https://github.com/vasude1/UMAT.git

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1 Finite Viscoelasticity

In order to describe the processes that are far from equilibrium, a visceoasltic model has been proposed in [1, 2] in a thermodynamically consistent way. In this model, the strain energy density is taken to be a function of the deformation as well as some internal variables that characterize the rate dependence of the material. Hence, the strain energy density can be written as

$$\psi = \hat{\psi}(\boldsymbol{C}, \boldsymbol{Q}_1, \boldsymbol{Q}_2, ..., \boldsymbol{Q}_n), \tag{1}$$

where $\boldsymbol{C}$ is the right Cauchy Green deformation tensor and $\boldsymbol{Q}_i$ are the internal variables. The evolution of internal variables are described by n equations of the form

$$\dot{\boldsymbol{Q}}_k = \hat{\boldsymbol{f}}_k(\boldsymbol{C}, \boldsymbol{Q}_1, \boldsymbol{Q}_2, ..., \boldsymbol{Q}_n). \tag{2}$$

The evolution equations and the expression for the internal energy shall satisfy the internal dissipation inequality

$$\frac{1}{2}\boldsymbol{S} : \dot{\boldsymbol{C}} - \dot{\psi} \geq 0. \tag{3}$$

Using a Maxwell type representation of the material, the Strain energy function can be split into equilibrium and non-equilibrium parts as

$$\psi = \psi_{EQ}(\bar{\boldsymbol{C}}) + \psi_{NEQ}(\boldsymbol{C}_e) = \psi_{EQ}(\bar{\boldsymbol{C}}) + \psi_{NEQ}(\boldsymbol{F}_i^{-T}\bar{\boldsymbol{C}}\boldsymbol{F}_i^{-1}) \tag{4}$$

where $\boldsymbol{C}_e$ is the elastic part of the deviatoric Cauchy Green tensor, $\bar{\boldsymbol{C}}$. It has been assumed that $\bar{\boldsymbol{F}} = J^{-\frac{1}{3}}\boldsymbol{F}$ admits a decomposition into an elastic and an inelastic part, $\bar{\boldsymbol{F}} = \boldsymbol{F}_e\boldsymbol{F}_i$. Using the above expressions in the internal dissipation inequality gives the expressions for stress as

$$\boldsymbol{S} = pJ\boldsymbol{C}^{-1} + J^{-2/3}DEV\left[\boldsymbol{S}_{EQ} + \boldsymbol{S}_{NEQ}\right]. \tag{5}$$

$\boldsymbol{S}_{EQ}$ and $\boldsymbol{S}_{NEQ}$ denote the elastic and viscous contributions to the total stress. DEV represents the deviatoric projection and the first term is a consequence of the incompressibility constraint. These stresses are given by

$$\boldsymbol{S}_{EQ} = 2\frac{\partial\psi_{EQ}}{\partial\bar{\boldsymbol{C}}} \tag{6}$$

$$\boldsymbol{S}_{NEQ} = 2\boldsymbol{F}_i^{-1}\frac{\partial\psi_{NEQ}}{\partial\boldsymbol{C}_e}\boldsymbol{F}_i^{-T} \tag{7}$$

and the deviatoric projector by

$$DEV\left[\bullet\right] = \bullet - \frac{\bullet : \boldsymbol{C}}{3}\boldsymbol{C}^{-1}. \tag{8}$$

The Kirchhoff stress is then given as

$$\boldsymbol{\tau} = \boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T = pJ\boldsymbol{I} + \boldsymbol{\tau}^{iso} = -pJ\boldsymbol{I} + \mathbb{P} : \bar{\boldsymbol{\tau}}, \tag{9}$$

where $\bar{\boldsymbol{\tau}} = \bar{\boldsymbol{\tau}}_{EQ} + \bar{\boldsymbol{\tau}}_{NEQ}$ are defines as

$$\bar{\boldsymbol{\tau}}_{EQ} = \frac{\partial\psi_{EQ}(\bar{\boldsymbol{b}})}{\partial\bar{\boldsymbol{b}}}\bar{\boldsymbol{b}}, \tag{10}$$

$$\bar{\boldsymbol{\tau}}_{NEQ} = \frac{\partial\psi_{NEQ}(\boldsymbol{b}_e)}{\partial\boldsymbol{b}_e}\boldsymbol{b}_e. \tag{11}$$

$\mathbb{P}$ is the deviatoric projector in the deformed configuration defined as

$$\mathbb{P} = \mathbb{I} - \frac{\boldsymbol{I} \otimes \boldsymbol{I}}{3}. \tag{12}$$

