{\Gamma' e_j}, \qquad j = 1, \ldots, n, \tag{15}$$

then the rank condition (12) can be separated into n rank conditions:

$$\operatorname{rank}\left\{\begin{bmatrix} R'_{\Gamma' e_j}(B') \\ \Lambda'_y \end{bmatrix}\right\} = m, \qquad j = 1, \ldots, n. \tag{16}$$

The corresponding order conditions provide condition 1.

Conditions 6–15

We start by rewriting (1) as

$$(y', x', \eta', \xi')\begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & I_q & 0 & 0 \\ -\Lambda_y & 0 & B & 0 \\ 0 & -\Lambda_x & -\Gamma & I_n \end{bmatrix} = (\epsilon', \delta', \zeta', \xi'), \tag{17}$$

which is a formulation similar to the one given by Wegge (1991). Let Σ be the covariance matrix of (y', x', η', ξ'), then

$$\begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & I_q & 0 & 0 \\ -\Lambda_y & 0 & B & 0 \\ 0 & -\Lambda_x & -\Gamma & I_n \end{bmatrix}' \Sigma \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & I_q & 0 & 0 \\ -\Lambda_y & 0 & B & 0 \\ 0 & -\Lambda_x & -\Gamma & I_n \end{bmatrix}$$
$$= \begin{bmatrix} \Phi_\epsilon & 0 & 0 & 0 \\ 0 & \Phi_\delta & 0 & 0 \\ 0 & 0 & \Phi_\zeta & 0 \\ 0 & 0 & 0 & \Phi_\xi \end{bmatrix}. \tag{18}$$

The covariance matrix Σ can be written as:

$$\Sigma = \begin{bmatrix} \Sigma_{(y',x')} & \begin{bmatrix} \Lambda_y & 0 \\ 0 & \Lambda_x \end{bmatrix}' \Sigma_{(\eta',\xi')} \\ \Sigma_{(\eta',\xi')} \begin{bmatrix} \Lambda_y & 0 \\ 0 & \Lambda_x \end{bmatrix} & \Sigma_{(\eta',\xi')} \end{bmatrix}, \tag{19}$$

with

$$\Sigma_{(\eta',\xi')} = \begin{bmatrix} \Phi_\eta & \Delta' \\ \Delta & \Phi_\xi \end{bmatrix}, \tag{20}$$

where Φ_η and Δ are defined implicitly.

The equations (18) are equations in the unknown elements of B, Γ, Λ_y, Λ_x, Φ_ζ, Φ_ξ, Φ_ϵ, Φ_δ, Δ and Φ_η, where also Φ_η is a symmetrical matrix. For solutions of (18), where $\Sigma_{(y',x')}$ is fixed, there exists a one-to-one relation between the elements of B, Γ, Λ_y, Λ_x, Φ_ζ, Φ_ξ, Φ_ϵ, Φ_δ, and the elements of B, Γ, Λ_y, Λ_x, Φ_ξ, Δ and Φ_η, since the matrix

$$\begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & I_q & 0 & 0 \\ -\Lambda_y & 0 & B & 0 \\ 0 & -\Lambda_x & -\Gamma & I_n \end{bmatrix}$$

is nonsingular. Therefore we may as well consider the identifiability of the parameters in the latter set.

On the assumption that the restrictions on the parameter vector ψ can be separated into restrictions on the separate parameter matrices we may write, without loss of generality:

$$R' = \begin{bmatrix} R'_{(B',\Gamma')'} & & & & & & 0 \\ & R'_{\Lambda_y} & & & & & \\ & & R'_{\Lambda_x} & & & & \\ & & & R'_{\Phi_\zeta} & & & \\ & & & & R'_{\Phi_\xi} & & \\ 0 & & & & & R'_{\Phi_\epsilon} & \\ & & & & & & R'_{\Phi_\delta} \end{bmatrix}; \quad c = \begin{pmatrix} c_{(B',\Gamma')'} \\ c_{\Lambda_y} \\ c_{\Lambda_x} \\ c_{\Phi_\zeta} \\ c_{\Phi_\xi} \\ c_{\Phi_\epsilon} \\ c_{\Phi_\delta} \end{pmatrix}. \tag{21}$$

The equations in (18) and the restrictions in (5) can now be given, in Table 3, in the form of nine sets of equations in terms of the parameters of the matrices B, Γ, Λ_y, Λ_x, Φ_ξ, Δ and Φ_η. Table 3 also gives the number and the maximum number of equations per set.

