Fundamentals of Wireless Communication

(D. Tse and P. Viswanath, *Fundamentals of Wireless Communications*.,)

Detection and Estimation in Additive Gaussian Noise

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Gaussian Random Variable

- **Scalar Real Gaussian Random Variable**

- A standard Gaussian random variable, w takes values over the real line and has the probability density function:

$$f(w) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-w^2}{2}\right), \qquad w \in \Re$$

- The standard Gaussian random variable is denoted by $\mathcal{N}(0,1)$.

- This random variable is called a standard Gaussian random variable.

Gaussian Random Variable

- A general Gaussian random variable, $\boldsymbol{x}$ is of the form:

$$x = \sigma w + \mu$$

- The random variable x is a one-to-one function of w and thus the probability density function is:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{(x-\mu)^2}{2\sigma^2}\right), \qquad x \in \Re$$

- Since the random variable is completely characterized by its mean and variance, we denote, x by $\mathcal{N}(\mu, \sigma^2)$.

Gaussian Random Variable

- **Note:** Gaussianity is preserved by linear transformations: linear combinations of independent Gaussian random variables are still Gaussian.
- If $x_1, x_{2,} \dots, x_n$ are independent and $x_i \sim \mathcal{N}(\mu_i, {\sigma_i}^2)$, then:

$$\sum_{i=1}^{n} c_i x_i \sim \mathrm{N}\left(\sum_{i=1}^{n} c_i \mu_i \sum_{i=1}^{n} {c_i}^2 {\sigma_i}^2\right)$$

Gaussian Random Variable

- **Real Gaussian Random Vector**
- A standard Gaussian random vector, $\boldsymbol{w}$ is a collection of n independent and identically distributed ($i.i.d$) standard Gaussian random variables $w_1, w_{2,} \dots, w_n$.
- The vector $\boldsymbol{w} = (w_1, w_{2,} \dots, w_n)^t$ takes values in the vector space $\Re^n$.
- The probability density function of $\boldsymbol{w}$ is :

$$f(w) = \frac{1}{(\sqrt{2\pi}\,)^n} \exp\left(\frac{-||\boldsymbol{w}||^2}{2}\right), \qquad w \in \Re$$

- In general, a Gaussian random vector is completely characterized by its mean μ and by the covariance matrix K; we denote the random vector by $\mathcal{N}(\mu, K\,)$.

Detection in Gaussian Noise

- Consider the real additive Gaussian noise channel: $\boldsymbol{y} = \boldsymbol{u} + \boldsymbol{w}$
- The transmit symbol u is equally likely to be $u_A \ or \ u_B$ ($u_A > u_B \in \Re$) and $w \sim \mathcal{N}\left(0, \frac{N_o}{2}\right)$ is real Gaussian noise.

  The detection problem involves making a decision on whether $u_A \ or \ u_B$ was transmitted based on the observation y.
- The optimal detector, with the least probability of making an erroneous decision, chooses the symbol that is most likely to have been transmitted given the received signal y, i.e., u_A is chosen if

$$\mathbb{P}\{\boldsymbol{u} = \boldsymbol{u_A} | \boldsymbol{y}\} \geq \mathbb{P}\{\boldsymbol{u} = \boldsymbol{u_B} | \boldsymbol{y}\}$$

Detection in Gaussian Noise

- Since the two symbols u_A, u_B are equally likely to have been transmitted, Bayes' rule lets us simplify this to the maximum likelihood (ML) receiver, which chooses the transmit symbol that makes the observation, y most likely. the decision rule is to choose u_A if

$$\frac{1}{\sqrt{\pi N_0}} exp\left(-\frac{(y-u_A)^2}{N_0}\right) \geq \frac{1}{\sqrt{\pi N_0}} exp\left(-\frac{(y-u_B)^2}{N_0}\right)$$