The dissipation inequality becomes

$$-\boldsymbol{\tau}_{NEQ} : \frac{1}{2}\mathcal{L}_v \boldsymbol{b}_e . \boldsymbol{b}_e^{-1} \geq 0 \tag{13}$$

where $\mathcal{L}_v \boldsymbol{b}_e$ denotes the Lie derivative of $\boldsymbol{b}_e$ defined as

$$\mathcal{L}_v \boldsymbol{b}_e = \bar{\boldsymbol{F}} \dot{\overline{\boldsymbol{C}_i^{-1}}} \bar{\boldsymbol{F}}^T \tag{14}$$

The expression 13 can be satisfied by specifying the evolution equation as

$$-\frac{1}{2}\mathcal{L}_v \boldsymbol{b}_e . \boldsymbol{b}_e^{-1} = \mathcal{V}^{-1} : \boldsymbol{\tau}_{NEQ} \tag{15}$$

where $\mathcal{V}$ is a fourth order isotropic positive definite tensor possibly a function of $\boldsymbol{b}_e$. A slightly different equation has been proposed by [1].

1.1 Integration of the evolution equation

The integration of equation 13 is carried out by a predictor-corrector type algorithm. In the elastic predictor step, the inelastic strains are taken to be fixed and so,

$$(\boldsymbol{C}_i^{-1})_{tr} = (\boldsymbol{C}_i^{-1})_{t_{n-1}} \implies \boldsymbol{b}_e^{tr} = \bar{\boldsymbol{F}} (\boldsymbol{C}_i^{-1})_{t_{n-1}} \bar{\boldsymbol{F}}^T. \tag{16}$$

In the inelastic corrector step, the total deformation is assumed to be held fixed and so, $\mathcal{L}_v \boldsymbol{b}_e = \dot{\boldsymbol{b}}_e$. Using this in equation 15 gives

$$\dot{\boldsymbol{b}}_e \boldsymbol{b}_e^{-1} = -2 \left[\mathcal{V}^{-1} : \boldsymbol{\tau}_{NEQ}\right] \tag{17}$$

The above equation can be integrated using the exponential mapping technique. The resulting expression is

$$\boldsymbol{b}_e = \exp\left[-2\int_{t_{n-1}}^{t_n} \mathcal{V}^{-1} : \boldsymbol{\tau}_{NEQ}\, \mathrm{d}t\right] \boldsymbol{b}_e^{tr} \tag{18}$$

$$(\boldsymbol{b}_e)_{t_n} \approx \exp\left[-2\Delta t (\mathcal{V}^{-1} : \boldsymbol{\tau}_{NEQ})_{t=t_n}\right] \boldsymbol{b}_e^{tr} \tag{19}$$

The above equation is first order accurate.

Since the material is assumed to be isotropic, $\boldsymbol{b}_e$, $\boldsymbol{b}_e^{tr}$ and hence $\boldsymbol{\tau}_{NEQ}^{iso}$ share the Eigen space. Since $\mathcal{V}^{-1}$ is Isotropic, equation 19 can be written in Eigen basis as

$$\lambda_{Ae}^2 = \exp\left[-\frac{\Delta t}{\eta_D}\mathrm{dev}(\tau_A)\right] (\lambda_{Ae}^2)_{tr}. \tag{20}$$

Taking logarithm of both sides,

$$\epsilon_{Ae} = -\frac{\Delta t}{2\eta_D}\mathrm{dev}(\tau_A) + (\epsilon_{Ae})_{tr}. \tag{21}$$

The above equation is non-linear if τ_A is a non-linear function of ϵ_e. Hence, Newton iterations are used to solve it as below.

Defining

$$r_A = \epsilon_{Ae} + \frac{\Delta t}{2\eta_D}\mathrm{dev}(\tau_A) - (\epsilon_{Ae})_{tr} = 0, \tag{22}$$

it can be solved by linearizing around $\epsilon_{Ae} = (\epsilon_{Ae})_k$ as

$$r_A + \frac{\partial r_A}{\partial \epsilon_{Be}} \Delta\epsilon_{Be} = 0 \implies K_{AB}\Delta\epsilon_{Be} = -r_A \tag{23}$$

where $K_{AB} = \dfrac{\partial r_A}{\partial \epsilon_{Be}}$. The above equation is solved to obtain $\Delta\epsilon_e$, which is then used to update the elastic strain as $(\epsilon_e)_{k+1} = (\epsilon_e)_k + \Delta\epsilon_e$.

