A plane stress implementation of Finite Viscoealstic model in Abaqus

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Abstract

Some abstract.

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1. Why this article?

Here are two sample references: [**? ?**].

2. The actual article

The Finite Linear Viscoelastic (FLV) model has been used in the previous chapters to describe the viscoelastic behavior of the bulk material. However, the energy dissipated in the bulk material during the propagation of a crack cannot be computed with that model as it does not involve an explicit expression for energy or dissipation. To the author's knowledge, a plane stress implementation of the model by [**?**] does not exist. Hence, the current chapter discusses its plane stress implementation and its use to compute the energy dissipated in the material during the propagation of a dynamic crack.

3. Thermodynamics

In order to describe the processes that are far from equilibrium, a viscoelastic model has been proposed in [**? ?**] in a thermodynamically consistent way. In this model, the strain energy density is taken to be a function of the deformation as well as some internal variables that characterize the rate dependence of the material. The strain energy density can be written as

$$\psi = \hat{\psi}(\boldsymbol{C}, \boldsymbol{Q}_1, \boldsymbol{Q}_2, ..., \boldsymbol{Q}_n), \tag{1}$$

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where $\boldsymbol{C}$ is the right Cauchy Green deformation tensor and $\boldsymbol{Q}_i$ are the internal variables. The evolution of internal variables are described by n equations of the form

$$\dot{\boldsymbol{Q}}_k = \hat{\boldsymbol{f}}_k(\boldsymbol{C}, \boldsymbol{Q}_1, \boldsymbol{Q}_2, ..., \boldsymbol{Q}_n). \tag{2}$$

The evolution equations and the expression for the internal energy shall satisfy the dissipation inequality

$$\mathcal{D} := \frac{1}{2}\boldsymbol{S} : \dot{\boldsymbol{C}} - \dot{\psi} \geq 0. \tag{3}$$

Using a Maxwell type spring-dash pot model to represent the material, the strain energy function can be split into equilibrium and non-equilibrium parts as

$$\psi := \psi_{EQ}(\bar{\boldsymbol{C}}) + \psi_{NEQ}(\boldsymbol{C}_e) = \psi_{EQ}(\bar{\boldsymbol{C}}) + \psi_{NEQ}(\boldsymbol{F}_i^{-T}\bar{\boldsymbol{C}}\boldsymbol{F}_i^{-1}), \tag{4}$$

where $\boldsymbol{C}_e$ is the elastic part of the deviatoric Cauchy Green tensor, $\bar{\boldsymbol{C}} := \bar{\boldsymbol{F}}^T\bar{\boldsymbol{F}}$. It has been assumed that $\bar{\boldsymbol{F}} = J^{-\frac{1}{3}}\boldsymbol{F}$ admits a decomposition into an elastic and an inelastic part, $\bar{\boldsymbol{F}} = \boldsymbol{F}_e\boldsymbol{F}_i$. Using the above expressions in the internal dissipation inequality gives the expressions for stress as

$$\boldsymbol{S} = -pJ\boldsymbol{C}^{-1} + J^{-2/3}\mathrm{DEV}\left\{\boldsymbol{S}_{EQ} + \boldsymbol{S}_{NEQ}\right\}. \tag{5}$$

$\boldsymbol{S}_{EQ}$ and $\boldsymbol{S}_{NEQ}$ denote the elastic and viscous contributions to the total stress. $\mathrm{DEV}\left\{\bullet\right\}$ represents the deviatoric projection and the first term is a consequence of the incompressibility constraint. These stresses are given by

$$\boldsymbol{S}_{EQ} := 2\frac{\partial\psi_{EQ}}{\partial\bar{\boldsymbol{C}}}, \tag{6}$$

$$\boldsymbol{S}_{NEQ} := 2\boldsymbol{F}_i^{-1}\frac{\partial\psi_{NEQ}}{\partial\boldsymbol{C}_e}\boldsymbol{F}_i^{-T}, \tag{7}$$

and the deviatoric projector by

$$\mathrm{DEV}\left\{\bullet\right\} := \bullet - \frac{\bullet : \boldsymbol{C}}{3}\boldsymbol{C}^{-1}. \tag{8}$$

The Kirchhoff stress is then obtained as

$$\boldsymbol{\tau} = \boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T = -pJ\boldsymbol{I} + \boldsymbol{\tau}^{iso} = -pJ\boldsymbol{I} + \mathbb{P} : \bar{\boldsymbol{\tau}}, \tag{9}$$

where $\bar{\boldsymbol{\tau}} = \bar{\boldsymbol{\tau}}_{EQ} + \bar{\boldsymbol{\tau}}_{NEQ}$ are defined as

$$\bar{\boldsymbol{\tau}}_{EQ} := 2\frac{\partial\psi_{EQ}(\bar{\boldsymbol{b}})}{\partial\bar{\boldsymbol{b}}}\bar{\boldsymbol{b}}, \tag{10}$$

$$\boldsymbol{\tau}_{NEQ} := 2\frac{\partial\psi_{NEQ}(\boldsymbol{b}_e)}{\partial\boldsymbol{b}_e}\boldsymbol{b}_e. \tag{11}$$

$\mathbb{P}$ is the deviatoric projector in the deformed configuration defined as

$$\mathbb{P} := \mathbb{I} - \frac{\boldsymbol{I} \otimes \boldsymbol{I}}{3}. \tag{12}$$

