FINITE VISCOELASTIC MODEL

In this chapter, the Finite Viscoelastic model (**Reese1998**; **Bergstrom1998**) will be presented and its implementation will be described in a plane stress setting. At first, the model will be presented in a 3D setting and the differences arising in the implementation in plane stress will be discussed.

CONTENTS

9.1 THERMODYNAMICS

The Finite Linear Viscoelastic (FLV) model has been used in the previous sections to describe the viscoelastic behavior of the bulk material. However, the energy dissipated in the bulk material during the propagation of a crack cannot be computed with that model as it does not involve an explicit expression for energy or dissipation. In fact, it has not been proved that the FLV model is thermodynamically consistent for finite strains **Govindjee1997**.

In order to describe the processes that are far from equilibrium, a visceoasltic model has been proposed in **Bergstrom1998**; **Reese1998** in a thermodynamically consistent way. In this model, the strain energy density is taken to be a function of the deformation as well as some internal variables that characterize the rate dependence of the material. The strain energy density can be written as

$$\psi = \hat{\psi}(\boldsymbol{C}, \boldsymbol{Q}_1, \boldsymbol{Q}_2, \ldots, \boldsymbol{Q}_n), \tag{9.1}$$

where $\boldsymbol{C}$ is the right Cauchy Green deformation tensor and $\boldsymbol{Q}_i$ are the internal variables. The evolution of internal variables are described by n equations of the form

$$\dot{\boldsymbol{Q}}_k = \hat{\boldsymbol{f}}_k(\boldsymbol{C}, \boldsymbol{Q}_1, \boldsymbol{Q}_2, \ldots, \boldsymbol{Q}_n). \tag{9.2}$$

The evolution equations and the expression for the internal energy shall satisfy the dissipation inequality

$$\mathcal{D} := \frac{1}{2}\boldsymbol{S} : \dot{\boldsymbol{C}} - \dot{\psi} \geq 0. \tag{9.3}$$

Using a Maxwell type spring-dash pot model to represent the material, the strain energy function can be split into equilibrium and non-equilibrium parts as

$$\psi := \psi_{EQ}(\bar{\boldsymbol{C}}) + \psi_{NEQ}(\boldsymbol{C}_e) = \psi_{EQ}(\bar{\boldsymbol{C}}) + \psi_{NEQ}(\boldsymbol{F}_i^{-T}\bar{\boldsymbol{C}}\boldsymbol{F}_i^{-1}) \tag{9.4}$$

where $\boldsymbol{C}_e$ is the elastic part of the deviatoric Cauchy Green tensor, $\bar{\boldsymbol{C}}$. It has been assumed that $\bar{\boldsymbol{F}} = J^{-\frac{1}{3}}\boldsymbol{F}$ admits a decomposition into an elastic and an inelastic part, $\bar{\boldsymbol{F}} = \boldsymbol{F}_e\boldsymbol{F}_i$ (**Dal2009**). Using the above expressions in the internal dissipation inequality gives the expressions for stress as

$$\boldsymbol{S} = -pJ\boldsymbol{C}^{-1} + J^{-2/3}\mathrm{DEV}\left\{\boldsymbol{S}_{EQ} + \boldsymbol{S}_{NEQ}\right\}. \tag{9.5}$$

$\boldsymbol{S}_{EQ}$ and $\boldsymbol{S}_{NEQ}$ denote the elastic and viscous contributions to the total stress. $\mathrm{DEV}\left\{\bullet\right\}$ represents the deviatoric projection and the first term is a consequence of the incompressibility constraint. These stresses are given by

$$\boldsymbol{S}_{EQ} := 2\frac{\partial\psi_{EQ}}{\partial\bar{\boldsymbol{C}}} \tag{9.6}$$

$$\boldsymbol{S}_{NEQ} := 2\boldsymbol{F}_i^{-1}\frac{\partial\psi_{NEQ}}{\partial\boldsymbol{C}_e}\boldsymbol{F}_i^{-T} \tag{9.7}$$

and the deviatoric projector by

$$\mathrm{DEV}\left\{\bullet\right\} := \bullet - \frac{\bullet : \boldsymbol{C}}{3}\boldsymbol{C}^{-1}. \tag{9.8}$$

