Information Diffusion in Grid-like Social Networks

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In this paper we study how fast information propagates through various grid-like social networks. We start by developing a model for information propagation in a general network and utilize time inhomogeneous Markov chains to provide a lower bound on the *diffusion rate* of information of any general network. Through this model, we observe that as nodes in a 4-regular grid graph start forming new edges with other nodes, the diffusion rate of the network increases exponentially. Finally, we perform several simulations and note that our initial results confirm our theoretical analysis.

1 INTRODUCTION

In our lives, every single one of us is a part of several social networks. These social networks, besides providing us with a means of communication with other individuals, perform another role; that of information distribution. Indeed, through our interactions with other people we obtain several pieces of information every day. This fact is actually extremely integral in our decision-making process, as we use this information to make decisions that may potentially improve our quality of life. Examples of such phenomena include asking one's friends and acquaintances who they believe to be the best doctor in a particular specialty, or who is their current car mechanic and how satisfied they are with them. These kind of questions provide the motivation to our work, which attempts to discern how fast this information can spread through a social network.

While this question is important in every setting, it is extremely difficult to analyze in the general case. For that reason, we focus mainly on grid-like network structures, in an attempt to obtain more precise results. Specifically, we consider a simple, 4-regular graph as a basis, and compare the information's *diffusion rate* in this graph with that of two other graphs, variants of the plain grid. We note, however, that our characterization of the diffusion rate of a graph – for which a closed-form characterization was, to the best of our knowledge, not known – is defined for any network structure, which is an important step in getting similar results in different, more general network structures.

Critical to the development of our model was the introduction of ideas from stochastic processes and ergodic theory. Specifically, we formulate the problem as a question about expected hitting times of a time inhomogeneous Markov chain, and use state-of-the-art techniques to develop novel theoretical results about this problem. While time homogeneous Markov chains have been studied a lot in literature and many standard results about them are known, their inhomogeneous variant has, as of yet, escaped most attempts made towards its analysis. Thus, not many results about them are known, which in turn makes our results, albeit particularly model-specific, one of few such developments in the field and adds to their significance.

Finally, we performed a series of simulations on these three network structures and oberved that, in most cases, the simulation results seem to agree with the results predicted by our model. We take this to be at least a partial confirmation that our approach is a correct and important step in understanding more about how information is spread through a social network.

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2 RELATED WORK

This work was mainly inspired by novel approaches taken in analyzing how technologies diffuse in social networks [Immorlica et al., 2007]. While technology adoption can certainly be viewed as an information diffusion problem, the constraints imposed on the model in most technology adoption problems make it difficult to obtain a general model that is useful in real-life passive information diffusion. Information diffusion in social networks has of course been studied in past literature, and there have been significant results, both theoretical [Jackson and Yariv, 2005, Morris, 2000, Valente, 1995] and experimental [Cooper, 1982, Rogers, 2003]. An active line of research in economics and mathematical sociology is concerned with modeling these types of diffusion processes as a coordination game on the social network [Blume, 1993, Ellison, 1993, Jackson and Yariv, 2005, Morris, 2000, Young, 2001].

Another approach to information diffusion is the problem of *influence maximization* where instead of analyzing the passive diffusion of information in the social network, one attempts to compute the optimal set of agents in the network which, if given the information, will trigger the maximum cascade of this information throughout the network. This problem, while NP-hard in the general case [Kempe et al., 2003], has attracted a lot of attention and there have been several recent attempts to utilize network structure in order to obtain strong theoretical guarantees [Seeman and Singer, 2013].

On the stochastic side, time inhomogeneous Markov chains have evaded theoretical analysis for decades. However, while early work in the field relied mostly on spectral techniques to provide guarantees on the asymptotic behavior of such Markov chains [Fleischer and Joffe, 1995, Sonin, 1996], more recent techniques utilize a combination of both spectral techniques and state-of-the-art results from stability theory [Saloff-Coste and Zuniga, 2009, Saloff-Coste and Ziga, 2007, 2010, 2011] to provide a better understanding into how non-homogeneous Markov chains behave. These results have very recently culminated in the development of highly significant quantitative upper bounds on the expected mixing and hitting times of non-homogeneous Markov chains [Douc et al., 2004, Saloff-Coste and Zuniga, 2010, Shen, 2013], which we rely upon in our theoretical analysis.

