2
The Two-Body Problem

So we grew together,
Like to a double cherry, seeming parted,
But yet an union in partition –
Two lovely berries moulded on one stem;
So, with two seeming bodies, but one heart.

William Shakespeare, *A Midsummer Night's Dream, II, ii*

2.1 Introduction

The two-body problem is perhaps the simplest, integrable problem in solar system dynamics. It concerns the interaction of two point masses moving under a mutual gravitational attraction described by Newton's universal law of gravitation, Eq. (1.1). The wide variety of masses in the solar system permits the orbits of most planets and satellites to be approximated by two-body motion, consisting of a smaller body moving around a much larger central body. The effects of other bodies can usually be treated as perturbations to the two-body system. For example, the path of Jupiter (mass $m_{\mathrm{J}} = 1.9 \times 10^{27}$ kg) around the Sun (mass $m_{\mathrm{Su}} = 2.0 \times 10^{30}$ kg $= 1000\, m_{\mathrm{J}}$) is basically an ellipse with the principal perturbations coming from the other planets, notably Saturn (mass $m_{\mathrm{Sa}} = 5.7 \times 10^{26}$ kg).

In Volume I of his *Principia*, Isaac Newton (1687) showed that only two types of central force could give rise to the observed elliptical motion of the planets. The first was a linear force directed towards the centre of the ellipse, and the second was an inverse square force directed towards one focus of the ellipse. However, only the second type of force can give rise to Kepler's empirical laws of planetary motion (Sect. 1.3). In this chapter we derive the basic equations of planetary motion and solve the two-body problem, showing how Kepler's laws arise.

2.2 Equations of Motion

Consider the motion of two masses m_1 and m_2 with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ referred to some origin O fixed in inertial space (see Fig. 2.1).

The vector $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ denotes the relative position of the mass m_2 with respect to m_1. The gravitational forces and the consequent accelerations experienced by the two masses are

$$\mathbf{F}_1 = +\mathcal{G}\frac{m_1 m_2}{r^3}\mathbf{r} = m_1\ddot{\mathbf{r}}_1 \qquad \text{and} \qquad \mathbf{F}_2 = -\mathcal{G}\frac{m_1 m_2}{r^3}\mathbf{r} = m_2\ddot{\mathbf{r}}_2 \tag{2.1}$$

respectively, where $\mathcal{G} = 6.67260 \times 10^{-11}\,\mathrm{N m^2 kg^{-2}}$ is the *universal gravitational constant*. Thus

$$m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = 0, \tag{2.2}$$

which can be integrated directly twice to give

$$m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 = \mathbf{a} \qquad \text{and} \qquad m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = \mathbf{a}t + \mathbf{b}, \tag{2.3}$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors. If $\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1 + m_2)$ denotes the position vector of the centre of mass, then Eqs. (2.3) can be written

$$\dot{\mathbf{R}} = \frac{\mathbf{a}}{m_1 + m_2} \qquad \text{and} \qquad \mathbf{R} = \frac{\mathbf{a}t + \mathbf{b}}{m_1 + m_2}. \tag{2.4}$$

This implies that either the centre of mass is stationary (if $\mathbf{a} = 0$) or it is moving with a constant velocity in a straight line with respect to the origin O. Note that this result is not specific to the inverse square law of force.

Now consider the motion of m_2 with respect to m_1. This allows us to simplify the problem without losing any of its essential features. In Sect. 2.7 we shall revert to considering motion in the centre of mass frame. Writing $\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1$, and using Eq. (2.1), we obtain

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} + \mu\frac{\mathbf{r}}{r^3} = 0, \tag{2.5}$$

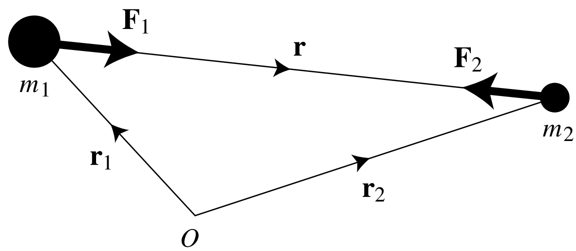

Fig. 2.1. A vector diagram for the forces acting on two masses, m_1 and m_2, with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$.

where $\mu = \mathcal{G}(m_1 + m_2)$. This is the *equation of relative motion*. In order to solve it and find the path of m_2 relative to m_1 we must first derive several constants of the motion.

Taking the vector product of $\mathbf{r}$ with Eq. (2.5) we have $\mathbf{r} \times \ddot{\mathbf{r}} = 0$, which can be integrated directly to give

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}, \tag{2.6}$$

where $\mathbf{h}$ is a constant vector perpendicular to both $\mathbf{r}$ and $\dot{\mathbf{r}}$. Hence the motion of m_2 about m_1 lies in a plane perpendicular to the direction defined by $\mathbf{h}$. This also implies that the position and velocity vectors always lie in the same plane (see Fig. 2.2). Equation (2.6) is commonly referred to as the *angular momentum integral*. However, although $h = |\mathbf{h}|$ for systems in which $m_2 \ll m_1$ is approximately equal to the orbital angular momentum per unit mass of the body m_2, it is not the actual angular momentum in the inertial system since this is calculated using the position and velocity vectors referred to the centre of mass. We consider this in more detail in Sect. 2.7.

Since $\mathbf{r}$ and $\dot{\mathbf{r}}$ always lie in the same plane (the *orbit plane*) it is natural that we now restrict ourselves to considering motion in that plane; the motion referred to an arbitrary reference frame is considered in Sect. 2.8. We now transform to a polar coordinate system (r, θ) referred to an origin centred on the mass m_1 and an arbitrary reference line corresponding to $\theta = 0$. Note that even though the centre of mass of m_1 and m_2 could be moving in inertial space, the direction of the reference line remains fixed. If we let $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ denote unit vectors along and perpendicular to the radius vector respectively, then the position, velocity, and acceleration vectors can be written in polar coordinates as

$$\mathbf{r} = r\hat{\mathbf{r}}, \qquad \dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}, \qquad \ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right)\right]\hat{\boldsymbol{\theta}}. \tag{2.7}$$

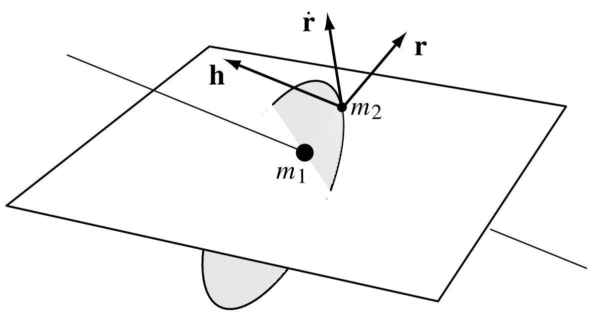

Fig. 2.2. The motion of m_2 with respect to m_1 defines an orbital plane (shaded region), because $\mathbf{r} \times \dot{\mathbf{r}}$ is a constant vector, $\mathbf{h}$, the angular momentum vector, and this is always perpendicular to the orbit plane.

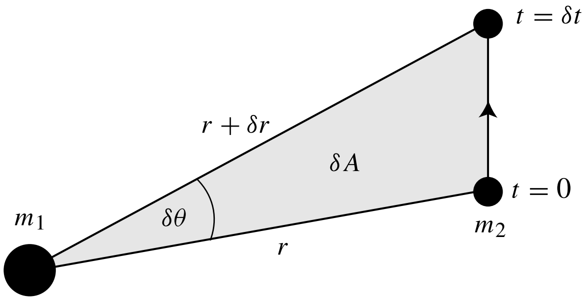

Fig. 2.3. The area δA swept out in a time δt as a position vector moves through an angle $\delta\theta$.