1.2 Computation of tangent

The tangent can be computed as

$$\mathfrak{C} = 2\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{C}}. \tag{24}$$

This has contributions from the pressure term as well as the deviatoric term. The latter will be dealt with here and the former will be dealt with later on. As a recollection, the deviatoric part is computed as $J^{-2/3}\left(\boldsymbol{S}^{iso} - \frac{\boldsymbol{S}^{iso}:\boldsymbol{C}}{3}\boldsymbol{C}^{-1}\right)$. The derivative of first term is computed as

$$DEV[\boldsymbol{S}_{IJ}]\frac{\partial J^{-2/3}}{\partial \boldsymbol{C}_{KL}} = -\frac{1}{3}J^{-2/3}DEV[\boldsymbol{S}_{IJ}]\boldsymbol{C}^{-1}_{KL}. \tag{25}$$

In tensor notation, $DEV[S]\otimes\dfrac{\partial J^{-2/3}}{\partial \boldsymbol{C}} = -\frac{1}{3}J^{-2/3}DEV[\boldsymbol{S}]\otimes\boldsymbol{C}^{-1}$. The next term is $\dfrac{\partial \boldsymbol{S}^{iso}}{\partial \boldsymbol{C}}$. Since $\boldsymbol{S}^{iso}$ is a function of $\bar{\boldsymbol{C}}$, the total derivative is computed using the chain rule.

$$\frac{\partial \boldsymbol{S}^{iso}_{IJ}}{\partial \boldsymbol{C}_{KL}} = \frac{\partial \boldsymbol{S}^{iso}_{IJ}}{\partial \bar{\boldsymbol{C}}_{MN}}\frac{\partial \bar{\boldsymbol{C}}_{MN}}{\partial \boldsymbol{C}_{KL}} \tag{26}$$

The second derivative, written using indicial notation is $J^{-2/3}\left[\mathbb{I}_{MNKL} - \frac{1}{3}\boldsymbol{C}_{MN}\boldsymbol{C}^{-1}_{KL}\right]$, where $\mathbb{I}$ is the fourth order identity tensor. using these in the above equation,

$$\frac{\partial \boldsymbol{S}^{iso}_{IJ}}{\partial \boldsymbol{C}_{KL}} = J^{-2/3}\left[\bar{\mathfrak{C}}_{IJKL} - \frac{1}{3}\bar{\mathfrak{C}}_{IJMN}\boldsymbol{C}_{MN}\boldsymbol{C}^{-1}_{KL}\right] \tag{27}$$

where $\bar{\mathfrak{C}} = \dfrac{\partial \boldsymbol{S}^{iso}}{\partial \bar{\boldsymbol{C}}}$. In tensor notation, $\dfrac{\partial \boldsymbol{S}^{iso}}{\partial \boldsymbol{C}} = J^{-2/3}\left[\bar{\mathfrak{C}} - \frac{\bar{\mathfrak{C}}:\boldsymbol{C}}{3}\otimes\boldsymbol{C}^{-1}\right]$. The next term is $\frac{\boldsymbol{S}^{iso}:\boldsymbol{C}}{3}\boldsymbol{C}^{-1}$. Its derivative is computed using the chain rule as

$$\frac{\partial\left(\frac{\boldsymbol{S}^{iso}:\boldsymbol{C}}{3}\boldsymbol{C}^{-1}\right)}{\partial \boldsymbol{C}} = \frac{\boldsymbol{S}^{iso}:\boldsymbol{C}}{3}\frac{\partial \boldsymbol{C}^{-1}}{\partial \boldsymbol{C}} + \boldsymbol{C}^{-1}\otimes\frac{\partial\frac{\boldsymbol{S}^{iso}:\boldsymbol{C}}{3}}{\partial \boldsymbol{C}}. \tag{28}$$

The derivative $\dfrac{\partial \boldsymbol{C}^{-1}}{\partial \boldsymbol{C}} = -\mathcal{I}$, where $\mathcal{I}_{IJKL} = \frac{1}{2}\left[\boldsymbol{C}^{-1}_{IK}\boldsymbol{C}^{-1}_{JL} + \boldsymbol{C}^{-1}_{IL}\boldsymbol{C}^{-1}_{JK}\right]$. The secon term is again evaluated using chain rule.

$$\frac{\partial\frac{\boldsymbol{S}^{iso}_{IJ}\boldsymbol{C}_{JI}}{3}}{\partial \boldsymbol{C}_{KL}} = \frac{1}{3}\left[\frac{\partial \boldsymbol{S}^{iso}_{IJ}}{\partial \boldsymbol{C}_{KL}}\boldsymbol{C}_{JI} + \boldsymbol{S}^{iso}_{IJ}\frac{\partial \boldsymbol{C}_{JI}}{\partial \boldsymbol{C}_{KL}}\right] \tag{29}$$

