Assignment 3: Billiard Simulation

1 Requirements

Write a billiard simulation according to the following requirements:

1. Create a table of realistic proportions. Add cushions and legs. Place the table on some ground. See here for table dimensions: https://en.wikipedia.org/wiki/Billiard_table#Dimensions

2. Add 8 billiard balls (ideally as wireframe models to observe the rolling motion) of realistic size:

   (a) Place the balls initially at random, non-overlapping positions on the table.

   (b) Move the balls according to a random velocity vector assigned to each ball. Due to friction the speed of each ball drops by 20% each second.

   (c) Make sure the balls are rolling without slip and not just sliding.

   (d) Take into account reflection off the cushions. The speed of a ball drops by 20% at each reflection due to loss of energy.

3. Add the texture images in the `PoolBallSkins` directory to the eight billiard balls.

4. Implement collisions between billiard balls by implementing elastic collisions between the balls (see section 2) and reducing the speed of a ball by 30 % at each of its collision.

5. Add a ceiling to the scene. Also add a spot light above the table whose position is indicated by a yellow light bulb. Connect the light bulb to the ceiling with some cord.

6. Add a shadow of the table visible on the ground and shadows of the billiard balls visible on the table.

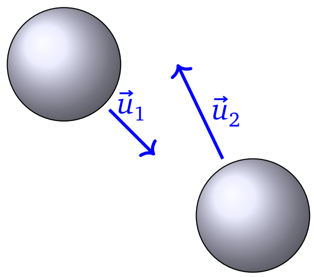
7. Take into account that at a later stage more balls might get added to the program by *not* copying and pasting the code for one ball eight times. Use a loop instead.

2 Elastic collision between balls of equal mass

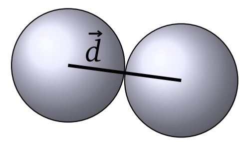
Elastic collision of two balls conserves the kinetic energy of the balls throughout the collision process, i.e. no energy is dissipated during the collision. Given the velocities of two balls *before* the collision, the task is to calculate the velocities of the balls *after* the collision. In- and outgoing velocities are denoted as follows:

- $\vec{u}_1, \vec{u}_2$: velocities of balls 1 and 2 *before* the collision
- $\vec{v}_1, \vec{v}_2$: velocities of balls 1 and 2 *after* the collision

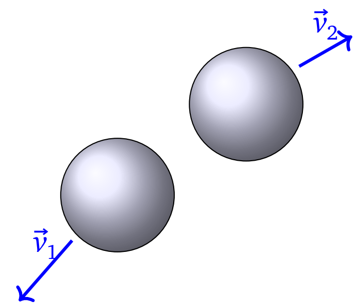
The collision process can be visualized as follows:



Before collision



At collision



After collision

Let $\vec{d}$ denote the vector between the centers of the balls at the moment of the collision. The collision process can be analyzed by decomposing the velocities into components parallel and perpendicular to $\vec{d}$:

$$\vec{u}_k = \vec{u}_{k,\parallel} + \vec{u}_{k,\perp} \qquad \text{with} \qquad \vec{u}_{k,\parallel} = \frac{\vec{u}_k \cdot \vec{d}}{|\vec{d}|^2}\,\vec{d} \tag{1}$$

for $k = 1,2$. The same decomposition can be done for the outgoing velocities $\vec{v}_k$.
These components change as follows during the collision:

- The velocity components *perpendicular* to $\vec{d}$ are unaffected by the collision, i.e.

$$\vec{v}_{k,\perp} = \vec{u}_{k,\perp} \qquad \text{for } k = 1,2$$

- The velocity components *parallel* to $\vec{d}$ follow the laws of elastic collision in one dimension (see next section, in particular equation (5)), i.e.

$$\vec{v}_{1,\parallel} = \vec{u}_{2,\parallel} \qquad \text{and} \qquad \vec{v}_{2,\parallel} = \vec{u}_{1,\parallel}$$

Inserting these results into the decomposition of the outgoing velocity $\vec{v}_1$ yields:

$$\vec{v}_1 = \vec{v}_{1,\parallel} + \vec{v}_{1,\perp} = \vec{u}_{2,\parallel} + \vec{u}_{1,\perp} = \vec{u}_{2,\parallel} + \vec{u}_1 - \vec{u}_{1,\parallel} = \vec{u}_1 - (\vec{u}_1 - \vec{u}_2)_\parallel$$

