$$d = (\sigma - \sigma^*)^T \frac{C^{-1}W}{2} (\sigma - \sigma^*) + (\varepsilon - \varepsilon^*) \frac{CW}{2} (\varepsilon - \varepsilon^*) - \eta^T B^T W \sigma -$$

$$\mu \ln s_u + \mu \ln s_\eta + \lambda_u \left(s_u - (Y_{le} - \psi(u))\right) +$$

$$\lambda_\eta \left(s_\eta - (Y_{le} - \psi(u+\eta))\right)$$

δu: $B^T C W (Bu - \varepsilon^*) + \lambda_u \{2 G^T G u\} + \lambda_\eta \{2 G^T G (u+\eta)\} = 0$

$\delta \eta$: $-B^T W(\sigma^* + CB\eta) + \lambda_\eta \{2G^TG(u+\eta)\} = 0$

$\delta \lambda_u$: $u^T G^T G u - Y_{le} + s_u = 0$

$\delta \lambda_\eta$: $(u+\eta)^T G^T G (u+\eta) - Y_{le} + s_\eta = 0.$

Considered here & later symmetrized.

δs_u: $\boxed{\lambda_u s_u = \mu}$ $\iff$ $\lambda_u - \mu/s_u = 0$

δs_η: $\boxed{\lambda_\eta s_\eta = -\mu.}$ $\iff$ $\lambda_\eta + \mu/s_\eta = 0$

Tangent matrix:

$$\begin{bmatrix}
B^TCWB + 2G^T\Lambda_u G + 2G^T\Lambda_\eta G & \lambda_\eta\{2G^TG\} & 2G^TGu & 2G^TG(u+\eta) & 0 & 0 \\
\lambda_\eta\{2G^TG\} & -B^TCWB + 2G^T\Lambda_\eta G & 0 & 2G^TG(u+\eta) & 0 & 0 \\
2G^TGu & 0 & 0 & 0 & 1 & 0 \\
2G^TG(u+\eta) & 2G^TG(u+\eta) & 0 & 0 & 0 & 1 \\
0 & 0 & s_u & 0 & \lambda_u & 0 \\
0 & 0 & 0 & s_\eta & 0 & \lambda_\eta
\end{bmatrix}$$

Residual: $\{ r_u \quad r_\eta \quad r_{\lambda_u} \quad r_{\lambda_\eta} \quad r_{s_u} \quad r_{s_\eta} \}^T$

Symmetrized tangent:

Tangent / residual } — last two rows — divide by s_u & s_η respectively

Tangent_Scaled:

$$
\begin{array}{cccccc}
\left[\begin{array}{cccccc}
B^T C W B + 2G^T \Lambda_u G + 2G^T \Lambda_\eta G & \lambda_\eta\{2G^T G\} & 2G^T G u & 2G^T G(u+\eta) & 0 & 0 \\
\lambda_\eta\{2G^T G\} & -B^T C W B + 2G^T \Lambda_\eta G & 0 & 2G^T G(u+\eta) & 0 & 0 \\
2G^T G u & 0 & 0 & 0 & 1 & 0 \\
2G^T G(u+\eta) & 2G^T G(u+\eta) & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & \frac{\lambda_u}{s_u} & 0 \\
0 & 0 & 0 & 1 & 0 & \frac{\lambda_\eta}{s_\eta}
\end{array}\right] \\
\boxed{u \;|\; \eta \;|\; \lambda_u \;|\; \lambda_\eta \;|\; s_u \;|\; s_\eta}
\end{array}
$$

Residual: $\left\{ r_u \quad r_\eta \quad r_{\lambda_u} \quad r_{\lambda_\eta} \quad \underbrace{\frac{r_{s_u}}{s_u}}_{\lambda_u - \mu/s_u} \quad \underbrace{\frac{r_{s_\eta}}{s_\eta}}_{\lambda_\eta + \mu/s_\eta} \right\}$

$$
\begin{bmatrix}
K_{uu} & K_{u\eta} & K_{u\lambda_u} & K_{u\lambda_\eta} & 0 & 0 \\
K_{\eta u} & K_{\eta\eta} & 0 & K_{\eta\lambda_\eta} & 0 & 0 \\
K_{\lambda_u u} & 0 & 0 & 0 & I & 0 \\
K_{\lambda_\eta u} & K_{\lambda_\eta \eta} & 0 & 0 & 0 & I \\
0 & 0 & I & 0 & \Lambda_u & 0 \\
0 & 0 & 0 & I & 0 & \Lambda_\eta
\end{bmatrix}
\begin{matrix}
\Delta u \\ \Delta \eta \\ \Delta \lambda_u \\ \Delta \lambda_\eta \\ \Delta s_u \\ \Delta s_\eta
\end{matrix}
$$