3 THEORETICAL ANALYSIS

In this section we present our main theoretical contributions. We start with some necessary definitions and background, before we present our model which is based on time inhomogeneous Markov chains. We then use this model to derive a characterization of the diffusion rate of information in any general graph. Finally, we specialize our model to several variants of grid-like graphs and compare their diffusion rate.

Consider n agents that interact with each other in a social network. Each agent i has a specific set of other agents that they interact with, which is called the *neighborhood* of i and is denoted by $\mathcal{N}_i$. In this network, each agent has a set of strategies available to them. Specifically, agent i proposes an *interaction frequency* $f_{ij} \in \mathbb{R}$ to every agent $j \in \mathcal{N}_i$. Similarly, each agent j proposes an interaction frequency f_{ji} to i. The frequency $f_{ij}^* = f_{ji}^*$ that they end up interacting at is simply

$$f_{ij}^* = \min\{f_{ij}, f_{ji}\}$$

However, in real social networks, the agents have different preferences for who to interact with. To capture this in our model, we assign w_{ij} to be the *weight* i places on j, or similarly how much i values the interaction with j. Further motivated by real social networks, we assign all agents a uniform "budget" of interaction frequency β that they cannot exceed.

In our social network, we assume that there exist some external *service providers* that provide some service of a certain quality to the agents. We assume a rather complicated framework, where each agent i has a specific *type*, denoted by $\theta_i \in [0,1]$ out of N_θ possible types, and there exist k distinct service providers that are not a part of the social network. Each service provider has a different quality of service for each agent type, however the quality they provide is fixed for all agents of the same type. This is again an attempt to model real-life social networks, where the quality of service an agent receives is based on specific attributes they possess which in turn classify the agent as being of a certain type. While we start byanalyzing the case where the agent types are drawn at random from the uniform distribution, we believe it is of great importance to also study cases where neighboring agents have a greater chance of being of the same type. This implies a clustering of agents' types, which is motivated by the homophily that real social networks exhibit in general.

We make the following assumption. Each agent i initially has a service provider chosen uniformly at random, and receives a specific quality $q_i \in [0,1]$ from them. Furthermore, i gets some utility by communicating with all agents in their neighborhood but also by learning about different service providers with higher quality than q_i for their type. We assume that, as i communicates with a neighbor j of the same type, for each unit of communication the probability that i learns j's quality is p, which is constant and uniform for all agents in the network. Therefore, since between time t and $t+1$ they interact $f^*_{ij}(t)$ times, the probability that i learns q_j is

$$1-(1-p)^{f^*_{ij}(t)}$$

Also, since agents of different type exhibit different qualities from the same service provider, we make the assumption that if j is a neighbor of i and $\theta_i \neq \theta_j$, then i is not interested in learning q_j and therefore agents of type different than θ_i contribute in i's utility simply through their interaction and not by letting i know about the existence of a service provider with higher quality. Our final assumption is that once i learns of a different service provider that offers higher quality service to agents with type θ_i, they immediately switch to that provider and start receiving the aforementioned higher quality.

Thus, we model i's utility by

$$u_i(t) = \sum_{j\in\mathcal{N}_i} w_{ij} f^*_{ij}(t)\left(\beta - f^*_{ij}(t)\right) + \sum_{j\in\mathcal{Z}_i} \mathbb{E}\left[\min\left(q_j - q_i, 0\right)\right]\left(1-(1-p)^{f^*_{ij}(t)}\right) \tag{1}$$

where $\mathcal{Z}_i$ is the subset of i's neighbors that have the same type as i. Our model is sequential in that at each time step t one agent is chosen at random and updates their proposals to those that maximize their utility at time t. This dynamics is called *best-response*, since i plays their best-response strategy to the strategies played by all other players, assuming that they remain fixed at these strategies, at least for time t. It is easy to see that to play their best-response strategy at time t, i has to solve the following convex optimization problem for variables $f_{ij}(t)$

$$\begin{aligned}
\max \quad & \sum_{j\in\mathcal{N}^i} w_{ij} f^*_{ij}(t)\left(\beta - f^*_{ij}(t)\right) + \sum_{j\in\mathcal{Z}_i} \mathbb{E}\left[\min\left(q_j - q_i, 0\right)\right]\left(1-(1-p)^{f^*_{ij}(t)}\right) \qquad (2)\\
s.t. \quad & \sum_{j\in\mathcal{N}^i} f_{ij} \leq \beta \\
& f_{ij} \geq 0, \;\; \forall j \in \mathcal{N}_i \\
& f_{ij} \leq f_{ji}, \;\; \forall j \in \mathcal{N}_i
\end{aligned}$$

This problem will in general be very difficult to solve. In order to make it more tractable, we consider an approximation to each agent's utility, where we replace

$$1-(1-p)^{f^*_{ij}(t)}$$

with

$$pf^*_{ij}(t)$$

This approximation makes sense only for $p << f^*_{ij}(t)$, therefore we limit our analysis to cases where p is significantly small, compared to the frequency proposals.