The dissipation inequality becomes

$$\mathcal{D} = -\boldsymbol{\tau}_{NEQ} : \frac{1}{2}\mathcal{L}_v \boldsymbol{b}_e . \boldsymbol{b}_e^{-1} \geq 0, \tag{13}$$

where $\mathcal{L}_v \boldsymbol{b}_e$ denotes the Lie derivative of $\boldsymbol{b}_e$ defined as

$$\mathcal{L}_v \boldsymbol{b}_e = \bar{\boldsymbol{F}} \dot{\overline{\boldsymbol{C}_i^{-1}}} \bar{\boldsymbol{F}}^T. \tag{14}$$

The expression 13 can be satisfied by specifying the evolution equation as

$$-\frac{1}{2}\mathcal{L}_v \boldsymbol{b}_e . \boldsymbol{b}_e^{-1} = \gamma_0 \mathbb{V}^{-1} : \boldsymbol{\tau}_{NEQ}. \tag{15}$$

where $\mathbb{V}$ is a fourth order isotropic positive definite tensor possibly a function of $\boldsymbol{b}_e$ and $\gamma_0 > 0$. A slightly different equation has been proposed by [**?**]. In fact, the model of [**?**] can be seen to be a special case of the model of [**?**].

4. Integration of the evolution equation

The integration of equation 13 is carried out by a predictor-corrector type algorithm. In the elastic predictor step, the inelastic strains are taken to be fixed and so,

$$(\boldsymbol{C}_i^{-1})_{tr} = (\boldsymbol{C}_i^{-1})_{t_{n-1}} \implies \boldsymbol{b}_e^{tr} = \bar{\boldsymbol{F}} (\boldsymbol{C}_i^{-1})_{t_{n-1}} \bar{\boldsymbol{F}}^T. \tag{16}$$

In the inelastic corrector step, the total deformation is assumed to be held fixed and so, $\mathcal{L}_v \boldsymbol{b}_e = \dot{\boldsymbol{b}}_e$. Using this in equation 15 gives

$$\dot{\boldsymbol{b}}_e \boldsymbol{b}_e^{-1} = -2\gamma_0 \left[\mathbb{V}^{-1} : \boldsymbol{\tau}_{NEQ}\right]. \tag{17}$$

The above equation can be integrated using the exponential mapping technique Weber1990. The resulting expression is

$$\boldsymbol{b}_e = \exp\left[-2\gamma_0 \int_{t_{n-1}}^{t_n} \mathbb{V}^{-1} : \boldsymbol{\tau}_{NEQ} \, \mathrm{d}t\right] \boldsymbol{b}_e^{tr}, \tag{18}$$

$$(\boldsymbol{b}_e)_{t_n} \approx \exp\left[-2\gamma_0 \Delta t (\mathbb{V}^{-1} : \boldsymbol{\tau}_{NEQ})_{t=t_n}\right] \boldsymbol{b}_e^{tr}. \tag{19}$$

The above equation is first order accurate.

Since the material is assumed to be isotropic, $\boldsymbol{b}_e$, $\boldsymbol{b}_e^{tr}$ and hence $\boldsymbol{\tau}_{NEQ}$ share the Eigen space. Since $\mathbb{V}^{-1} := \frac{1}{2\eta_D}\left[\mathbb{I} - \frac{\boldsymbol{I} \otimes \boldsymbol{I}}{3}\right]$ is isotropic, equation 19 can be written in Eigen basis as

$$\lambda_{Ae}^2 = \exp\left[-\frac{\gamma_0 \Delta t}{\eta_D} \mathrm{dev}(\tau_A)\right] (\lambda_{Ae}^2)_{tr}. \tag{20}$$

Taking logarithm of both sides,

$$\epsilon_{Ae} = -\frac{\gamma_0 \Delta t}{2\eta_D} \mathrm{dev}(\tau_A) + (\epsilon_{Ae})_{tr}, \tag{21}$$

where $\epsilon_{Ae} = \ln \lambda_{Ae}$, $(\epsilon_{Ae})_{tr} = \ln(\lambda_{Ae})_{tr}$. The above equation is non-linear if τ_A is a non-linear function of ϵ_e. Hence, Newton iterations are used to solve it as below.

Defining

$$r_A :- \epsilon_{Ae} + \frac{\gamma_0 \Delta t}{2\eta_D} \mathrm{dev}(\tau_A) - (\epsilon_{Ae})_{tr} = 0, \tag{22}$$

it can be solved by linearizing around $\epsilon_{Ae} = (\epsilon_{Ae})_k$ as

$$r_A + \frac{\partial r_A}{\partial \epsilon_{Be}} \Delta\epsilon_{Be} = 0 \implies K_{AB} \Delta\epsilon_{Be} = -r_A. \tag{23}$$

where $K_{AB} = \dfrac{\partial r_A}{\partial \epsilon_{Be}}$. The above equation is solved to obtain $\Delta\epsilon_e$, which is then used to update the elastic strain as $(\epsilon_e)_{k+1} = (\epsilon_e)_k + \Delta\epsilon_e$.