The first term has already been evaluated earlier. The second term can be seen to be $\boldsymbol{S}^{iso}_{IJ}\dfrac{\partial \boldsymbol{C}_{JI}}{\partial \boldsymbol{C}_{KL}} = \boldsymbol{S}^{iso}_{KL}$. In tensor notation,

$$\frac{\partial\left(\frac{\boldsymbol{S}^{iso}:\boldsymbol{C}}{3}\boldsymbol{C}^{-1}\right)}{\partial \boldsymbol{C}} = -\frac{\boldsymbol{S}^{iso}:\boldsymbol{C}}{3}\mathcal{I} + \frac{1}{3}\boldsymbol{C}^{-1}\otimes\left\{\boldsymbol{C}:J^{-2/3}\left[\bar{\mathfrak{C}} - \frac{\bar{\mathfrak{C}}:\boldsymbol{C}}{3}\otimes\boldsymbol{C}^{-1}\right] + \boldsymbol{S}^{iso}\right\}. \tag{30}$$

Combining all the above,

$$\frac{1}{2}\mathfrak{C} = -\frac{1}{3}J^{-2/3}DEV[\boldsymbol{S}]\otimes\boldsymbol{C}^{-1} + J^{-2/3}\left[\bar{\mathfrak{C}} - \frac{\bar{\mathfrak{C}}:\boldsymbol{C}}{3}\otimes\boldsymbol{C}^{-1}\right] + \frac{\boldsymbol{S}^{iso}:\boldsymbol{C}}{3}\mathcal{I} - \frac{1}{3}\boldsymbol{C}^{-1}\otimes\left\{\boldsymbol{C}:J^{-2/3}\left[\bar{\mathfrak{C}} - \frac{\bar{\mathfrak{C}}:\boldsymbol{C}}{3}\otimes\boldsymbol{C}^{-1}\right] + \boldsymbol{S}^{iso}\right\} \tag{31}$$

Using $\boldsymbol{S}^{iso} = DEV[\boldsymbol{S}] + \frac{\boldsymbol{S}^{iso}:\boldsymbol{C}}{3}\boldsymbol{C}^{-1}$, the above equation can be refined as

$$\frac{1}{2}\mathfrak{C} = \frac{1}{2}\mathfrak{C}^0 - \frac{1}{3}J^{-2/3}DEV[\boldsymbol{S}]\otimes\boldsymbol{C}^{-1} - -\frac{1}{3}J^{-2/3}\boldsymbol{C}^{-1}\otimes DEV[\boldsymbol{S}] + \frac{\boldsymbol{S}^{iso}:\boldsymbol{C}}{3}\left[\mathcal{I} - \frac{1}{3}\boldsymbol{C}^{-1}\otimes\boldsymbol{C}^{-1}\right] \tag{32}$$

where

$$\frac{1}{2}\mathfrak{C}^0 = J^{-2/3}\left[\bar{\mathfrak{C}} - \frac{\bar{\mathfrak{C}}:\boldsymbol{C}}{3}\otimes\boldsymbol{C}^{-1} - \boldsymbol{C}^{-1}\otimes\frac{\bar{\mathfrak{C}}:\boldsymbol{C}}{3} + \frac{\boldsymbol{C}:\bar{\mathfrak{C}}:\boldsymbol{C}}{9}\boldsymbol{C}^{-1}\otimes\boldsymbol{C}^{-1}\right] \tag{33}$$

The push forward of the tangent to spatial configuration is

$$\mathfrak{c} = \phi_*\left[\mathfrak{C}\right] = \mathbb{P}:\bar{\mathfrak{c}}:\mathbb{P} - \frac{1}{3}J^{-2/3}\left[\boldsymbol{\tau}^{iso}\otimes\boldsymbol{I} + \boldsymbol{I}\otimes\boldsymbol{\tau}^{iso}\right] + \mathrm{tr}(\bar{\boldsymbol{\tau}})\mathbb{P}. \tag{34}$$

where

$$\bar{\mathfrak{c}}_{ijkl} = (\det\boldsymbol{F})^{-1}\boldsymbol{F}_{iI}\boldsymbol{F}_{jJ}\boldsymbol{F}_{kK}\boldsymbol{F}_{lL}\bar{\mathfrak{C}}_{IJKL} \tag{35}$$

In the above expression, $\bar{\mathfrak{C}} = 2\dfrac{\partial\boldsymbol{S}^{iso}}{\partial\bar{\boldsymbol{C}}}$. This can be evaluated by observing that

$$\dot{\boldsymbol{S}}^{iso} = \frac{\partial\boldsymbol{S}^{iso}}{\partial\bar{\boldsymbol{C}}}\dot{\bar{\boldsymbol{C}}}. \tag{36}$$