Substituting the expression for $\dot{\mathbf{r}}$ into Eq. (2.6) gives $\mathbf{h} = r^2\dot{\theta}\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is a unit vector perpendicular to the plane of the orbit forming a right-handed triad with $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. Hence

$$h = r^2\dot{\theta}\,. \tag{2.8}$$

Consider the motion of the mass m_2 during a time interval δt (see Fig. 2.3). At time $t = 0$ it has polar coordinates $(r,\ \theta)$, while at time $t + \delta t$ its polar coordinates have changed to $(r + \delta r,\ \theta + \delta\theta)$. The area swept out by the radius vector in time δt is

$$\delta A \approx \frac{1}{2} r\,(r + \delta r)\sin\delta\theta \approx \frac{1}{2} r^2 \delta\theta, \tag{2.9}$$

where we have neglected second- and higher-order terms in the small quantities. Hence, by dividing each side by δt and taking the limit as $\delta t \to 0$ we have

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2} r^2 \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{2} h. \tag{2.10}$$

Since h is a constant this implies that equal areas are swept out in equal times and hence Eq. (2.10) is the mathematical form of Kepler's second law of planetary motion (see Sect. 1.3). Note that this does not require an inverse square law of force, but only that the force is directed along the line joining the two masses.

2.3 Orbital Position and Velocity

We obtain a scalar equation for the relative motion by substituting the expression for $\ddot{\mathbf{r}}$ from Eq. (2.7) into Eq. (2.5); comparing the $\hat{\mathbf{r}}$ components gives

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}\,. \tag{2.11}$$

To solve this equation and find r as a function of θ we need to make the substitution $u = 1/r$ and to eliminate the time by making use of the constant $h = r^2\dot{\theta}$. By differentiating r with respect to time, we obtain

$$\dot{r} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\theta}\dot{\theta} = -h\frac{\mathrm{d}u}{\mathrm{d}\theta} \qquad \text{and} \qquad \ddot{r} = -h\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\dot{\theta} = -h^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} \tag{2.12}$$

and hence Eq. (2.11) can be written

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{\mu}{h^2}\,. \tag{2.13}$$

This is a second-order, linear differential equation with a general solution

$$u = \frac{\mu}{h^2}\left[1 + e\cos(\theta - \varpi)\right], \tag{2.14}$$

where e (an amplitude) and ϖ (a phase) are two constants of integration. Substituting back for r we have

$$r = \frac{p}{1 + e\cos(\theta - \varpi)}, \tag{2.15}$$

which is the general equation of a conic in polar coordinates where e is the *eccentricity* and p is the *semilatus rectum* given by

$$p = h^2/\mu\,. \tag{2.16}$$

The four possible conics are:

$$\begin{array}{lll} \text{circle:} & e = 0, & p = a; \\ \text{ellipse:} & 0 < e < 1, & p = a(1 - e^2); \\ \text{parabola:} & e = 1, & p = 2q; \\ \text{hyperbola:} & e > 1, & p = a(e^2 - 1), \end{array} \tag{2.17}$$

where the constant a is the *semi-major axis* of the conic. In the special case of the parabola p is defined in terms of q, the distance to the central mass at closest approach. The conic section curves derive their name from the curves formed by the intersection of various planes with the surface of a cone (see Fig. 2.4).

The type of conic is determined by the angle the plane makes with the horizontal. If the plane is horizontal, that is, perpendicular to the axis of symmetry of the cone, then the resulting curve is a circle. If the angle is less than the slope angle of the cone then an ellipse results, whereas if the plane is parallel to the slope of the cone a parabola results. A hyperbola results if the angle the plane makes with the horizontal is anywhere between the slope angle of the cone and the vertical.

In the context of the two-body problem the path of a planet about the Sun is elliptical and closed in inertial space (see Fig. 2.5), and hence Kepler's first law of planetary motion (see Sect. 1.3) is a consequence of the inverse square law of force. Note that the mass m_1 fills one focus of the ellipse while the other focus is empty.

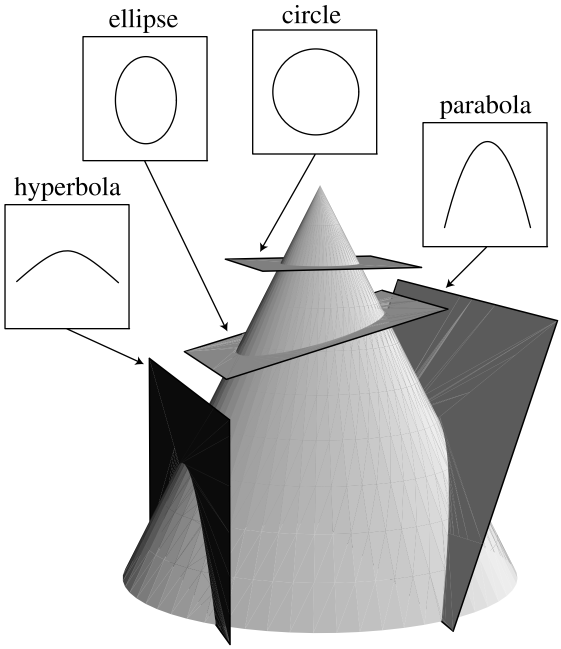

Fig. 2.4. The intersections of planes at different angles with the surface of a cone form the family of curves known as the conic sections.

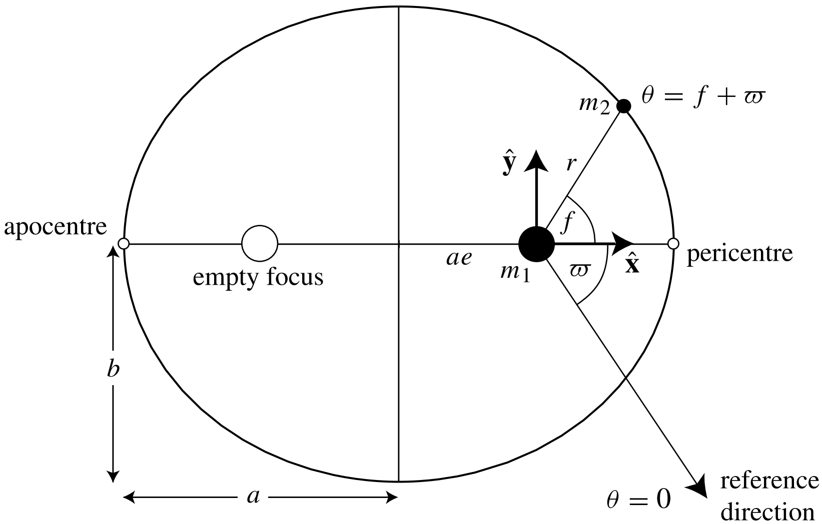

Fig. 2.5. The geometry of the ellipse of semi-major axis a, semi-minor axis b, eccentricity e, and longitude of pericentre ϖ.

Although a large number of cometary orbits have $e \approx 1$, most permanent members of the solar system have $e \ll 1$. The notable exceptions among the planets are Pluto ($e = 0.25$) and Mercury ($e = 0.21$), while Nereid ($e = 0.75$), a moon of Neptune, has the largest eccentricity of any known natural satellite. Consequently, throughout most of this book we concentrate on elliptical motion. In this case $p = a(1 - e^2)$ and the quantities a and e are related by

$$b^2 = a^2(1 - e^2), \tag{2.18}$$

where b is the *semi-minor axis* of the ellipse (see Fig. 2.5); we also have

$$r = \frac{a(1 - e^2)}{1 + e\cos(\theta - \varpi)}\,. \tag{2.19}$$

In celestial mechanics it is customary to use the term longitude when referring to any angle that is measured with respect to a reference line fixed in inertial space. The angle θ is called the *true longitude*. A simple inspection of Eq. (2.19) shows that the minimum and maximum values of the orbital radius are $r_{\mathrm{p}} = a(1 - e)$ and $r_{\mathrm{a}} = a(1 + e)$, which occur when $\theta = \varpi$ and $\theta = \varpi + \pi$ respectively. These points in the orbit are called the *pericentre* (or *periapse*) and the *apocentre* (or *apoapse*) respectively, although other names can be used for particular systems (e.g., perihelion, perijove, periselenium). Note that the distance of either focus from the centre of the ellipse is ae.