The Kirchhoff stress is then obtained as

$$\boldsymbol{\tau} = \boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T = -pJ\boldsymbol{I} + \boldsymbol{\tau}^{iso} = -pJ\boldsymbol{I} + \mathbb{P} : \bar{\boldsymbol{\tau}}, \tag{9.9}$$

where $\bar{\boldsymbol{\tau}} = \bar{\boldsymbol{\tau}}_{EQ} + \bar{\boldsymbol{\tau}}_{NEQ}$ are defined as

$$\bar{\boldsymbol{\tau}}_{EQ} := \frac{\partial \psi_{EQ}(\bar{\boldsymbol{b}})}{\partial \bar{\boldsymbol{b}}} \bar{\boldsymbol{b}}, \tag{9.10}$$

$$\boldsymbol{\tau}_{NEQ} := \frac{\partial \psi_{NEQ}(\boldsymbol{b}_e)}{\partial \boldsymbol{b}_e} \boldsymbol{b}_e. \tag{9.11}$$

$\mathbb{P}$ is the deviatoric projector in the deformed configuration defined as

$$\mathbb{P} := \mathbb{I} - \frac{\boldsymbol{I} \otimes \boldsymbol{I}}{3}. \tag{9.12}$$

The dissipation inequality becomes

$$\mathcal{D} = -\boldsymbol{\tau}_{NEQ} : \frac{1}{2} \mathcal{L}_v \boldsymbol{b}_e . \boldsymbol{b}_e^{-1} \geq 0 \tag{9.13}$$

where $\mathcal{L}_v \boldsymbol{b}_e$ denotes the Lie derivative of $\boldsymbol{b}_e$ defined as

$$\mathcal{L}_v \boldsymbol{b}_e = \bar{\boldsymbol{F}} \dot{\overline{\boldsymbol{C}_i^{-1}}} \bar{\boldsymbol{F}}^T \tag{9.14}$$

The expression 9.13 can be satisfied by specifying the evolution equation as

$$-\frac{1}{2} \mathcal{L}_v \boldsymbol{b}_e . \boldsymbol{b}_e^{-1} = \gamma_0 \mathcal{V}^{-1} : \boldsymbol{\tau}_{NEQ} \tag{9.15}$$

where $\mathcal{V}$ is a fourth order isotropic positive definite tensor possibly a function of $\boldsymbol{b}_e$ and $\gamma_0 > 0$. A slightly different equation has been proposed by **Bergstrom1998**.

9.2 INTEGRATION OF THE EVOLUTION EQUATION

The integration of equation 9.13 is carried out by a predictor-corrector type algorithm. In the elastic predictor step, the inelastic strains are taken to be fixed and so,

$$(\boldsymbol{C}_i^{-1})_{tr} = (\boldsymbol{C}_i^{-1})_{t_{n-1}} \implies \boldsymbol{b}_e^{tr} = \bar{\boldsymbol{F}} (\boldsymbol{C}_i^{-1})_{t_{n-1}} \bar{\boldsymbol{F}}^T. \tag{9.16}$$

In the inelastic corrector step, the total deformation is assumed to be held fixed and so, $\mathcal{L}_v \boldsymbol{b}_e = \dot{\boldsymbol{b}}_e$. Using this in equation 9.15 gives

$$\dot{\boldsymbol{b}}_e \boldsymbol{b}_e^{-1} = -2 \left[\mathcal{V}^{-1} : \boldsymbol{\tau}_{NEQ} \right] \tag{9.17}$$

The above equation can be integrated using the exponential mapping technique (**Weber1990**). The resulting expression is

$$\boldsymbol{b}_e = \exp\left[-2 \int_{t_{n-1}}^{t_n} \mathcal{V}^{-1} : \boldsymbol{\tau}_{NEQ} \, \mathrm{d}t \right] \boldsymbol{b}_e^{tr} \tag{9.18}$$

$$(\boldsymbol{b}_e)_{t_n} \approx \exp\left[-2\Delta t (\mathcal{V}^{-1} : \boldsymbol{\tau}_{NEQ})_{t=t_n} \right] \boldsymbol{b}_e^{tr} \tag{9.19}$$

The above equation is first order accurate.

