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How to Assess a Model's Testability and Identifiability

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Formal definitions are given of the following intuitive concepts: (a) A model is *quantitatively testable* if its predictions are highly precise and narrow. (b) A model is *identifiable* if the values of its parameters can be ascertained from empirical observations. (c) A model is *redundant* if the values of some parameters can be deduced from others or if the values of some observables can be deduced from others. Various rules of thumb for nonredundant models are examined. The Counting Rule states that a model is quantitatively testable if and only if it has fewer parameters than observables. This rule can be safely applied only to identifiable models. If a model is unidentifiable, one must apply a generalization of the Counting Rule known as the Jacobian Rule. This rule states that a model is quantitatively testable if and only if the maximum rank (i.e., the number of linearly independent columns) of its Jacobian matrix (i.e., the matrix of partial derivatives of the function that maps parameter values to the predicted values of observables) is smaller than the number of observables. The Identifiability Rule states that a model is identifiable if and only if the maximum rank of its Jacobian matrix equals the number of parameters. The conclusions provided by these rules are only presumptive. To reach definitive conclusions, additional analyses must be performed. To illustrate the foregoing, the quantitative testability and identifiability of linear models and of discrete-state models are analyzed.

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1. RELATIVE VS ABSOLUTE ASSESSMENTS OF A MODEL'S ADEQUACY

The other papers in this special issue of the *Journal of Mathematical Psychology* are concerned with *model comparison and selection*. Given a collection of models, their concern is to find the one model that, in some sense, is better than all the other models. Therefore, they are concerned with assessments of a model's adequacy *relative* to other models.

1.1. Model Testability

In contrast, this paper is concerned with assessing a model's adequacy *without regard to other models*. In other words, this paper is concerned with an *absolute* assessment of a model's adequacy.

Suppose that we have a model that makes a prediction about the outcome of a particular experiment and that experiment has now been run. How impressed should we be with the performance of the model? Clearly, we should not be impressed with the model if the model's prediction disagrees with the outcome of the experiment. Also, we should not be impressed with the model if the model's prediction is so imprecise that, no matter what the outcome of the experiment, it could not disagree with the experiment. What should impress us is if, although the experiment was capable of showing the model's prediction to be wrong, nevertheless the experiment's outcome was in accord with the model's prediction. The more precise the model's prediction, the more impressed we should be that the model's prediction was correct.

In this paper, we review some earlier work of ours (Bamber & van Santen, 1985) in which we proposed a two-level classification of the precision of a model's prediction. If a model's prediction was highly precise by our criterion, the model was said to be *quantitatively testable*. If the model's prediction was not highly precise but was at least falsifiable, the model was said to be *qualitatively testable*.

This work has the following relevance to the problem of model comparison. Suppose that we wish to compare two models. Both have made correct predictions about the outcome of a particular experiment. If one model is quantitatively testable and the other only qualitatively testable, then we should be more impressed with the former than the latter.

1.2. Model Identifiability

As a byproduct of our approach to assessing a model's testability, we also obtain information relevant to the model's identifiability. Hence, the issue of model identifiability is also discussed in this paper.

Given a parametric model, we may set all of its parameters to constants, thus creating a no-parameter model. We may create as many no-parameter models as there are possible parameter settings. A parametric model is said to be *identifiable* if, given any experimental outcome in accordance with the model's prediction, there is only one no-parameter model, created from the parametric model, that is consistent with the experiment's outcome. If there can be multiple such no-parameter models, the parametric model is said to be *unidentifiable*.

Thus, the problem of model identifiability is related to the problem of model selection at the level of no-parameter models. A parametric model is identifiable if, given a successful prediction, it is always possible to identify *exactly one* of its no-parameter models as superior to *all* the others.

2. INTRODUCTORY EXAMPLE CONCERNING MODEL TESTABILITY

Some models make much more precise predictions about the result of an experiment than other models do. For example, consider the two following models of gravitation. Let d denote the distance in meters traveled by a dropped ball after it has fallen for t seconds. The *constant-acceleration model* states that

$$d = \tfrac{1}{2} g t^2, \tag{1}$$

where gravitational acceleration $g > 0$ is a parameter that varies from one location to another on the surface of the Earth and other planets. The *positive-acceleration model* states that, if a ball is dropped, its velocity always increases. Specifically, this model states that, if we let v denote the average velocity up to time t, then

$$d = vt \text{ where } v \text{ increases as } t \text{ increases.} \tag{2}$$

Suppose that we do the following *one-second two-second experiment* to test these two models. We drop a ball and measure the distance that it has fallen at one second and at two seconds. Let d_1 and d_2 denote the distances fallen after one and two seconds and let v_1 and v_2 denote the average velocity during the first second and during the first two seconds. Using Eq. (1), it is seen that the constant-acceleration model predicts:

$$(d_1, d_2) = (0.5g, 2.0g) \qquad \text{where} \quad g > 0. \tag{3}$$

And, using Eq. (2), it is seen that the positive-acceleration model predicts:

$$(d_1, d_2) = (v_1, 2v_2) \qquad \text{where} \quad v_2 > v_1 > 0.$$

There is a striking difference in the precision of the predictions of the constant-acceleration and positive-acceleration models for the one-second two-second experiment. The constant-acceleration model makes the *exact* prediction that $d_2/d_1 = 4$, whereas the positive-acceleration model makes the looser prediction that $d_2/d_1 > 2$.

This paper will be concerned with differentiating models that make precise predictions from those that make loose predictions.

3. MODELS

It is useful to distinguish *general models* that can be used to derive predictions for a variety of experiments from *experiment-specific models* that make predictions for

only a single experiment. For example, Eq. (1) describes the constant-acceleration model in full generality and can be used to derive predictions for a variety of experiments. In contrast, Eq. (3) describes a version of the constant-acceleration model that makes a prediction specifically for the one-second two-second experiment and for no other experiment.

Throughout this paper, it is assumed that we are concerned with predicting the outcome of a single experiment and, therefore, the term *model* will refer to an experiment-specific model rather than to a general model.

3.1. Definition of Model

In an experiment, the values of a set of quantities, to be called *observables*, are measured. Let the observables measured in the experiment be denoted by the column vector $\mathbf{y} = [y_1, ..., y_n]' \in \Re^n$. (In this notation, if $\mathbf{x}$ is a row vector, then $\mathbf{x}'$ denotes the column vector obtained by transposing $\mathbf{x}$.) Among the vectors $\mathbf{y} \in \Re^n$, some are conceivable results of the experiment and some are not. For example, it is not conceivable that a reaction time would be negative. Let the set of all vectors that are conceivable outcomes of the experiment be called the experiment's *outcome space* and be denoted $\mathscr{O}$.

Let the values of a model's parameters be denoted as a column vector $\boldsymbol{\theta} = [\theta_1, ..., \theta_m]'$. Let the set of all parameter vectors that represent conceivable combinations of parameter values be called the model's *parameter domain* and be denoted $\mathscr{D}$.

For each parameter vector in $\mathscr{D}$, the model predicts an outcome in $\mathscr{O}$. In other words, the model implicitly or explicitly specifies a function that maps the parameter domain $\mathscr{D}$ into the outcome space $\mathscr{O}$. Let this function be called the model's *prediction function* and let it be denoted f^{P}.