5. Plane stress formulation

As already mentioned, to the author's knowledge, a plane stress implementation of the FV model does not exist. This and the further sections discuss this implementation. *It shall be noted that no changes to the model will be made. Rather, all the expressions for the stresses and the tangents will be rewritten so that they can be computed only using the in-plane components of the deformation gradient ($\boldsymbol{F}$) and its elastic part ($\boldsymbol{F}_e$).*

5.1. Stress and tangent computation

5.1.1. Stress computation

In plane stress scenario and for an incompressible material, the computation of stress can be simplified. The value of p in the equation 9 can be found by using the condition that $\tau_{33} = 0$. This condition can be imposed separately for the elastic and the viscous branches and the results can be combined. Beginning with the elastic branch, the term $\dfrac{\partial \psi}{\partial \bar{\boldsymbol{b}}} \bar{\boldsymbol{b}}$ can be computed as

$$\frac{\partial \psi}{\partial \bar{\boldsymbol{b}}} \bar{\boldsymbol{b}} = \left[\frac{\partial \psi}{\partial I_1} \boldsymbol{I} + \frac{\partial \psi}{\partial I_2} (I_1 \boldsymbol{I} - \boldsymbol{b}) \right] \boldsymbol{b}, \tag{24}$$

where the incompressibility of the material has been taken into account. The deviatoric projection of the above term is

$$\mathbb{P} : \left(\frac{\partial \psi}{\partial \bar{\boldsymbol{b}}} \bar{\boldsymbol{b}} \right) = \left[\frac{\partial \psi}{\partial I_1} \boldsymbol{I} + \frac{\partial \psi}{\partial I_2} (I_1 \boldsymbol{I} - \boldsymbol{b}) \right] \boldsymbol{b} - \frac{1}{3} \left[\frac{\partial \psi}{\partial I_1} I_1 + 2 \frac{\partial \psi}{\partial I_2} I_2 \right] \boldsymbol{I}. \tag{25}$$

The contribution of the elastic branch to p, denoted p_e, can be written as (using $\tau_{33} = 0$)

$$p_e = 2 \left[\frac{\partial \psi}{\partial I_1} b_{33} + \frac{\partial \psi}{\partial I_2} (I_1 b_{33} - b_{33}^2) \right] - \frac{2}{3} \left[\frac{\partial \psi}{\partial I_1} I_1 + 2 \frac{\partial \psi}{\partial I_2} I_2 \right]. \tag{26}$$

The total elastic part of the stress can then be found as

$$-p_e\boldsymbol{I} + 2\mathbb{P} : \left(\frac{\partial\psi}{\partial\bar{\boldsymbol{b}}}\bar{\boldsymbol{b}}\right) = 2\frac{\partial\psi}{\partial I_1}(\boldsymbol{b} - b_{33}\boldsymbol{I}) + 2\frac{\partial\psi}{\partial I_2}\left(I_1(\boldsymbol{b} - b_{33}\boldsymbol{I}) - (\boldsymbol{b}^2 - b_{33}^2\boldsymbol{I})\right). \tag{27}$$

The above equation, written with its components restricted to within the plane, can be seen to be

$$\boldsymbol{\tau}^e = 2\frac{\partial\psi}{\partial I_1}(\boldsymbol{b}^{2d} - b_{33}\boldsymbol{I}^{2d}) + 2\frac{\partial\psi}{\partial I_2}\left(I_1(\boldsymbol{b}^{2d} - b_{33}\boldsymbol{I}^{2d}) - ((\boldsymbol{b}^{2d})^2 - b_{33}^2\boldsymbol{I}^{2d})\right). \tag{28}$$

$\boldsymbol{b}^{2d}$ is the restriction of $\boldsymbol{b}$ to within the plane. As a consequence of plane stress assumption, $\boldsymbol{b}$ has been assumed to be of the form $\boldsymbol{b} = \begin{bmatrix} \boldsymbol{b}_{2d} & \boldsymbol{o} \\ \boldsymbol{o}^T & b_{33} \end{bmatrix}$. Expressing the first and the second invariants in terms of in-plane components as (realizing that $b_{33} = 1/\det\boldsymbol{b}^{2d}$)

$$I_1 = \mathrm{tr}(\boldsymbol{b}^{2d}) + 1/\det(\boldsymbol{b}^{2d}), \tag{29}$$

$$I_2 = \frac{1}{2}\left[(I_1(\boldsymbol{b}^{2d}))^2 - (\boldsymbol{b}^{2d} : \boldsymbol{b}^{2d} + 1/\det(\boldsymbol{b}^{2d})^2)\right], \tag{30}$$

the term $\boldsymbol{b}^{2d} - b_{33}\boldsymbol{I}^{2d}$ in the equation 28 can now be simply written as $\dfrac{\partial I_1}{\partial\boldsymbol{b}^{2d}}\boldsymbol{b}^{2d}$ and the term $I_1(\boldsymbol{b}^{2d} - b_{33}\boldsymbol{I}^{2d}) - ((\boldsymbol{b}^{2d})^2 - b_{33}^2\boldsymbol{I}^{2d})$ as $\dfrac{\partial I_2}{\partial\boldsymbol{b}^{2d}}\boldsymbol{b}^{2d}$. The total stress in the equation 28 then simply becomes

$$\boldsymbol{\tau}^e = 2\left[\frac{\partial\psi}{\partial I_1}\frac{\partial I_1}{\partial\boldsymbol{b}^{2d}} + \frac{\partial\psi}{\partial I_2}\frac{\partial I_2}{\partial\boldsymbol{b}^{2d}}\right]\boldsymbol{b}^{2d} = 2\frac{\partial\psi}{\partial\boldsymbol{b}^{2d}}\boldsymbol{b}^{2d}. \tag{31}$$

By a similar exercise for the viscous branches, the viscous contribution to the total stress becomes

$$\boldsymbol{\tau}^v = 2\frac{\partial\psi}{\partial\boldsymbol{b}_e^{2d}}\boldsymbol{b}_e^{2d}. \tag{32}$$

Hence the equations 5 and 9 become

$$\boldsymbol{\tau} = \boldsymbol{\tau}^e + \boldsymbol{\tau}^v, \quad \text{and} \tag{33}$$

$$\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{\tau}\boldsymbol{F}^{-T}. \tag{34}$$

