IE 598 JG Games, Markets and Mathematical Programming
ColorfulCarathéodory $\in$ PPAD
A simpler proof using LCPs

Rucha Kulkarni
ruchark2@illinois.edu

Vasilis Livanos
livanos3@illinois.edu

1 Introduction

Carathéodory's theorem is a classical statement in discrete geometry [3]. Generally, it appears in the class of theorems that prove statements of the form: *If subsets of some set have a property P, then the entire set also has the property.* Helly's and Radon's theorems are some examples of other statements of this form. Later Barany [1], in search of a mathematical game, discovered what he termed a "colorful" version of the theorem. This version, called the "Colorful Carathéodory theorem", then found applications in diverse areas of mathematics and computer science.

The Colorful Carathéodory theorem states that if there are $d+1$ sets $C^i, i \in [d+1]$ of $d+1$ vertices each in $\mathbb{R}^d$ such that the origin lies in the convex hull of each of these sets, then there exists a choice of $d+1$ vertices S, with exactly one vertex chosen from each set C^i such that the origin lies in the convex hull of this set.

For visualizing the theorem, one can think all vertices of one input set C^i are assigned one color, which we denote by i. Then given $d+1$ monochromatic sets with the origin $\mathbf{0}$ in the convex hull of each, the Colorful Carathéodory theorem proves the existence of a set S of $d+1$ vertices of distinct colors, with $\mathbf{0}$ in the convex hull of S. We call such a set *panchromatic.*

The original Carathéodory's theorem itself has several applications in various areas. For instance, it has been used in information theory to bound the channel capacity or the rate at which information can be reliably transmitted over a communication channel [5], in control systems theory to get approximate solutions of PDEs [6] and in mathematics to characterize the spectral set of a compact and convex set of real matrices [12]. The colorful version, while originally used in discrete geometry, later led to the development of the algorithmic theory of colorful linear programming by Barany et al [8]. This found applications in game theory, operations research and combinatorics, notably the formulation of a Nash equilibrium of a bimatrix game as a colorful linear programming problem [8].

Because of its diverse uses, the natural computational version of the theorem of finding a panchromatic vertex set, given a suitable input set, attracted interest from the TCS community. Its connections with the Nash equilibrium problem led to conjectures that game theoretic complexity classes might help in resolving the complexity of the problem. This conjecture was proven to hold true, when Meunier et al proved the problem belonged in PPAD [7].

We now think game theory can further help in this classification, by supplying more elegant proof techniques for determining the problem's complexity. Specifically, we aim to find an alternative simpler proof of membership of the Colorful Carathéodory Problem (ColorfulCarathéodory) in PPAD. The proof idea is to design a Linear Complementarity Program (LCP) for the problem, and prove that if Lemke's algorithm is implemented on the LCP, the resulting path followed by the algorithm does not diverge on a secondary ray. This Lemke's path, if proven to be finite, will be a valid instance of the characteristic PPAD-Complete problem EndOfLine. This describes a reduction from ColorfulCarathéodory to EndOfLine, proving ColorfulCarathéodory to be in PPAD.

Our project thus involves a study of the Colorful Carathéodory problem, and attempts to formulate LCPs for the program so that the path taken by Lemke's algorithm, when applied on these LCPs, does not diverge on a secondary ray. We describe in detail our attempts to formulate the LCP, as well as the drawbacks of each attempt.

Presentation: Section 2 describes the background and the notation required for the technical description. Section 3 is the main part of the project where we outline our attempts, culminating in a formulation of a correct LCP for the problem. This LCP however always leads to a secondary ray, and we provide a formal proof of that fact in section 4. In section 5 we conclude with possible future attempts to improve our work, which could eventually lead to a complete proof for the problem using this technique.

2 Preliminaries

We first define the Colorful Carathéodory theorem and the computational problem ColorfulCarathéodory, the main focus of our project, followed by a short overview of game-theoretic complexity classes, ending with a description of Linear Complementarity Programs, and the pivoting technique called Lemke's algorithm for solving them.