The model is consistent with a conceivable experimental outcome $\mathbf{y} \in \mathscr{O}$ if and only if there exists a conceivable parameter vector $\boldsymbol{\theta} \in \mathscr{D}$ such that $\mathbf{y} = f^{\mathrm{P}}(\boldsymbol{\theta})$. Thus, the set of conceivable experimental outcomes with which the model is consistent is $f^{\mathrm{P}}(\mathscr{D})$. In other words, the model predicts that the experiment's outcome will belong to the set $f^{\mathrm{P}}(\mathscr{D})$. Let this set be called the model's *prediction range* and let it be denoted $\mathscr{P}$.

The preceding is summarized in the following formal definition.

DEFINITION 1. A *model* $\mathscr{M}$ is an ordered triple $(\mathscr{D}, f^{\mathrm{P}}, \mathscr{O})$ where, for positive integers m and n, $\mathscr{D} \subseteq \Re^m$, $\mathscr{O} \subseteq \Re^n$, and f^{P} is a function defined on $\mathscr{D}$ such that $f^{\mathrm{P}}(\mathscr{D}) \subseteq \mathscr{O}$. $\mathscr{D}$ is called the *parameter domain*, $\mathscr{O}$ is called the *outcome space*, f^{P} is called the *prediction function*; and $f^{\mathrm{P}}(\mathscr{D})$ is called the *prediction range* and is denoted $\mathscr{P}$.

Remark. We have defined a model's parameter domain, outcome space, and prediction range to be subsets of $\Re^k$ for some $k \geqslant 1$. In this paper, the elements of $\Re^k$ may be taken to be either k-element column vectors $[x_1, ..., x_k]'$ or ordered k-tuples $(x_1, ..., x_k)$, whichever is more convenient.

To simplify discussion, this paper will ignore models in which $\mathscr{D}$ or $\mathscr{O}$ contains only one point.

EXAMPLE: THE CONSTANT-ACCELERATION MODEL. Consider once again the one-second two-second experiment. If we consider it inconceivable that a dropped object would do anything other than move downward, then the set of conceivable outcomes for the experiment is:

$$\mathcal{O}_{12} = \{(d_1, d_2) : 0 < d_1 < d_2\}.$$

The constant-acceleration model has one parameter, namely, gravitational acceleration g. The model's parameter domain is

$$\mathcal{D}_{\text{ca}} = \{g \in \Re : g > 0\}.$$

The model's prediction function f^{p}_{ca} maps each $g \in \mathcal{D}_{\text{ca}}$ to the point $(d_1, d_2) = (0.5g, 2.0g) \in \mathcal{O}_{12}$. The model's prediction range is

$$\mathcal{P}_{\text{ca}} = f^{\text{p}}_{\text{ca}}(\mathcal{D}_{\text{ca}}) = \{(d_1, 4d_1) : d_1 > 0\}.$$

EXAMPLE: THE POSITIVE-ACCELERATION MODEL. This model has the same outcome space as the constant-acceleration model. The positive-acceleration model has two parameters: v_1, the average velocity during the first second, and v_2, the average velocity during the first two seconds. According to the model, the average velocity always increases. Thus, the model's parameter domain is

$$\mathcal{D}_{\text{pa}} = \{(v_1, v_2) : 0 < v_1 < v_2\}.$$

The model's prediction function f^{p}_{pa} maps each (v_1, v_2) in the parameter domain to the point $(d_1, d_2) = (v_1, 2v_2)$ in the outcome space. The model's prediction range is

$$\mathcal{P}_{\text{pa}} = f^{\text{p}}_{\text{pa}}(\mathcal{D}_{\text{pa}}) = \{(d_1, d_2) : 0 < d_1 < d_2/2\}.$$

3.2. Redundant vs Nonredundant Models

A model has *redundant parameters* (or *redundant observables*) if, given the values of a proper subset of its parameters (or observables), the values of the remaining parameters (or observables) can be deduced. A *redundant model* is one that has either redundant parameters or redundant observables.

EXAMPLE. Consider a model whose observables are the probabilities of n mutually exclusive and exhaustive events. These observables are redundant because they sum to one and, thus, the probability of any one of the events can be deduced from the probabilities of the other $n-1$ events.

Put loosely, if a redundant model and a nonredundant model are "the same" except for the presence of redundant parameters and/or redundant observables in the former model, then we say that the latter model is a *nonredundant edition* of the former.

EXAMPLE. Consider an experiment in which we repeatedly toss a pair of identical coins and observe the probabilities of zero, one, or two heads on each pair of

tosses. These we denote p_0, p_1, and p_2, respectively. We propose a model based upon the assumption that, within each pair of coin tosses, each toss is independent of the other. The model has two parameters, the probability p_h of a head and the probability p_t of a tail. The model predicts:

$$(p_0, p_1, p_2) = (p_t^2, 2p_t p_h, p_h^2).$$

A nonredundant edition of this model having only one parameter, p_h, and only two observables, p_0 and p_1, may be constructed. This new model predicts:

$$(p_0, p_1) = ((1 - p_h)^2, 2(1 - p_h)\, p_h).$$

A model is said to have *m nonredundant parameters* and *n nonredundant observables* if it has a nonredundant edition with *m* parameters and *n* observables. (Commonly encountered synonyms for *nonredundant parameters* and *observables* are *independent parameters* and *observables* and *free parameters* and *observables*.)

The above definition of the number of nonredundant parameters and observables is a bit careless. The problem is that *m* and *n* need not be unique. For example, consider a model having three parameters θ_1, θ_2, and θ_3 and whose parameter domain is

$$\{(0, 0, 0), (1, 0, 1), (2, 1, 0), (3, 1, 1)\}.$$

Note that θ_1 (interpreted as a decimal number) always equals $\theta_2\theta_3$ (interpreted as a binary number), i.e., $0 = 00$, $1 = 01$, $2 = 10$, $3 = 11$. So, from the original model, we may construct a nonredundant edition having only one parameter, namely, θ_1. Alternatively, we may construct a nonredundant edition having two parameters, namely, θ_2 and θ_3.

Despite this example, in practice, the potential nonuniqueness of the number of nonredundant parameters and observables does not seem to be a problem. As far as we are aware, in all models that have been proposed as serious science, the numbers of nonredundant parameters and observables are both unique.

4. TESTABILITY OF MODELS

4.1. Testability Definitions

DEFINITION 2. A model $(\mathscr{D}, f^{\mathrm{p}}, \mathscr{O})$ is *qualitatively testable*, abbreviated *QlT*, if $\mathscr{P}$ is a *proper* subset of $\mathscr{O}$; otherwise (i.e., if $\mathscr{P} = \mathscr{O}$), the model is *untestable*.

Thus, a model is QlT if there exists a conceivable experimental outcome with which the model is not consistent. In other words, a model is QlT if and only it is falsifiable.

We now consider *quantitative testability*. Although Bamber and van Santen (1985) distinguished between weak and strong forms of quantitative testability, we shall not explain that distinction in this paper. For ordinary sorts of models, the two forms of quantitative testability are equivalent; it is only in rather strange

models where the two forms differ. In this paper, the term *quantitative testability* is equivalent to *weak quantitative testability* in Bamber and van Santen (1985).

Notation. If A is a subset of a finite-dimensional Euclidean space (i.e., a subset of $\Re^k$ for some positive integer k), then $\lambda(A)$ denotes the Lebesgue measure of A. Recall that, in the real line, Lebesgue measure corresponds to length; in $\Re^2$, it corresponds to area; in $\Re^3$, it corresponds to volume; etc. (Throughout this paper, it is assumed that all the sets under consideration are Lebesgue measurable.)