Different System:

$$K_{uu}\,\Delta u + K_{u\eta}\,\Delta\eta + K_{u\lambda_u}\,\Delta\lambda_u + K_{u\lambda_\eta}\,\Delta\lambda_\eta = -r_u$$

$$K_{\eta u}\,\Delta u + K_{\eta\eta}\,\Delta\eta + K_{\eta\lambda_\eta}\,\Delta\lambda_\eta = -r_\eta$$

$$\left.\begin{aligned}
&K_{\lambda_u u}\,\Delta u + \Delta s_u = -r_{\lambda_u} \\
&K_{\lambda_\eta u}\,\Delta u + K_{\lambda_\eta \eta}\,\Delta\eta + \Delta s_\eta = -r_{\lambda_\eta}
\end{aligned}\right\} \quad a, b$$

$$\left.\begin{aligned}
&\Delta\lambda_u + \Lambda_u\,\Delta s_u = -r_{s_u} \\
&\Delta\lambda_\eta + \Lambda_\eta\,\Delta s_\eta = -r_{s_\eta}
\end{aligned}\right\} \Rightarrow
\begin{aligned}
&\Delta\lambda_u = -r_{s_u} - \Lambda_u\,\Delta s_u \\
&\Delta\lambda_\eta = -r_{s_\eta} - \Lambda_\eta\,\Delta s_\eta
\end{aligned}$$

Rewrite System:

$K_{uu}\,\Delta u + K$

$$K_{uu}\,\Delta u + K_{u\eta}\,\Delta\eta - K_{u\lambda_u}\Lambda_u\,\Delta s_u - K_{u\lambda_\eta}\Lambda_\eta\,\Delta s_\eta =$$

$$-r_u + K_{u\lambda_u} r_{s_u} + K_{u\lambda_\eta} r_{s_\eta}$$

$$K_{\eta u}\,\Delta u + K_{\eta\eta}\,\Delta\eta - K_{\eta\lambda_\eta}\Lambda_\eta\,\Delta s_\eta = -r_\eta + K_{\eta\lambda_\eta} r_{s_\eta}$$

$$K_{\lambda_u u}\,\Delta u + \Delta s_u = -r_{\lambda_u}$$

$$K_{\lambda_\eta u}\,\Delta u + K_{\lambda_\eta\eta}\,\Delta\eta + \Delta s_\eta = -r_{\lambda_\eta}$$

Matrix:

$$\begin{pmatrix} K_{uu} & K_{u\eta} & -K_{u\lambda_u}\Lambda_u & -K_{u\lambda_\eta}\Lambda_\eta \\ K_{\eta u} & K_{\eta\eta} & 0 & -K_{\eta\lambda_\eta}\Lambda_\eta \\ K_{\lambda_u u} & 0 & I & 0 \\ K_{\lambda_\eta u} & K_{\lambda_\eta\eta} & 0 & I \end{pmatrix} \begin{matrix} \Delta u \\ \Delta\eta \\ \Delta s_u \\ \Delta s_\eta \end{matrix}$$

$$= \begin{pmatrix} -r_u + K_{u\lambda_u} r_{s_u} + K_{u\lambda_\eta} r_{s_\eta} \\ -r_\eta + K_{\eta\lambda_\eta} r_{s_\eta} \\ -r_{\lambda_u} \\ \dots \end{pmatrix}$$