The above equations are restricted to in-plane components and the superscript $2d$ has been eliminated for convenience. The total stress in the equation 33 can be written for the case of multiple (say, N) viscous branches simply as $\boldsymbol{\tau} = \boldsymbol{\tau}^e + \sum_{i=1}^N(\boldsymbol{\tau}^v)^{(i)}$, where each of the $(\boldsymbol{\tau}^v)^{(i)}$ s now denote the viscous stress in the corresponding viscous arm, defined as $(\boldsymbol{\tau}^v)^{(i)} := 2\dfrac{\partial\psi^{(i)}}{\partial(\boldsymbol{b}_e^{2d})^{(i)}}(\boldsymbol{b}_e^{2d})^{(i)}$. $(\boldsymbol{b}_e^{2d})^{(i)}$ is the left cauchy green tensor in the i^{th} viscous arm.

The two invariants written in terms of principal stretches become

$$I_1 = \lambda_A^2 + \lambda_B^2 + \lambda_C^2 = \lambda_A^2 + \lambda_B^2 + 1/\lambda_A^2\lambda_B^2, \tag{35}$$

$$I_2 = \lambda_A^2\lambda_B^2 + \lambda_B^2\lambda_C^2 + \lambda_C^2\lambda_A^2 = \lambda_A^2\lambda_B^2 + 1/\lambda_A^2 + 1/\lambda_B^2, \tag{36}$$

where $\lambda_C^2 = 1/\lambda_A^2\lambda_B^2$ has been used.

Using the above, the Kirchhoff stress can be written in principal basis to be

$$\tau_A = 2\lambda_A^2\frac{\partial\psi}{\partial\lambda_A^2} = 2\lambda_A^2\left[\frac{\partial\psi}{\partial I_1}\frac{\partial I_1}{\partial\lambda_A^2} + \frac{\partial\psi}{\partial I_2}\frac{\partial I_2}{\partial\lambda_A^2}\right], \tag{37}$$

$$\tau_B = 2\lambda_B^2\frac{\partial\psi}{\partial\lambda_B^2} = 2\lambda_B^2\left[\frac{\partial\psi}{\partial I_1}\frac{\partial I_1}{\partial\lambda_B^2} + \frac{\partial\psi}{\partial I_2}\frac{\partial I_2}{\partial\lambda_B^2}\right], \tag{38}$$

and $\tau_C = 0$ as a consequence of the plane stress assumption. The partial derivatives of the invariants can be evaluated as

$$\frac{\partial I_1}{\partial\lambda_A^2} = 1 - 1/\lambda_A^4\lambda_B^2, \quad \frac{\partial I_1}{\partial\lambda_B^2} = 1 - 1/\lambda_A^2\lambda_B^4, \tag{39}$$

$$\frac{\partial I_2}{\partial\lambda_A^2} = \lambda_B^2 - 1/\lambda_A^4, \quad \frac{\partial I_2}{\partial\lambda_B^2} = \lambda_A^2 - 1/\lambda_B^4. \tag{40}$$

The stresses can be expressed in global Cartesian basis by using

$$\boldsymbol{\tau} = \tau_1\boldsymbol{n}_1 \otimes \boldsymbol{n}_1^T + \tau_2\boldsymbol{n}_2 \otimes \boldsymbol{n}_2^T, \tag{41}$$

where $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are the eigen vectors of $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T$. As a recollection, the stresses obtained this way are the total stresses, $\boldsymbol{\tau} = -pJ\boldsymbol{I} + \boldsymbol{\tau}^{iso} = -pJ\boldsymbol{I} + \mathbb{P} : \bar{\boldsymbol{\tau}}$.

Stresses can be similarly computed in viscous branches where λ is replaced by λ_e, which are the eigen values of $\boldsymbol{b}_e = \boldsymbol{F}_e\boldsymbol{F}_e^T$ and the ψ replaced by the strain energy of the corresponding viscous arm.

5.1.2. Integration of evolution equation

For viscous branches, the evolution equation remains same even in the plane stress scenario. The residual can be written, similar to the equation 22 as

$$r_A = \epsilon_{Ae} + \frac{\gamma_0\Delta t}{2\eta_D}\mathrm{dev}(\tau_A) - (\epsilon_{Ae})_{tr} = 0, \tag{42}$$

Since plane stress condition is assumed to prevail, only the in-plane components of the above equation are considered. Also, the deviatoric part of the Kirchhoff stress can be expressed as $\mathrm{dev}(\boldsymbol{\tau}) = \boldsymbol{\tau} + p\boldsymbol{I}$, where p is the Lagrange multiplier that enforces incompressibility, which is found by using the condition that $\tau_3 = 0$. $\boldsymbol{\tau}$ can be evaluated by using the procedure in the previous section. The expression for p can be seen to be

$$p = -\frac{\tau_1 + \tau_2}{3}. \tag{43}$$

The above equation can be obtained by taking the trace of the equation 9 and realizing that $\tau_3 = 0$ as a consequence of plane stress assumption and that trace of the deviatoric projector is 0. The in-plane evolution equations then become

$$r_1 = \epsilon_{1e} + \frac{\gamma_0 \Delta t}{2\eta_D}(\tau_1 + p) - (\epsilon_{1e})_{tr} = 0, \tag{44}$$

$$r_2 = \epsilon_{2e} + \frac{\gamma_0 \Delta t}{2\eta_D}(\tau_2 + p) - (\epsilon_{2e})_{tr} = 0. \tag{45}$$

In the third direction, the evolution equation becomes $\epsilon_{3e} + \frac{\gamma_0\Delta t}{2\eta_D} p - (\epsilon_{3e})_{tr} = 0$. It can be shown that solving the first two equations will result in the third equation being satisfied automatically. Adding the equations 44 and 45 will result in $\epsilon_{1e} + \epsilon_{2e} + \frac{\gamma_0\Delta t}{2\eta_D}(\tau_1 + \tau_2 + 2p) - [(\epsilon_{1e})_{tr} + (\epsilon_{2e})_{tr}] = 0$. This, in conjunction with the assumption of incompressibility and plane stress condition, results in $\epsilon_{3e} + \frac{\gamma_0\Delta t}{2\eta_D} p - (\epsilon_{3e})_{tr} = 0$, which is the third equation.