DEFINITION 3. A model $(\mathscr{D}, f^{\mathrm{P}}, \mathscr{O})$ is *quantitatively testable*, abbreviated *QntT*, if $\lambda(\mathscr{O}) > 0$ and if $\lambda(\mathscr{P}) = 0$. The model is *non-QntT* if $\lambda(\mathscr{P}) > 0$.

This definition may be paraphrased as follows. Suppose that we consider a set $\mathscr{S}$ to be large if $\lambda(\mathscr{S}) > 0$ and to be small if $\lambda(\mathscr{S}) = 0$. Then a model is QntT if $\mathscr{O}$ is large but $\mathscr{P}$ is small. The model is non-QntT if $\mathscr{O}$ and $\mathscr{P}$ are both large. Note, however, that, if $\mathscr{O}$ and $\mathscr{P}$ are both small, then the model is not classifiable as either QntT or non-QntT.

Frequently, if a model is not classifiable as either QntT or non-QntT, its non-redundant editions will either all be classified QntT or all be classified non-QntT. In such circumstances, we may give the model a courtesy classification of QntT or non-QntT, as appropriate.

It is the intent of the above definitions that the experimental confirmation of a QntT model should be much more impressive than the experimental confirmation of a model that is merely QlT. Put loosely, for a model to be QntT, the probability that a point randomly selected from the outcome space $\mathscr{O}$ will lie in the prediction range $\mathscr{P}$ must be zero. To be more precise: Suppose that $\lambda(\mathscr{O}) > 0$. Let Pr be *any* probability measure that meets the following two criteria: (a) $\Pr(\mathscr{O}) = 1$, (b) Pr has a density function that is *greater than zero* everywhere in $\mathscr{O}$. (The significance of criterion (b) is that the probability measure Pr should not *a priori* exclude any point in $\mathscr{O}$ from being randomly selected.) Then the model is QntT if and only if $\Pr(\mathscr{P}) = 0$.

Why is this important? Imagine that the model is based upon false principles and that Nature selects an experimental outcome at random from $\mathscr{O}$. If the model is QntT, it is extremely unlikely that this randomly selected outcome will be consistent with the model's prediction. Hence, an experimental confirmation of the model is unlikely to occur by chance.

EXAMPLE: LINEAR MODEL. Let x be an independent variable and y a dependent variable. Consider the general model that states that y is a linear function of x. In other words, letting θ_1 and θ_x denote parameters, $y = \theta_1 + \theta_x x$. An experiment is performed in which x is set to the values $x_1 < \cdots < x_n$ and the corresponding values of y are observed. The experiment-specific model constructed from the general model has $\Re^n$ as its outcome space; its parameter domain is $\Re^2$; its prediction function maps each $[\theta_1, \theta_x]' \in \Re^2$ to $[\theta_1 + \theta_x x_1, ..., \theta_1 + \theta_x x_n]' \in \Re^n$. If $n \geqslant 3$, this experiment-specific model is QntT but, if $n = 2$, the model is untestable. On the other hand, if the model is modified by restricting the parameter domain so that θ_x must always be positive, then the model predicts that $y_1 < y_2$ and, thus, the model is QlT when $n = 2$.

4.2. Evaluating Whether a Model Is QntT

4.2.1. The Counting Rule

There is a well-known rule of thumb (which we call the *Counting Rule*) for ascertaining when a model is testable. The rule does not specify what is meant by *testable*. It is our view that the users of the rule have interpreted *testable* to mean some intuitive notion similar to QntT. Hence, we formulate the Counting Rule as: *For a model to be QntT, the number of nonredundant parameters should be less than the number of nonredundant observables*; *otherwise, the model is non-QntT.*

Unfortunately, as will be seen, the Counting Rule is overly conservative; it classifies some QntT models as non-QntT.

4.2.2. The Jacobian Rule

DEFINITION 4. If r is the largest number of linearly independent column vectors that can be selected from the matrix $\mathbf{A}$, then r is called the *rank* of $\mathbf{A}$ and is denoted $\rho[\mathbf{A}]$.

Consider the *nonredundant* model $\mathscr{M} = (\mathscr{D}, f^{\mathrm{P}}, \mathscr{O})$. Recall that any $\boldsymbol{\theta} \in \mathscr{D}$ may be written in the form

$$\boldsymbol{\theta} = [\theta_1, ..., \theta_m]'.$$

Define real-valued functions $f_1^{\mathrm{P}}, ..., f_n^{\mathrm{P}}$ such that, at each $\boldsymbol{\theta} \in \mathscr{D}$,

$$f^{\mathrm{P}}(\boldsymbol{\theta}) = [f_1^{\mathrm{P}}(\boldsymbol{\theta}), ..., f_n^{\mathrm{P}}(\boldsymbol{\theta})]'.$$

Take any point $\boldsymbol{\theta}^0 \in \mathscr{D}$ at which f^{P} is differentiable. Then the *Jacobian matrix of* f^{P} evaluated at $\boldsymbol{\theta}^0$ is defined to be the $n \times m$ matrix of partial derivatives

$$\mathbf{J}_{f^{\mathrm{P}}}(\boldsymbol{\theta}^0) = \begin{bmatrix} \partial f_1^{\mathrm{P}}(\boldsymbol{\theta}^0)/\partial\theta_1 & \cdots & \partial f_1^{\mathrm{P}}(\boldsymbol{\theta}^0)/\partial\theta_m \\ \vdots & & \vdots \\ \partial f_n^{\mathrm{P}}(\boldsymbol{\theta}^0)/\partial\theta_1 & \cdots & \partial f_n^{\mathrm{P}}(\boldsymbol{\theta}^0)/\partial\theta_m \end{bmatrix}.$$

If f^{P} is differentiable at $\boldsymbol{\theta}^0$, then it will be approximated near $\boldsymbol{\theta}^0$ by the function $f_{\boldsymbol{\theta}^0}^{\mathrm{P}}$ defined by

$$f_{\boldsymbol{\theta}^0}^{\mathrm{P}}(\boldsymbol{\theta}) = f^{\mathrm{P}}(\boldsymbol{\theta}^0) + [\mathbf{J}_{f^{\mathrm{P}}}(\boldsymbol{\theta}^0)](\boldsymbol{\theta} - \boldsymbol{\theta}^0).$$

The model $\mathscr{M}_{\boldsymbol{\theta}^0} = (\Re^m, f_{\boldsymbol{\theta}^0}^{\mathrm{P}}, \Re^n)$ is called the *affine approximation* at $\boldsymbol{\theta}^0$ to the model $\mathscr{M} = (\mathscr{D}, f^{\mathrm{P}}, \mathscr{O})$. Note that, like $\mathscr{M}$, $\mathscr{M}_{\boldsymbol{\theta}^0}$ is nonredundant. It is readily shown that $\mathscr{M}_{\boldsymbol{\theta}^0}$ is QntT if and only if $\rho[\mathbf{J}_{f^{\mathrm{P}}}(\boldsymbol{\theta}^0)] < n$. Note that, if $\rho[\mathbf{J}_{f^{\mathrm{P}}}(\boldsymbol{\theta}^0)] < n \leqslant m$, then the model $\mathscr{M}_{\boldsymbol{\theta}^0}$ will be QntT even though the Counting Rule (mistakenly) says that it is non-QntT.