Table 4 indicates which parameters appear in which equations. That is, Table 4 shows the structure of the Jacobian matrix if we differentiate the left-hand side of the equations in Table 3 with respect to the elements in (B', Γ'), Φ_ξ, Δ, Φ_η, Λ_y and Λ_x, respectively.

Order conditions for identifiability can now be found by considering subsets of matrices from $\{(B', \Gamma'), \Phi_\xi, \Delta, \Phi_\eta, \Lambda_y, \Lambda_x\}$. The number of parameters in these matrices should not be larger than the number of equations in which they appear. For example, there are pm parameters in Λ_y; these parameters appear in the equation sets 5, 7 and 8. So, we find that the number of restrictions on Λ_y and Φ_ϵ should be at least equal to $pm - pq$; and the number of free parameters in Λ_y and Φ_ϵ should not exceed $pm + \frac{1}{2}p(p+1) - (pm - pq) = \frac{1}{2}p(p+1) + pq$.

Table 3
Nine sets of identifying equations

Set	Equations	# eqtns	Max. # eqtns.
1	$R'_{(B',\Gamma')}\ \text{vec}\begin{bmatrix} B \\ \Gamma \end{bmatrix} = c_{(B',\Gamma')}$	$r_{(B',\Gamma')}$	$m^2 + mn$
2	$R'_{\Phi_\xi}\text{vec}(\Phi_\xi) = c_{\Phi_\xi}$	r_{Φ_ξ}	$\frac{1}{2}n(n+1)$
3	$\text{vec}\left\{(\Delta, \Phi_\xi)\begin{bmatrix} B \\ -\Gamma \end{bmatrix}\right\} = 0$	mn	mn
4	$R'_{\Phi_\zeta}\text{vec}\left\{(B', -\Gamma')\begin{bmatrix} \Phi_\eta & 0 \\ 0 & -\Phi_\xi \end{bmatrix}\begin{bmatrix} B \\ -\Gamma \end{bmatrix}\right\} = c_{\Phi_\zeta}$	r_{Φ_ζ}	$\frac{1}{2}m(m+1)$
5	$R'_{\Lambda_y}\text{vec}(\Lambda_y) = c_{\Lambda_y}$	r_{Λ_y}	mp
6	$R'_{\Lambda_x}\text{vec}(\Lambda_x) = c_{\Lambda_x}$	r_{Λ_x}	nq
7	$\text{vec}\{\Lambda'_x \Delta \Lambda_y - E(xy')\} = 0$	pq	pq
8	$R_{\Phi_\epsilon}\text{vec}\{\Lambda'_y \Phi_\eta \Lambda_y - \text{E}(yy')\} = c_{\Phi_\epsilon}$	r_{Φ_ϵ}	$\frac{1}{2}p(p+1)$
9	$R'_{\Phi_\delta}\text{vec}\{\Lambda'_x \Phi_\xi \Lambda_x - E(xx')\} = c_{\Phi_\delta}$	r_{Φ_δ}	$\frac{1}{2}q(q+1)$

However, we can find stronger conditions. Table 3 and Table 4 have been constructed so that each diagonal block of Table 4 has a number of rows that is not larger than its number of columns. Consequently, we may find an even stronger condition by considering, in addition to the parameters in Λ_y, also the parameters in (B', Γ') and Δ and Φ_η. That way we find that the number of restrictions on B, Γ, Λ_y, Φ_ζ and Φ_ϵ should be at least equal to $\frac{1}{2}m(m+1) + m^2 + mp + mn - pq$; or, alternatively, the number of free parameters in these matrices should not exceeed $\frac{1}{2}p(p+1) + pq$.