2.1 The Colorful Carathéodory Theorem

Theorem 2.1 (Colorful Carathéodory Theorem)**.** *Given $d+1$ sets $C^i, i \in [d+1]$ of $d+1$ vertices each in $\mathbb{R}^d$ such that the origin lies in the convex hull of each of these sets, there exists a choice of $d+1$ vertices S, with exactly one vertex chosen from each set C^i, so that the origin lies in the convex hull of this set.*

Definition 2.2 (ColorfulCarathéodory)**.** *Given as input $d+1$ sets $C^i,\ i \in [d+1]$ of $d+1$ vertices in $\mathbb{R}^d$, such that $\mathbf{0} \in conv(C^i)\ \forall i \in [d+1]$, the Colorful Carathéodory problem (ColorfulCarathéodory) asks to find a panchromatic set $S = \mathbf{v}^1, \mathbf{v}^2 \cdots \mathbf{v}^{d+1}$ where $\mathbf{v}^i \in C^i \quad \forall i \in [d+1]$, such that $\mathbf{0} \in conv(S)$.*

Because of the theorem, the existence of a solution to the problem is always known. We now sketch a pivoting algorithm to find a solution to an instance of the problem: Start with any panchromatic choice of vertices S_1 from the $d+1$ sets and check if the origin lies in the convex hull of this set. If it does not, there is at least one color that prevents the origin from doing so. That is, there exists at least one vertex $\mathbf{v}_1^i$ in S_1 such that the origin and this vertex lie on opposite sides of the hyperplane formed by the remaining vertices in S_1. Let this vertex belong to the color set C^i. Now because a solution to the problem exists, there will be at least one vertex $\mathbf{v}_2^i$ in C^i other than $\mathbf{v}_1^i$, such that the origin and this new vertex from C^i lie on the same side of the aforementioned hyperplane. We now pivot to this vertex, and propose the new colorful choice $S_2 = S_1 \cup \{\mathbf{v}_2^i\} \setminus \{\mathbf{v}_1^i\}$. It can be proved that the distance of the origin to the nearest hyperplane formed by selecting any d out of the $d+1$ vertices in a colorful set is always lesser than that in the previous choice. As a solution with distance 0 exists, and there are a finite number of colorful choices, repeatedly pivoting will eventually converge to a solution.

2.2 Complexity Classes of Total Functions

The traditional complexity classes of P and NP seem insufficient in capturing the complexity of problems like Colorful Carathéodory and Nash equilibrium computation. As the existence of solution to these problems is known, they trivially lie in NP, but any of these problems, if proven NP-hard, will imply $\mathsf{NP} = \mathsf{coNP}$ [10]. Also, the input to these problems can be specified so that the underlying space of solutions is exponentially large, thus simple polynomial time algorithms too seem difficult. This class of problems, where a solution is known to exist, is called TFNP (Total Functional NP).

TFNP, by itself, is a large class, and most likely is semantic. This means there seems to be no formal way to encode the characteristic property that “a solution exists”, and create an automata for this class. This led Papadimitriou to define several complexity subclasses of TFNP, characterized by the nature of the proof of existence to the problems of that class.

One of these classes, relevant to our discussion, is PPAD (Polynomial Parity Argument for Directed Graphs). Defined by Papadimitriou [10], PPAD is the set of all problems in $\mathsf{NP} \cap \mathsf{coNP}$ that are guaranteed to have a solution, whose proof of existence is the following combinatorial statement

Every directed graph has an even number of odd degree nodes.

That is, all problems that can be reduced to the problem of finding *another* odd degree node in a graph, belong in PPAD. An equivalent version of PPAD, more commonly used, assumes the in-degree and out-degree of every node in the graph to be at most 1, with one node of in-degree 0 specified.