The equations 44 and 45 are solved iteratively using Newton method.

$$r_A^{(k+1)} = r_A^{(k)} + \frac{\partial r_A}{\partial \epsilon_{Be}} \Delta\epsilon_{Be} = 0. \tag{46}$$

This in turn requires the evaluation of $K_{AB} = \dfrac{\partial r_A}{\partial \epsilon_{Be}}$. This can be evaluated as

$$K_{AB} = \frac{\partial r_A}{\partial \epsilon_{Be}} = \delta_{AB} + \frac{\gamma_0 \Delta t}{2\eta_D}\left(\frac{\partial \tau_A}{\partial \epsilon_{Be}} + \frac{\partial p}{\partial \epsilon_{Be}}\right). \tag{47}$$

The pressure derivative can be computed from equation 43 as

$$\frac{\partial p}{\partial \epsilon_{Ae}} = -\frac{1}{3}\left(\frac{\partial \tau_1}{\partial \epsilon_{Ae}} + \frac{\partial \tau_2}{\partial \epsilon_{Ae}}\right). \tag{48}$$

The computation of derivative $\dfrac{\partial \tau_A}{\partial \epsilon_{Be}}$ can be carried out as shown in the following sections.

5.2. Tangent computation

5.2.1. For elastic branch

The computation of tangent first involves the computation of $\mathfrak{C} = 2\dfrac{\partial \boldsymbol{S}}{\partial \boldsymbol{C}}$. The derivative can be computed by noting that Bonet2001

$$\dot{\boldsymbol{S}} = \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{C}} : \dot{\boldsymbol{C}}. \tag{49}$$

Since $\boldsymbol{C} = \sum_{i=1}^{2} \lambda_i^2 \boldsymbol{N}_i \otimes \boldsymbol{N}_i$, $\dot{\boldsymbol{C}} = \sum_{i=1}^{2}\left[\dfrac{\partial \lambda_i^2}{\partial t}\boldsymbol{N}_i \otimes \boldsymbol{N}_i + \lambda_i^2 \dot{\boldsymbol{N}}_i \otimes \boldsymbol{N}_i + \lambda_i^2 \boldsymbol{N}_i \otimes \dot{\boldsymbol{N}}_i\right]$. $\dot{\boldsymbol{N}}_i = \sum_{j=1}^{2} W_{ij}\boldsymbol{N}_j$, where $W_{ij} = -W_{ji}$ are the components of a skew symmetric tensor. Hence

$$\dot{\boldsymbol{C}} = \sum_{i=1}^{2} \frac{\partial \lambda_i^2}{\partial t}\boldsymbol{N}_i \otimes \boldsymbol{N}_i + \sum_{i,j=1, i\neq j}^{2} W_{ij}(\lambda_i^2 - \lambda_j^2)\boldsymbol{N}_i \otimes \boldsymbol{N}_j. \tag{50}$$

As a consequence of isotropy, $\boldsymbol{S}$ and $\boldsymbol{C}$ share the eigen vectors. Hence, following the same procedure,

$$\dot{\boldsymbol{S}} = \sum_{i,j=1}^{2} 2\frac{\partial^2\psi}{\partial\lambda_i^2\partial\lambda_j^2}\frac{\partial\lambda_j^2}{\partial t}\boldsymbol{N}_i\otimes\boldsymbol{N}_i + \sum_{i,j=1,i\neq j}^{2} W_{ij}(S_i - S_j)\boldsymbol{N}_i\otimes\boldsymbol{N}_j. \tag{51}$$

The tangent can hence be written as

$$\mathfrak{C} = \sum_{i,j=1}^{2} 4\frac{\partial^2\psi}{\partial\lambda_i^2\partial\lambda_j^2}\boldsymbol{N}_i\otimes\boldsymbol{N}_i\otimes\boldsymbol{N}_j\otimes\boldsymbol{N}_j + \sum_{i,j=1,i\neq j}^{2}\frac{S_i - S_j}{\lambda_i^2-\lambda_j^2}(\boldsymbol{N}_i\otimes\boldsymbol{N}_j\otimes\boldsymbol{N}_i\otimes\boldsymbol{N}_j + \boldsymbol{N}_i\otimes\boldsymbol{N}_j\otimes\boldsymbol{N}_j\otimes\boldsymbol{N}_i). \tag{52}$$