Since $\bar{\boldsymbol{C}} \overset{\text{sum}}{=} \bar{\lambda}_i^2\boldsymbol{N}_i\otimes\boldsymbol{N}_i$, $\dot{\bar{\boldsymbol{C}}} = \dfrac{\mathrm{d}\bar{\lambda}_i^2}{\mathrm{d}t}\boldsymbol{N}_i\otimes\boldsymbol{N}_i + \bar{\lambda}_i^2\dot{\boldsymbol{N}}_i\otimes\boldsymbol{N}_i + \bar{\lambda}_i^2\boldsymbol{N}_i\otimes\dot{\boldsymbol{N}}_i$. $\dot{\boldsymbol{N}}_i = \sum_{j=1}^3 W_{ij}\boldsymbol{N}_j$, where $W_{ij} = -W_{ji}$ are components of a skew symmetric tensor. Hence

$$\dot{\bar{\boldsymbol{C}}} = \sum_{i=1}^{3}\frac{\mathrm{d}\bar{\lambda}_i^2}{\mathrm{d}t}\boldsymbol{N}_i\otimes\boldsymbol{N}_i + \sum_{i,j=1,i\neq j}^{3} W_{ij}(\bar{\lambda}_i^2 - \bar{\lambda}_j^2)\boldsymbol{N}_i\otimes\boldsymbol{N}_j \tag{37}$$

$\boldsymbol{S}^{iso} \overset{\text{sum}}{=} S_i\boldsymbol{N}_i\otimes\boldsymbol{N}_i$. Then similarly,

$$\dot{\boldsymbol{S}^{iso}} = \sum_{i,j=1}^{3} 2\frac{\partial^2\psi}{\partial\bar{\lambda}_i^2\partial\bar{\lambda}_j^2}\frac{\mathrm{d}\bar{\lambda}_j^2}{\mathrm{d}t}\boldsymbol{N}_i\otimes\boldsymbol{N}_i + \sum_{i,j=1,i\neq j}^{3} W_{ij}(S_i - S_j)\boldsymbol{N}_i\otimes\boldsymbol{N}_j \tag{38}$$

Hence,

$$\bar{\mathfrak{C}} \overset{\text{sum}}{=} 4\frac{\partial^2\psi}{\partial\bar{\lambda}_i^2\partial\bar{\lambda}_j^2}\boldsymbol{N}_i\otimes\boldsymbol{N}_i\otimes\boldsymbol{N}_j\otimes\boldsymbol{N}_j + \sum_{i,j=1,i\neq j}^{3}\frac{S_i - S_j}{\bar{\lambda}_i^2 - \bar{\lambda}_j^2}(\boldsymbol{N}_i\otimes\boldsymbol{N}_j\otimes\boldsymbol{N}_i\otimes\boldsymbol{N}_j + \boldsymbol{N}_i\otimes\boldsymbol{N}_j\otimes\boldsymbol{N}_j\otimes\boldsymbol{N}_i). \tag{39}$$

Pushing it forward gives

$$\begin{aligned}\bar{\mathfrak{c}} \overset{\text{sum}}{=} & \frac{\partial^2\psi}{\partial\ln\bar{\lambda}_i\partial\ln\bar{\lambda}_j}\boldsymbol{n}_i\otimes\boldsymbol{n}_i\otimes\boldsymbol{n}_j\otimes\boldsymbol{n}_j - 2\sigma_i\boldsymbol{n}_i\otimes\boldsymbol{n}_i\otimes\boldsymbol{n}_i\otimes\boldsymbol{n}_i \\ & + \sum_{i,j=1,i\neq j}^{3}\frac{\sigma_i\bar{\lambda}_j^2 - \sigma_j\bar{\lambda}_i^2}{\bar{\lambda}_i^2 - \bar{\lambda}_j^2}(\boldsymbol{n}_i\otimes\boldsymbol{n}_j\otimes\boldsymbol{n}_i\otimes\boldsymbol{n}_j + \boldsymbol{n}_i\otimes\boldsymbol{n}_j\otimes\boldsymbol{n}_j\otimes\boldsymbol{n}_i).\end{aligned} \tag{40}$$

For the viscous branches, instead of computing $\bar{\mathfrak{C}}_{IJKL} = \dfrac{\partial\boldsymbol{S}_{IJ}}{\partial\bar{\boldsymbol{C}}_{KL}}$ and pushing it forward by $\boldsymbol{F}$ to obtain $\bar{\mathfrak{c}}$, the following will be used.

In the elastic trial state, since the inelastic strain is held fixed, the viscous stresses can be taken to be a function of the elastic trail state. $\bar{\boldsymbol{F}}^n = \boldsymbol{F}_e^{tr}\boldsymbol{F}_i^{n-1} \implies \bar{\boldsymbol{C}}^n = (\boldsymbol{F}_i^{n-1})^T\boldsymbol{C}_e^{tr}\boldsymbol{F}_i^{n-1}$. Hence, the stresses can be treated as a function fo trial state. Since

$$\boldsymbol{S}_{NEQ} = \bar{\boldsymbol{F}}^{-1}\bar{\boldsymbol{\tau}}_{NEQ}\bar{\boldsymbol{F}}^{-T} = (\boldsymbol{F}_i^{n-1})^{-1}.\underbrace{(\boldsymbol{F}_e^{tr})^{-1}\bar{\boldsymbol{\tau}}_{NEQ}(\boldsymbol{F}_e^{tr})^{-T}}_{\tilde{\boldsymbol{S}}_{NEQ}}.(\boldsymbol{F}_i^{n-1})^{-T} \tag{41}$$

