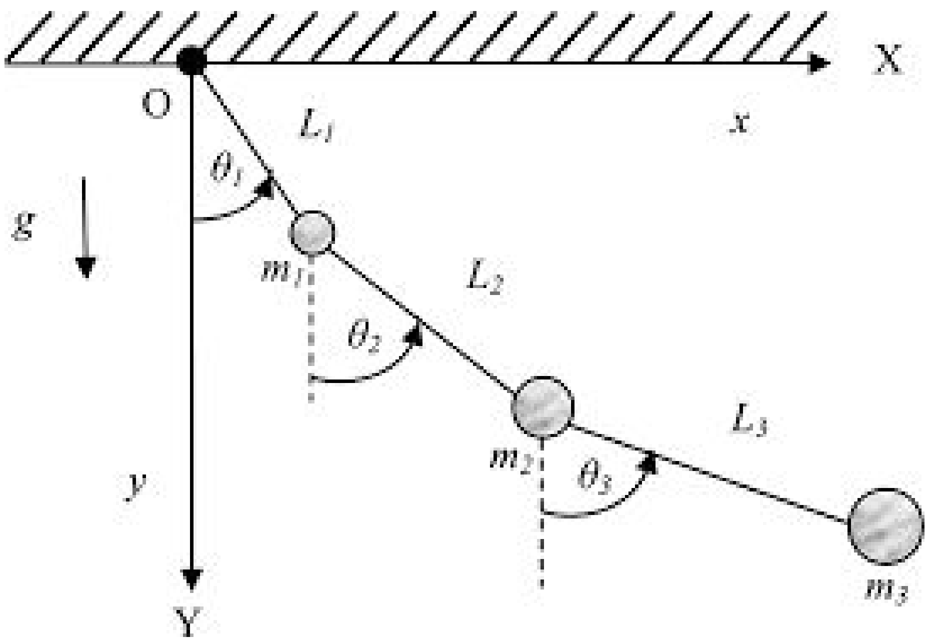
Triple Pendulum

- Vasu Bansal, 160776
- Vishal Singh, 160804

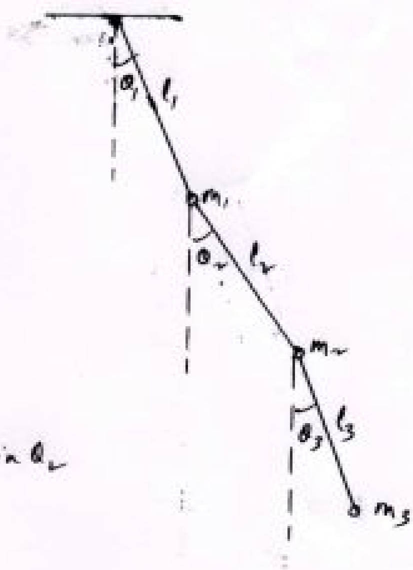
Overview : A triple pendulum is a pendulum with a double pendulum attached to its end.This can be regarded as an open chain mechanism having one end of a link rigidly attached to a support and the other end exhibiting a rich dynamic behaviour governed by the initial conditions of the setup and upon the field in which the system is kept.



The triple pendulum considered in our case is taken as massless rods and the entire mass of the pendulum is assumed to reside in the bobs as is also depicted in the diagrams. In this simulation project, we have tried to simulate the motion of a triple pendulum under a user desired field (which could be acceleration due to gravity) and also allowed for the user to set the lengths of the 3 rods along with the masses of the bobs. The simulation also allows to change the initial conditions of angular positions of the three pendulums. The entire simulation is created through a Python code which integrates the differential equations of motion (shown in the next section) through a specific time step.