While the solution to (2) is not trivial, we assume that all agents are able to solve it and to calculate their best-response. Furthermore, it should be clear that since $p > 0$, given infinite time, all agents eventually learn the service provider that offers the highest quality for their type in their connected component, through their communication with the agents of the same type. But this immediately raises another question; what is the rate of diffusion for this information and, more importantly, how is it affected by – seemingly – small changes in the network structure? Before we are ready to provide an answer to this question, or even provide a clear definition of the diffusion rate in a graph, we have to introduce our Markov chain model, which will make all such concepts significantly easier to analyze.

3.1 Markov Chain Model

For a specific time t, we define two sets of agents below

$$\mathcal{I}(t) = \{i \mid q_i \geq q_j \quad \forall j : \theta_j = \theta_i\}$$

$$\mathcal{F}(t) = \{i \mid \exists j \in \mathcal{I}(t) \cap \mathcal{N}_i : \theta_j = \theta_i \land q_j > q_i\}$$

It is understood that the agents in $\mathcal{I}(t)$ are the ones that possess the information at time t, while the agents in $\mathcal{F}(t)$ are the agents which do not possess the information at time t, but have a neighbor that does. Equivalently, $\mathcal{F}(t)$ is the set of agents to which the information may spread next. Next, we define a random variable X_t which represents the number of agents who have the service provider of the highest quality. In our model, we assume that $X_0 = 1$, in the worst-case, and we want to calculate the information diffusion rate of G defined below.

Definition 3.1 (Diffusion Rate). Consider a graph G of n agents. Then the **diffusion rate** of information in G is

$$\gamma_G = \frac{1}{\min_{t>0}\{X_t = n\}}$$

Consider a Markov chain on X_t that is essentially a path graph of size n, enhanced with self-loops. Our initial state is $X_0 = 1$ and, if we define the event A^k_t as

$$A^k_t = \{X_t \geq k \mid X_{t-1} = k-1\}$$

for $k, t \geq 1$, then we would like to calculate the hitting time of the event A^n_t which is the expected minimum t for which this event comes true for the first time. While normally we can analyze such hitting times on Markov chains with ease, our inhomogeneous Markov chain provides an extra level of difficulty, due to the fact that the transition probabilities vary with time. For that analysis, we need to define

$$C_t = \left\{\{i,j\} \in E(G) \mid i \in \mathcal{I}(t), j \in \mathcal{F}(t), \theta_i = \theta_j\right\}$$

as the set of all possible edges through which the information could leak through at time t.

Next, we present one of our main contributions, a lower bound on the information diffusion rate for any graph G, or equivalently, an upper bound on the expected mixing time of our Markov chain on G

Theorem 3.2. *Consider a graph G of n agents. Then, if*

$$\delta = \min_{t,\{i,j\}\in C_t} pf^*_{ij}(t) > 0$$

the diffusion rate of G is

$$\gamma_G \geq \frac{1}{n \cdot \sum_{t=1}^{\infty} t \left(1 - (1-\delta)^{|C_t|}\right) \cdot \prod_{\lambda=1}^{t-1} (1-\delta)^{|C_\lambda|}}$$

Proof. We know that

$$\Pr\left[A_t^k\right] \geq 1 - \prod_{\{i,j\} \in C_t} \left(1 - p f_{ij}^*(t)\right)$$

for any $k \geq 1$. We also know by [Saloff-Coste and Zuniga, 2010] that if we can lower bound the non-zero interactions between the agents by a fixed constant $\delta > 0$, then our time inhomogeneous Markov chain converges almost surely to its (unique) stationary distribution. Thus, for

$$\delta = \min_{t, \{i,j\} \in C_t} p f_{ij}^*(t)$$

we get

$$\Pr\left[A_t^k\right] \geq 1 - \prod_{\{i,j\} \in C_t} (1-\delta) = 1 - (1-\delta)^{|C_t|} \tag{3}$$