- Which can be simplified further as:

$$|y-u_A| < |y-u_B|$$

Detection in Gaussian Noise

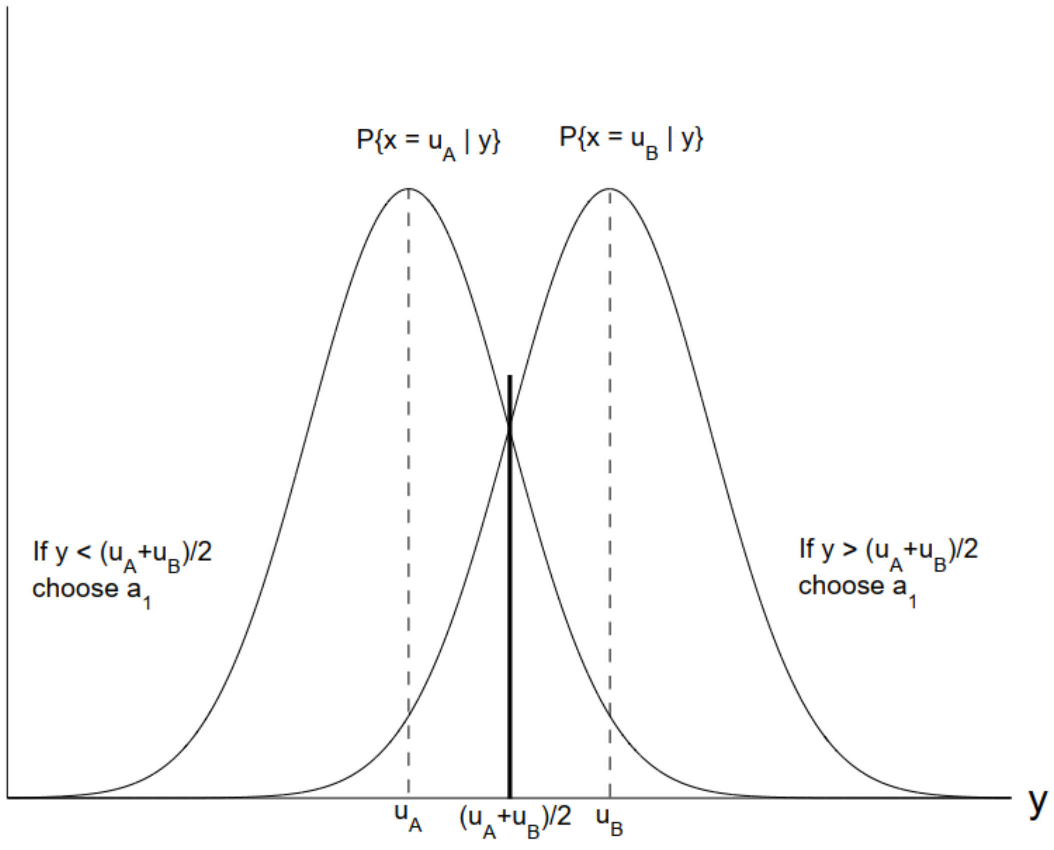


Figure A.4: The ML rule is to choose the symbol that is *closest* to the received symbol.

Detection in Vector Space

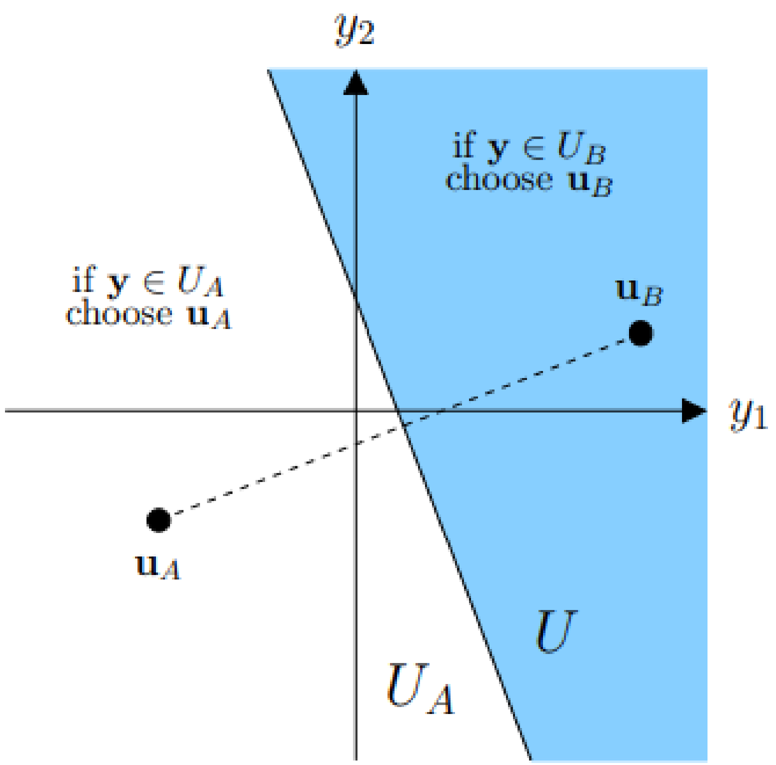
- Now consider detecting the transmit vector u equally likely to be $u_A \; or \; u_B$ (both elements of $\Re^n$). The received vector is: $\boldsymbol{y} = \boldsymbol{u} + \boldsymbol{w}$
- The ML decision rule is to choose u_A if