Condition 9 in Table 1 is even stronger. It also requires that the number of free parameters in the matrices B, Γ, Λ_y, Φ_ζ and Φ_ϵ does not exceed $\frac{1}{2}p(p+1) + pn$. This latter condition cannot be derived from Table 4 directly. In

Table 4
The Structure of the Jacobian matrix

Set	Parameters						# Eqtns.	Max. # eqtns.
	(B', Γ')	Φ_ξ	Δ	Φ_η	Λ_y	Λ_x		
1	....						$r_{(B',\Gamma')}$	$m^2 + mn$
2		......					r_{Φ_ξ}	$\frac{1}{2}n(n+1)$
3	....	......	...				mn	mn
4	....	......		......			r_{Φ_ζ}	$\frac{1}{2}m(m+1)$
5					....		r_{Λ_y}	mp
6						....	r_{Λ_x}	nq
7			...		....	....	pq	pq
8				......	....		r_{Φ_ϵ}	$\frac{1}{2}p(p+1)$
9		......				....	r_{Φ_δ}	$\frac{1}{2}q(q+1)$
# Pars	$m^2 + mn$	$\frac{1}{2}n(n+1)$	mn	$\frac{1}{2}m(m+1)$	pm	qn		

order to derive the conditions of Table 1, we have to consider in more detail the matrices entering the Jacobian matrix in the blocks indicated in Table 4.

A necessary and sufficient condition for the local identifiability of almost all parameter points is that the Jacobian matrix is of full column rank for regular points. In this derivation of order conditions we use only limited information, which consists of the order of the separate parameter matrices and the number of restrictions on these matrices; without considering an exact specification of the restrictions. Under this information the maximum rank of the separate parameter matrices is determined by their number of rows and columns. This holds as well for most of the matrices entering the Jacobian matrix, in the blocks indicated in Table 4. However, the blocks corresponding to equation set 7 form an exception. It may happen that these latter matrices are not of full (row or column) rank due to the order of the parameter matrices.

In more detail, if we differentiate $\operatorname{vec}\{\Lambda'_x \Delta \Lambda_y - E(xy')\}$ with respect to the parameters in $\operatorname{vec}(\Delta)$, $\operatorname{vec}(\Lambda_y)$ and $\operatorname{vec}(\Lambda'_x)$, we find the blocks corresponding to set 7:

$$\left(\Lambda'_y \otimes \Lambda'_x,\ I_p \otimes \Lambda'_x \Delta,\ \Lambda'_y \Delta' \otimes I_q\right). \tag{22}$$

Due to the order of the matrices $\Lambda_y(m \times p)$, $\Lambda_x(n \times q)$ and $\Delta(n \times m)$, we find

$$\begin{aligned}
&\operatorname{rank}\left(\Lambda'_y \otimes \Lambda'_x\right) \leqslant \min(pq,\ pn,\ mq,\ mn),\\
&\operatorname{rank}\left(\Lambda'_y \otimes \Lambda'_x,\ I_p \otimes \Lambda'_x \Delta\right) \leqslant \min(pq,\ pn),\\
&\operatorname{rank}\left(\Lambda'_y \otimes \Lambda'_x,\ \Lambda'_y \Delta' \otimes I_q\right) \leqslant \min(pq,\ mq),\\
&\operatorname{rank}\left(I_p \otimes \Lambda'_x \Delta\right) \leqslant \min(pq,\ pn,\ pm),\\
&\operatorname{rank}\left(\Lambda'_y \Delta' \otimes I_q\right) \leqslant \min(pq,\ mq,\ nq),
\end{aligned} \tag{23}$$

So if we consider the parameters in (B', Γ'), Δ, Φ_η and Λ_y, which appear in sets 1, 3, 4, 5, 7 and 8, then the pq equations in set 7 should be counted as $\min(pq, pn)$ independent equations in the elements of Δ and Λ_y. Thus we find that the number of restrictions on B, Γ, Λ_y, Φ_ζ and Φ_ϵ should be at least equal to $\frac{1}{2}m(m+1) + m^2 + mp + mn - \min(pq, pn)$; or, alternatively, the number of free parameters in these matrices should not exceed $\frac{1}{2}p(p+1) + \min(pq, pn)$, which is condition 9 in Table 1.

The following lines give, for each of the conditions 6–15, the set of considered parameter matrices and the resulting order condition.