The angle ϖ (pronounced "curly pi") is called the *longitude of pericentre*. Although this is a constant for the two-body problem, it can vary with time when additional perturbations are introduced (see Chaps. 3, 6–8). It is usually more convenient to refer the angular coordinate to the pericentre rather than to the arbitrary reference line. This leads to the introduction of the angle $f = \theta - \varpi$ (see Fig. 2.5), which is called the *true anomaly*. Since ϖ is constant the path is closed and the angular position is described by f or θ, which are 2π-periodic variables. Hence Eq. (2.19) can be written

$$r = \frac{a(1 - e^2)}{1 + e\cos f}\,. \tag{2.20}$$

Using a Cartesian coordinate system centred on the central mass with the x axis pointing towards the pericentre (see Fig. 2.5), the components of the position vector are

$$x = r\cos f \qquad \text{and} \qquad y = r\sin f\,. \tag{2.21}$$

In one orbital period T the area swept out by a radius vector is simply the area $A = \pi ab$ enclosed by the ellipse. From Eq. (2.10) this area has to equal $hT/2$ and hence, given that $h^2 = \mu a(1 - e^2)$,

$$T^2 = \frac{4\pi^2}{\mu}a^3, \tag{2.22}$$

which corresponds to Kepler's third law of planetary motion (see Sect. 1.3). Note that the period of the orbit is independent of e and is a function of μ and a only.

Consider the case of two objects of mass m and m', orbiting a central object of mass m_c. Let the orbiting objects have semi-major axes a and a' and orbital periods T and T'. Equation (2.22) gives

$$\frac{m_c + m}{m_c + m'} = \left(\frac{a}{a'}\right)^3 \left(\frac{T'}{T}\right)^2. \tag{2.23}$$

In the case of planets orbiting the Sun we have $m, m' \ll m_c$ and hence $(a/a')^3 \approx (T/T')^2$. Therefore, if a and T denote the values of the semi-major axis and the period of the Earth's orbit and the unit of length is taken to be the *astronomical unit AU* (1 AU is the approximate semi-major axis of the Earth's orbit) and the unit of time is taken to be the year (the approximate period of the Earth's orbit), we have $T' \approx a'^{3/2}$.

If any solar system object (e.g., an asteroid or a comet) has a small natural or artificial satellite, then observations of the distance and period of the satellite can be used with Kepler's third law to derive an estimate of the mass of the object. Consider Eq. (2.22) applied to the Sun–object and object–satellite systems. Let m_c, m, and m' now denote the masses of the Sun, object, and satellite respectively with similar definitions for the semi-major axes and orbital periods. This gives

$$\frac{m + m'}{m_c + m} \approx \frac{m}{m_c} = \left(\frac{a'}{a}\right)^3 \left(\frac{T}{T'}\right)^2, \tag{2.24}$$

where we have taken $m' \ll m$ and $m \ll m_c$. This means that the mass of the object can be estimated from the orbital properties of its satellite.

Figure 2.6 shows an image of the asteroid (243) Ida and its moon Dactyl taken by the *Galileo* spacecraft on its way to Jupiter. Direct estimates of the masses of asteroids are notoriously difficult because of their small size (the largest asteroid, (1) Ceres, has a diameter of 913 km) and hence their small perturbations on other objects. Although analysis of the *Viking* data (Standish & Hellings 1989) has permitted mass determinations of the larger objects due to their direct perturbations on the orbit of Mars, similar calculations for the smaller asteroids are almost impossible. However, observations of Dactyl's motion using images obtained by the *Galileo* spacecraft have resulted in an estimated density of 2.6 ± 0.5 g cm^{-3} (Belton et al. 1995).

Since the angle θ covers 2π radians in one orbital period we can define the "average" angular velocity, or the *mean motion*, n as

$$n = \frac{2\pi}{T} \tag{2.25}$$

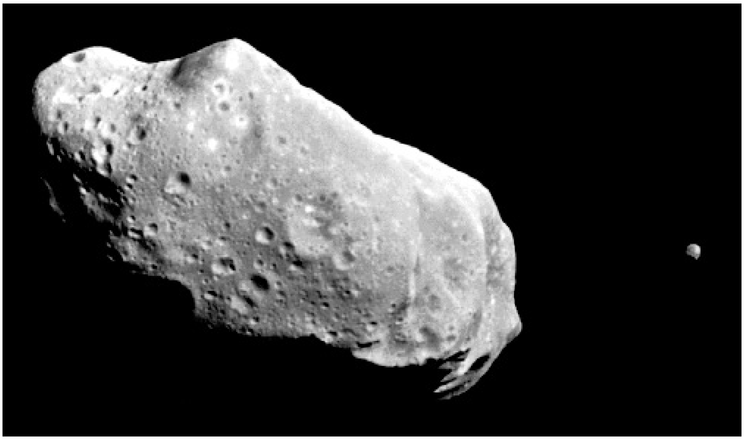

Fig. 2.6. An image of the asteroid (243) Ida and its moon Dactyl taken by the *Galileo* spacecraft on 28th August 1993. Ida is approximately $56 \times 24 \times 21$ km in size and its moon Dactyl is about 1.4 km across. *(Image courtesy of NASA/JPL.)*

and we can write

$$\mu = n^2 a^3 \quad \text{and} \quad h = na^2\sqrt{1-e^2} = \sqrt{\mu a(1-e^2)}\,. \tag{2.26}$$

Although the mean motion is constant in the two-body problem, the actual angular velocity $\dot{f}$ of the orbiting body *is* a function of the longitude.

We can derive another constant of the motion by taking the scalar product of $\dot{\mathbf{r}}$ with Eq. (2.5) and using the expressions for $\mathbf{r}$ and $\dot{\mathbf{r}}$ from Eq. (2.7). This gives the scalar equation

$$\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} + \mu \frac{\dot{r}}{r^2} = 0, \tag{2.27}$$

which can be integrated to give

$$\frac{1}{2}v^2 - \frac{\mu}{r} = C, \tag{2.28}$$

where $v^2 = \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$ is the square of the velocity and C is a constant of the motion. Equation (2.28), often called the *vis viva integral*, shows that the orbital energy per unit mass is conserved. Thus the two-body problem has four constants of the motion: the energy integral C and the three components of the angular momentum integral, $\mathbf{h}$. Note that it is also possible to express these constants in different forms such as the orbital elements, or quantities such as the eccentricity vector (see Question 2.4).