DIFFERENTIAL EQUATIONS FOR TRIPLE PENDULUM

- Rods are massless
- Acceleration due to gravity $= -g\hat{j}$



$x_1 = l_1 \sin\theta_1 \quad ; \quad y_1 = -l_1\cos\theta_1$

$x_2 = l_1\sin\theta_1 + l_2\sin\theta_2 \quad ; \quad y_2 = -l_1\cos\theta_1 - l_2\cos\theta_2$

$x_3 = l_1\sin\theta_1 + l_2\sin\theta_2 + l_3\sin\theta_3$

$y_3 = -l_1\cos\theta_1 - l_2\cos\theta_2 - l_3\cos\theta_3$

$\dot{x}_1 = l_1\dot{\theta}_1\cos\theta_1 \quad ; \quad \dot{y}_1 = l_1\dot{\theta}_1\sin\theta_1$

$\dot{x}_2 = l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2 \quad ; \quad \dot{y}_2 = l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2$

$\dot{x}_3 = l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2 + l_3\dot{\theta}_3\cos\theta_3$

$\dot{y}_3 = l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2 + l_3\dot{\theta}_3\sin\theta_3$

- Kinetic Energy $T = \frac{1}{2}\left(m_1v_1^2 + m_2v_2^2 + m_3v_3^2\right)$

$$= \frac{1}{2}\left[m_1(\dot{x}_1^2+\dot{y}_1^2) + m_2(\dot{x}_2^2+\dot{y}_2^2) + m_3(\dot{x}_3^2+\dot{y}_3^2)\right]$$

$$= \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right]$$

$$+ \frac{1}{2}m_3\Big[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + l_3^2\dot{\theta}_3^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + 2l_2l_3\dot{\theta}_2\dot{\theta}_3\cos(\theta_2-\theta_3)$$

$$+ 2l_3l_1\dot{\theta}_1\dot{\theta}_3\cos(\theta_3-\theta_1)\Big]$$

- Potential Energy $V = m_1gy_1 + m_2gy_2 + m_3gy_3$

$$= -(m_1+m_2+m_3)g\,l_1\cos\theta_1 - (m_2+m_3)g\,l_2\cos\theta_2 - m_3g\,l_3\cos\theta_3$$

- Lagrangian $\mathcal{L} = T - V$

$$\therefore \mathcal{L} = \frac{l_1^2\dot{\theta}_1^2}{2}(m_1+m_2+m_3) + \frac{l_2^2\dot{\theta}_2^2}{2}(m_2+m_3) + \frac{l_3^2\dot{\theta}_3^2}{2}m_3 + l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)(m_2+m_3)$$

$$+ l_2l_3\dot{\theta}_2\dot{\theta}_3\cos(\theta_2-\theta_3)m_3 + l_3l_1\dot{\theta}_3\dot{\theta}_1\cos(\theta_3-\theta_1)m_3 + (m_1+m_2+m_3)gl_1\cos\theta_1$$

$$+ (m_2+m_3)gl_2\cos\theta_2 + m_3gl_3\cos\theta_3$$

(A)

- Lagrange's equations are given by:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_{ni}$$

- q_i = generalized coordinate $(\theta_1, \theta_2, \theta_3)$
- Q_{ni} = nonconservative generalized force (0 in our case)

(B)

- $$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = (m_1+m_2+m_3)\, l_1^2 \dot{\theta}_1 + l_1 l_2 \dot{\theta}_2 \cos(\theta_1-\theta_2)(m_2+m_3) + l_3 l_1 \dot{\theta}_3 \cos(\theta_3-\theta_1) m_3$$

- $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) = (m_1+m_2+m_3)\, l_1^2 \ddot{\theta}_1 + l_1 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2)(m_2+m_3) + l_3 l_1 \ddot{\theta}_3 \cos(\theta_3-\theta_1) m_3$$
$$- l_1 l_2 \dot{\theta}_2(\dot{\theta}_1-\dot{\theta}_2)\sin(\theta_1-\theta_2)(m_2+m_3) - l_3 l_1 \dot{\theta}_3(\dot{\theta}_3-\dot{\theta}_1)\sin(\theta_3-\theta_1) m_3$$

- $$\frac{\partial \mathcal{L}}{\partial \theta_1} = - l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2)(m_2+m_3) + l_3 l_1 \dot{\theta}_3 \dot{\theta}_1 \sin(\theta_3-\theta_1) m_3$$
$$-(m_1+m_2+m_3)\, g\, l_1 \sin\theta_1$$