Condition 6
ϕ_η: $r_{\Phi_\zeta} + r_{\Phi_\epsilon} \geqslant \frac{1}{2}m(m+1)$.
Condition 7
(B', Γ') and Δ: $r_{(B', \Gamma')} + mn + r_{\Phi_\zeta} + \min(pq, pn, mq, mn) \geqslant m^2 + 2mn$.
Condition 8
(B', Γ'), Δ and Φ_η: $r_{(B', \Gamma')} + mn + r_{\Phi_\zeta} + \min(pq, pn, mq, mn) + r_{\Phi_\epsilon} \geqslant m^2 + 2mn + \frac{1}{2}m(m+1)$.

As $r_{\Lambda_y} \leqslant mp$, condition 9 implies $r_{(B',\Gamma')} + r_{\Phi_\zeta} + r_{\Phi_\epsilon} \geqslant m^2 + mn + \frac{1}{2}m(m+1) - \min(pq, pn)$. So only $r_{(B',\Gamma')} + r_{\Phi_\zeta} + r_{\Phi_\epsilon} \geqslant m^2 + mn + \frac{1}{2}m(m+1) - \min(mq, mn)$ is necessary.

Condition 9

(B', Γ'), Δ, Φ_η and Λ_y: $r_{(B',\Gamma')} + mn + r_{\Phi_\zeta} + r_{\Lambda_y} + \min(pq, pn) + r_{\Phi_\epsilon} \geqslant m^2 + 2mn + \frac{1}{2}m(m+1) + mp$.

Condition 10

(B', Γ'), Φ_ξ and Δ: $r_{(B',\Gamma')} + r_{\Phi\xi} + mn + r_{\Phi_\zeta} + \min(pq, pn, mq, mn) + r_{\Phi_\delta} \geqslant m^2 + 2mn + \frac{1}{2}n(n+1)$.

As $r_{\Lambda_x} \leqslant nq$, it follows from condition 11 that only $r_{(B',\Gamma')} + r_{\Phi_\xi} + r_{\Phi_\zeta} + r_{\Phi_\delta} \geqslant m^2 + mn + \frac{1}{2}n(n+1) - \min(mn, pn)$ is necessary.

Condition 11

(B', Γ'), Φ_ξ, Δ and Λ_x: $r_{(B',\Gamma')} + r_{\Phi_\xi} + mn + r_{\Phi_\zeta} + r_{\Lambda_x} + \min(pq, mq) + r_{\Phi_\delta} \geqslant m^2 + 2mn + \frac{1}{2}n(n+1) + nq$.

Condition 12

(B', Γ'), Φ_ξ and Φ_η: $r_{(B',\Gamma')} + r_{\Phi_\xi} + mn + r_{\Phi_\zeta} + r_{\Phi_\epsilon} + r_{\Phi_\delta} \geqslant m^2 + mn + \frac{1}{2}n(n+1) + \frac{1}{2}m(m+1)$.

Condition 13

(B', Γ'), Φ_ξ, Δ, Φ_η and Λ_y: $r_{(B',\Gamma')} + r_{\Phi_\xi} + mn + r_{\Phi_\zeta} + r_{\Lambda_y} + \min(pq, pn) + r_{\Phi_\epsilon} + r_{\Phi_\delta} \geqslant m^2 + 2mn + \frac{1}{2}n(n+1) + \frac{1}{2}m(m+1) + mp$.

It follows from condition 15 that only $r_{(B',\Gamma')} + r_{\Phi_\xi} + r_{\Phi_\zeta} + r_{\Lambda_y} + r_{\Phi_\epsilon} + r_{\Phi_\delta} \geqslant m^2 + mn + \frac{1}{2}n(n+1) + \frac{1}{2}m(m+1) + mp - pn$ is necessary.

Condition 14

(B', Γ'), Φ_ξ, Δ, Φ_η and Λ_x: $r_{(B',\Gamma')} + r_{\Phi_\xi} + mn + r_{\Phi_\zeta} + r_{\Lambda_x} + \min(pq, mq) + r_{\Phi_\epsilon} + r_{\Phi_\delta} \geqslant m^2 + 2mn + \frac{1}{2}n(n+1) + \frac{1}{2}m(m+1) + nq$.

It follows from condition 15 that only $r_{(B',\Gamma')} + r_{\Phi_\xi} + r_{\Phi_\zeta} + r_{\Lambda_x} + r_{\Phi_\epsilon} + r_{\Phi_\delta} \geqslant m^2 + mn + \frac{1}{2}n(n+1) + \frac{12}{2}m(m+1) + nq - mq$ is necessary.