By finding another expression for v^2 we can derive an expression for C. Since ϖ is fixed we have $\dot{\theta} = \mathrm{d}(f + \varpi)/\mathrm{d}t = \dot{f}$ and using the definition of $\dot{\mathbf{r}}$ from Eq. (2.7) gives

$$v^2 = \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \dot{r}^2 + r^2 \dot{f}^2\,. \tag{2.29}$$

Differentiating Eq. (2.20) we have

$$\dot{r} = \frac{r\,\dot{f}\,e \sin f}{1 + e \cos f}\,. \tag{2.30}$$

Using $r^2 \dot{f} = h = na^2\sqrt{1-e^2}$, we can write

$$\dot{r} = \frac{na}{\sqrt{1-e^2}}\,e \sin f \tag{2.31}$$

and

$$r\dot{f} = \frac{na}{\sqrt{1-e^2}}(1 + e \cos f), \tag{2.32}$$

so that Eq. (2.29) can be written

$$v^2 = \frac{n^2a^2}{1-e^2}(1 + 2\,e \cos f + e^2) = \frac{n^2a^2}{1-e^2}\left(\frac{2a(1-e^2)}{r} - (1-e^2)\right). \tag{2.33}$$

Hence

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right). \tag{2.34}$$

Consequently, the velocity of the orbiting body is a maximum at pericentre ($f = 0$) and a minimum at apocentre ($f = \pi$). The respective values are

$$v_{\mathrm{p}} = na\sqrt{\frac{1+e}{1-e}} \quad \text{and} \quad v_{\mathrm{a}} = na\sqrt{\frac{1-e}{1+e}}\,. \tag{2.35}$$

We can also find the x and y components of the velocity vector by taking the time derivatives of the expressions for x and y in Eq. (2.21) and substituting the expressions for $\dot{r}$ and $r\dot{f}$ given in Eqs. (2.31) and (2.32). This gives

$$\begin{aligned} \dot{x} &= -\frac{na}{\sqrt{1-e^2}}\,\sin f, \\ \dot{y} &= +\frac{na}{\sqrt{1-e^2}}\,(e + \cos f)\,. \end{aligned} \tag{2.36}$$

By comparing Eq. (2.34) with Eq. (2.28) we see that the energy constant can be written as

$$C = -\frac{\mu}{2a}, \tag{2.37}$$

and hence the energy of an elliptical orbit is a function of its semi-major axis alone and is independent of the eccentricity. Similar quantities can be defined for parabolic and hyperbolic orbits. It can be shown that

$$C_{\mathrm{para}} = 0, \quad \text{and} \quad C_{\mathrm{hyper}} = \frac{\mu}{2a}\,. \tag{2.38}$$

2.4 The Mean and Eccentric Anomalies

In the previous section we showed that, given the value of the true anomaly f, we can calculate the orbital radius and velocity of a body provided we know the eccentricity and semi-major axis of its orbit. However, in practice we usually want to calculate the location of a body at a given time and our solution to the two-body problem (Eq. (2.20)) does not contain the time explicitly. Although f and r are functions of t, we have not shown the nature of this dependence, although it is obviously nonlinear for $e \neq 0$.

Ideally we would like to make use of an angle that is not only 2π-periodic but also a linear function of the time. This will be particularly useful later on when we have to calculate time averages of various quantities. Using our definition of the mean motion n in Eq. (2.25) we can define the *mean anomaly* M by

$$M = n(t - \tau), \tag{2.39}$$

where the constant τ is the *time of pericentre passage*. Although M has the dimensions of an angle, and it increases linearly with time at a constant rate equal to the mean motion, it has no simple geometrical interpretation. However, from our definition of M and Eq. (2.20) it is clear that when $t = \tau$ (pericentre passage), $M = f = 0$, and when $t = \tau + T/2$ (apocentre passage), $M = f = \pi$; similar relationships will hold for additive multiples of the orbital period T.

Although M has no simple geometrical interpretation, it can be related to an angle that does. Consider a circumscribed circle, radius a, that is concentric with an orbital ellipse of semi-major axis a and eccentricity e (see Fig. 2.7). A line perpendicular to the major axis of the ellipse is extended through the point on the orbit and intersects the circle. We can define E, the *eccentric anomaly*, to be the angle between the major axis of the ellipse and the radius from the centre

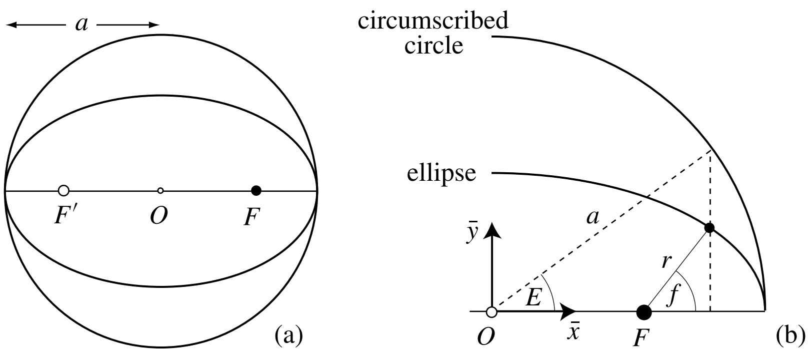

Fig. 2.7. (a) The circumscribed, concentric circle has a radius a equal to the semi-major axis of the ellipse. (b) The relationship between the true anomaly f and the eccentric anomaly E.

to the intersection point on the circumscribed circle. Hence, $E = 0$ corresponds to $f = 0$ and $E = \pi$ corresponds to $f = \pi$.

The equation of a centred ellipse in rectangular coordinates is

$$\left(\frac{\bar{x}}{a}\right)^2 + \left(\frac{\bar{y}}{b}\right)^2 = 1. \tag{2.40}$$

But from Fig. 2.7 we have $\bar{x} = a\cos E$ and therefore $\bar{y}^2 = b^2 \sin^2 E$ and hence from Eq. (2.18), $\bar{y} = a\sqrt{1-e^2}\sin E$. Thus the projections of r in the horizontal and vertical directions are

$$x = a(\cos E - e) \qquad \text{and} \qquad y = a\sqrt{1-e^2}\sin E \tag{2.41}$$

(cf. Eq. (2.21)). By adding the squares of these expressions and then taking the square root we have

$$r = a(1 - e\cos E) \tag{2.42}$$

and

$$\cos f = \frac{\cos E - e}{1 - e\cos E}. \tag{2.43}$$

We can derive a simpler relationship between E and f by writing

$$1 - \cos f = \frac{(1+e)(1-\cos E)}{1 - e\cos E}, \qquad 1 + \cos f = \frac{(1-e)(1+\cos E)}{1 - e\cos E}. \tag{2.44}$$

Using the standard double angle formulae, these equations can be written as

$$2\sin^2\frac{f}{2} = \frac{1+e}{1-e\cos E}2\sin^2\frac{E}{2}, \qquad 2\cos^2\frac{f}{2} = \frac{1-e}{1-e\cos E}2\cos^2\frac{E}{2} \tag{2.45}$$

and hence

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}. \tag{2.46}$$

Thus, knowing E, we can determine r and f uniquely from Eqs. (2.42) and (2.43), since E and f will always lie in the same half of the ellipse. However, to locate a body in its orbit at some time t, we need to derive a relationship between M and E.

Using $v^2 = \dot{r}^2 + (r\dot{f})^2$ and Eqs. (2.32) and (2.34), we have

$$\dot{r}^2 = n^2a^3\left(\frac{2}{r} - \frac{1}{a}\right) - \frac{n^2a^4(1-e^2)}{r^2}. \tag{2.47}$$

Hence

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{na}{r}\sqrt{a^2e^2 - (r-a)^2}. \tag{2.48}$$

This can then be integrated by making the substitution

$$r - a = -ae\cos E \tag{2.49}$$

from Eq. (2.42). Hence Eq. (2.48) can be written as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{n}{1 - e\cos E} . \tag{2.50}$$

This equation can also be derived by differentiating Eq. (2.41) to find $\dot{x}$ and $\dot{y}$, finding $\mathbf{r} \times \dot{\mathbf{r}}$ and equating the magnitude, h, with $na^2\sqrt{1-e^2}$. The resulting equation, Eq. (2.50), can be easily integrated to give

$$n(t - \tau) = E - e\sin E \tag{2.51}$$

where we have taken τ to be the constant of integration and used the boundary condition $E = 0$ when $t = \tau$. Hence, from Eq. (2.39) we have

$$M = E - e\sin E . \tag{2.52}$$

This is *Kepler's equation* and its solution is fundamental to the problem of finding the orbital position at a given time. At a particular time t we can (i) find M from Eq. (2.39), (ii) solve Eq. (2.52) for E, (iii) use Eq. (2.41), or Eqs. (2.43) and (2.20), to find r and f.