The tangent is then computed as

$$\bar{\mathfrak{c}}_{ijkl} = (\det\boldsymbol{F})^{-1}(\boldsymbol{F}_e^{tr})_{iI}(\boldsymbol{F}_e^{tr})_{jJ}(\boldsymbol{F}_e^{tr})_{kK}(\boldsymbol{F}_e^{tr})_{lL}\frac{\partial(\tilde{\boldsymbol{S}}_{NEQ})_{IJ}}{\partial(\boldsymbol{C}_e^{tr})_{KL}}. \tag{42}$$

$\tilde{\boldsymbol{S}}_{NEQ}$, written in Eigen basis is

$$\tilde{\boldsymbol{S}}_{NEQ} = \sum_{A=1}^{n} \frac{\tau_A}{(\lambda_{Ae})^2_{tr}} \tilde{\boldsymbol{N}}_A \otimes \tilde{\boldsymbol{N}}_A. \tag{43}$$

It is to be noted that the τ_A in the above equation is a function of ϵ_e and so, the derivative is computed using chain rule. Defining C^{alg}_{AC} as $\frac{\tau_A}{\epsilon_{Be}} K^{-1}_{BC}$, where K_{BC} has been defined earlier during the description of Newton iterations, the final expression for the tangent can be obtained as (written in Eigen basis)

$$\bar{\mathfrak{c}} \overset{\text{sum}}{=} \tilde{L}_{ABCD} (\lambda_{Ae})_{tr} (\lambda_{Be})_{tr} (\lambda_{Ce})_{tr} (\lambda_{De})_{tr} \boldsymbol{n}_A \otimes \boldsymbol{n}_B \otimes \boldsymbol{n}_C \otimes \boldsymbol{n}_D, \tag{44}$$

where

$$\begin{aligned}\tilde{\mathscr{L}} = &\sum_{A=1}^{3} \sum_{B=1}^{3} \left(\frac{1}{(\lambda_{Ae})^2_{tr} (\lambda_{Be})^2_{tr}} (C^{alg}_{AB} - 2\tau_A \delta_{AB}) \tilde{\boldsymbol{N}}_A \otimes \tilde{\boldsymbol{N}}_A \otimes \tilde{\boldsymbol{N}}_B \otimes \tilde{\boldsymbol{N}}_B \right) \\ &+ \sum_{A=1, A\neq B}^{3} \sum_{B=1}^{3} \frac{(\tilde{S}_{NEQ})_B - (\tilde{S}_{NEQ})_A}{(\lambda_{Be})^2_{tr} - (\lambda_{Ae})^2_{tr}} (\tilde{\boldsymbol{N}}_A \otimes \tilde{\boldsymbol{N}}_B \otimes \tilde{\boldsymbol{N}}_A \otimes \tilde{\boldsymbol{N}}_B + \tilde{\boldsymbol{N}}_A \otimes \tilde{\boldsymbol{N}}_B \otimes \tilde{\boldsymbol{N}}_B \otimes \tilde{\boldsymbol{N}}_A) \\ &\overset{\text{sum}}{=} \tilde{L}_{ABCD} \tilde{\boldsymbol{N}}_A \otimes \tilde{\boldsymbol{N}}_B \otimes \tilde{\boldsymbol{N}}_C \otimes \tilde{\boldsymbol{N}}_D.\end{aligned} \tag{45}$$

It shall be noted that the tangent computed has been expanded in the Eigen basis and should be rotated to bring it back to the global Cartesian basis. The rotation is performed as follows.

Since $\boldsymbol{b}$ and $(\boldsymbol{b}_e)_{tr}$ share the eigen space, the eigen vectors of either of the tensors satisfy

$$(\boldsymbol{b}_e)_{tr} \boldsymbol{n} = (\lambda_e)^2_{tr} \boldsymbol{n} \tag{46}$$