Condition 15

(B', Γ'), Φ_ξ, Δ, Φ_η, Λ_y and Λ_x: $r_{(B',\Gamma')} + r_{\Phi_\xi} + mn + r_{\Phi_\zeta} + r_{\Lambda_y} + r_{\Lambda_x} + pq + r_{\Phi_\epsilon} + r_{\Phi_\delta} \geqslant m^2 + 2mn + \frac{1}{2}n(n+1) + \frac{1}{2}m(m+1) + mp + nq$.

All other conditions that can be derived in this way, by considering other sets of parameter matrices, are implied by the conditions 6–15.

Condition 3

If the restrictions on l rows of $(B', \Gamma', \Phi_\zeta)$ can be separated from the remaining restrictions, then the elements of the l rows of (B', Γ') appear only in a limited number of equations of the equation sets 1, 3 and 4 of Table 3. Let the l rows be given by $N'(B', \Gamma, \Phi_\zeta)$, where N consists of l different columns of I_m. The equations are then given, in an obvious notation, by:

$$\begin{aligned} &R'_{(B',\Gamma')'N}\operatorname{vec}\left\{\begin{pmatrix} B \\ \Gamma \end{pmatrix} N\right\} = c_{(B',\Gamma')'N}, \\ &(\Delta, \Phi_\xi)\begin{pmatrix} B \\ -\Gamma \end{pmatrix} N = 0, \\ &R'_{\Phi_\zeta N}\operatorname{vec}\left\{(B', -\Gamma')\begin{bmatrix} \Phi_\eta & 0 \\ 0 & -\Phi_\xi \end{bmatrix}\begin{bmatrix} B \\ -\Gamma \end{bmatrix} N\right\} = c_{\Phi_\zeta N}. \end{aligned} \tag{24}$$

The l rows of $N'(B', \Gamma')$ have $l(m+n)$ parameters; the number of equations is equal to the number of restrictions on $N'(B', \Gamma', \Phi_\zeta) + ln$. Thus we find a lower bound on the number of restrictions on l rows of $(B', \Gamma', \Phi_\zeta)$ as given in condition 3. This bound can also be formulated in terms of an upper bound on the number of free parameters by recognizing that $N'\Phi_\zeta$ has $lm - \frac{1}{2}l(l-1)$ parameters.

Conditions 4 and 5

In order to derive conditions 4 and 5, we use an approach similar to the one that has been used to derive condition 3. Let l rows of $(\Lambda'_y, \Phi_\epsilon)$ be given by $S'(\Lambda'_y, \Phi_\epsilon)$, where S consists of l different columns of I_p. If the restrictions on $S'(\Lambda'_y, \Phi_\epsilon)$ can be separated from the remaining restrictions, then the elements of $S'(\Lambda'_y)$ appear only in the following equations of Table 3, which are given in an obvious notation as:

$$\begin{aligned} &R'_{\Lambda_y S}\mathrm{vec}(\Lambda_y S) = c_{\Lambda_y S}, \\ &\Lambda'_x \Delta \Lambda_y S - E(xy')S = 0, \\ &R'_{\Phi_\epsilon S}\mathrm{vec}\{\Lambda'_y \Phi_\eta \Lambda_y S - E(yy')S\} = c_{\Phi_\epsilon S}. \end{aligned} \tag{25}$$

The l rows of $S'\Lambda'_y$ have lm parameters; the number of equations in (25) is equal to the number of restriction on $S'(\Lambda'_y, \Phi_\epsilon) + lq$. However, the number of independent equations may be less than this number.

Consider the equations in $\Lambda'_x \Delta \Lambda_y S - E(xy')S = 0$. Differentiating $\mathrm{vec}(\Lambda'_x \Delta \Lambda_y S)$ with respect to the elements of $\mathrm{vec}(\Lambda_y S)$ gives a Jacobian matrix of the form: $I_l \otimes \Lambda'_x \Delta$. The rank of this matrix is bounded not only by the number of its rows, lq, but also, as we have seen in (23), by the numbers ln and lm. Therefore, the number of independent equations in $\Lambda'_x \Delta \Lambda_y S - E(xy')S = 0$ is not larger than $l\{\min(m, n, q)\}$. Consequently, we find the condition that the number of restrictions on l rows of $(\Lambda'_y, \Phi_\epsilon)$ should not be less than $l\{m - \min(m, n, q)\}$, which implies condition 4. Condition 5 can be found in a similar way, by considering columns of Λ_x instead of Λ_y.