Its push forward to the spatial configuration can be seen to be

$$\begin{aligned}\mathfrak{c} = &\sum_{i,j=1}^{2}(C_{ij} - 2\sigma_i\delta_{ij})\boldsymbol{n}_i\otimes\boldsymbol{n}_i\otimes\boldsymbol{n}_j\otimes\boldsymbol{n}_j\\ &+\sum_{i,j=1,i\neq j}^{2}\frac{\sigma_i\lambda_j^2-\sigma_j\lambda_i^2}{\lambda_i^2-\lambda_j^2}(\boldsymbol{n}_i\otimes\boldsymbol{n}_j\otimes\boldsymbol{n}_i\otimes\boldsymbol{n}_j + \boldsymbol{n}_i\otimes\boldsymbol{n}_j\otimes\boldsymbol{n}_j\otimes\boldsymbol{n}_i),\end{aligned} \tag{53}$$

where $C_{ij} = \dfrac{\partial^2\psi}{\partial\ln\lambda_i\partial\ln\lambda_j} = \dfrac{\partial\tau_i}{\partial\epsilon_j}$ and $\sigma_i = \tau_i$, since the material is incompressible. The components of the above fourth order tensor can be stored in a matrix as

$$[\mathfrak{c}] = \begin{bmatrix}1111 & 1122 & 1112\\ 2211 & 2222 & 2212\\ 1211 & 1222 & 1212\end{bmatrix}_{\boldsymbol{n}_1,\boldsymbol{n}_2}. \tag{54}$$

The components of the tangent can be converted to Cartesian basis by using the transformation Reese1995

$$[\mathfrak{c}]_{(\boldsymbol{e}_1,\boldsymbol{e}_2)} = [P][\mathfrak{c}]_{(\boldsymbol{n}_1,\boldsymbol{n}_2)}[P]^T, \tag{55}$$

where

$$[P] = \begin{bmatrix}Q_{11}^2 & Q_{12}^2 & 2Q_{11}Q_{12}\\ Q_{21}^2 & Q_{22}^2 & 2Q_{21}Q_{22}\\ Q_{11}Q_{21} & Q_{12}Q_{22} & Q_{11}Q_{22}+Q_{12}Q_{21}\end{bmatrix}. \tag{56}$$

Here, Q_{ij}s are the elements of $[Q]$ matrix which is the transpose of $[\tilde{Q}]$, $[Q] = [\tilde{Q}]^T$. The columns of $[\tilde{Q}]$ matrix are the components of eigen vectors of $\boldsymbol{b}$ in cartesian basis.

The tangent to be supplied to abaqus ($\mathcal{C}^{(JK)}$) corresponds to the Jaumann rate of the Kirchhoff stress Nguyen2016 written as

$$\overset{\nabla(JK)}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} + \boldsymbol{\tau}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{\tau} = \mathcal{C}^{(JK)} : \boldsymbol{d}. \tag{57}$$

$\mathcal{C}^{(JK)}$ is related to $\mathfrak{c}$ as

$$\mathcal{C}^{(JK)}_{ijkl} = \mathfrak{c}_{ijkl} + \frac{1}{2}(\sigma_{ij}\delta_{kl} + \sigma_{kl}\delta_{ij} + \sigma_{il}\delta_{jk} + \sigma_{jk}\delta_{il}). \tag{58}$$

The computation of c requires the computation of $\frac{\partial \tau_i}{\partial \epsilon_j}$, which can be carried out as follows

$$\frac{\partial \tau_i}{\partial \epsilon_j} = 2\lambda_j^2 \frac{\partial \tau_i}{\partial \lambda_j^2}, \quad \text{i,j=1,2.} \tag{59}$$

$$\frac{\partial \tau_i}{\partial \lambda_j^2} = 2\left[\frac{\partial^2 \psi}{\partial \lambda_j^2 \partial I_1}\lambda_i^2 \frac{\partial I_1}{\partial \lambda_i^2} + \frac{\partial \psi}{\partial I_1}\frac{\partial}{\partial \lambda_j^2}\left(\lambda_i^2 \frac{\partial I_1}{\partial \lambda_i^2}\right) + \frac{\partial^2 \psi}{\partial \lambda_j^2 \partial I_2}\lambda_i^2 \frac{\partial I_2}{\partial \lambda_i^2} + \frac{\partial \psi}{\partial I_2}\frac{\partial}{\partial \lambda_j^2}\left(\lambda_i^2 \frac{\partial I_2}{\partial \lambda_i^2}\right)\right]. \tag{60}$$

The partial derivatives can be further evaluated as

$$\frac{\partial^2 \psi}{\partial \lambda_i^2 \partial I_j} = \frac{\partial^2 \psi}{\partial I_1 \partial I_j}\frac{\partial I_1}{\partial \lambda_i^2} + \frac{\partial^2 \psi}{\partial I_2 \partial I_j}\frac{\partial I_2}{\partial \lambda_i^2} \tag{61}$$

$$\frac{\partial}{\partial \lambda_A^2}\left(\lambda_A^2 \frac{\partial I_1}{\partial \lambda_A^2}\right) = 1 + 1/\lambda_A^4\lambda_B^2, \quad \frac{\partial}{\partial \lambda_B^2}\left(\lambda_A^2 \frac{\partial I_1}{\partial \lambda_A^2}\right) = 1/\lambda_A^2\lambda_B^4, \tag{62}$$

$$\frac{\partial}{\partial \lambda_A^2}\left(\lambda_A^2 \frac{\partial I_2}{\partial \lambda_A^2}\right) = \lambda_B^2 + 1/\lambda_A^4, \quad \frac{\partial}{\partial \lambda_B^2}\left(\lambda_A^2 \frac{\partial I_2}{\partial \lambda_A^2}\right) = \lambda_A^2. \tag{63}$$

Remark. *The total stress and tangent computation in the case of plane stress condition for incompressible materials involves the computation of derivatives* ***after*** *enforcing all the material (incompressibility) and geometric (plane stress) constraints.*

In case of equal eigen values $\lambda_1 = \lambda_2$, the second term of equation 53 takes a $\frac{0}{0}$ form and so, L'Hospital's rule is used to compute it.