Consider the following conjecture: *The nonredundant model $\mathscr{M}$ is QntT if and only if its affine approximation at $\boldsymbol{\theta}^0$ is QntT for every $\boldsymbol{\theta}^0 \in \mathscr{D}$; otherwise, the model is*

non-QntT. This conjecture may be reformulated as follows. First, define the *maximum rank of the Jacobian matrix* for the model $\mathscr{M}$ to be

$$\max \rho[\mathbf{J}_{\mathscr{M}}] = \max_{\boldsymbol{\theta}^0 \in \mathscr{D}} \rho[\mathbf{J}_{f^{\mathrm{p}}}(\boldsymbol{\theta}^0)].$$

Then, using this definition, our conjecture may be restated: *A nonredundant model $\mathscr{M}$ having n observables is QntT if and only if*

$$\max \rho[\mathbf{J}_{\mathscr{M}}] < n; \tag{4}$$

otherwise, the model is non-QntT. We call this conjecture the *Jacobian Rule*.

Obviously, to apply the Jacobian Rule, it is necessary to ascertain the maximum rank of the Jacobian matrix. Methods for finding the Jacobian matrix's maximum rank are discussed in Bamber and van Santen (1985, pp. 457–459). For the maximum Jacobian-matrix rank of some models related to the discrete-state models discussed later in this paper, see Geiger, Heckerman, and Meek (1996).

Note that the Jacobian Rule and the Counting Rule can disagree. Specifically, if $\max \rho[\mathbf{J}_{\mathscr{M}}] < n \leqslant m$, where m is the number of parameters in the model, then the Jacobian Rule will assert that the model $\mathscr{M}$ is QntT whereas the Counting Rule will assert that the model is non-QntT. However, we have already seen that, in the case of the model $(\mathfrak{R}^m, f^{\mathrm{p}}_{\boldsymbol{\theta}_0}, \mathfrak{R}^n)$, the Counting Rule can be incorrect. So, if the Counting Rule is not trustworthy, does that mean that the Jacobian Rule is trustworthy?

4.2.3. When to Trust the Jacobian Rule

The Jacobian Rule is only a *rule of thumb*. It can give a *presumptive* answer to the question of whether or not a model is QntT. If one desires a *definitive* answer, one should check whether the model satisfies additional conditions that were stated by Bamber and van Santen (1985).

Case 1: *The Jacobian Rule says* "*yes*." Suppose that Eq. (4) holds and thus, according to the Jacobian Rule, the model should be QntT. Theorem 9 of Bamber and van Santen (1985), which is based upon Sard's theorem, gives additional conditions which, if satisfied by the model, will guarantee that it is QntT. The first condition is that $\mathscr{O}$ should have non-zero Lebesgue measure in $\mathfrak{R}^n$. (This condition will be automatically satisfied if the model's outcome space contains a solid sphere of any radius greater than zero.) The remaining condition says, roughly speaking, that f^{p} must be extendible to an open set containing $\mathscr{D}$ and, furthermore, this extension must be sufficiently smooth.

Case 2: *The Jacobian Rule says* "*no*." According to the Jacobian Rule, the model should be non-QntT if Eq. (4) does not hold (i.e., if the Jacobian matrix has rank n at some point in $\mathscr{D}$). Proposition 10 of Bamber and van Santen (1985) says that, if the Jacobian matrix has rank n, not just at any point in $\mathscr{D}$, but at an interior point of $\mathscr{D}$ where f^{p} is continuously differentiable, then the model is non-QntT.

5. IDENTIFIABILITY OF MODELS

5.1. Definition of Identifiability

Roughly speaking, a model is *identifiable* if the values of its parameters can be ascertained from empirical observations. Here is a formal definition.

DEFINITION 5. A model $(\mathscr{D}, f^{\mathrm{p}}, \mathscr{O})$ is *identifiable* if its prediction function f^{p} is one-to-one. The model is *locally identifiable* at a point $\boldsymbol{\theta}^0 \in \mathscr{D}$ if f^{p} is one-to-one when restricted to points within some distance $\varepsilon > 0$ of $\boldsymbol{\theta}^0$. If a model is not identifiable, then it is *unidentifiable*.

When a model is unidentifiable, this does not mean that observing the outcome of an experiment provides us with *no* information about the model's parameters. Rather it means the outcome of the experiment provides us with only *partial* rather than *complete* information about the model's parameters. If $\mathbf{y} \in \mathscr{P}$ is the outcome of the experiment, then the set of all $\boldsymbol{\theta} \in \mathscr{D}$ such that $f^{\mathrm{p}}(\boldsymbol{\theta}) = \mathbf{y}$ will contain more than one point but will typically be smaller than $\mathscr{D}$.

If a model is not identifiable, then one cannot expect to use empirical data to ascertain a *unique* value for its parameter vector. Nevertheless, one can use Bayesian procedures to obtain a *posterior distribution* for the parameter vector (Chechile, 1977).

EXAMPLE OF AN UNIDENTIFIABLE MODEL. Luce (1959) proposed a model for choices made in paired-comparison experiments. In such an experiment, there is a set of m objects: $\mathrm{Obj}_1, ..., \mathrm{Obj}_m$. There are $m(m-1)/2$ observable quantities measured in the experiment. These are, for $1 \leqslant i < j \leqslant m$, the probability p_{ij} that the subject will select Obj_i when offered a choice of Obj_i or Obj_j. The model has m parameters, $\theta_1, ..., \theta_m$, each of which is a positive number that may be interpreted as representing the attractiveness of the corresponding object. The model predicts that, for $1 \leqslant i < j \leqslant m$,

$$p_{ij} = \theta_i/(\theta_i + \theta_j).$$

Obviously, if a particular m-tuple of parameter values $(\theta_1^0, ..., \theta_m^0)$ correctly predicts all the observed p_{ij}s then, for every $\alpha > 0$, the m-tuple of parameter values $(\alpha\theta_1^0, ..., \alpha\theta_m^0)$ will also correctly predict the observed p_{ij}s. Thus, the Luce choice model is not identifiable.

Not only is the Luce choice model not identifiable, it is not locally identifiable anywhere. Take any $\boldsymbol{\theta}^0 = (\theta_1^0, ..., \theta_m^0)$ in the parameter domain and any $\varepsilon > 0$. It is always possible to choose $\alpha \neq 1$ sufficiently close to one so that $\alpha\boldsymbol{\theta}^0$ is within distance ε of $\boldsymbol{\theta}^0$ and yet $\boldsymbol{\theta}^0$ and $\alpha\boldsymbol{\theta}^0$ predict the same p_{ij}s.

EXAMPLE OF A LOCALLY IDENTIFIABLE MODEL. Consider a model whose parameter domain is $\Re$ and whose prediction function is given by $f^{\mathrm{p}}(\theta) = (\cos\theta, \sin\theta) \in \Re^2$. This model is not identifiable because, for any integer k, $f^{\mathrm{p}}(\theta) = f^{\mathrm{p}}(\theta + 2k\pi)$. However, the model is locally identifiable at every $\theta^0 \in \Re$. To show this, note that f^{p} is one-to-one when restricted to the open interval $(\theta^0 - \pi/2, \theta^0 + \pi/2)$.

Reparameterization. Sometimes, an unidentifiable model can be reparameterized to make it identifiable. For example, the Luce choice model can be reduced from m to $m-1$ parameters by arbitrarily setting $\theta_m = 1$, thus, effectively eliminating that parameter. (Setting $\theta_m = 1$ is equivalent to multiplying all parameter values by $\alpha = 1/\theta_m$.) Once θ_m has been eliminated as a parameter, the remaining parameters are identifiable.