Since the material is assumed to be isotropic, $\boldsymbol{b}_e$, $\boldsymbol{b}_e^{tr}$ and hence $\boldsymbol{\tau}_{NEQ}$ share the Eigen space. Since $\mathcal{V}^{-1} := \frac{1}{2\eta_D} \left[\mathbb{I} - \frac{\boldsymbol{I} \otimes \boldsymbol{I}}{3} \right]$ is isotropic, equation 9.19 can be written in Eigen basis as

$$\lambda_{Ae}^2 = \exp\left[-\frac{\Delta t}{\eta_D} \mathrm{dev}(\tau_A) \right] (\lambda_{Ae}^2)_{tr}. \tag{9.20}$$

Taking logarithm of both sides,

$$\epsilon_{Ae} = -\frac{\Delta t}{2\eta_D}\text{dev}(\tau_A) + (\epsilon_{Ae})_{tr}, \tag{9.21}$$

where $\epsilon_{Ae} = \ln \lambda_{Ae}$, $(\epsilon_{Ae})_{tr} = \ln(\lambda_{Ae})_{tr}$. The above equation is non-linear if τ_A is a non-linear function of ϵ_e. Hence, Newton iterations are used to solve it as below.

Defining

$$r_A := \epsilon_{Ae} + \frac{\Delta t}{2\eta_D}\text{dev}(\tau_A) - (\epsilon_{Ae})_{tr} = 0, \tag{9.22}$$

it can be solved by linearizing around $\epsilon_{Ae} = (\epsilon_{Ae})_k$ as

$$r_A + \frac{\partial r_A}{\partial \epsilon_{Be}}\Delta\epsilon_{Be} = 0 \implies K_{AB}\Delta\epsilon_{Be} = -r_A \tag{9.23}$$

where $K_{AB} = \dfrac{\partial r_A}{\partial \epsilon_{Be}}$. The above equation is solved to obtain $\Delta\epsilon_e$, which is then used to update the elastic strain as $(\epsilon_e)_{k+1} = (\epsilon_e)_k + \Delta\epsilon_e$.

9.3 PLANE STRESS FORMULATION

Stress computation

Since the material is incompressible and plane stress conditions are assumed to prevail, the out of plane component of $\boldsymbol{C}$ can be expressed in terms of the in-plane components as

$$C_{33} = 1/\det(\tilde{\boldsymbol{C}}) \tag{9.24}$$

Hence, the two invariants written in principal stretches become

$$I_1 = \lambda_A^2 + \lambda_B^2 + \lambda_C^2 = \lambda_A^2 + \lambda_B^2 + 1/\lambda_A^2\lambda_B^2, \tag{9.25}$$

$$I_2 = \lambda_A^2\lambda_B^2 + \lambda_B^2\lambda_C^2 + \lambda_C^2\lambda_A^2 = \lambda_A^2\lambda_B^2 + 1/\lambda_A^2 + 1/\lambda_B^2, \tag{9.26}$$

where $\lambda_C^2 = 1/\lambda_A^2\lambda_B^2$ has been used.