$$\lim_{\lambda_2 \to \lambda_1} \frac{\sigma_1 \lambda_2^2 - \sigma_2 \lambda_1^2}{\lambda_1^2 - \lambda_2^2} = \frac{1}{2}\left[\frac{\partial^2 \psi}{\partial \epsilon_2 \partial \epsilon_2} - \frac{\partial^2 \psi}{\partial \epsilon_1 \partial \epsilon_2}\right] - \sigma_2 = \frac{1}{2}\left[\frac{\partial \tau_2}{\partial \epsilon_2} - \frac{\partial \tau_2}{\partial \epsilon_1}\right] - \sigma_2. \tag{64}$$

5.2.2. For viscous branches

For the viscous branches, the following procedure will be used to compute the tangent similar to as in [**?**]. All the stress and strain components are now restricted to within the plane.

In the elastic trial state, since the inelastic strain is held fixed, $\boldsymbol{F}^n = \boldsymbol{F}_e^{tr}\boldsymbol{F}_i^{n-1} \implies \boldsymbol{C}^n = (\boldsymbol{F}_i^{n-1})^T \boldsymbol{C}_e^{tr} \boldsymbol{F}_i^{n-1}$. Hence,

$$\frac{\partial S_{IJ}}{\partial C_{KL}} = \frac{\partial S_{IJ}}{\partial (C_e^{tr})_{\alpha\beta}}\frac{\partial (C_e^{tr})_{\alpha\beta}}{\partial C_{KL}} = \left((\boldsymbol{F}_i^{n-1})^{-1}\right)_{K\alpha}\left((\boldsymbol{F}_i^{n-1})^{-1}\right)_{L\beta}\frac{\partial S_{IJ}}{\partial (C_e^{tr})_{\alpha\beta}}. \tag{65}$$

where the symmetry of $\boldsymbol{C}_e^{tr}$ has been used. Since in viscous branches,

$$\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{\tau}\boldsymbol{F}^{-T} = (\boldsymbol{F}_i^{n-1})^{-1}.\underbrace{(\boldsymbol{F}_e^{tr})^{-1}\boldsymbol{\tau}(\boldsymbol{F}_e^{tr})^{-T}}_{\tilde{\boldsymbol{S}}}.(\boldsymbol{F}_i^{n-1})^{-T}, \tag{66}$$

the stress derivative can be further refined as

$$\frac{\partial S_{IJ}}{\partial (C_e^{tr})_{\alpha\beta}} = \left((\boldsymbol{F}_i^{n-1})^{-1}\right)_{I\gamma}\left((\boldsymbol{F}_i^{n-1})^{-1}\right)_{J\delta}\frac{\partial \tilde{S}_{\gamma\delta}}{\partial (C_e^{tr})_{\alpha\beta}}. \tag{67}$$

Hence,

$$2\frac{\partial S_{IJ}}{\partial C_{KL}} = 2\left((\boldsymbol{F}_i^{n-1})^{-1}\right)_{I\gamma}\left((\boldsymbol{F}_i^{n-1})^{-1}\right)_{J\delta}\left((\boldsymbol{F}_i^{n-1})^{-1}\right)_{K\alpha}\left((\boldsymbol{F}_i^{n-1})^{-1}\right)_{L\beta}\frac{\partial \tilde{S}_{\gamma\delta}}{\partial (C_e^{tr})_{\alpha\beta}}. \tag{68}$$

The push-forward of above by $\boldsymbol{F}$ results in

$$\mathfrak{c}_{ijkl} = 2(\det \boldsymbol{F})^{-1} F_{iI}F_{jJ}F_{kK}F_{lL}\frac{\partial S_{IJ}}{\partial C_{KL}} = 2(\boldsymbol{F}_e^{tr})_{i\gamma}(\boldsymbol{F}_e^{tr})_{j\delta}(\boldsymbol{F}_e^{tr})_{k\alpha}(\boldsymbol{F}_e^{tr})_{l\beta}\frac{\partial \tilde{S}_{\gamma\delta}}{\partial (C_e^{tr})_{\alpha\beta}}. \tag{69}$$

$\tilde{\boldsymbol{S}}$, written in Eigen basis is

$$\tilde{\boldsymbol{S}} = \sum_{A=1}^{2}\frac{\tau_A}{(\lambda_{Ae})_{tr}^2}\tilde{\boldsymbol{N}}_A \otimes \tilde{\boldsymbol{N}}_A. \tag{70}$$

It is to be noted that the τ_A in the above equation is a function of ϵ_e. The development from here is similar to that used to arrive at the equation 53 except that C_{ij} in that equation will be replaced by C_{ij}^{alg}, which will be defined below. A crucial factor is that $\boldsymbol{b}_e$ and $(\boldsymbol{b}_e)_{tr}$ share the same eigen space as a consequence of isotropy.

$$\mathfrak{c} = \sum_{i,j=1}^{2}(C_{ij}^{alg} - 2\sigma_i\delta_{ij})\boldsymbol{n}_i \otimes \boldsymbol{n}_i \otimes \boldsymbol{n}_j \otimes \boldsymbol{n}_j$$
$$+ \sum_{i,j=1,i\neq j}^{2}\frac{\sigma_i(\lambda_j)_{tr}^2 - \sigma_j(\lambda_i)_{tr}^2}{(\lambda_i)_{tr}^2 - (\lambda_j)_{tr}^2}(\boldsymbol{n}_i \otimes \boldsymbol{n}_j \otimes \boldsymbol{n}_i \otimes \boldsymbol{n}_j + \boldsymbol{n}_i \otimes \boldsymbol{n}_j \otimes \boldsymbol{n}_j \otimes \boldsymbol{n}_i). \tag{71}$$