So far we have defined the true longitude (θ), the true anomaly (f), the mean anomaly (M), the eccentric anomaly (E), and the longitude of pericentre (ϖ). To complete this set we define the *mean longitude* λ by

$$\lambda = M + \varpi . \tag{2.53}$$

Therefore λ is a linear function of time and, since it is derived from M, it has no geometrical interpretation, except in the special case of a circular orbit. It is important to note that all longitudes (θ, ϖ, λ) are defined with respect to a common, arbitrary reference direction (see Fig. 2.5).

Colwell (1993) points out that papers have been published about the solution of Kepler's equation in virtually every decade since 1650 and that many eminent scientists have attempted solutions. Kepler's equation cannot be solved directly because it is transcendental in E and therefore, apart from the trivial solutions $E = j\pi$ when $M = j\pi$ for integer j, it is impossible to express E as a simple function of M. We briefly consider two iterative techniques: one producing a series solution and the other a numerical solution. In each case we assume that M and E are expressed in radians.

We can derive a series solution by using an iterative method of the form

$$E_{i+1} = M + e\sin E_i, \qquad i = 0, 1, \ldots, \tag{2.54}$$

where we take $E_0 = M$ as our first approximation. Using the formula $\sin(A + B) = \sin A\cos B + \cos A\sin B$ and the fact that $\sin x \approx x - \frac{1}{6}x^3 + \mathcal{O}(x^5)$ and

$\cos x \approx 1 - \frac{1}{2}x^2 + \mathcal{O}(x^4)$ for small x, we obtain

$$\begin{aligned}
E_1 &= M + e \sin M, \\
E_2 &= M + e \sin(M + e \sin M) \approx M + e \sin M + \frac{1}{2}e^2 \sin 2M, \\
E_3 &= M + e \sin(M + e \sin M + \frac{1}{2}e^2 \sin 2M) \\
&\approx M + \left(e - \frac{1}{8}e^3\right) \sin M + \frac{1}{2}e^2 \sin 2M + \frac{3}{8}e^3 \sin 3M
\end{aligned} \tag{2.55}$$

for the first three steps, where we introduced only one additional term in e at each step. It is clear from this approach that the final series for $E - M$ will have the form

$$E - M = \sum_{s=1}^{\infty} b_s(e) \sin sM, \tag{2.56}$$

where the lowest order term in $b_s(e)$ is $\mathcal{O}(e^s)$. The form of Eq. (2.56) suggests that we have expressed $E - M$ as a Fourier sine series in M. We study such useful expansions for elliptical motion in more detail in Sect. 2.5, where we derive expressions for the $b_s(e)$ terms.

It is important to note that the series solution of Kepler's equation diverges for values of $e > 0.6627434$ (see Hagihara 1970, for a detailed explanation). Not only is this property a fundamental limitation to deriving a useful series solution to Kepler's equation, it also has important implications for other series, such as the planetary disturbing function (see Chap. 6), that make use of this solution. However, numerical solutions of Kepler's equation are not affected by this limitation.

Danby (1988) gives a variety of numerical methods for solving Kepler's equation. By writing Eq. (2.52) as

$$f(E) = E - e \sin E - M \tag{2.57}$$

we can use the Newton–Raphson method to find the root of the nonlinear equation $f(E) = 0$. The iteration scheme is

$$E_{i+1} = E_i - \frac{f(E_i)}{f'(E_i)}, \qquad i = 0, 1, 2, \ldots, \tag{2.58}$$

where $f'(E_i) = \mathrm{d}f(E_i)/\mathrm{d}E_i = 1 - e \cos E_i$. Danby (1988) points out that the convergence of the Newton–Raphson scheme is quadratic but that quartic convergence is also possible with a modified scheme. Using Danby's notation and a Taylor series expansion we can write

$$0 = f(E_i + \epsilon_i) = f(E_i) + \epsilon_i f'(E_i) + \frac{1}{2}\epsilon_i^2 f''(E_i) + \frac{1}{6}\epsilon_i^3 f'''(E_i) + \mathcal{O}(\epsilon_i^4)\,. \tag{2.59}$$

Neglecting the higher order terms in ϵ_i we can write

$$0 = f_i + \delta_i f_i' + \frac{1}{2}\delta_i^2 f_i'' + \frac{1}{6}\delta_i^3 f_i''', \tag{2.60}$$

where $f_i = f(E_i)$, $f_i' = f'(E_i)$, etc. Hence

$$\delta_i = -\frac{f_i}{f_i' + \frac{1}{2}\delta_i f_i'' + \frac{1}{6}\delta_i^2 f_i'''} . \tag{2.61}$$

This can be solved for δ_i by defining

$$\delta_{i1} = -\frac{f_i}{f_i'}, \quad \delta_{i2} = -\frac{f_i}{f_i' + \frac{1}{2}\delta_{i1} f_i''}, \quad \delta_{i3} = -\frac{f_i}{f_i' + \frac{1}{2}\delta_{i2} f_i'' + \frac{1}{6}\delta_{i2}^2 f_i'''} \tag{2.62}$$

and then using the iteration scheme

$$E_{i+1} = E_i + \delta_{i3} . \tag{2.63}$$

Although this method has more arithmetic operations per iteration than the standard Newton–Raphson scheme given above, it is more efficient since (a) it can be programmed to make use of quantities that have already been calculated at each iteration and (b) it will converge faster.

An important consideration in either of these numerical schemes is a suitable starting value, E_0. Obviously for small e we have $E \approx M$ and so $E_0 = M$ seems appropriate. However, this guess is only correct in the cases where $e = 0$ or M is a multiple of π. Danby (1988) points out that, by first reducing M to the range $0 \leq M \leq 2\pi$, the initial guess

$$E_0 = M + \operatorname{sign}(\sin M)\, ke, \qquad 0 \leq k \leq 1 \tag{2.64}$$

has a better chance of being correct and improves the convergence; the recommended value is $k = 0.85$. Further details of this and other methods are discussed in Danby & Burkardt (1983), Burkardt & Danby (1983), and Danby (1987).

The solution of Kepler's equation to find E for a given value of M allows the calculation of the position and velocity at any time t for an object in an elliptical orbit. If the object has a position vector $\mathbf{r}_0 = \mathbf{r}(t_0)$ and a velocity vector $\mathbf{v}_0 = \mathbf{v}(t_0)$ at time t_0 then this process can be simplified by the introduction of two special functions and their time derivatives. Provided that the initial vectors $\mathbf{r}_0$ and $\mathbf{v}_0$ are not parallel, $\mathbf{r}(t)$ can be written as

$$\mathbf{r}(t) = f(t, t_0)\mathbf{r}_0 + g(t, t_0)\mathbf{v}_0, \tag{2.65}$$

where $f(t, t_0)$ and $g(t, t_0)$ are referred to as the *f and g functions.*

Separating the x and y components we have

$$x = f(t, t_0)x_0 + g(t, t_0)\dot{x}_0 \qquad \text{and} \qquad y = f(t, t_0)y_0 + g(t, t_0)\dot{y}_0, \tag{2.66}$$

where we have taken $\mathbf{r}_0 = (x_0, y_0)$ and $\mathbf{v}_0 = (\dot{x}_0, \dot{y}_0)$. This gives two simultaneous linear equations, which we can solve for f and g. The solution is

$$f(t, t_0) = \frac{x\dot{y}_0 - y\dot{x}_0}{x_0\dot{y}_0 - y_0\dot{x}_0} \qquad \text{and} \qquad g(t, t_0) = \frac{yx_0 - xy_0}{x_0\dot{y}_0 - y_0\dot{x}_0}. \tag{2.67}$$

Since $\cos f = x/r$ and $\sin f = y/r$ we can write Eq. (2.36) in terms of the eccentric anomaly instead of the true anomaly. This gives

$$\dot{x} = -\frac{na^2}{r}\sin E \qquad \text{and} \qquad \dot{y} = \frac{na^2\sqrt{1-e^2}}{r}\cos E\,. \tag{2.68}$$