The following computational problem, which we denote the EndOfLine problem, naturally arises from the definition of PPAD

Definition 2.3 (EndOfLine Problem)**.** *Given two circuits termed P and S that take as input an n-bit string and output unique n-bit strings such that $S(u) = v \Leftrightarrow P(v) = u \quad \forall u, v \in \{0,1\}^n$, and one string $s = 0^n$, such that $P(s) = \varnothing$, find a string $w \neq s$ such that $P(w) = \varnothing$ or $S(w) = \varnothing$.*

Informally, the above definition describes a directed graph where every node has in-degree and out-degree at most 2, and one source node $\mathbf{0}$ of in-degree 0, and asks to find another odd degree node (source or sink).

The Colorful Carathéodory problem was recently proven to belong in PPAD [7]. We will describe an attempt at a simpler proof using the game theoretic techniques of Linear Complementarity Programs and Lemke’s algorithm. We continue with a brief overview of these concepts.

2.3 Linear Complementarity Problems

3 An LCP for ColorfulCarathéodory

The main purpose of this section is to provide an LCP that captures the solutions of ColorfulCarathéodory and that is correct. The existence of such an LCP already is a strong indication that ColorfulCarathéodory $\in$ PPAD. However, this result requires that the LCP has no secondary rays and Lemke's algorithm can be applied to it, which as we will see is not the case for our designed LCP.

We start off by introducing a set of necessary and sufficient constraints that capture the solutions to ColorfulCarathéodory. Specifically, given $d+1$ sets of $d+1$ vertices where, assuming an ordering of the vertices of each color, we denote the jth vertex of color i by $\mathbf{v}_j^i$, we consider the coefficients of these vertices in a possible convex combination, and denote them by a_j^i. Consider a set S of vertices that is a solution to ColorfulCarathéodory. Then, by the definition of the problem, we know that

- At a solution, if one coefficient of a color is strictly positive, then all other coefficients of the same color are equal to zero. We capture this property by introducing the following complementarity constraint

$$\forall i, j \qquad a_j^i \geq 0 \qquad \perp \qquad \sum_{k \neq j} a_k^i \geq 0$$

  This property assures that at most one coefficient is strictly positive in a solution. Note that we do not force exactly one coefficient to be positive because of the possibility of $\mathbf{0}$ lying in the boundary of $conv(S)$ which is a subspace of dimension lesser than d.

- At a solution, $\mathbf{0} \in conv(S)$. However, we do not know *a priori* which a_j^i are going to be non-zero. Therefore, we capture this property by imposing the following equality constraint

$$\sum_{i=1}^{d+1} \sum_{j=1}^{d+1} a_j^i \mathbf{v}_j^i = \mathbf{0}$$

- The a_j^i define a convex combination of $\mathbf{v}_j^i$, therefore we have to impose the following equality constraint as well

$$\sum_{i=1}^{d+1} \sum_{j=1}^{d+1} a_j^i = 1$$

While we are on the right track with the complementarity constraints, this set of conditions are far from being a correct LCP for ColorfulCarathéodory, mainly for two reasons. First of all, we have two additional equality constraints that are not incorporated in the standard LCP (M, q) format. Furthermore we note that if we set one $a_j^i > 0$, by the complementarity constraint of a_j^i, we get $\sum_{k \neq j} a_k^i = 0$, but we also get $\sum_{k' \neq k} a_{k'}^i > 0$ from the other inequalities of color i, which imply $a_k^i = 0 \quad \forall k \neq j$. Therefore, simply by changing one variable from zero to non-zero, we make $d+1$ inequalities tight in $\mathbb{R}^d$. This fact implies that we have degeneracy in our system of inequalities which, if not fixed, will prove to be a problem when we try to apply Lemke's algorithm to our LCP since in a degenerate LCP Lemke's algorithm could potentially cycle between vertices and never converge to a solution.