If the components of eigen vectors of $(\boldsymbol{b}_e)_{tr}$ in global cartesian basis are written as columns of a matrix $\tilde{Q}$, and $Q = \tilde{Q}^T$, the rotation matrix is defined as

$$[P] = \begin{bmatrix} Q_{11}^2 & Q_{12}^2 & Q_{13}^2 & 2Q_{11}Q_{12} & 2Q_{12}Q_{13} & 2Q_{13}Q_{11} \\ Q_{21}^2 & Q_{22}^2 & Q_{23}^2 & 2Q_{21}Q_{22} & 2Q_{22}Q_{23} & 2Q_{23}Q_{21} \\ Q_{31}^2 & Q_{32}^2 & Q_{33}^2 & 2Q_{31}Q_{32} & 2Q_{32}Q_{33} & 2Q_{33}Q_{31} \\ Q_{11}Q_{21} & Q_{12}Q_{22} & Q_{13}Q_{23} & Q_{11}Q_{22} + Q_{12}Q_{21} & Q_{12}Q_{23} + Q_{13}Q_{22} & Q_{13}Q_{21} + Q_{11}Q_{23} \\ Q_{21}Q_{31} & Q_{22}Q_{32} & Q_{23}Q_{33} & Q_{21}Q_{32} + Q_{22}Q_{31} & Q_{22}Q_{33} + Q_{23}Q_{32} & Q_{23}Q_{31} + Q_{21}Q_{23} \\ Q_{31}Q_{11} & Q_{32}Q_{12} & Q_{33}Q_{13} & Q_{31}Q_{12} + Q_{32}Q_{11} & Q_{32}Q_{13} + Q_{33}Q_{12} & Q_{33}Q_{11} + Q_{31}Q_{13} \end{bmatrix} \tag{47}$$

The tangent, in global cartesian basis, is obtained as

$$[\bar{\mathfrak{c}}]_{(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)} = [P][\bar{\mathfrak{c}}]_{(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)} [P]^T. \tag{48}$$

The components of $\bar{\mathfrak{c}}$ are stored in a matrix with components as

$$[\bar{\mathfrak{c}}] = \begin{bmatrix} 1111 & 1122 & 1133 & 1112 & 1123 & 1113 \\ 2211 & 2222 & 2233 & 2212 & 2223 & 2213 \\ 3311 & 3322 & 3333 & 3312 & 3323 & 3313 \\ 1211 & 1222 & 1233 & 1212 & 1223 & 1213 \\ 2311 & 2322 & 2333 & 2312 & 2323 & 2313 \\ 1311 & 1322 & 1333 & 1312 & 1323 & 1313 \end{bmatrix} \tag{49}$$

and that of a (symmetric) second order tensor, $\boldsymbol{\tau}$, as

$$[\boldsymbol{\tau}] = \begin{bmatrix} 11 & 22 & 33 & 12 & 23 & 13 \end{bmatrix}^T \tag{50}$$

1.3 Plane stress version of FV Model

The expression for Kirchhoff Stress is given by

$$\boldsymbol{\tau} = Jp\boldsymbol{I} + \mathbb{P} : \bar{\boldsymbol{\tau}}, \tag{51}$$

where

$$\bar{\boldsymbol{\tau}} = \bar{\boldsymbol{\tau}}_{EQ} + \bar{\boldsymbol{\tau}}_{NEQ} \tag{52}$$

Under Plane stress conditions, for perfectly incompressible material, the expression for pressure can be established by using the condition that $\tau_{33} = 0$. This results in

$$Jp = -\left((\bar{\tau}_3)_{EQ} - \frac{1}{3}\operatorname{tr}(\bar{\boldsymbol{\tau}}_{EQ})\right) - \left((\bar{\tau}_3)_{NEQ} - \frac{1}{3}\operatorname{tr}(\bar{\boldsymbol{\tau}}_{NEQ})\right) \tag{53}$$

It can be seen that as a consequence of the plane stress assumption, the pressure becomes a function of the total and elastic strains through stresses. Hence, they contribute to the material tangent as well. Differentiating the first term of stress, $Jp\boldsymbol{C}^{-1}$ with respect to $\boldsymbol{C}$,

$$\mathfrak{C}^{vol} = 2\frac{\partial Jp\boldsymbol{C}^{-1}}{\partial \boldsymbol{C}} = 2Jp\frac{\partial \boldsymbol{C}^{-1}}{\partial \boldsymbol{C}} + 2\boldsymbol{C}^{-1} \otimes \frac{\partial Jp}{\partial \boldsymbol{C}} \tag{54}$$

The first derivative has been evaluated earlier as $-\mathcal{I}$. The above expression is pushed forward as earlier to obtain the volumetric contribution to tangent $\mathfrak{c}^{vol} = \phi_* \left[\mathfrak{C}^{vol}\right]$. The push forward of $\mathcal{I}$ is the fourth order identity tensor, $\mathbb{I} = \phi_* [\mathcal{I}]$. The second term, when written in terms of Eigen basis is

$$\boldsymbol{C}^{-1} \otimes \frac{\partial Jp}{\partial \boldsymbol{C}} = \left(\sum_{i=1}^{3} \frac{1}{\lambda_i^2}\boldsymbol{N}_i \otimes \boldsymbol{N}_i\right) \otimes \left(\sum_{j=1}^{3} \frac{\partial pJ}{\partial \lambda_j^2}\boldsymbol{N}_j \otimes \boldsymbol{N}_j\right) \tag{55}$$