Substituting Eqs. (2.41) and (2.68), with appropriate expressions for x_0, y_0, $\dot{x}_0$, and $\dot{y}_0$, and making use of Eqs. (2.42) and (2.51) gives

$$\begin{aligned} f(t, t_0) &= \frac{a}{r_0}\left\{\cos(E - E_0) - 1\right\} + 1, \\ g(t, t_0) &= (t - t_0) + \frac{1}{n}\left\{\sin(E - E_0) - (E - E_0)\right\}. \end{aligned} \tag{2.69}$$

The velocity at time t can be written as

$$\mathbf{v}(t) = \dot{f}\mathbf{r}_0 + \dot{g}\mathbf{v}_0, \tag{2.70}$$

where $\dot{f}$ and $\dot{g}$, the partial derivatives of f and g with respect to time, can be derived from Eq. (2.67) by making use of the expression for $\dot{E}$ given in Eq. (2.50). We then have

$$\begin{aligned} \dot{f}(t, t_0) &= -\frac{a^2}{rr_0}n\sin(E - E_0), \\ \dot{g}(t, t_0) &= \frac{a}{r}\left\{\cos(E - E_0) - 1\right\} + 1. \end{aligned} \tag{2.71}$$

The use of the f and g functions means that once E is known from the solution of Kepler's equation, we can readily find $\mathbf{r}$ and $\mathbf{v}$. Although we have formulated the expressions for the scalar quantities f, g, $\dot{f}$, and $\dot{g}$ by considering motion in the plane of the orbit, the formulae are equally applicable in other reference systems. In particular the f and g functions obviate the need to transform to and from a coordinate system in the orbital plane to one in a more general three-dimensional reference frame (see Sect. 2.8). This introduces considerable computational savings in numerical work.

2.5 Elliptic Expansions

Since there are so few integrable problems in solar system dynamics, frequently we have to resort to approximations in order to achieve a practical solution to a particular problem. The small quantities inherent in most branches of solar system dynamics are the eccentricity and inclination (the angle the orbit plane makes with a reference plane) of an orbit. For example, in Sect. 2.4 we have shown how Kepler's equation can be solved by means of a series in powers of the eccentricity. In Chapter 6 we deal with an expansion of the perturbing potential experienced by one planet or satellite due to another. In that case the expansion is in terms of the eccentricity and inclination of the bodies involved. Throughout this book we make use of a number of expansions. Typically we make an expansion, neglect higher order terms in some quantity, and then apply the resulting series to a problem of interest. In this section we derive a number of fundamental expansions that will be needed later on.

In the previous section we saw how it was possible to derive a simple series solution for E in terms of M. We can now formalise the result we obtained. If we write Eq. (2.52) as $E - M = e \sin E$ then, since $E - M$ is an odd periodic function, it can be expanded as a Fourier sine series,

$$e \sin E = \sum_{s=1}^{\infty} b_s(e) \sin sM, \tag{2.72}$$

where the coefficients $b_s(e)$ are given by

$$\begin{aligned} b_s(e) &= \frac{2}{\pi} \int_0^{\pi} e \sin E \, \sin sM \, \mathrm{d}M \\ &= \left[-\frac{2}{s\pi} e \sin E \cos sM \right]_0^{\pi} + \frac{2}{s\pi} \int_0^{\pi} \cos sM \, \mathrm{d}(e \sin E). \end{aligned} \tag{2.73}$$

The first part of this equation evaluates to zero and by using Kepler's equation to write $\mathrm{d}(e \sin E) = \mathrm{d}(E - M)$ we have

$$b_s(e) = -\frac{2}{s\pi} \int_0^{\pi} \cos sM \, \mathrm{d}M + \frac{2}{s\pi} \int_0^{\pi} \cos sM \, \mathrm{d}E. \tag{2.74}$$

The first integral evaluates to zero and we can use Kepler's equation again to obtain

$$b_s(e) = \frac{2}{s\pi} \int_0^{\pi} \cos(sE - se \sin E) \, \mathrm{d}E. \tag{2.75}$$

This integral can be written in terms of a standard function called the *Bessel function* of the first kind (see, e.g., Bowman 1958). We can write

$$b_s(e) = \frac{2}{s} J_s(se), \tag{2.76}$$

where

$$J_s(se) = \frac{1}{\pi} \int_0^{\pi} \cos(sE - se \sin E) \, \mathrm{d}E \tag{2.77}$$

is the Bessel function. For positive values of s, we can write

$$J_s(x) = \frac{1}{s!} \left(\frac{x}{2} \right)^s \sum_{\beta=0}^{\infty} (-1)^{\beta} \frac{(x/2)^{2\beta}}{\beta!(s+1)(s+2)\ldots(s+\beta)}. \tag{2.78}$$

This series is absolutely convergent for all values of x. The series for $J_s(x)$ for $s = 1, \ldots, 5$ including terms up to $\mathcal{O}(x^5)$ are given below:

$$
\begin{aligned}
J_1(x) &= \frac{1}{2}x - \frac{1}{16}x^3 + \frac{1}{384}x^5 + \mathcal{O}(x^7), \\
J_2(x) &= \frac{1}{8}x^2 - \frac{1}{96}x^4 + \mathcal{O}(x^6), \\
J_3(x) &= \frac{1}{48}x^3 - \frac{1}{768}x^5 + \mathcal{O}(x^7), \\
J_4(x) &= \frac{1}{384}x^4 + \mathcal{O}(x^6), \\
J_5(x) &= \frac{1}{3840}x^5 + \mathcal{O}(x^7).
\end{aligned}
\tag{2.79}
$$

We can now write the solution of Kepler's equation as

$$
\begin{aligned}
E &= M + 2\sum_{s=1}^{\infty} \frac{1}{s} J_s(se) \sin sM \\
&= M + e \sin M + e^2 \left(\frac{1}{2} \sin 2M \right) + e^3 \left(\frac{3}{8} \sin 3M - \frac{1}{8} \sin M \right) \\
&\quad + e^4 \left(\frac{1}{3} \sin 4M - \frac{1}{6} \sin 2M \right) + \mathcal{O}(e^5),
\end{aligned}
\tag{2.80}
$$

which is consistent with our result in Sect. 2.4. It is important to repeat the warning, mentioned in Sect. 2.4, that although this series solution rapidly converges for small values of e, the series is divergent for values of $e > 0.6627434$. This implies that all the series in this section that make use of this series are also divergent for sufficiently large values of e.

In addition to the series solution of Kepler's equation we will also need a number of other series expansions, all of which can be expressed in terms of Bessel functions. In particular we need series for r/a, $\cos E$, $(a/r)^3$, $\sin f$, $\cos f$, and $f - M$. The derivation of the results that follow are given in Brouwer & Clemence (1961).

The series for r/a is given by

$$
\begin{aligned}
\frac{r}{a} &= 1 + \frac{1}{2}e^2 - 2e \sum_{s=1}^{\infty} \frac{1}{s^2} \frac{\mathrm{d}}{\mathrm{d}e} J_s(se) \cos sM \\
&= 1 - e \cos M + \frac{e^2}{2} (1 - \cos 2M) + \frac{3e^3}{8} (\cos M - \cos 3M) \\
&\quad + \frac{e^4}{3} (\cos 2M - \cos 4M) + \mathcal{O}(e^5).
\end{aligned}
\tag{2.81}
$$

This series is used in Sect. 2.6 in the guiding centre approximation and in Sect. 6.3 and 6.5 in our expansion of the planetary disturbing function.