5. The derivation of Table 2

Conditions 16 and 20

These conditions are related to conditions 4 and 5, respectively. With respect to condition 16, we return to the equations (25) that were used to derive condition 4.

The last set of equations in (25), i.e.

$$R'_{\Phi_\epsilon S}\mathrm{vec}\{\Lambda'_y \Phi_\eta \Lambda_y S - E(yy')S\} = c_{\Phi_\epsilon S}, \tag{26}$$

corresponds to the restrictions on l rows of Φ_ϵ, which are assumed to be separable from the remaining restrictions. Let there be $r_{\Phi_\epsilon S}$ such restrictions, i.e. $\mathrm{rank}(R_{\Phi_\epsilon S}) = r_{\Phi_\epsilon S}$, then the number of independent equations in the unknown elements of $\Lambda_y S$ in (26) is not larger than $\min(r_{\Phi_\epsilon S}, d)$; where d is the dimension of the manifold described by $\mathrm{vec}(\Lambda'_y \Phi_\eta \Lambda_y S)$ as a function of the ml elements of $\Lambda_y S$.

If we write $Z = \Phi_\eta^{1/2} \Lambda_y \mathrm{S}$, while the remaining columns of $\Phi_\eta^{1/2} \Lambda_y$ are collected in a matrix A, then we may apply the lemma in the Appendix. We find that

$$\begin{aligned} d = {} & p\{\min(p, m)\} - \tfrac{1}{2}\min(p, m)\{\min(p, m) - 1\} \\ & - (p - l)\{\min(p - l, m)\} + \tfrac{1}{2}\min(p - l, m)\{\min(p - l, m) - 1\}. \end{aligned} \tag{27}$$

As a lower bound on the number of restrictions on l rows Λ'_y we find the number $l\{m - \min(n, q)\} - \min(r_{\Phi_\epsilon S}, d)$. If $p \leqslant m$, then $d = \frac{1}{2}l(l + 1) + (p - l)l$. However, $r_{\Phi_\epsilon S} \leqslant \frac{1}{2}l(l + 1) + (p - l)l$. So, in case $p \leqslant m$, $\min(r_{\Phi_\epsilon S}, d) = r_{\Phi_\epsilon S}$, and we simply find condition 4.

If $m \leqslant p - l$, we find $d = ml$. Now suppose $ml \leqslant r_{\Phi_\epsilon S}$, then the condition would say that the number of restrictions on l rows of Λ'_y should be larger than a negative number, which is not informative. If, on the other hand, $ml > r_{\Phi_\epsilon S}$, then we would find, again, condition 4.

If, finally, $p > m > p - l$, then $d = ml - \frac{1}{2}\{m - (p - l)\}\{m - (p - l) - 1\}$. Suppose $d < r_{\Phi_\epsilon S}$, then we find condition 16, which will be stronger than condition 4. If, on the other hand, $d \geqslant r_{\Phi_\epsilon S}$, then we would find, again, condition 4, which in this case is stonger than condition 16.

Condition 20 can be found in a similar way.

Conditions 17–19, and 21–24

In the derivation sofar we used the number of restrictions on Φ_ϵ and Φ_δ as an upper bound for the number of independent equations in the equation sets 8 and 9 of Table 3:

$$R'_{\Phi_\epsilon}\mathrm{vec}\{\Lambda'_y \Phi_\eta \Lambda_y - E(yy')\} = c_{\Phi_\epsilon}, \tag{28}$$

and

$$R'_{\Phi_\delta}\mathrm{vec}\{\Lambda'_x \Phi_\xi \Lambda_x - E(xx')\} = c_{\Phi_\delta}. \tag{29}$$

However, the number of independent equations in (28), with respect to all $pm + \frac{1}{2}m(m + 1)$ elements in Λ_y and Φ_η, is not larger than $\min(r_{\Phi_\epsilon}, d)$; where d is now the dimension of the manifold described by $\mathrm{vec}(\Lambda'_y \Phi_\eta \Lambda_y)$ as a function of the $pm + \frac{1}{2}m(m + 1)$ elements of Λ_y and Φ_η.