$$||y - u_A|| < ||y - u_B||$$

- the same nearest neighbor rule.

Detection in a Vector Space

- Geometrically, this says that the decision regions are the two sides of the hyperplane perpendicular to the vector, $u_B - u_A$.
- An error occurs when the received vector lies on the side of the hyperplane opposite to the transmit vector.



Alternative View

- To see how we could have reduced the vector detection problem to the scalar one. We can write the transmit vector u as:

$$u = x(u_A - u_B) + \frac{1}{2}(u_A + u_B)$$

- where the information is in the scalar x, which is equally likely to be $\pm {}^{1}/_{2}$

$$y - \frac{1}{2}(u_A + u_B) = x(u_A - u_A) + w$$

Alternative View

- We observe that the transmit symbol (a scalar x) is only in a specific direction:

$$v = (u_A - u_B)/\ ||u_A - u_B||$$

- Therefore, projecting the received vector along the signal direction v provides all the necessary information for detection:

$$\tilde{y} = v^t \left(y - \frac{1}{2}(u_A + u_B) \right)$$

Alternative View

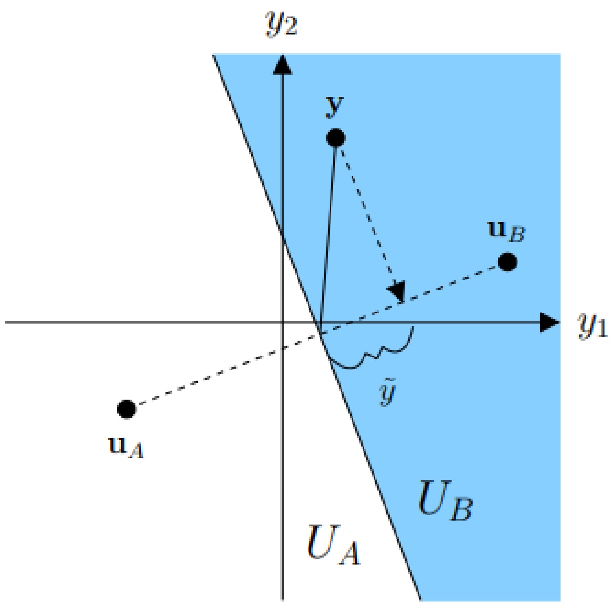


Figure A.6: Projecting the received vector $\mathbf{y}$ onto the signal direction $\mathbf{v}$ reduces the vector detection problem to the scalar one.

Estimation in Gaussian Noise

- Consider a zero mean real transmit signal x embedded in independent additive real Gaussian noise $\left(w \sim \mathcal{N}\left(0, {}^{N_0}/_2\right)\right)$

$$y = x + w$$

- Suppose we wish to come up with an estimate ˆ x of x and we use the mean squared error (MSE) to evaluate the performance:

$$MSE = \mathbb{E}[(x - x)^2]$$

- The estimate that yields the smallest mean squared error is the classical conditional mean operator:

$$x = \mathbb{E}[x|y]$$

Estimation in Gaussian Noise

- The classical conditional mean operator has the important orthogonality property: the error is independent of the observation. In particular, this implies that :