As another example, consider serial-processing models for memory-scanning experiments (e.g., Sternberg, 1966). One way to parameterize such models is use three parameters: a, b, and c. Without going into details, suffice it to say that, in an experiment using memorized lists of one, two, and three items, the model's prediction function maps each (a, b, c) in the parameter domain to $(a+b+c,\ a+2b+c,\ a+3b+c)$ in the outcome space. Consequently, given an experimental outcome in accordance with the model's prediction, it is possible to ascertain the values of b and of $a+c$. However, there is no way to ascertain how the sum $a+c$ decomposes into a and c. For that reason, it is common to reparameterize the model so that it has two parameters, b and d, where d is equivalent to $a+c$. Under the new parameterization, the model is identifiable.

5.2. Evaluating Whether a Model Is Identifiable

Just as the Jacobian matrix is relevant to ascertaining whether a model is QntT, that matrix is also relevant to ascertaining whether the model is identifiable.

The rank of the Jacobian matrix must be either (a) equal to the number of parameters or (b) smaller. What is critical for identifiability is whether (a) or (b) holds. Smith (1998, Appendix A) has shown that, if a model's prediction function is continuously differentiable at a point $\boldsymbol{\theta}^0$ (which, therefore, must be an interior point of the model's parameter domain) and if the prediction function's Jacobian matrix evaluated at $\boldsymbol{\theta}^0$ has rank equal to the number of parameters in the model, then the model is locally identifiable at that point. Bamber and van Santen have shown a converse result. Put loosely, if the maximum rank of a model's Jacobian matrix is less than the number of parameters in the model, then the model is not identifiable—see Proposition 20 of Bamber and van Santen (1985) for details.

Based on the above mathematical results, we formulate a rule of thumb, which we name the *Identifiability Rule*. This states: *The maximum rank of the Jacobian matrix being equal to the number of parameters suggests that the model is at least locally identifiable*; *the maximum rank of the Jacobian matrix being smaller than the number of parameters suggests that the model is unidentifiable*.

5.3. Identifiability and the Counting Rule

Put loosely, the Counting Rule is reliable only for identifiable models. Thus, within a broad class of well-behaved models, the following results hold. First, if a model is identifiable, then the Counting Rule and the Jacobian Rule are equivalent (Bamber & van Santen, 1985, Proposition 21). Second, if the Counting Rule and the Jacobian Rule disagree about whether a model is QntT (in which case the Jacobian Rule should take precedence), then the model is *not* identifiable (Bamber & van Santen, 1985, Proposition 20).

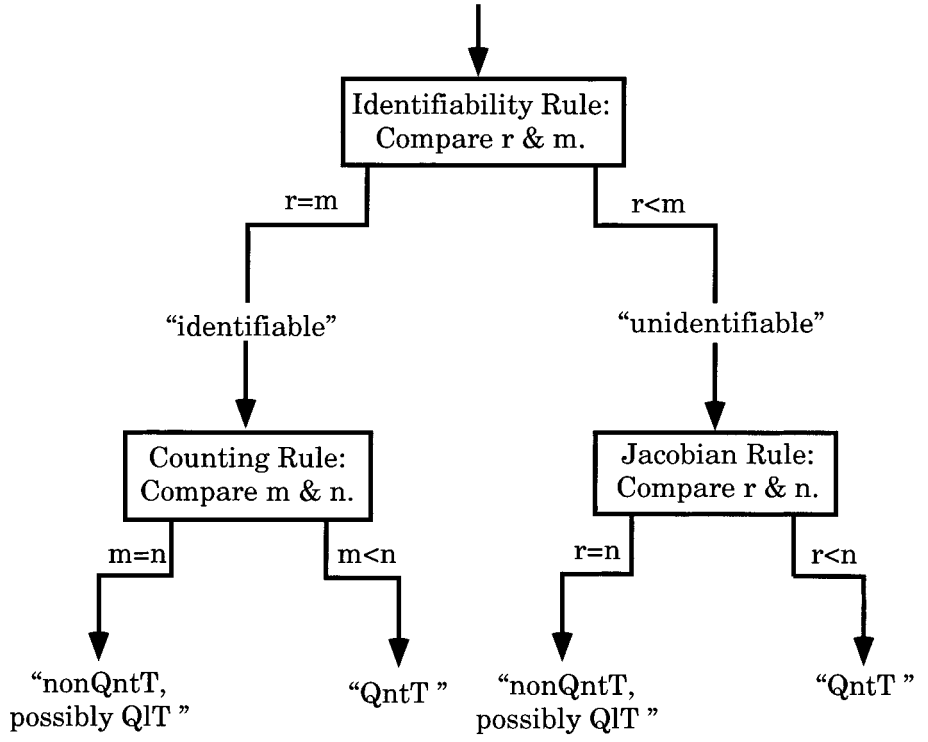

FIG. 1. Provisional evaluation of a nonredundant model's identifiability and quantitative testability. Here r denotes the maximum rank of the Jacobian matrix for the model's prediction function; m denotes the number of parameters; n denotes the number of observables. Recall that $r \leqslant m, n$.

5.4. Combining the Rules of Thumb

Three rules of thumb have now been discussed: the Counting Rule, the Jacobian Rule, and the Identifiability Rule. The conclusions produced by these rules are presumptive only; to reach definitive conclusions, additional analyses must be performed. Figure 1 shows how to combine the three rules of thumb into a procedure for provisionally evaluating a model's identifiability and quantitative testability.

6. COMPARISON OF MODEL PROPERTIES

Testability. When we assert that a model $(\mathscr{D}, f^{\mathrm{P}}, \mathscr{O})$ is QlT or QntT, we are making an assertion about the relationship between the model's prediction range $\mathscr{P} = f^{\mathrm{P}}(\mathscr{D})$ and its outcome space $\mathscr{O}$. A model is QlT if and only if $\mathscr{P}$ is a proper subset of $\mathscr{O}$. A model is QntT if and only if, roughly speaking, $\mathscr{P}$ is so small compared to $\mathscr{O}$ that a point selected at random from $\mathscr{O}$ has zero probability of lying in $\mathscr{P}$.

Identifiability. When we assert that a model is identifiable, we are making an assertion about the model's prediction function f^{P}. A model is identifiable if and only if its prediction function is one-to-one.

Nonredundant parameters and/or observables. When we assert that a model's parameters are nonredundant, we are making an assertion about the model's parameter domain $\mathscr{D}$. A model's parameters are redundant if, knowing some of the coordinates of a point within $\mathscr{D}$, one can deduce the remaining coordinates. Analogously, when we assert that a model's observables are redundant, we are making an assertion about the model's outcome space $\mathscr{O}$.

6.1. Combinations of Testability and Identifiability

Consider the following examples of combinations of testability and identifiability.

Models that are both QntT and identifiable. These are the most commonly encountered type of model. The constant-acceleration model is QntT and identifiable.

Models that are QntT but unidentifiable. As an example, consider the memory-scanning model with three parameters: a, b, and c.

The Luce (1959) model for paired comparison judgments for $m \geqslant 3$ objects is QntT but unidentifiable. (The Counting Rule mistakenly states that the Luce model is QntT only if $m \geqslant 4$.)

Crowther, Batchelder, and Hu (1995, Theorem 1) have shown that the two-factor, two-category fuzzy logic model of perception is not identifiable. In fact, their proof shows that the model is not even locally identifiable. However, the model is QntT.

Later in this paper, we will discuss discrete-state models that are QntT but not identifiable.

Models that are identifiable but untestable. Such models are commonly employed in experiments where (a) we want to ascertain the values of one or more parameters of a model and (b) we have faith that the model is correct and, consequently, have no need to test it.