(C)

- $$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = l_2^2 \dot{\theta}_2 (m_2+m_3) + l_1 l_2 \dot{\theta}_1 \cos(\theta_1-\theta_2)(m_2+m_3) + l_2 l_3 \dot{\theta}_3 \cos(\theta_2-\theta_3) m_3$$

- $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) = l_2^2 \ddot{\theta}_2 (m_2+m_3) + l_1 l_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2)(m_2+m_3) + l_2 l_3 \ddot{\theta}_3 \cos(\theta_2-\theta_3) m_3$$
$$- l_1 l_2 \dot{\theta}_1(\dot{\theta}_1-\dot{\theta}_2)\sin(\theta_1-\theta_2)(m_2+m_3) - l_2 l_3 \dot{\theta}_3(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3) m_3$$

- $$\frac{\partial \mathcal{L}}{\partial \theta_2} = l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2)(m_2+m_3) - l_2 l_3 \dot{\theta}_2 \dot{\theta}_3 \sin(\theta_2-\theta_3) m_3 - (m_2+m_3)\, g\, l_2 \sin\theta_2$$

(D)

- $$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_3} = l_3^2 \dot{\theta}_3 m_3 + l_2 l_3 \dot{\theta}_2 \cos(\theta_2-\theta_3) m_3 + l_3 l_1 \dot{\theta}_1 \cos(\theta_3-\theta_1) m_3$$

- $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_3}\right) = l_3^2 \ddot{\theta}_3 m_3 + l_2 l_3 \ddot{\theta}_2 \cos(\theta_2-\theta_3) m_3 + l_3 l_1 \ddot{\theta}_1 \cos(\theta_3-\theta_1) m_3$$
$$- l_2 l_3 \dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3) m_3 - l_3 l_1 \dot{\theta}_1(\dot{\theta}_3-\dot{\theta}_1)\sin(\theta_3-\theta_1) m_3$$

- $$\frac{\partial \mathcal{L}}{\partial \theta_3} = l_2 l_3 \dot{\theta}_2 \dot{\theta}_3 \sin(\theta_2-\theta_3) m_3 - l_3 l_1 \dot{\theta}_3 \dot{\theta}_1 \sin(\theta_3-\theta_1) m_3 - m_3\, g\, l_3 \sin\theta_3$$

Combining (A), (B), (C) & (D),

$$(m_1+m_2+m_3)\,l_1^2\,\ddot{\theta}_1 + l_1 l_2 \cos(\theta_1-\theta_2)(m_2+m_3)\,\ddot{\theta}_2 + l_3 l_1 \cos(\theta_3-\theta_1)\,m_3\,\ddot{\theta}_3 - l_1 l_2\,\dot{\theta}_2(\dot{\theta}_1-\dot{\theta}_2)\sin(\theta_1-\theta_2)(m_2+m_3)$$
$$- l_3 l_1\,\dot{\theta}_3(\dot{\theta}_3-\dot{\theta}_1)\sin(\theta_3-\theta_1)\,m_3 + l_1 l_2\,\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)(m_2+m_3) - l_3 l_1\,\dot{\theta}_3\dot{\theta}_1\sin(\theta_3-\theta_1)\,m_3$$
$$+ (m_1+m_2+m_3)\,g\,l_1\sin\theta_1 = 0 \quad \text{———} \quad (1)$$

$$l_1 l_2 \cos(\theta_1-\theta_2)(m_2+m_3)\,\ddot{\theta}_1 + l_2^2(m_2+m_3)\,\ddot{\theta}_2 + l_2 l_3\,\ddot{\theta}_3\cos(\theta_2-\theta_3)\,m_3 - l_1 l_2\,\dot{\theta}_1(\dot{\theta}_1-\dot{\theta}_2)\sin(\theta_1-\theta_2)(m_2+m_3)$$
$$- l_2 l_3\,\dot{\theta}_3(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)\,m_3 - l_1 l_2\,\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)(m_2+m_3) + l_2 l_3\,\dot{\theta}_2\dot{\theta}_3\sin(\theta_2-\theta_3)\,m_3$$
$$+ (m_2+m_3)\,g\,l_2\sin\theta_2 = 0 \quad \text{———} \quad (2)$$