We can use the fact that $\cos E = (1 - r/a)/e$ and the series for (r/a) to derive the series for $\cos E$. It is given by

$$\begin{aligned}
\cos E &= -\frac{1}{2}e + 2\sum_{s=1}^{\infty}\frac{1}{s^2}\frac{\mathrm{d}}{\mathrm{d}e}J_s(se)\cos sM \\
&= \cos M + \frac{e}{2}\left(\cos 2M - 1\right) + \frac{3e^2}{8}\left(\cos 3M - \cos M\right) \\
&\quad + e^3\left(\frac{1}{3}\cos 4M - \frac{1}{3}\cos 2M\right) \\
&\quad + e^4\left(\frac{5}{192}\cos M - \frac{45}{128}\cos 3M + \frac{125}{384}\cos 5M\right) + \mathcal{O}(e^5).
\end{aligned} \tag{2.82}$$

We can also use the series for r/a to derive the series for $(a/r)^3$. It is given by

$$\begin{aligned}
\left(\frac{a}{r}\right)^3 &= 1 + 3e\cos M + e^2\left(\frac{3}{2} + \frac{9}{2}\cos 2M\right) \\
&\quad + e^3\left(\frac{27}{8}\cos M + \frac{53}{8}\cos 3M\right) \\
&\quad + e^4\left(\frac{15}{8} + \frac{7}{2}\cos 2M + \frac{77}{8}\cos 4M\right) + \mathcal{O}(e^5).
\end{aligned} \tag{2.83}$$

This series is used in Sect. 6.3 in our expansion of the planetary disturbing function.

The series for $\sin f$ and $\cos f$ are given by

$$\begin{aligned}
\sin f &= 2\sqrt{1-e^2}\sum_{s=1}^{\infty}\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}e}J_s(se)\sin sM \\
&= \sin M + e\sin 2M + e^2\left(\frac{9}{8}\sin 3M - \frac{7}{8}\sin M\right) \\
&\quad + e^3\left(\frac{4}{3}\sin 4M - \frac{7}{6}\sin 2M\right) \\
&\quad + e^4\left(\frac{17}{192}\sin M - \frac{207}{128}\sin 3M + \frac{625}{384}\sin 5M\right) + \mathcal{O}(e^5)
\end{aligned} \tag{2.84}$$

and

$$\begin{aligned}
\cos f &= -e + \frac{2(1-e^2)}{e}\sum_{s=1}^{\infty}J_s(se)\cos sM \\
&= \cos M + e(\cos 2M - 1) + \frac{9e^2}{8}\left(\cos 3M - \cos M\right) \\
&\quad + \frac{4e^3}{3}\left(\cos 4M - \cos 2M\right) \\
&\quad + e^4\left(\frac{25}{192}\cos M - \frac{225}{128}\cos 3M + \frac{625}{384}\cos 5M\right) + \mathcal{O}(e^5).
\end{aligned} \tag{2.85}$$

These series are used in Sect. 5.4 in our study of spin–orbit resonance and in Sect. 6.5 as part of the expansion of the planetary disturbing function.

We can derive a series for $f - M$, also called the *equation of the centre.* From Eqs. (2.20) and (2.32) we obtain

$$r^2 \dot{f} = na^2(1 - e^2)^{1/2}. \tag{2.86}$$

Using $\mathrm{d}M = n\,\mathrm{d}t$, $r = a(1 - e\cos E)$, and Kepler's equation, we obtain

$$\mathrm{d}f = \frac{\sqrt{1-e^2}}{(1 - e\cos E)^2}\mathrm{d}M = \sqrt{1-e^2}\left(\frac{\mathrm{d}E}{\mathrm{d}M}\right)^2 \mathrm{d}M. \tag{2.87}$$

This can be integrated term by term using the series solution for Kepler's equation to give

$$\begin{aligned} f - M = {} & 2e\sin M + \frac{5}{4}e^2 \sin 2M + e^3\left(\frac{13}{12}\sin 3M - \frac{1}{4}\sin M\right) \\ & + e^4\left(\frac{103}{96}\sin 4M - \frac{11}{24}\sin 2M\right) + \mathcal{O}(e^5). \end{aligned} \tag{2.88}$$

This series is used in Sect. 2.6 in an analysis of the guiding centre approximation and in Sect. 5.2 where we investigate tidal de-spinning.

Lagrange developed a useful method for inverting series expansions, which has applications to this work. He showed that if a variable z can be expressed as a function of ζ of the form

$$\zeta = z + e\phi(\zeta) \qquad (e < 1) \tag{2.89}$$

then ζ can also be expressed as a function of z using the relationship

$$\zeta = z + \sum_{j=1}^{\infty} \frac{e^j}{j!}\frac{\mathrm{d}^{j-1}}{\mathrm{d}z^{j-1}}\left[\phi(z)\right]^j. \tag{2.90}$$

For example, to obtain the expression for f in terms of M, given in Eq. (2.88), we start from Kepler's second law, Eq. (2.8), and Eq. (2.26) giving,

$$h = r^2\dot{f} = na^2(1 - e^2)^{1/2}. \tag{2.91}$$

Integrating this equation and substituting Eq. (2.20) for r gives

$$M = (1 - e^2)^{3/2}\int_0^f \frac{\mathrm{d}f}{(1 + e\cos f)^2}. \tag{2.92}$$

Expanding binomially and integrating term by term, we obtain

$$M = f - 2e\sin f + \frac{3}{4}e^2\sin 2f + \mathcal{O}(e^3). \tag{2.93}$$

This can be written as

$$f = M + e\left(2\sin f - \frac{3}{4}e\sin 2f + \cdots\right). \tag{2.94}$$

Lagrange's inverse theorem then gives

$$f = M + \sum_{j=1}^{\infty} \frac{e^j}{j!} \frac{\mathrm{d}^{j-1}}{\mathrm{d}M^{j-1}} \left[2\sin M - \frac{3}{4} e \sin 2M + \cdots \right]^j , \qquad (2.95)$$

which agrees with Eq. (2.88) after expansion. We make frequent use of Lagrange's method in Sect. 3.6 where we derive the locations of the collinear equilibrium points in the circular restricted problem.

2.6 The Guiding Centre Approximation

In many applications in solar system dynamics the eccentricity is very small and approximations that are accurate to order e are useful, particularly in some systems that are best viewed in a rotating reference frame. This approach is also appropriate when considering systems such as perturbed motion in the vicinity of equilibrium points (Sect. 3.8), the effects of planetary oblateness on near-circular, near-equatorial orbits (Sect. 6.11), and its applications to planetary rings (Chapter 10). In all these cases it is useful to characterise the extent of the departure from circular motion.

In the *guiding centre approximation,* the motion of a particle P moving in an elliptical orbit about a focus F (see Fig. 2.8) is viewed in a reference frame that is centred on a point G, the guiding centre, that rotates about the focus in a circle of radius a equal to the particle's semi-major axis, with angular speed equal to the particle's mean motion n.

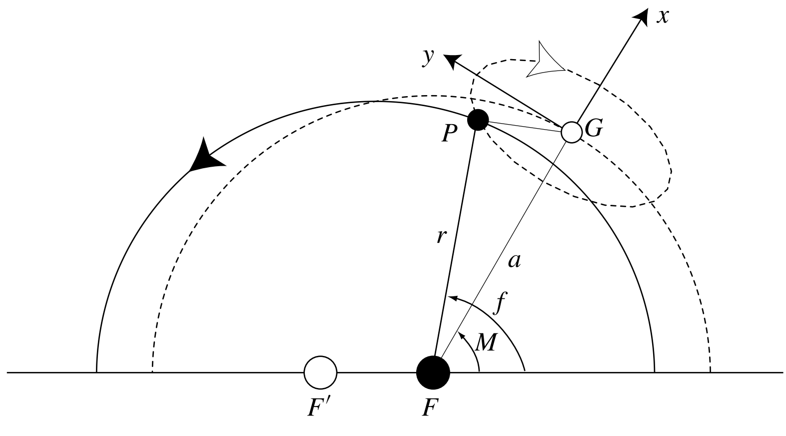

Fig. 2.8. The relationship between the true anomaly f and the mean anomaly M in the guiding centre approximation. G denotes the guiding centre, P the particle, F the focus, and F' the empty focus. The guiding centre moves on a circle of radius a centred on F.