If we write $Z = \Phi_\eta^{1/2} \Lambda_y$, then we may apply the lemma in the Appendix, with $l = p$. We find:

$$d = p\{\min(p, m)\} - \tfrac{1}{2}\min(p, m)\{\min(p, m) - 1\}. \tag{30}$$

If $p \leqslant m$, then $d = \frac{1}{2}p(p + 1) \geqslant r_{\Phi_\epsilon}$; in which case the number of independent equations in (28) is simply bounded by r_{Φ_ϵ}: the total number of equations in (28). However, if $p > m$, then $d = pm - \frac{1}{2}m(m - 1) = \frac{1}{2}m(m + 1) + m(\mathrm{p} - m)$, which may be a smaller number than r_{Φ_ϵ}.

Consequently, the conditions found sofar by considering, among others, both Φ_η and Λ_y can be reformulated if $p > m$. That is, in conditions 9, 13 and 15, r_{Φ_ϵ} may be replaced by $\frac{1}{2}m(m+1) + m(p-m)$ if $p > m$. Thus we find conditions 17, 18 and 19 in Table 2. If $p > m$ and $r_{\Phi_\epsilon} > \frac{1}{2}m(m+1) + m(p-m)$, then these latter conditions are stronger than the conditions 9, 13 and 15, respectively.

In a similar way we may consider the equations in (29). If $q > n$, we find that in conditions 11, 14 and 15, r_{Φ_δ} may be replaced by $\frac{1}{2}n(n+1) + n(q-n)$. Thus we find conditions 21, 22 and 23 in Table 2, which are stronger than the former conditions if $q > n$ and $r_{\Phi_\delta} > \frac{1}{2}n(n+1) + n(q-n)$.

Finally, if both $p > m$ and $q > n$, then in Condition 15 r_{Φ_ϵ} may be replaced by $\frac{1}{2}m(m+1) + m(p-m)$, and r_{Φ_δ} may be replaced by $\frac{1}{2}n(n+1) + n(q-n)$, resulting in Condition 24.

Appendix

Lemma. *Consider the function* $F(Z) = \mathrm{vec}\{(Z, A)'(Z, A)\}$, *where* Z *is an* $m \times l$ *matrix, and* A *is an* $m \times (p-l)$ *matrix, So* $F\colon \mathbb{R}^{ml} \to \mathbb{R}^{p^2}$, *and* $F(Z)$ *describes a differentiable manifold of dimension* d, *say, in* $\mathbb{R}^{p^2}$. *Then*

$$d = p\{\min(p, m)\} - \tfrac{1}{2}\min(p, m)\{\min(p, m) - 1\} - (p-l)\{\min(p-l, m)\}$$
$$+ \tfrac{1}{2}\min(p-l, m)\{\min(p-l, m) - 1\}.$$

Proof. If $p \leqslant m$, then (Z, A) can be written as PD, where $P'P = I_p$ and D is lower triangular ($e_i'De_j = 0$ if $j > i$), while the last $p - l$ columns of D do not depend on Z. As $(Z, A)'(Z, A) = D'D$, we find that $d + \frac{1}{2}(p-l)(p-l+1) = \frac{1}{2}p(p+1)$, or $d = \frac{1}{2}l(l+1) + (p-l)l$, which corresponds to the equality in the Lemma.

If $p > m$, then (Z, A) can be written as QF, where $Q'Q = QQ' = I_m$, and the last m columns of F form a lower triangular matrix, while the last $p - l$ columns of F do not depend on Z. Furthermore $(Z, A)'(Z, A) = F'F$. If, moreover, $m \leqslant p - l$, then $d + m(p-l) - \frac{1}{2}m(m-1) = mp - \frac{1}{2}m(m-1)$. So $d = ml$. If, on the other hand, we have $p > m > p - l$, then $d + \frac{1}{2}(p-l)(p-l+1) = mp - \frac{1}{2}m(m-1)$. So $d = ml - \frac{1}{2}\{m - (p-l)\}\{m - (p-l) - 1\}$. Both results agree with the lemma.

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