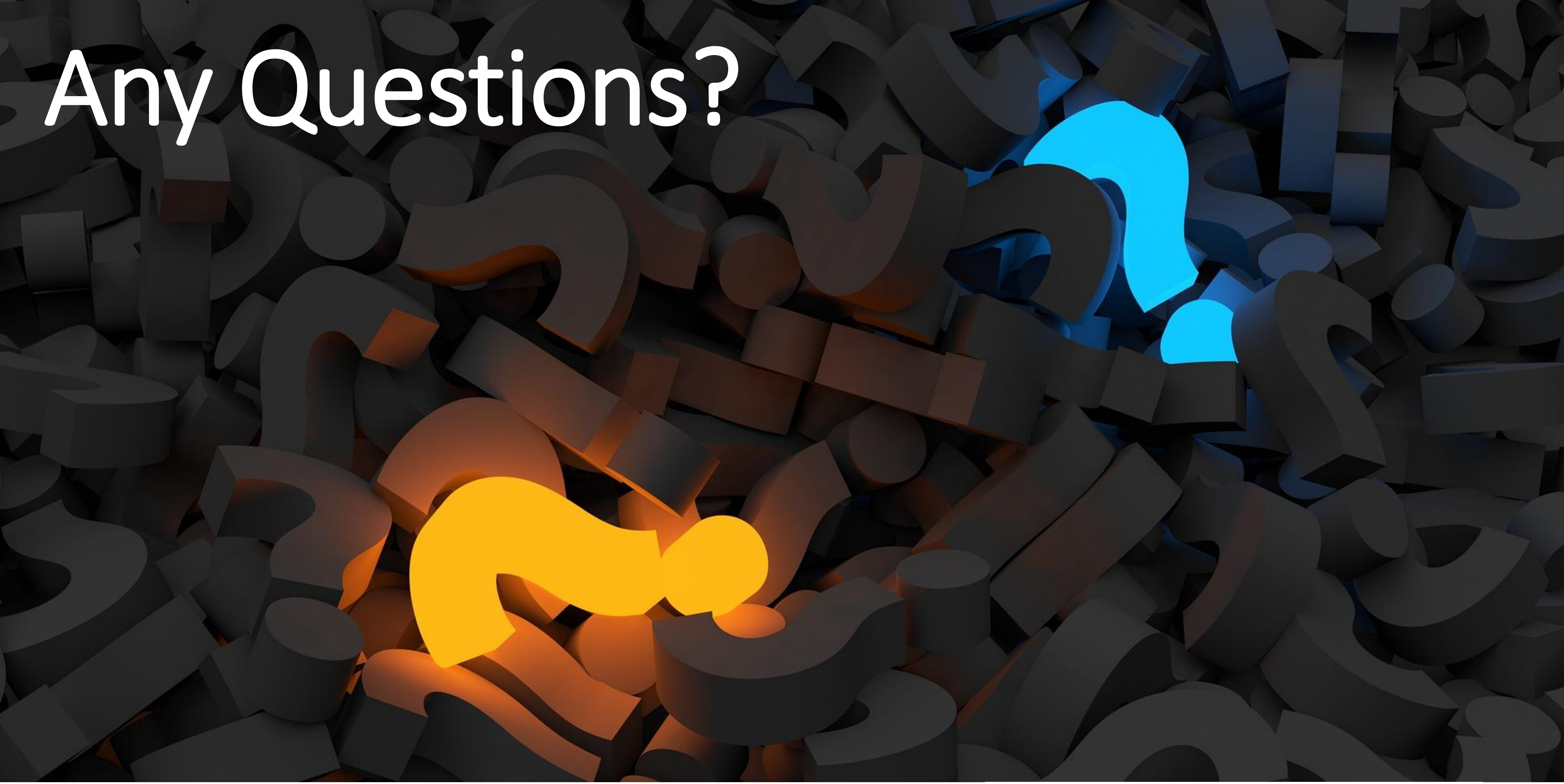
$$\mathbb{E}[(x - x)y] = 0$$

- The conditional mean operator is some complicated nonlinear function of y. To simplify the analysis, we use the restricted class of linear estimates that minimize the MSE.

- The corresponding minimum mean squared error (MMSE) is:

$$MMSE = \frac{\mathbb{E}[x^2]\,{}^{N_0}\!/_2}{\mathbb{E}[x^2] + {}^{N_0}\!/_2}$$

Any Questions?



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