Thankfully, it is easy to get around both problems with a simple change, and get an LCP formulation for ColorfulCarathéodory. To do this, we utilize the two equality constraints and solve them for a_1^i. Hence forth, a_1^i will not be variables, but expressions whose value rely on the rest of variables $a_j^i \quad 2 \leq j \leq d+1$. To keep track of this fact and avoid any confusion, we introduce in our notation the values $le^i \triangleq a_1^i$. If we denote by $(\mathbf{v}_j^i)^k$ the coordinate of $\mathbf{v}_j^i$ in dimension k, then the values of le^i are calculated by solving the following linear system

$$\begin{bmatrix} (\mathbf{v}_1^1)^1 & (\mathbf{v}_1^2)^1 & \dots & (\mathbf{v}_1^{d+1})^1 \\ (\mathbf{v}_1^1)^2 & (\mathbf{v}_1^2)^2 & \dots & (\mathbf{v}_1^{d+1})^2 \\ \vdots & \vdots & & \vdots \\ (\mathbf{v}_1^1)^d & (\mathbf{v}_1^2)^d & \dots & (\mathbf{v}_1^{d+1})^d \\ 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} le^1 \\ le^2 \\ \vdots \\ le^d \\ le^{d+1} \end{bmatrix} = \begin{bmatrix} -\sum_i \sum_{j\neq 1} a_j^i (\mathbf{v}_j^i)^1 \\ -\sum_i \sum_{j\neq 1} a_j^i (\mathbf{v}_j^i)^2 \\ \vdots \\ -\sum_i \sum_{j\neq 1} a_j^i (\mathbf{v}_j^i)^d \\ 1 - \sum_i \sum_{j\neq 1} a_j^i \end{bmatrix} \tag{1}$$

We are now able to write the following LCP for ColorfulCarathéodory

$$\forall i, j \neq 1 \quad le^i + \textstyle\sum_{k\neq 1,j} a_k^i \geq 0 \quad \perp \quad a_j^i \geq 0 \tag{2}$$

However, we immediately notice a problem in our current formulation that could potentially lead to an incorrect solution of ColorfulCarathéodory. Since the le^i are now calculated by solving a linear system of equations, we fail to enforce the necessary condition $a_1^i \geq 0$, and the le^i could potentially be negative. To solve this new problem, we incorporate a new set of variables and inequalities, one for each color, to bound the le^i from 0. We call these new variables γ^i. Thus, we tweak (2) to the following LCP

$$\begin{array}{lrcl} \forall i & le^i - \gamma^i \geq 0 & \perp & \gamma^i \geq 0 \\ \forall i, j \neq 1 & le^i + \sum_{k\neq 1,j} a_k^i \geq 0 & \perp & a_j^i \geq 0 \end{array} \tag{3}$$

The above LCP succeeds in capturing the set of necessary and sufficient conditions for a solution to ColorfulCarathéodory, a fact which we formalize with the following theorem

Theorem 3.1 (Correctness of LCP (3))**.** *Consider an instance of* ColorfulCarathéodory *with* $C^i = \{\mathbf{v}_1^i, \mathbf{v}_2^i, \cdots, \mathbf{v}_{d+1}^i\} \quad \forall i \in [d+1]$ *as the input vertices and* $le^i \, \forall i \in [d+1]$ *which are calculated by the linear system* (1)*. Then, every solution to the Linear Complementarity Program defined in* (3) *is also a solution to this instance of* ColorfulCarathéodory*.*

Proof. □

4 Existence of a Secondary Ray

In the previous section, we provided a correct LCP formulation of ColorfulCarathéodory with (3). This fact alone is a strong indication that we could potentially prove Colorful-Carathéodory $\in$ PPAD via LCPs. However, to complete our proof, we would have to prove that Lemke's algorithm, implemented in LCP (3) always converges to a solution. In other words, we would have to prove that no secondary ray exists in the polyhedron which defines the feasible space of our LCP.