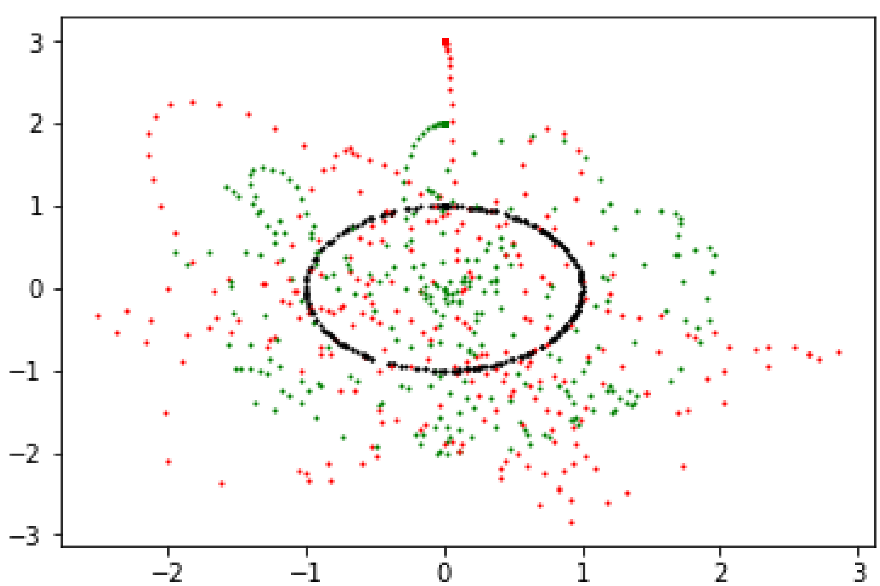
$$l_3 l_1\,\ddot{\theta}_1\cos(\theta_3-\theta_1)\,m_3 + l_2 l_3\,\ddot{\theta}_2\cos(\theta_2-\theta_3)\,m_3 + l_3^2\,\ddot{\theta}_3\,m_3 - l_2 l_3\,\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_3)\sin(\theta_2-\theta_3)\,m_3$$
$$- l_3 l_1\,\dot{\theta}_1(\dot{\theta}_3-\dot{\theta}_1)\sin(\theta_3-\theta_1)\,m_3 - l_2 l_3\,\dot{\theta}_2\dot{\theta}_3\sin(\theta_2-\theta_3)\,m_3 + l_3 l_1\,\dot{\theta}_3\dot{\theta}_1\sin(\theta_3-\theta_1)\,m_3$$
$$+ m_3\,g\,l_3\sin\theta_3 = 0 \quad \text{———} \quad (3)$$

(1), (2) & (3) are the Differential equations for a triple Pendulum

The above equations are rearranged to form separate second order differential equations in theta1, theta2, theta3. These are then converted into two first order differential equations (in the code), because Python's inbuilt solver 'odeint' can solve only first order ODEs.

Butterfly Effect : We have also demonstrated the Butterfly Effect (read more at https://en.wikipedia.org/wiki/Butterfly_effect). The video was created by composition of three separate videos. Then a free software 'OpenShot' was used to composite them. The pendulums had initial angles of (90,90,90), (89.5,89.5,89.5) and (90.5,90.5,90.5). As we can see they show similar motion initially but it changes drastically after sometime. This due to the large dependence on the initial conditions in a chaotic system. Read more at https://simple.wikipedia.org/wiki/Chaos_theory.

Why is it called the butterfly effect ? This effect grants the power to cause a hurricane in China to a butterfly flapping its wings in New Mexico. It may take a very long time, but the connection is real. If the butterfly had not flapped its wings at just the right point in space/time, the hurricane would not have happened. A more rigorous way to express this is that small changes in the initial conditions lead to drastic changes in the results. Our lives are an ongoing demonstration of this principle. Who knows what the long-term effects of teaching millions of kids about chaos and fractals will be?



Scatter Plot when the pendulum is released from the vertically up position.