For example, suppose that an engineer wishes to ascertain the resistance R of a resistor. An experiment is done in which a known voltage V is applied across the resistor and the amperage I of the current through the resistor is measured. An experiment-specific model based on Ohm's law predicts $I = V/R$, where R is the model's only parameter. This model is identifiable but not testable. However, the model's lack of testability is of no concern to an engineer who is confident that Ohm's law is correct.

Models that are untestable and unidentifiable. The Luce (1959) model for paired comparison judgments for $m = 2$ objects is both untestable and unidentifiable.

7. EXAMPLE: LINEAR MODELS

Suppose that the dependent variable y is some unknown function of the independent variables w and x. Consider a *general model* (i.e., a theoretical formulation that can be used to derive predictions for a variety of experiments) that states that

y is a *linear* function of w and x. In other words, there exist parameters $\theta_1, \theta_w, \theta_x \in \Re$ such that, for all $w, x \in \Re$,

$$y = \theta_1 + \theta_w w + \theta_x x. \tag{5}$$

To test this general model, an experiment is done in which the independent variables are set to n distinct pairs of values, denoted $(w_1, x_1), ..., (w_n, x_n)$, and the corresponding values of the dependent variable $y_1, ..., y_n$ are observed.

From the general model, we derive an *experiment-specific model*. This model has three parameters, $\theta_1, \theta_w, \theta_x$, each of which can take any value in $\Re$. Hence, the model's parameter domain is $\Re^3$. In addition, n observables, $y_1, ..., y_n$, each of which can take any value in $\Re$, are measured in the experiment. Hence, the model's outcome space is $\Re^n$. Let the model's prediction function be denoted $f^{\mathrm{p}}_{\mathrm{lin}}$. Then, from Eq. (5), we see that, for all $(\theta_1, \theta_w, \theta_x) \in \Re^3$,

$$f^{\mathrm{p}}_{\mathrm{lin}}(\theta_1, \theta_w, \theta_x) = \begin{bmatrix} \theta_1 + \theta_w w_1 + \theta_x x_1 \\ \vdots \\ \theta_1 + \theta_w w_n + \theta_x x_n \end{bmatrix} \in \Re^n.$$

Let this experiment-specific model be denoted $\mathscr{M}_{\mathrm{lin}} = (\Re^3, f^{\mathrm{p}}_{\mathrm{lin}}, \Re^n)$. Note that this model is nonredundant.

Remark. The values of the independent variables $w_1, ..., w_n$ and $x_1, ..., x_n$ are neither model parameters nor observed experimental outcomes. Rather, their role is to describe the design of the experiment.

7.1. Jacobian Matrix

The model's Jacobian matrix has the same value no matter what parameter values it is evaluated at. Specifically,

$$\mathbf{J}_{f^{\mathrm{p}}_{\mathrm{lin}}}(\theta_1, \theta_w, \theta_x) = \begin{bmatrix} 1 & w_1 & x_1 \\ \vdots & \vdots & \vdots \\ 1 & w_n & x_n \end{bmatrix}.$$

Hence, the Jacobian matrix attains its maximum rank at all values of $(\theta_1, \theta_w, \theta_x)$. The size of the maximum rank depends upon whether or not the independent-variable pairs $(w_1, x_1), ..., (w_n, x_n)$ are arranged in $\Re^2$ in such a way that a single line contains all of them. Assuming that $n > 1$, the maximum Jacobian-matrix rank for the model is:

$$\max \rho[\mathbf{J}_{\mathscr{M}_{\mathrm{lin}}}] = \begin{cases} 3 & \text{if } (w_1, x_1), ..., (w_n, x_n) \text{ are noncollinear,} \\ 2 & \text{if } (w_1, x_1), ..., (w_n, x_n) \text{ are collinear.} \end{cases}$$

7.2. Testability and Identifiability

In the following, let $\rho_{\mathbf{J}}$ denote the rank of the Jacobian matrix. Let the number of model parameters, which is always three, be denoted m. The number of observables (i.e., the number of y observations) has already been denoted n. Recall that the model $\mathscr{M}_{\text{lin}}$ is nonredundant. The following results may be demonstrated.

(a) The Identifiability Rule correctly indicates whether or not the model is identifiable. Specifically, the model is identifiable if and only if (w_1, x_1), ..., (w_n, x_n) are noncollinear. But, these points are noncollinear if and only if $\rho_{\mathbf{J}} = 3 = m$; they are collinear if and only if $\rho_{\mathbf{J}} = 2 < 3 = m$.

(b) When the model is identifiable, the Counting Rule correctly indicates whether or not the model is QntT. Specifically, when (w_1, x_1), ..., (w_n, x_n) are noncollinear, the model is QntT if and only if $n > 3 = m$.

(c) When the model is unidentifiable, the Jacobian Rule correctly indicates whether or not the model is QntT. Specifically, when (w_1, x_1), ..., (w_n, x_n) are collinear, the model is QntT if and only if $n > 2 = \rho_{\mathbf{J}}$.

These results concerning the Identifiability, Counting, and Jacobian Rules generalize to linear models having more than two independent variables.

7.3. Adequacy of the Experiment

Consider the following special case of collinearity. Suppose that the values of w_1, ..., w_n are all distinct, whereas the values of x_1, ..., x_n are all identical. Given this design, the experiment fails to differentiate general models in which x has a linear effect on y from general models in which x has a nonlinear effect on y. Nevertheless, for $n > 2$, the experiment-specific model $\mathscr{M}_{\text{lin}}$ is QntT. Thus, even when an experiment-specific model is QntT, an experiment may do a poor job of differentiating general models.

8. EXAMPLE: DISCRETE-STATE MODELS

8.1. Description of Experiment and Model

On each trial of an experiment, a stimulus drawn from the set $\{S_s : s = 1, ..., N_S\}$ is presented to the subject who is then required to make a response from the set $\{R_r : r = 1, ..., N_R\}$. The observations collected in the experiment consist of estimates of the conditional probabilities:

$$\Pr(R_r \mid S_s) \text{ for all possible } R_r \text{ and } S_s. \tag{6}$$

Consider the following *discrete-state* model for the above experiment. According to this model, the subject performs his/her task in two stages. In the first stage, upon presentation of the stimulus, the subject forms a judgment about the stimulus. This judgment is drawn from a set of discrete judgments $\{J_j : j = 1, ..., N_J\}$. The model does not specify what these judgments are, but it does specify that the

number of them is N_J. The judgment is then passed on to a second stage which determines which response to make. It is assumed that, given the judgment, the response is *conditionally independent* of the stimulus. This means that

$$\Pr(R_r \mid J_j \,\&\, S_s) = \Pr(R_r \mid J_j) \tag{7}$$

for all possible S_s, J_j, and R_r. Let $\mathscr{M}(N_J)$ denote the version of the model in which there are N_J possible judgments.

EXAMPLE. Consider the following signal-detection experiment. The stimuli consist of either no signal or a signal having one of several different intensities. The subject's response is either a *yes* or a *no* bundled together with a confidence rating. The model $\mathscr{M}(2)$ postulates that the subject is limited to making one of two possible judgments. These may be called *suspect-signal* and *suspect-nonsignal.* The stimulus stochastically determines whether the subject enters the suspect-signal or the suspect-nonsignal state. This state stochastically determines the subject's *yes* or *no* selection plus confidence rating.