Since we now know that our Markov chain converges, we can calculate the expected hitting time of A_t^k for any k

$$\mathbb{E}_t\left[A_t^k\right] = \sum_{t=1}^{\infty} t \Pr\left[A_t^k\right] \cdot \prod_{\lambda=1}^{t-1} \left(1 - \Pr\left[A_\lambda^k\right]\right) \tag{4}$$

$$\mathbb{E}_t\left[A_t^k\right] = \sum_{t=1}^{\infty} t \left(1 - (1-\delta)^{|C_t|}\right) \cdot \prod_{\lambda=1}^{t-1} (1-\delta)^{|C_\lambda|}$$

Finally, since the information can spread at most n times, we know that there are only n possible values of k in A_t^k. Thus, the expected time it will take for the information to spread to all n agents is

$$\mathbb{E}_t\left[A_t^n\right] \leq n \cdot \sum_{t=1}^{\infty} t \left(1 - (1-\delta)^{|C_t|}\right) \cdot \prod_{\lambda=1}^{t-1} (1-\delta)^{|C_\lambda|}$$

and we can provide a lower bound on the diffusion rate of G

$$\gamma_G \geq \frac{1}{n \cdot \sum_{t=1}^{\infty} t \left(1 - (1-\delta)^{|C_t|}\right) \cdot \prod_{\lambda=1}^{t-1} (1-\delta)^{|C_\lambda|}} \qquad \square$$

3.2 Diffusion Rate in Grid-like Graphs

Suppose G is a 4-regular graph of n nodes where the vertex are arranged in a 2-dimensional grid, and every edge is either vertical or horizontal. We call this graph a *plain grid*, and we use it as a basis for our analysis of its variants. We denote the diffusion rate of the plain grid by γ_{PG}. Next, we define two variants of the plain grid and compare their diffusion rates.

3.2.1 ***The unbiased grid***. In the *unbiased grid*, we extend the previous graph model by allowing each node i to form one long-range edge with some other node j that does not immediately communicate with i through the grid structure. However, the probability distribution for this edge formation is not random, but instead is proportional to $\frac{1}{d^2}$, where d is the Manhattan distance, or equivalently the L_1 norm, defined as

$$d(i,j) = |i.x - j.x| + |i.y - j.y|$$

where $i.x$ denotes the row and $i.y$ denotes the column of agent i in the grid. Note here that the edge formation distribution does not depend on the types of agents, therefore, for sufficiently large N_θ, we expect that most agents will not form an edge with an agent of the same type. However, for those that will, the probability that they learn of a service provider with higher quality for their type rises. We formalize this intuition with the following theorem

Theorem 3.3. *Consider an instance PG of the plain grid graph with n nodes. Then, the expected increase of $|C_t|$ in the corresponding unbiased grid UG created from PG is*

$$\mathbb{E}\left[|C_t^{UG}| - |C_t^{PG}|\right] = \frac{\left(\sum_{k=2}^{2n} \frac{1}{k^2}\right) |\mathcal{I}(t)| \left(1 - \frac{|\mathcal{I}(t)|}{n}\right)}{N_\theta}$$

Proof. The probability for a new edge $e = \{i, j\}$ to be in C_t is

$$\Pr\left[e \in C_t\right] = \Pr\left[\theta_i = \theta_j\right] \cdot \Pr\left[j \notin \mathcal{I}(t) \mid \theta_i = \theta_j\right] \tag{5}$$

Assuming the distribution of agents' types is uniform, we have

$$\Pr\left[\theta_i = \theta_j\right] = \frac{1}{N_\theta}$$

and by using the fact that the events $\{\theta_i = \theta_j\}$ and $\{j \notin \mathcal{I}(t)\}$ are independent in the unbiased grid, we get

$$\Pr\left[j \notin \mathcal{I}(t) \mid \theta_i = \theta_j\right] = \Pr\left[j \notin \mathcal{I}(t)\right] = \left(\sum_{k=2}^{2n} \frac{1}{k^2}\right) \left(1 - \frac{|\mathcal{I}(t)|}{n}\right)$$

Combining the above equalities, we get

$$\Pr\left[e \in C_t\right] = \frac{1}{N_\theta} \left(\sum_{k=2}^{2n} \frac{1}{k^2}\right) \left(1 - \frac{|\mathcal{I}(t)|}{n}\right)$$

because $\Pr\left[j \notin \mathcal{I}(t)\right]$ is calculated by fixing a new neighbor j of i (with probability $\frac{1}{d^2}$ for $d \geq 2$), and then calculating the probability that $j \notin \mathcal{I}(t)$.