If we transform to a rectangular coordinate system centred on G, then the coordinates of P are

$$x = r\cos(f - M) - a \qquad \text{and} \qquad y = r\sin(f - M). \tag{2.96}$$

To order e, the expansion of $f - M$ from Eq. (2.88) is

$$f - M \approx 2e\sin M. \tag{2.97}$$

Hence

$$x \approx -ae\cos M \qquad \text{and} \qquad y \approx 2ae\sin M \tag{2.98}$$

and

$$\frac{x^2}{(ae)^2} + \frac{y^2}{(2ae)^2} \approx 1. \tag{2.99}$$

It follows that while G moves about F in a circle of radius a with mean motion n and period $2\pi/n$, P moves about G in the opposite sense on a 2:1 ellipse of semi-major axis $2ae$, semi-minor axis ae, and period $2\pi/n$. The motion of P with respect to F is a Lissajou figure obtained by the superposition of two harmonic motions with a common frequency n, a phase difference of $\pi/2$, and a 2:1 amplitude ratio.

At this level of approximation there are two other features of the motion that are worth noting. The distance R of P from the centre of the ellipse (see Fig. 2.9) can be obtained from

$$R^2 = r^2 + (ae)^2 + 2aer\cos f. \tag{2.100}$$

Hence

$$R \approx a\left(1 - \frac{1}{2}e^2\sin^2 f\right) \approx a\left(1 - \frac{1}{2}e^2\sin^2 M\right) \tag{2.101}$$

since, from Eq. (2.97), $f = M + \mathcal{O}(e)$. Thus, to order e, the path of P is a circle with centre at O. Therefore the path and the circumscribed circle (see Fig. 2.9) coincide and thus the angle $P\hat{O}F$ is the eccentric anomaly E. In fact, as we show below, in this approximation the angle $P\hat{F}'F$, where F' is the empty focus, is the mean anomaly M.

Consider the true elliptical path of P and denote the angle $F\hat{F}'P$ by g. Applying the cosine rule to the triangle $FF'P$ we obtain

$$r^2 = (2a - r)^2 + 4(ae)^2 - 4ae(2a - r)\cos g. \tag{2.102}$$

Hence

$$\cos g = \frac{(1 - r/a) + e^2}{e(1 - r/a) + e}. \tag{2.103}$$

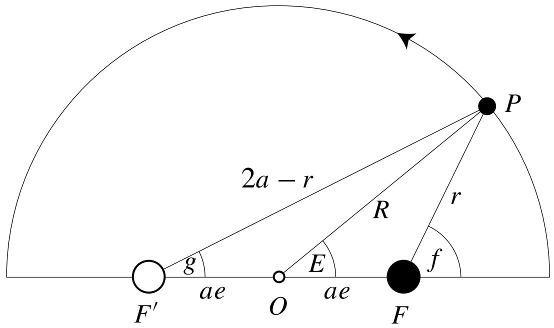

Fig. 2.9. The relationships among the true, mean, and eccentric anomalies in the guiding centre approximation. Note that the diagram exaggerates the actual case: In reality the eccentricity is small and F and F' are close to O.

Here we have used the fact that since F and F' are the foci of the ellipse, $FP + F'P = 2a$. From the expansion of r/a in Eq. (2.81) we obtain

$$1 - \frac{r}{a} \approx e \cos M - \frac{1}{2}e^2(1 - \cos 2M) - \frac{3}{8}e^3(\cos M - \cos 3M) \qquad (2.104)$$

and hence

$$\cos g \approx \cos M - \frac{1}{8}e^2(\cos M - \cos 3M) + \mathcal{O}(e^3), \qquad (2.105)$$

so that to $\mathcal{O}(e)$ we have $g = M$. Therefore the line joining the orbiting mass to the empty focus must rotate at the same rate as the mean motion of the orbiting mass.

This result has an interesting consequence if we apply it to the motion of a satellite that has a spin period equal to its orbital period (a *synchronously rotating* satellite). Since the line drawn from the satellite to the empty focus rotates with frequency n, equal to the mean motion, it follows that a synchronously rotating satellite rotates with one face pointing towards the empty focus of its orbit. This proves to be useful in understanding the origin of the librational tide on a synchronously rotating satellite like the Moon or Io. It also follows that the line drawn from the guiding centre to the central mass is parallel to the line joining the orbiting mass to the empty focus (Fig. 2.10).

It is interesting to note that Ptolemy's scheme for the motion of the Sun about the Earth had the Sun moving in a circle with uniform angular motion about an equant with the Earth displaced from the centre of the circle. If we place the Earth at the focus F and identify the equant with the point F', then we see that Ptolemy's scheme was accurate to order e. The triumph of Kepler was to produce a theory that was accurate to order e^2 (Hoyle 1974).

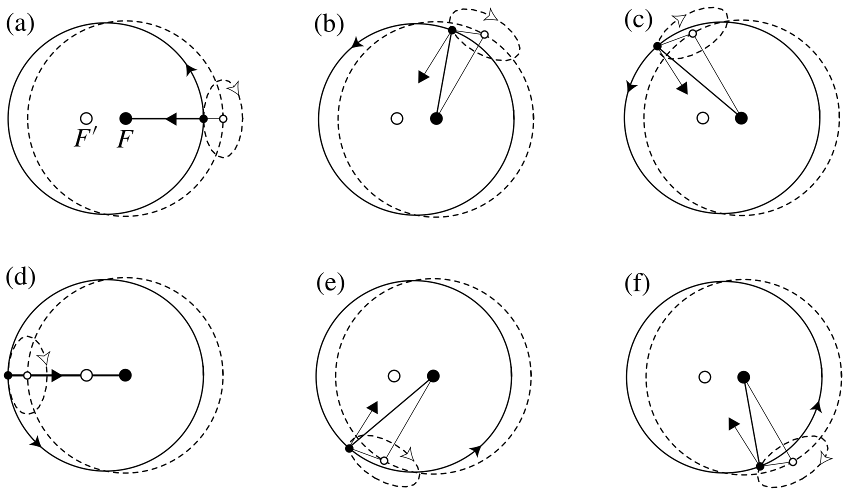

Fig. 2.10. An illustration of the guiding centre approximation for an ellipse of eccentricity $e = 0.2$. The position of the orbiting mass (small filled circle) with respect to the central mass (large filled circle) and the empty focus (large white circle) is shown at equal intervals of mean anomaly M. (a) $M = 0$, (b) $M = \pi/3$, (c) $M = 2\pi/3$, (d) $M = \pi$, (e) $M = 4\pi/3$, and (f) $M = 5\pi/3$. The solid curve denotes the keplerian ellipse while the dashed circle with a radius equal to the semi-major axis of the ellipse denotes the path of the guiding centre; the circle is centred on the primary focus of the ellipse.

2.7 Barycentric Orbits

We have shown that, with suitable starting conditions, the motion of the mass m_2 with respect to the mass m_1 describes a conic section in space. We now return to the formulation of the two-body problem using a centre of mass coordinate system with origin at the point O' (see Fig. 2.11).

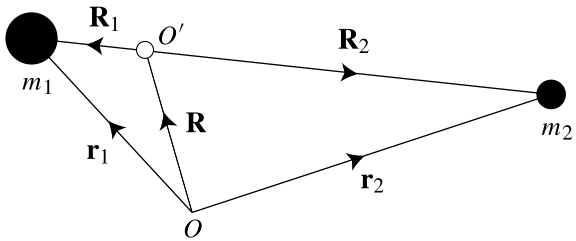

Fig. 2.11. The position vectors of the two masses with respect to the origin, O, and with respect to the centre of mass, O'.