$\Lambda_u = \lambda_u / s_u$

$\Lambda_\eta = \lambda_\eta / \dots$

$$\begin{pmatrix} -r_{\lambda_\eta} \end{pmatrix}$$

$$\Delta\lambda_u = -r_{s_u} - \Lambda_u \Delta s_u$$

$$\Rightarrow$$

$$\Delta\lambda_\eta = -r_{s_\eta} - \Lambda_\eta \Delta s_\eta$$

Further reduction:

$$K_{\lambda_u u}\Delta u + \Delta s_u = -r_{\lambda_u}; \quad K_{\lambda_\eta u}\Delta u + K_{\lambda_\eta \eta}\Delta\eta + \Delta s_\eta = -r_{\lambda_\eta}$$

First two eqns:

$$K_{uu}\Delta u + K_{u\eta}\Delta\eta - K_{u\lambda_u}\Lambda_u\left\{-r_{\lambda_u} - K_{\lambda_u u}\Delta u\right\}$$

$$- K_{u\lambda_\eta}\Lambda_\eta\left\{-r_{\lambda_\eta} - K_{\lambda_\eta u}\Delta u - K_{\lambda_\eta\eta}\Delta\eta\right\} =$$

$$-r_u + K_{u\lambda_u} r_{s_u} + K_{u\lambda_\eta} r_{s_\eta}$$

$$K_{\eta u}\Delta u + K_{\eta\eta}\Delta\eta - K_{\eta\lambda_\eta}\overset{\Lambda_\eta}{\left(\left\{-r_{\lambda_\eta} - K_{\lambda_\eta u}\Delta u - K_{\lambda_\eta\eta}\Delta\eta\right\}\right.}$$

$$= -r_\eta + K_{\eta\lambda_\eta} r_{s_\eta}$$

$$\begin{bmatrix} K_{uu} + K_{u\lambda_u}\overset{\Lambda_u}{(} K_{\lambda_u u} + K_{u\lambda_\eta}\Lambda_\eta K_{\lambda_\eta u} & K_{u\eta} + K_{u\lambda_\eta}\Lambda_\eta K_{\lambda_\eta\eta} \end{bmatrix}$$

$$\left[K_{\eta u} + K_{\eta\lambda_\eta}\Lambda_\eta K_{\lambda_\eta u} \qquad K_{\eta\eta} + K_{\eta\lambda_\eta}\Lambda_\eta K_{\lambda_\eta \eta} \right]$$

$$\begin{Bmatrix} \Delta u \\ \Delta\eta \end{Bmatrix} = \begin{matrix} -r_u + K_{u\lambda_u} r_{\delta u} + K_{u\lambda_\eta} r_{\delta\eta} - K_{u\lambda_u}\Lambda_u r_{\lambda_u} \\ - K_{u\lambda_\eta}\Lambda_\eta r_{\lambda_\eta} \\ \\ -r_\eta + K_{\eta\lambda_\eta} r_{\delta\eta} - K_{\eta\lambda_\eta}\Lambda_\eta r_{\lambda_\eta} \end{matrix}$$

Then $\Delta\delta_u = -r_{\lambda_u} - K_{\lambda_u u}\Delta u.$

$$\Delta\delta_\eta = -r_{\lambda_\eta} - K_{\lambda_\eta u}\Delta u - K_{\lambda_\eta \eta}\Delta\eta. \qquad \frac{\partial r_{\delta u}}{\partial\delta u}.$$

And

$$\Delta\lambda_u = -r_{\delta u} - \Lambda_u \Delta\delta_u \qquad \Lambda_u = \lambda/\delta_u$$

$$\Delta\lambda_\eta = -r_{\delta\eta} - \Lambda_\eta \Delta\delta_\eta \qquad \Lambda_\eta = \lambda_\eta/\delta_\eta$$

$(\partial r_{\delta\eta}/\partial\delta_\eta)$

Reduction with arc length:

DOFS: $\{u_f \;\lambda_f \;\eta_f\}$ — Contributions from Δy &

$$\begin{Bmatrix} \Delta u \\ \Delta\lambda \\ \Delta\eta \end{Bmatrix} = \begin{matrix} -r_u + K_{u\lambda_u} r_{\delta u} + K_{u\lambda_\eta} r_{\delta\eta} - K_{u\lambda_u}\Lambda_u r_{\lambda_u} \\ - K_{u\lambda_\eta}\Lambda_\eta r_{\lambda_\eta} \\ -r_\eta + K \ldots \end{matrix}$$

$$\ldots K_{l\lambda_u} r_{\delta_u} + K_{l\lambda_\eta} r_{\delta_\eta} - K_{l\lambda_u} \Lambda_u r_u - K_{l\lambda_\eta} \Lambda_\eta r_{\lambda_\eta}$$

$$-r_\eta + K_{\eta\lambda_\eta} r_{\delta_\eta} - K_{\eta\lambda_\eta} \Lambda_\eta r_{\lambda_\eta}.$$

Then $\Delta\delta_u = -r_{\lambda_u} - K_{\lambda_u u} \Delta u - K_{\lambda_u l} \Delta l$ $\qquad \dfrac{\partial \delta_u}{\partial \delta_u}$

$$\Delta\delta_\eta = -r_{\lambda_\eta} - K_{\lambda_\eta u} \Delta u - K_{\lambda_\eta \eta} \Delta\eta - K_{\lambda_\eta l} \Delta l$$

And $\Delta\lambda_u = -r_{\delta_u} - \Lambda_u \Delta\delta_u$ $\qquad \Lambda_u = \lambda/\delta_u$

$$\Delta\lambda_\eta = -r_{\delta_\eta} - \Lambda_\eta \Delta\delta_\eta \qquad \Lambda_\eta = \lambda_\eta/\delta_\eta$$

$$\left[\begin{array}{c|c}
\underbrace{K_{uu} + K_{u\lambda_u} \Lambda_u K_{\lambda_u u} + K_{u\lambda_\eta} \Lambda_\eta K_{\lambda_\eta u}}_{\partial d/\partial u_f} & \overbrace{\underbrace{K_{u\eta} + K_{u\lambda_\eta} \Lambda_\eta K_{\lambda_\eta \eta}}_{\partial d/\partial \eta_f}}^{\partial r_u/\partial y} \\
K_{\eta u} + K_{\eta\lambda_\eta} \Lambda_\eta K_{\lambda_\eta u} \;\; \dfrac{\partial r_\eta}{\partial y} & K_{\eta\eta} + K_{\eta\lambda_\eta} \Lambda_\eta K_{\lambda_\eta \eta}
\end{array}\right]$$

$$\begin{Bmatrix} \Delta u \\ \Delta\lambda \\ \Delta\eta \end{Bmatrix} = \begin{Bmatrix} -r_u + K_{u\lambda_u} r_{\delta_u} + K_{u\lambda_\eta} r_{\delta_\eta} - K_{u\lambda_u} \Lambda_u r_{\lambda_u} - K_{u\lambda_\eta} \Lambda_\eta r_{\lambda_\eta} \\ -r_\lambda + K_{l\lambda_u} r_{\delta_u} + K_{l\lambda_\eta} r_{\delta_\eta} - K_{l\lambda_u} \Lambda_u r_u - K_{l\lambda_\eta} \Lambda_\eta r_{\lambda_\eta} \\ -r_\eta + K_{\eta\lambda_\eta} r_{\delta_\eta} - K_{\eta\lambda_\eta} \Lambda_\eta r_{\lambda_\eta} \end{Bmatrix}$$

Then

Then $\Delta \delta_u = -r_{\lambda_u} - K_{\lambda_u u_f} \Delta u_f - K_{\lambda_u dl} \Delta l$

$$\Delta \delta_\eta = -r_{\lambda_\eta} - K_{\lambda_\eta u} \Delta u - K_{\lambda_\eta \eta} \Delta \eta - K_{\lambda_\eta l}\, dl$$

$\frac{\partial r_{\delta u}}{\partial \delta u}$

And

$$\Delta \lambda_u = -r_{\delta u} - \Lambda_u \Delta \delta u \qquad \Lambda_u = \lambda/\delta u$$

$$\Delta \lambda_\eta = -r_{\delta\eta} - \Lambda_\eta \Delta \delta\eta \qquad \Lambda_\eta = \lambda_\eta/\delta\eta$$

$\partial r_{\delta\eta}/\partial \delta\eta$