Inserting the explicit form of the parallel components (1) gives the final result:

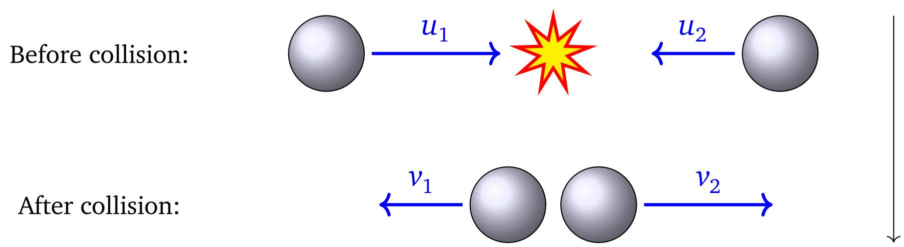
$$\boxed{\vec{v}_1 = \vec{u}_1 - \frac{\vec{d} \cdot (\vec{u}_1 - \vec{u}_2)}{|\vec{d}|^2}\,\vec{d}}$$

In a similar way one obtains

$$\boxed{\vec{v}_2 = \vec{u}_2 + \frac{\vec{d} \cdot (\vec{u}_1 - \vec{u}_2)}{|\vec{d}|^2}\,\vec{d}}$$

Elastic collision in one dimension

In one dimension all velocities are scalar values. Moreover, the masses of the two balls are assumed to be identical and will be denoted by m.



The aim of this analysis is to calculate the outgoing velocities given the incoming velocities. The whole process is determined by the following two principles:

Momentum conservation: The momentum of a ball with mass m and velocity v is defined to be $m \cdot v$. The principle of momentum conservation states that the sum of the momenta of the incoming balls is identical to the sum of the momenta of the outgoing balls:

$$\begin{aligned} mu_1 + mu_2 &= mv_1 + mv_2 \\ \iff u_1 + u_2 &= v_1 + v_2 \end{aligned} \tag{2}$$

Energy conservation: The kinetic energy[1] of a ball with mass m and velocity v is $E = \frac{1}{2}mv^2$.

The principle of energy conservation states that the sum of the energies of the incoming balls is identical to the sum of the energies of the outgoing balls:

$$\begin{aligned} \frac{1}{2}mu_1^2 + \frac{1}{2}mu_2^2 &= \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 \\ \iff u_1^2 + u_2^2 &= v_1^2 + v_2^2 \\ \iff u_1^2 - v_1^2 = (u_1+v_1)(u_1-v_1) &= v_2^2 - u_2^2 = (v_2+u_2)(v_2-u_2) \end{aligned} \tag{3}$$

Dividing equation (3) by equation (2) in the form $u_1 - v_1 = v_2 - u_2$ leads to

$$u_1 + v_1 = u_2 + v_2 \tag{4}$$

Equations (2) and (4) are two equations for the two unknowns v_1 and v_2 which can be solved to yield

$$\boxed{v_1 = u_2\,, \qquad v_2 = u_1} \tag{5}$$

i.e. the velocities are simply exchanged. This is a special case of a more general collision formula between two balls of unequal mass, see for instance https://en.wikipedia.org/wiki/Elastic_collision#One-dimensional_Newtonian.

A simple edge cases to test the result is a moving ball hitting a ball at rest, i.e. $u_1 \neq 0$, $u_2 = 0$. This gives

$$v_1 = 0\,, \qquad v_2 = u_1 \neq 0.$$

The initially moving ball rests after the collision while the ball initially at rest moves away with the incoming velocity. See https://commons.wikimedia.org/wiki/File:Elastischer_sto%C3%9F.gif for a visualization.

[1] This is only true for a sliding ball. For a rolling ball the kinetic energy is $E = \frac{7}{10}mv^2$, assuming the ball is solid and made of a homogeneous material. But since the prefactor of the kinetic energy cancels out of the calculation this makes no difference for the collision dynamics.

3 Coding style

- Stick to the coding style guide which can be found in the Readme file for chapter 3 in the gitlab repository.
- It is your choice whether you load the `three.js` library as a module or just as a JavaScript text file.

4 Handing in the solution

No group work allowed. Every course participant has to write her or his *own* code!
Implement your entire solution within the two files `Assignment1.html` and `Assignment1.js`. Feel free to change these files as you like but do not add any further files.
The deadline for submission is the **10th of January 2021**.