Therefore, we understand that the increase in $|C_t|$ on expectation is

$$\mathbb{E}\left[|C_t^{UG}| - |C_t^{PG}|\right] = \frac{\left(\sum_{k=2}^{2n} \frac{1}{k^2}\right) |\mathcal{I}(t)| \left(1 - \frac{|\mathcal{I}(t)|}{n}\right)}{N_\theta} \tag{6}$$

□

Note here that the diffusion rate γ_G depends exponentially on $|C_t|$. Therefore, we understand that the increase in diffusion rate between the unbiased and the plain grids is exponential. This fact alone is surprising, but we continue by generalizing it to a larger family of graphs.

3.2.2 ***The biased grid.*** This previous observation naturally leads us to study a generalization of the previous graph model which we call the *biased grid*. In this graph structure, agents again form long-range edges to other agents in the grid with probablities proportional to $\frac{1}{d^2}$, but they are biased towards forming an edge with an agent of the same type as them. Therefore, we assume that, when an agent i forms a long-range edge, agents with type θ_i are b times more likely to be chosen by i than agents of different type. We call this parameter b, which directly affects the graph's diffusion rate, the *bias* of G.

In the biased case, the probability that an edge e is in C_t changes, as the events $\{\theta_i = \theta_j\}$ and $\{j \notin \mathcal{I}(t)\}$ are not independent anymore. It turns out that the expected increase in the diffusion rate of the biased grid compared to the unbiased grid is again exponential and behaves in a nice linear fashion depending on the bias b, as is demonstrated by the following theorem

Theorem 3.4. *Consider an instance PG of the plain grid graph with n nodes. Then, the expected increase of $|C_t|$ in the corresponding biased grid BG with bias b created from PG is*

$$\mathbb{E}\left[|C_t^{BG}| - |C_t^{PG}|\right] = \frac{b \cdot \left(\sum_{k=2}^{2n} \frac{1}{k^2}\right) |\mathcal{I}(t)| \left(1 - \frac{|\mathcal{I}(t)|}{n}\right)}{N_\theta} = b \cdot \mathbb{E}\left[|C_t^{UG}| - |C_t^{PG}|\right]$$

Proof. As stated above, the events $\{\theta_i = \theta_j\}$ and $\{j \notin \mathcal{I}(t)\}$ are not independent anymore, thus the probability that an edge e is in C_t changes. We know that

$$\Pr\left[\{i,j\}\text{is formed}\right] = \begin{cases} \frac{b}{d^2}, & \theta_i = \theta_j \\ \frac{1}{d^2}, & \theta_i \neq \theta_j \end{cases}$$

Utilizing the above equation, we can write

$$\Pr\left[e \in C_t\right] = \frac{1}{N_\theta} \left(\sum_{k=2}^{2n} \frac{b}{k^2}\right) \left(1 - \frac{|\mathcal{I}(t)|}{n}\right)$$

Thus

$$\mathbb{E}\left[|C_t^{BG}| - |C_t^{PG}|\right] = \frac{b \cdot \left(\sum_{k=2}^{2n} \frac{1}{k^2}\right) |\mathcal{I}(t)| \left(1 - \frac{|\mathcal{I}(t)|}{n}\right)}{N_\theta} = b \cdot \mathbb{E}\left[|C_t^{UG}| - |C_t^{PG}|\right] \qquad \square$$

We observe that, once again, the increase between the biased and the plain grids in $|C_t|$ – thus also in γ_G – is exponential. However, the increase in diffusion rate between the biased and unbiased grids is also exponential! Specifically, for $b = 2$, we have

$$\mathbb{E}\left[|C_t^{BG}| - |C_t^{UG}|\right] = \mathbb{E}\left[|C_t^{UG}| - |C_t^{PG}|\right]$$

thus (because γ_G depends exponentially on $|C_t|$), if we observe an increase of a in the diffusion rate between the unbiased and plain grids, we expect to see an increase of a^2 (or a^b in general) between the biased and plain grids. Equivalently, we expect to see an increase of a (or a^{b-1} in general) between the biased and unbiased grids.

4 SIMULATIONS

5 CONCLUSION

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