8.2. Predictions from Model

Expressions for the conditional probabilities (6) estimated in the experiment are readily derived for the model $\mathscr{M}(N_J)$. From (7) it follows that, for all possible R_r and S_s,

$$\Pr(R_r \mid S_s) = \sum_{j=1}^{N_J} \Pr(J_j \mid S_s) \Pr(R_r \mid J_j). \tag{8}$$

This collection of Eqs. (8) will now be re-expressed as a matrix equation.

Terminology. A matrix whose elements are all non-negative and which sum to one in each row is called a *probability matrix.*

Let $\mathbf{\Omega}$ denote the $N_S \times N_R$ probability matrix in which the element at the intersection of row s and column r is $\Pr(R_r \mid S_s)$. Let $\mathbf{\Pi_1}$ denote the $N_S \times N_J$ probability matrix in which the element at the intersection of row s and column j is $\Pr(J_j \mid S_s)$. Let $\mathbf{\Pi_2}$ denote the $N_J \times N_R$ probability matrix in which the element at the intersection of row j and column r is $\Pr(R_r \mid J_j)$. Then, $\mathbf{\Omega}$ is a matrix that characterizes the subject's observable performance, $\mathbf{\Pi_1}$ is a matrix of parameters that characterizes the operation of the first stage, and $\mathbf{\Pi_2}$ is a matrix of parameters that characterizes the operation of the second stage. Then, (8) may be rewritten:

$$\mathbf{\Omega} = \mathbf{\Pi_1 \Pi_2}.$$

Thus, the model $\mathscr{M}(N_J)$ may be described as follows: The model's outcome space is the set of all $N_S \times N_R$ probability matrices $\mathbf{\Omega}$, the model's parameter domain is the set of all $(\mathbf{\Pi_1}, \mathbf{\Pi_2})$ pairs where $\mathbf{\Pi_1}$ is a $N_S \times N_J$ probability matrix and $\mathbf{\Pi_2}$ is a $N_J \times N_R$ probability matrix, and the model's prediction function maps each $(\mathbf{\Pi_1}, \mathbf{\Pi_2})$ pair to the product $\mathbf{\Pi_1 \Pi_2}$. It is, of course, a redundant model.

8.3. Qualitative Testability

It is well known that the number of linearly independent columns in a matrix is the same as the number of linearly independent rows (Eves, 1966, Theorem 2.7.1). Hence, if $\mathbf{C}$ is an $m \times p$ matrix, then $\rho[\mathbf{C}] \leqslant m, p$. Recall that $\mathbf{\Omega}$ is a $N_S \times N_R$ probability matrix. So, if $N_S \wedge N_R$ denotes the minimum of N_S and N_R, then $\rho[\mathbf{\Omega}] \leqslant N_S \wedge N_R$. Moreover, it can be shown that this rank $N_S \wedge N_R$ is achieved by some $N_S \times N_R$ probability matrices.

If $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ and $n \times p$ matrices such that $\mathbf{C} = \mathbf{AB}$, then $\rho[\mathbf{C}] \leqslant \rho[\mathbf{A}]$ (Eves, 1966, Theorem 2.7.6) and, therefore, $\rho[\mathbf{C}] \leqslant n$. According to the model $\mathscr{M}(N_J)$, $\mathbf{\Omega} = \mathbf{\Pi_1 \Pi_2}$, where $\mathbf{\Pi_1}$ and $\mathbf{\Pi_2}$ are $N_S \times N_J$ and $N_J \times N_R$ matrices. Thus, the model predicts that the rank of $\mathbf{\Omega}$ cannot be larger than N_J. But recall that there do exist $N_S \times N_R$ probability matrices having rank $N_S \wedge N_R$. So, if $N_J < N_S \wedge N_R$, then there exist $\mathbf{\Omega}$s which, if observed in an experiment, would falsify the model $\mathscr{M}(N_J)$; thus, that model is QlT. Moreover, although we shall not present the proof here, it can be shown that the converse also holds. Thus, we have:

THEOREM 6. *The model $\mathscr{M}(N_J)$ is QlT if and only if $N_J < N_S \wedge N_R$.*

8.4. Quantitative Testability

As defined earlier, the model $\mathscr{M}(N_J)$ is redundant and its outcome space has Lebesgue measure zero. Therefore, it is not classifiable as either QntT or non-QntT. Consequently, we will investigate whether it can be given a courtesy classification. A nonredundant edition of $\mathscr{M}(N_J)$ will be examined—it does not matter which one; our conclusions will be unaffected by the particular nonredundant edition that we choose. Let the meaning of the symbol $\mathscr{M}(N_J)$ be changed; from now on, let it denote our chosen nonredundant edition.

Let N_{par} and N_{obs} respectively denote the number of parameters and the number of observables in the nonredundant $\mathscr{M}(N_J)$. Then

$$\begin{aligned} N_{\text{par}} &= N_S(N_J - 1) + N_J(N_R - 1), \\ N_{\text{obs}} &= N_S(N_R - 1). \end{aligned} \tag{9}$$

Furthermore, it can also be shown that

$$\max \rho[\mathbf{J}_{\mathscr{M}(N_J)}] = \begin{cases} N_{\text{obs}} - (N_S - N_J)(N_R - N_J) & \text{if } N_J < N_S \wedge N_R \\ N_{\text{obs}} & \text{otherwise.} \end{cases} \tag{10}$$

The Jacobian Rule *suggests* that the model $\mathscr{M}(N_J)$ is QntT if and only if $\max \rho[\mathbf{J}_{\mathscr{M}(N_J)}] < N_{\text{obs}}$, in other words, if and only if $N_J < N_S \wedge N_R$. In fact, this is correct. By applying Theorem 9 and Proposition 10 of Bamber and van Santen (1985), the following theorem can be proved.

THEOREM 7. *The model $\mathscr{M}(N_J)$ is QntT if and only if it is QlT, i.e., if and only if $N_J < N_S \wedge N_R$.*

Inaccuracy of the Counting Rule. Take $N_S = 3$, $N_J = 2$, and $N_R = 3$. Then, Theorem 7 shows that $\mathscr{M}(N_J)$ is QntT even though, according to the Counting Rule, it should not be so.

Interpretation of Theorem 7. This theorem may be interpreted as follows. The input to the first stage has N_S possible states, the information passed from the first stage to the second stage has N_J possible states, and the output from the second stage has N_R possible states. Thus, the model $\mathscr{M}(N_J)$ is QntT if and only if the information passed from the first stage to the second stage has to go through a bottleneck having fewer states than either the input to the first stage or the output from the second stage.

State-trace analysis. With regard to the key role played by a bottleneck, $\mathscr{M}(N_J)$ is similar to state-trace analysis. In the state-trace model of Fig. 5(b) in Bamber (1979), a two-dimensional independent-variable space maps into a one-dimensional latent-variable space which, in turn, maps into a two-dimensional dependent-variable space. It is the one-dimensional bottleneck between the two-dimensional input and output that makes it possible to derive any predictions from the model.

8.5. Identifiability

Comparing Eqs. (10) and (9), it is seen that, for $N_J > 1$, the maximum rank of the Jacobian matrix is smaller than the number of parameters in the model $\mathscr{M}(N_J)$. Since the model satisfies all the conditions needed to apply Proposition 20 of Bamber and van Santen (1985), it follows from that proposition that the model is not identifiable.

So, for $1 < N_J < N_S \wedge N_R$, the model $\mathscr{M}(N_J)$ is not identifiable, but it is QntT. When we empirically test this model, our empirical results cannot tell us the values of the model's parameters, but they do tell us whether or not there is a bottleneck between the stimulus input and the response output.