A push forward of above gives

$$\begin{aligned}\phi_*\left[\boldsymbol{C}^{-1} \otimes \frac{\partial Jp}{\partial \boldsymbol{C}}\right] &= \left(\sum_{i=1}^{3} \frac{1}{\lambda_i^2}\boldsymbol{F}\boldsymbol{N}_i \otimes \boldsymbol{F}\boldsymbol{N}_i\right) \otimes \left(\sum_{j=1}^{3} \frac{\partial pJ}{\partial \lambda_j^2}\boldsymbol{F}\boldsymbol{N}_j \otimes \boldsymbol{F}\boldsymbol{N}_j\right) \\ &= \left(\sum_{i=1}^{3} \boldsymbol{n}_i \otimes \boldsymbol{n}_i\right) \otimes \left(\sum_{j=1}^{3} \lambda_j^2\frac{\partial pJ}{\partial \lambda_j^2}\boldsymbol{n}_j \otimes \boldsymbol{n}_j\right)\end{aligned} \tag{56}$$

The quantity $\lambda_j^2 \dfrac{\partial pJ}{\partial \lambda_j^2}$ can be seen to be equal to $\frac{1}{2}\dfrac{\partial pJ}{\partial \log \lambda_j} = \frac{1}{2}\dfrac{\partial pJ}{\partial \epsilon_j}$. The pressure term has been evaluated in equation 53. Pressure term is a function of the deviatoric part of Cauchy strain tensor or in terms of Hencky strain, $\bar{\epsilon}$. Hence

$$\frac{\partial pJ}{\partial \epsilon_j} = \frac{\partial pJ}{\partial \bar{\epsilon}_m}\frac{\partial \bar{\epsilon}_m}{\partial \epsilon_j} = \frac{\partial pJ}{\partial \bar{\epsilon}_j} - \frac{1}{3}\left(\frac{\partial pJ}{\partial \bar{\epsilon}_1} + \frac{\partial pJ}{\partial \bar{\epsilon}_2} + \frac{\partial pJ}{\partial \bar{\epsilon}_3}\right) \tag{57}$$

The derivative w.r.t $\bar{\epsilon}$ can be seen to be

$$\frac{\partial pJ}{\partial \bar{\epsilon}_j} = -\left[\frac{\partial \bar{\tau}_3}{\partial \bar{\epsilon}_j} - \frac{1}{3}\left(\frac{\partial \bar{\tau}_1}{\partial \bar{\epsilon}_j} + \frac{\partial \bar{\tau}_2}{\partial \bar{\epsilon}_j} + \frac{\partial \bar{\tau}_3}{\partial \bar{\epsilon}_j}\right)\right] = -\left(C_{3j}^{alg} - \frac{1}{3}(C_{1j}^{alg} + C_{2j}^{alg} + C_{3j}^{alg})\right) \tag{58}$$

Combining all the above gives

$$\mathfrak{c}_{ijkl}^{vol} \overset{(\boldsymbol{n}_1,\boldsymbol{n}_2,\boldsymbol{n}_3)}{=} -2Jp\mathbb{I}_{ijkl} + \delta_{ij}\frac{\partial pJ}{\partial \epsilon_k}\delta_{kl}(\text{no sum}) \tag{59}$$

It shall be observed that the viscous contribution to the pressure can differentiated the same way as earlier. Instead of differentiating with $\boldsymbol{C}$ and pushing it forward with $\boldsymbol{F}$, it is differentiated with respect to $\boldsymbol{C}_e^{tr}$ and pushed forward with $\boldsymbol{F}_e^{tr}$. The results still remain the same.

1.4 Quadratic Hencky strain energy density

If the strain energy density of the viscous branches are represented by a quadratic energy density based on Hencky strains, that is

$$\psi_{NEQ} = \frac{\mu_{NEQ}}{2} \left(\epsilon_{1e}^2 + \epsilon_{2e}^2 + \epsilon_{3e}^2 \right), \tag{60}$$

the Kirchhoff stress becomes $\tau_i = \mu \epsilon_{ie}$. The equation 22 hence becomes

$$r_A = \epsilon_{Ae} + \frac{\mu_{NEQ} \Delta t}{2\eta_D} \left(\epsilon_{Ae} - \frac{1}{3}(\epsilon_{Ae} + \epsilon_{Be} + \epsilon_{Ce}) \right) - (\epsilon_{Ae})_{tr} = 0, \tag{61}$$

which form an algebraic system of equations that can be solved without the Newton iterations that were described earlier.

References

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[2] S. Reese, S. Govindjee, A theory of finite viscoelasticity and numerical aspects, International Journal of Solids and Structures 35 (26-27) (1998) 3455–3482. doi:10.1016/S0020-7683(97)00217-5.