Unfortunately, this is not the case for our current LCP formulation. We can easily prove that there always exists a secondary ray, and Lemke's algorithm implemented (3) is never guaranteed to converge. To demonstrate this fact, we perform a comprehensive analysis of all possible secondary rays of (3) and distinguish the actual secondary ray from the rest. We begin by extending (3) to the LCP used by Lemke's algorithm, with covering vector $\boldsymbol{c}$, where $c^i_j \in \mathbb{R}$

$$\begin{array}{lrcl} \forall i & le^i - \gamma^i + c^i_1 z \geq 0 & \perp & \gamma^i \geq 0 \\ \forall i, j \neq 1 & le^i + \sum_{k \neq 1, j} a^i_k + c^i_j z \geq 0 & \perp & a^i_j \geq 0 \end{array} \tag{4}$$

We consider a tuple of all variable types of our problem (a, γ, z), which will denote the direction of a potential ray, and assign a sign to each position in the following way

- $+$: If at least one variable of that type is increasing in the current ray.
- 0: If no variable of that type is changing in the current ray.
- $-$: Otherwise.

We show that all rays are finite, except for one, namely $(0, +, +)$. In the following cases, we use $*$ in place of a sign to denote any possible sign for that set of variables.

- $(-, *, *), (*, -, *)$: None of these rays can be a secondary ray, since if at least one a^i_j or one γ^i keeps decreasing, then at some point we will violate the right complementarity inequalities in (4).
- $(+, *, -), (+, *, 0)$: From the linear system of equations that provides the form of the le^i, we have that

  $$le^{d+1} = 1 - \sum_{i=1}^{d+1} \sum_{j=2}^{d+1} a^i_j - \sum_{i=1}^{d} le^i$$

  For any case where at least one a^i_j increases and z does not increase, eventually $le^{d+1} < 0$. This cannot happen in a solution, since it violates the first complementarity condition of (4).
- $(+, *, +)$: In this case we have that at least one a^i_j si increasing, therefore the corresponding complementarity constraint has to be tight. Therefore, we get

$$le^i + \sum_{k\neq 1,j} a_k^i + c_j^i z = 0$$
$$le^i = -\sum_{k\neq 1,j} a_k^i - c_j^i z$$

We substitute this equality in the left part of the first complementarity constraint and get

$$le^i - \gamma^i + c_1^i z \geq 0 - \sum_{k\neq 1,j} a_k^i - \gamma^i + (c_1^i - c_j^i)z \geq 0$$

Now we can set $c_1^i < c_j^i \quad \forall i \in [d+1]$ and utilize the facts that $a_k^i \geq 0$, $\gamma^i \geq 0$ and that z is increasing to see that we immediately violate the above inequality and reach a contradiction. Therefore this ray is finite and cannot be a secondary ray.

- $(0,0,0)$: No variable is changing in this case, therefore this case does not represent a ray.
- $(0,0,-)$: This is our primary ray and the ray that Lemke's algorithm starts from.
- $(0,0,+)$: This is our primary ray again, but we traverse it in the opposite direction this time. This cannot be a secondary ray, since Lemke's algorithm cannot cycle back to the primary ray in order to traverse it in the opposite direction for a non-degenerate LCP, and we have that (4) is a non-degenerate LCP.
- $(0,+,-)$: Since no a_j^i is changing in this case, we have that all le^i will also stay fixed. Therefore, if we keep increasing at least one γ^i while decreasing z, then we either reach a case where $z = 0$ and Lemke's algorithm has found a solution, therefore it stops, or at some point we will violate the left side of the first complementarity condition of (4) for that i.
- $(0,+,0)$: In this case, all variables stay fixed except for at least one or more γ^i. Thus, no le^i is changing, and when $\gamma^i > le^i + c_1^i z$, we will violate the left side of the first complementarity condition of (4) for that i.
- $(0,+,+)$: In this case, we have that at least one $\gamma^i > 0$. Therefore, the corresponding complementarity constraint has to be tight

  $$le^i - \gamma^i + c_1^i z = 0 le^i = \gamma^i - c_1^i z$$

  Replacing this equality in the left side of the second complementarity constraint, we get

  $$\gamma^i + \sum_{k\neq 1,j} a_k^i + (c_j^i - c_1^i)z \geq 0$$

We immediately see that since $c_j^i > c_1^i$, we violate no constraints and we can keep increasing γ^i and z simultaneously without ever converging to a solution. Thus, unfortunately, this case represents a secondary ray and Lemke's algorithm implemented on (4) is not guaranteed to converge.