9. ALTERNATIVE ENCODINGS OF OUTCOMES

Whenever an experiment is done, the experimenter must choose, from an infinite set of possibilities, a method of numerically encoding the outcome of the experiment. For the most part in this paper, we have regarded the experimenter's choice of numeric encoding method as an integral part of the experiment itself and have not considered the possibility that some other encoding method might have been used.

Our implicit justification for accepting without question the experimenter's encoding method was our belief that the experimenter's choice of encoding method did not matter. In other words, a model's classification as QntT or non-QntT would be invariant across all reasonable encoding schemes.

[Unreasonable encoding schemes are another matter. For example, there exist pathological functions that map a line segment onto sets of positive Lebesgue measure in $\Re^2$ (Gelbaum & Olmsted, 1964, Chapter 10, Examples 5, 6, and 10). By using a pathological function to transform a reasonable encoding scheme into an

unreasonable one, it is a simple matter to convert a model from QntT to non-QntT and *vice versa.*]

There has been one exception to our uncritical acceptance of the experimenter's encoding method. That exception occurs when a redundant model is unclassifiable as QntT or non-QntT but all of its nonredundant editions are QntT. In that case, we give the original redundant model the courtesy classification of QntT. The justification for such courtesy classifications is that, in an experiment having redundant observables, it is merely a convention as to whether all of the observables are reported or only a nonredundant subset of them. The choice of a reporting convention should not influence whether a model is classified QntT or non-QntT.

However, this principle has wider application. We would like our classification of a model's testability to be independent of any scientist's judgment of what was an appropriate method for numerically encoding an experiment's outcome.

Unfortunately, under our present definition, the scientist's choice of encoding method can influence whether a model is *classifiable or unclassifiable* as regards the QntT vs non-QntT distinction. This is demonstrated in the following two examples in which a model is unclassifiable under one encoding scheme but is classifiable under another.

Direction model. Consider an experiment in which the observable is a spatial direction in the horizontal plane. One experimenter chooses to report the outcome of the experiment as an angle ϕ in radians, with the outcome space being the half-closed half-open interval $[0, 2\pi)$. Suppose that the model predicts the outcome $\phi = \pi/2$. Then, for this experimenter, the model is QntT. A second experimenter chooses to report the outcome of the experiment in the form $(\cos\phi, \sin\phi) \in \Re^2$. For the second experimenter, the outcome space is the unit-radius circle centered on the origin. Because this outcome space has Lebesgue measure zero in $\Re^2$, the model is not classifiable as either QntT or non-QntT. (Moreover, because the model's observables are not redundant, there is no opportunity of assigning a courtesy classification.)

Bouncing-ball model. Consider an experiment in which the observable is the number of times that a dropped ball bounces. One experimenter reports the outcome of the experiment in the form of a nonnegative integer equal to the number of bounces. For this experimenter, the outcome space $\mathcal{O}$ is the set of nonnegative integers and the prediction range $\mathcal{P}$ is a nonempty subset of the outcome space. On intuitive grounds, this experimenter *ought* to classify the model as non-QntT. (The reason is that, if I is an integer selected randomly from $\mathcal{O}$ and if no integer in $\mathcal{O}$ is *a priori* excluded from being selected, then necessarily $\Pr(I \in \mathcal{P}) > 0$.) However, when the formal definition (Definition 3) is applied, it is seen that the outcome space has Lebesgue measure zero in $\Re$ and, thus, the model is not classifiable as either QntT or non-QntT. (However, it is QlT, assuming that $\mathcal{P}$ is a proper subset of $\mathcal{O}$.)

In contrast, a second experimenter uses a stochastic method of encoding the outcome of the experiment. This experimenter reports the experimental outcome in the form of a real number equal to the sum of (a) the integer number of bounces plus

(b) a real number randomly sampled from the half-closed half-open interval $[0, 1)$. Note that this stochastic encoding of the experiment's outcome loses no information. For example, if the second experimenter reports that the outcome of the experiment was 2.7362..., then we know that the ball bounced twice. For the second experimenter, the outcome space is the set of nonnegative real numbers; this has infinite Lebesgue measure in $\Re$. If $\mathscr{P}_1$ denotes the first experimenter's prediction range, then the second experimenter's prediction range is $\mathscr{P}_2 = \bigcup_{k \in \mathscr{P}_1} [k, k+1)$. Because $\mathscr{P}_1$ cannot be empty, $\mathscr{P}_2$ must have Lebesgue measure greater than zero. Hence, for the second experimenter, the model is non-QntT.

These examples show that it is possible to find (a) a reasonable model and (b) a pair of reasonable encoding schemes such that the model is classifiable (as QntT or non-QntT) under one encoding scheme but is unclassifiable under the other. We wish to emphasize, however, that we know of no cases where a reasonable model is QntT under one reasonable encoding scheme but is non-QntT under another. Indeed, it is our opinion that such cases do not exist.

This raises an interesting question for future research: *Can the opinion expressed at the end of the last paragraph be justified by rigorous mathematics*?

Recommendations. (a) If a serious scientific model is classified as either QntT or non-QntT under Definition 3, accept that classification. We believe that this classification would be invariant across all encoding schemes that anyone would seriously propose to use in an experiment. (b) If such a model is not classifiable under Definition 3, try another encoding scheme.

10. SUMMARY

Definition. A *model*, as distinct from a *general model*, predicts the outcome of one experiment only. Specifically, a model is an ordered triple $(\mathscr{D}, f^{\mathfrak{p}}, \mathscr{O})$, where $\mathscr{D}$ is the set of conceivable parameter vectors and the function $f^{\mathfrak{p}}$ maps $\mathscr{D}$ into the set $\mathscr{O}$ of conceivable outcome vectors. The model predicts that the outcome of the experiment will belong to the set $f^{\mathfrak{p}}(\mathscr{D})$.

Model properties. A model is *nonredundant* provided that knowing the values of some of its parameters or of some of its observables does not make it possible to deduce the values of the others. A model is *qualtitatively testable* (QlT) if the model is falsifiable. A model is *quantitatively testable* (QntT) if $\mathscr{O}$ has positive Lebesgue measure whereas $f^{\mathfrak{p}}(\mathscr{D})$ has Lebesgue measure zero. In a QntT model, an experiment's outcome is unlikely to agree with the model by chance because an outcome randomly selected from $\mathscr{O}$ has zero probability of belonging to $f^{\mathfrak{p}}(\mathscr{D})$. A model is *identifiable* if the values of its parameters can be ascertained from empirical observations.

Rules of thumb. The Counting Rule states that a nonredundant model is QntT if and only if it has fewer parameters than observables. However, this rule can be safely applied only to identifiable models. If a model is unidentifiable, one must apply a generalization of the Counting Rule known as the Jacobian Rule. This rule states that a nonredundant model is QntT if and only if the maximum rank of the

Jacobian matrix of f^{p} is smaller than the number of observables in the experiment. The Identifiability Rule states that a nonredundant model is identifiable if and only if the maximum Jacobian-matrix rank equals the number of parameters in the model.

The above rules of thumb give only presumptive conclusions regarding a model's testability and identifiability. In order to reach definitive conclusions, additional analyses must be performed. These rules were shown to give correct answers when applied to linear models and to discrete-state models.

Research issue. An issue in need of further research is whether a model's classification as QntT or non-QntT is invariant across different numerical encodings of an experiment's outcome.

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