

# Oracle-Augmented Prophet Inequalities & How Much You Can Win by Cheating!

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Joint work with



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$$X_1 \sim U[4, 6]$$

$$X_2 \sim U[2, 8]$$

$$X_3 \sim U[0, 9]$$

$$X_4 = \begin{cases} 450 & \text{w.p. } \frac{1}{100} \\ 0 & \text{otherwise} \end{cases}$$



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4.88



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3.23



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$$X_4 = \begin{cases} 450 & \text{w.p. } \frac{1}{100} \\ 0 & \text{otherwise} \end{cases}$$

A man with a large, curly beard and a golden halo above his head is seated at a roulette table in a grand, ornate casino. He is wearing a dark blue velvet robe with gold embroidery. The roulette wheel is positioned between his hands, and a large stack of gold coins sits on the green felt surface. In the foreground, another man in a dark suit and bow tie is seated at the same table, looking towards the bearded man. The background is filled with other players at slot machines and roulette tables under numerous large, glowing chandeliers. A white rectangular box contains the following mathematical expression:

$$X_4 = \begin{cases} 450 & \text{w.p. } \frac{1}{100} \\ 0 & \text{otherwise} \end{cases}$$





4.88

0

6.67

3.23





6.67



$$\mathbb{E} \left[ \max_i X_i \right]$$



A man in a suit sits at a desk in a grand, ornate room filled with slot machines. He is looking up at a large screen displaying a bearded deity with a halo. The deity is pointing towards the screen. The text on the screen reads:

$$\text{Set } T = 1/2 \cdot \mathbb{E}[\max_i X_i]$$

Accept first  $X_i \geq T$

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[Kleinberg, Weinberg, '12]

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[Wittmann, '95]

[Kleinberg, Weinberg, '12]



$$\Pr \left[ \text{Gambler} \leftarrow \max_i X_i \right]$$

?

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*(concentration)*

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*[Esfandiari, HajiAghayi,  
Lucier, Mitzenmacher,'20]*



Top - 1 - of - k

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$$\geq 1 - \frac{1}{k+1} \quad [AS'00, AGS'02]$$

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$$\begin{aligned} &\geq 1 - e^{-k/6} \\ &\leq 1 - k^{-2k} \end{aligned} \quad [EFN'18]$$

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Auctions w/ Overbooking







A bearded man with a halo, dressed in a white robe, stands at a roulette table, holding a small orange box. A roulette wheel is visible on the table. A woman in a white dress stands behind him. The background shows ornate ceiling lights and a grand interior.

$$X_1 \sim U[4, 6]$$



4.88



6.67





A grand interior of a casino, featuring ornate gold chandeliers and slot machines. In the center, three figures stand under a large, ornate frame. On the left is a woman in a white gown. In the middle is a man with a long white beard, wearing a white robe. On the right is a man in a black tuxedo pointing upwards. A large, ornate frame surrounds the text "Questions?".

Questions?



$$\max \frac{\mathbb{E}[\text{ALG}]}{\mathbb{E}[\max_i X_i]}$$



$$\max \Pr \left[ \text{ALG} \leftarrow \max_i X_i \right]$$



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Oracle<sub>k</sub>  
≠

Top-1-of-(k + 1)

Oracle<sub>k</sub>  
≡

Top-1-of-(k + 1)

Oracle<sub>1</sub>  $\neq$  Top-1-of-2

$$X_1 = 1 \quad X_2 = \begin{cases} 1 + \varepsilon & \text{w.p. } \frac{1}{2} - \varepsilon \\ 0 & \text{otherwise} \end{cases}$$
$$X_3 = \begin{cases} \frac{1}{\varepsilon} & \text{w.p. } \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Top-1-of-2} \approx 1 + 1 = 2$$

$$\text{Oracle}_1 \approx \frac{1}{2} + 1 = \frac{3}{2}$$



$$\max \frac{\mathbb{E}[\text{ALG}]}{\mathbb{E}[\max_i X_i]}$$

$$\max \Pr \left[ \text{ALG} \leftarrow \max_i X_i \right]$$

Oracle<sub>k</sub>

↑

Top-1-of-(k + 1)

Oracle<sub>k</sub>

≡

Top-1-of-(k + 1)



$$\max \Pr \left[ \text{ALG} \leftarrow \max_i X_i \right]$$
$$\text{Oracle}_k \equiv \text{Top-1-of-}(k+1)$$

$$\max \frac{\mathbb{E}[\text{ALG}]}{\mathbb{E}[\max_i X_i]}$$

$$\begin{array}{ccc} \text{Oracle}_k & & \text{ALG} \rightarrow \geq \\ \not\equiv & & \downarrow \\ \text{Top-1-of-}(k+1) & & \text{ALG} \rightarrow \geq \end{array}$$

# Main Results

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Oracle<sub>k</sub>(Secretary, IID)

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Top-1-of- $k$ (Secretary, IID)

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# Techniques

## Sharding

$X_i \rightarrow \max\{Y_1, \dots, Y_\ell\}$  (*IID*)

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Poissonization

$$\max\{Y_1, \dots, Y_\ell\} \equiv \text{Poi}(\lambda)$$

# Techniques

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Poissonization

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Stochastic Dominance

$$\forall x \quad \Pr[\text{ALG} \geq x] \geq c \cdot \Pr[\text{Prophet} \geq x]$$

# Upper Bound

$$X_1 = 1 \quad X_2 = \begin{cases} 1 + \varepsilon & \text{w.p. } \frac{1}{2} - \varepsilon \\ 0 & \text{otherwise} \end{cases}$$
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$$\implies 1 - \frac{1}{2^{k+1}}$$

# Upper Bound

$$X_1 = 1 \quad X_2, \dots, X_{n-1} \sim \text{Poi}(\xi_k)$$

$$X_n = \begin{cases} \frac{1}{\varepsilon} & \text{w.p. } \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

$$\xi_k \implies \Pr[\text{Poi}(\xi_k) = 0] = \Pr[\text{Poi}(\xi_k) > k]$$

$$\xi_k = \frac{k}{e} + o(k)$$

$$1 - \frac{1}{2^{k+1}} \implies 1 - e^{-\xi_k}$$



$$X_1 \sim U[0, 9]$$

$$X_2 \sim U[0, 9]$$

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$$X_4 \sim U[0, 9]$$

## IID Idea

Select all  $\geq T$   
+ Chernoff bound

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max of  $\pi_1 \pi_2 \dots \pi_n$

changes  $O(\log n)$  times  
 $\implies$  can set  $T$  higher

$$\geq 1 - O(k^{-k/5}) \quad \leq 1 - O(k^{-k})$$

# Open Questions

Prophet, Gen

Prophet, IID

Secretary, Gen

Secretary, IID

Oracle  $\rightarrow$   $\Pi ?$     $\Pi ?$   
 $\Pi ?$     $\Pi ?$

Robustness?  $\implies$  ML Predictions



Thank You  
Estonia!