In the above, $C_{AC}^{alg} = \dfrac{\partial \tau_A}{\partial (\epsilon_{Ce})_{tr}}$. Since τ_As are a function of ϵ_es, the derivative is computed using chain rule.

$$\frac{\partial \tau_A}{\partial (\epsilon_{Ce})_{tr}} = \frac{\partial \tau_A}{\partial \epsilon_{Be}}\frac{\partial \epsilon_{Be}}{\partial (\epsilon_{Ce})_{tr}}. \tag{72}$$

The derivative $\dfrac{\partial \epsilon_{Be}}{\partial (\epsilon_{Ce})_{tr}}$ can be computed by realizing that the equations $r_B = 0$ are satisfied at the end of local Newton iterations and hence are valid at all the times during the global Newton iterations. Hence, during the global Newton iterations, $\dfrac{\partial r_B}{\partial (\epsilon_{Ce})_{tr}} = 0$ as well. Hence,

$$\frac{\partial \epsilon_{Be}}{\partial (\epsilon_{Ce})_{tr}} = K_{BC}^{-1}, \tag{73}$$

where K_{BC} is defined in equation 47.

The expression for dissipation becomes

$$\mathcal{D} = \frac{1}{\eta_D}\mathrm{dev}\{\boldsymbol{\tau}_{NEQ}\} : \mathrm{dev}\{\boldsymbol{\tau}_{NEQ}\} = (\boldsymbol{\tau} + p\boldsymbol{I}) : (\boldsymbol{\tau} + p\boldsymbol{I}) + p^2, \tag{74}$$

which can be seen to be positive since $\eta_D > 0$. In the above equation, $\boldsymbol{\tau}$ is as evaluated in the equation 41.

5.3. Implementation details

The above has been implemented into a `UMAT` subroutine of Abaqus. The subroutine computes the updated stress and tangent for a given time step and also updates the internal variables. The implementation details will be discussed in this section.

The deformation gradient is obtained as an input to the subroutine from Abaqus. The internal variables ($\boldsymbol{C}_i$) at the beginning of the step are stored in an array. The working of the subroutine can be seen below.

Data: $\boldsymbol{F}^n$, $\boldsymbol{C}_i^{n-1}$, Δt
Result: $\boldsymbol{\tau}^n$, $\boldsymbol{C}_i^n$, $\mathcal{C}^{(\mathcal{GK})}$, ψ^n, $\mathcal{D}^n$
$(\boldsymbol{b}_e)_{tr} = \boldsymbol{F}^n(\boldsymbol{C}_i^{n-1})^{-1}(\boldsymbol{F}^n)^T$;
Compute $(\lambda_e)_{tr}$s and $\boldsymbol{n}_A$s so that $(\boldsymbol{b}_e)_{tr} = \sum_{A=1}^{2}(\lambda_{Ae})_{tr}^2\boldsymbol{n}_A \otimes \boldsymbol{n}_A$;
Define $(\epsilon_{Ae})_{tr} = \ln((\lambda_{Ae})_{tr})$;
$k = 0$, $\epsilon_{Ae} \leftarrow (\epsilon_{Ae})_{tr}$;
do

- Compute $\boldsymbol{\tau}$ and p from ϵ_{Ae};
- $r_A :- \epsilon_{Ae} + \frac{\Delta t}{2\eta}(\tau_A + p) - (\epsilon_{Ae})_{tr} = 0$;
- Compute $\dfrac{\partial \tau_A}{\partial \epsilon_{Be}}, \dfrac{\partial p}{\partial \epsilon_{Be}}$;
- $K_{AB} = \delta_{AB} + \frac{\Delta t}{2\eta}\left(\dfrac{\partial \tau_A}{\partial \epsilon_{Be}} + \dfrac{\partial p}{\partial \epsilon_{Be}}\right)$;
- $\Delta\epsilon_{Ae}^k = -K_{AB}^{-1}r_A$;
- $\epsilon_{Ae}^{k+1} \leftarrow \epsilon_{Ae}^k + \Delta\epsilon_{Ae}^k$;
- $k \leftarrow k + 1$

while $\|r_A\| > TOL$;
Update $\boldsymbol{\tau}^n$, p, $\lambda_{Ae} = \exp(\epsilon_{Ae})$, and K_{AB};
$\boldsymbol{b}_e = \sum_{A=1}^{2}\lambda_{Ae}^2\boldsymbol{n}_A \otimes \boldsymbol{n}_A$, $\boldsymbol{b}_e^{-1} = \sum_{A=1}^{2}\lambda_{Ae}^{-2}\boldsymbol{n}_A \otimes \boldsymbol{n}_A$;
$\boldsymbol{C}_i^n = (\boldsymbol{F}^n)^T\boldsymbol{b}_e^{-1}\boldsymbol{F}^n$;
Compute $\mathcal{C}^{(\mathcal{GK})}$ using $\boldsymbol{C}^{alg}$, τ_A, and λ_{Ae};
Compute ψ^n and $\mathcal{D}^n$;

Algorithm 1: Steps followed in `UMAT` for Finite Viscoelastic model

As can be seen, the internal variables are updated and the strain energy density and dissipation are computed with the updated values of $\boldsymbol{b}_e$ and $\boldsymbol{\tau}$. The value of TOL has been chosen to be 10^{-5}.

The `C++` implementation of the model can be found at [**?**] and in the appendix **??**.

6. Some checks

7. Some results

8. Conclusion

Please accept this article.

References