5 Conclusion

Our project's aim was to analyze the possibility of proving that ColorfulCarathéodory lies in PPAD via the use of a standard game-theoretic technique, the Linear Complementarity Program. Our objective was twofold; first provide a correct LCP formulation for ColorfulCarathéodory which was not known to exist, and second prove that Lemke's algorithm converges on that LCP. In the preliminaries we showed why that was enough to prove that ColorfulCarathéodory lies in PPAD.

While in the end we failed to achieve both our objectives, we succeded in the first one, which provides both a strong indication that our method could potentially work and motivation for us to keep working on this project. There are several ways we could proceed from here and tweak our LCP to attain both correctness and convergence of Lemke's algorithm.

First of all, observe that we did not utilize at all one specific property of the input, that the origin $\mathbf{0}$ lies inside the convex hull of every "monochromatic" set C^i. In other words, that a solution to our problem always exists. It is difficult to imagine an LCP formulation on which Lemke's algorithm converges that does not make proper use of this property. While we tried to incorporate it in our set of complementary inequalities, we currently do not possess a correct way of doing so.

Furthermore, there is a vast amount of literature for this problem [2, 4, 9, 11] where many known properties of the theorem are described. Another way to approach the LCP formulation is to make use of these properties. As an example of such a property, it is known that every input vertex $\mathbf{v}_j^i$ is part of a solution to the problem. Perhaps by utilizing this and other known properties, we could arrive more easily at a corerct LCP formulation for ColorfulCarathéodory and finally be able to provide a clear, concise, simple and well-understood proof that ColorfulCarathéodory $\in$ PPAD.

References

[1] Imre Bárány. A generalization of carathéodory's theorem. *Discrete Mathematics*, 40(2):141 – 152, 1982.

[2] Imre Bárány and Jiri Matousek. Quadratically many colorful simplices, 2007.

[3] C. Carathéodory. Über den variabilitätsbereich der koeffizienten von potenzreihen, die gegebene werte nicht annehmen. *Mathematische Annalen*, 64(1):95–115, Mar 1907.

[4] Antoine Deza, Sui Huang, Tamon Stephen, and Tamas Terlaky. Colourful simplicial depth. *Discrete & Computational Geometry*, 35(4):597–615, May 2006.

[5] RadhaKrishna Ganti. Application of caratheodory's theorem in information theory. January 26 2012.

[6] Holly, Konstanty, Orewczyk, and Joanna. Applications of the carathéodory theorem to pdes. *Annales Polonici Mathematici*, 73(1):1–27, 2000.

[7] Frédéric Meunier, Wolfgang Mulzer, Pauline Sarrabezolles, and Yannik Stein. The rainbow at the end of the line - A PPAD formulation of the colorful carathéodory theorem with applications. *CoRR*, abs/1608.01921, 2016.

[8] Frédéric Meunier and Pauline Sarrabezolles. Colorful linear programming, nash equilibrium, and pivots. *CoRR*, abs/1409.3436, 2014.

[9] Wolfgang Mulzer and Yannik Stein. Computational aspects of the colorful caratheodory theorem. *CoRR*, abs/1412.3347, 2014.

[10] Christos H. Papadimitriou. On the Complexity of the Parity Argument and Other Inefficient Proofs of Existence. *J. Comput. Syst. Sci.*, 48(3):498–532, June 1994.

[11] Tamon Stephen and Hugh Thomas. A quadratic lower bound for colourful simplicial depth. *Journal of Combinatorial Optimization*, 16(4):324–327, Nov 2008.

[12] Monov Vladimir. An application of caratheodory's theorem to the spectral set problem for convex matrix sets. 2001.