IE 598 JG Games, Markets and Mathematical Programming
ColorfulCarathéodory *in* PPAD
A simpler proof using LCPs

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1 Introduction

The Colorful Carathéodory theorem is a classical result in graph theory. The theorem states that given $d+1$ sets $C^i, i \in [d+1]$ of $d+1$ vertices each in R^d, so that the origin lies in the convex hull of each of these sets, there exists a set of $d+1$ vertices S, such that exactly one vertex from each set C^i belongs to S, and the origin lies in the convex hull of this set.

For visualizing the theorem, one can think all vertices of one input set C^i are assigned one color, which we denote by i. Then given $d+1$ monochromatic sets with the origin in the convex hull of each, the Colorful Carathéodory theorem proves the existence of a set of $d+1$ vertices of distinct colors, with the origin in the convex hull of this set. We call such a set *panchromatic*.

The natural computational problem of finding a panchromatic vertex set has been proven to lie in PPAD [1]. We propose to find an alternative simpler proof of membership of the Colorful Carathéodory Problem (ColorfulCarathéodory) in PPAD. The proof idea is to design a Linear Complementarity Program (LCP) for the problem, and prove that if Lemke's algorithm is implemented on the LCP, the resulting path followed by the algorithm converges. This path, if finite, is an instance of the characteristic PPAD-Complete problem EndOfLine. Thus its a valid reduction from ColorfulCarathéodory to EndOfLine, proving ColorfulCarathéodory to be in PPAD.

2 Related Work

3 Preliminaries

3.1 The Colorful Carathéodory Theorem

3.2 Complexity Classes of Total Functions

3.3 Linear Complementarity Problems

4 An LCP for ColorfulCarathéodory

The main purpose of this section is to provide an LCP that captures the solutions of ColorfulCarathéodory and that is correct. The existence of such an LCP already is a strong indication that ColorfulCarathéodory $\in$ PPAD. However, this result requires that the LCP has no secondary rays and Lemke's algorithm can be applied to it, which as we will see is not the case for our designed LCP.

We start off by introducing a set of necessary and sufficient constraints that capture the solutions to ColorfulCarathéodory. Specifically, given $d+1$ sets of $d+1$ vertices where we denote the jth vertex of color i by $\mathbf{v}_j^i$, we consider the coefficients of these vertices in a possible convex combination, and denote them by a^i_j. Consider a set S of vertices that is a solution to ColorfulCarathéodory. Then, by the definition of the problem, we know that

- At a solution, if one coefficient of a color is strictly positive, then all other coefficients of the same color are equal to zero. We capture this property by introducing the following complementarity constraint

$$\forall i, j \qquad a^i_j \geq 0 \qquad \perp \qquad \sum_{k \neq j} a^i_k \geq 0$$

  This property assures that at most one coefficient is strictly positive in a solution. Note that we do not force exactly one coefficient to be positive because of the possibility of $\mathbf{0}$ lying in the boundary of $conv(S)$ which is a subspace of dimension lesser than d.

- At a solution, $\mathbf{0} \in conv(S)$. However, we do not know *a priori* which a^i_j are going to be non-zero. Therefore, we capture this property by imposing the following equality constraint

$$\sum_{i=1}^{d+1} \sum_{j=1}^{d+1} a^i_j \mathbf{v}^i_j = \mathbf{0}$$

- The a^i_j define a convex combination of $\mathbf{v}^i_j$, therefore we have to impose the following equality constraint as well

$$\sum_{i=1}^{d+1} \sum_{j=1}^{d+1} a^i_j = 1$$

5 Existence of a Secondary Ray

6 Conclusion

References

[1] Frédéric Meunier, Wolfgang Mulzer, Pauline Sarrabezolles, and Yannik Stein. The rainbow at the end of the line - A PPAD formulation of the colorful carathéodory theorem with applications. *CoRR*, abs/1608.01921, 2016.