As already mentioned in the previous chapter, stresses can be evaluated directly by the differentiation of the strain energy functional by treating it to be a function of in-plane strain components. Using the above, the Kirchhoff stress can be written in principal basis to be

$$\tau_A = 2\lambda_A^2\frac{\partial\psi}{\partial\lambda_A^2} = 2\lambda_A^2\left[\frac{\partial\psi}{\partial I_1}\frac{\partial I_1}{\partial\lambda_A^2} + \frac{\partial\psi}{\partial I_2}\frac{\partial I_2}{\partial\lambda_A^2}\right], \tag{9.27}$$

$$\tau_B = 2\lambda_B^2\frac{\partial\psi}{\partial\lambda_B^2} = 2\lambda_B^2\left[\frac{\partial\psi}{\partial I_1}\frac{\partial I_1}{\partial\lambda_B^2} + \frac{\partial\psi}{\partial I_2}\frac{\partial I_2}{\partial\lambda_B^2}\right], \tag{9.28}$$

and $\tau_C = 0$ as a consequence of the plane stress assumption. The partial derivatives of the invariants can be evaluated as

$$\frac{\partial I_1}{\partial\lambda_A^2} = 1 - 1/\lambda_A^4\lambda_B^2, \quad \frac{\partial I_1}{\partial\lambda_B^2} = 1 - 1/\lambda_A^2\lambda_B^4 \tag{9.29}$$

$$\frac{\partial I_2}{\partial\lambda_A^2} = \lambda_B^2 - 1/\lambda_A^4, \quad \frac{\partial I_2}{\partial\lambda_B^2} = \lambda_A^2 - 1/\lambda_B^4. \tag{9.30}$$

The stresses can be expressed in global Cartesian basis by using

$$\boldsymbol{\tau} = \tau_1\boldsymbol{n}_1 \otimes \boldsymbol{n}_1^T + \tau_2\boldsymbol{n}_2 \otimes \boldsymbol{n}_2^T, \tag{9.31}$$

where $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are the eigen vectors of $\boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T$. As a recollection, the stresses obtained this way are the total stresses, $\boldsymbol{\tau} = -pJ\boldsymbol{I} + \boldsymbol{\tau}^{iso} = -pJ\boldsymbol{I} + \mathbb{P} : \bar{\boldsymbol{\tau}}$.

Stresses can be similarly computed in viscous branches where λ is replaced by λ_e, which are the eigen values of $\boldsymbol{b}_e = \boldsymbol{F}_e\boldsymbol{F}_e^T$.

Integration of evolution equation

For viscous branches, the evolution equation remains same even in the plane stress scenario. The residual can be written, similar to the equation 9.22 as

$$r_A = \epsilon_{Ae} + \frac{\Delta t}{2\eta_D}\mathrm{dev}(\tau_A) - (\epsilon_{Ae})_{tr} = 0, \tag{9.32}$$

Since plane stress condition is assumed to prevail, only the in-plane components of the above equation are considered. Also, the deviatoric part of the Kirchhoff stress can be expressed as $\mathrm{dev}(\boldsymbol{\tau}) = \boldsymbol{\tau} + p\boldsymbol{I}$, where p is the Lagrange multiplier that enforces incompressibility, which is found by using the condition that $\tau_3 = 0$. $\boldsymbol{\tau}$ can be evaluated by using the procedure in the previous section. The expression for p can be seen to be

$$\begin{aligned} p = 2\left(\frac{\partial\psi}{\partial I_1} + \frac{\partial\psi}{\partial I_2}(I_1 - \lambda_C^2)\right)\lambda_C^2 - \frac{2}{3}\frac{\partial\psi}{\partial I_1}I_1 \\ - \frac{2}{3}\frac{\partial\psi}{\partial I_2}\left[(I_1 - \lambda_A^2)\lambda_A^2 + (I_1 - \lambda_B^2)\lambda_B^2 + (I_1 - \lambda_C^2)\lambda_C^2\right], \end{aligned} \tag{9.33}$$

where $\lambda_C^2 = 1/\lambda_A^2\lambda_B^2$. The in-plane evolution equations then become

$$r_1 = \epsilon_{1e} + \frac{\Delta t}{2\eta_D}(\tau_1 + p) - (\epsilon_{1e})_{tr} = 0, \tag{9.34}$$

$$r_2 = \epsilon_{2e} + \frac{\Delta t}{2\eta_D}(\tau_2 + p) - (\epsilon_{2e})_{tr} = 0. \tag{9.35}$$

In the third direction, the evolution equation becomes $\epsilon_{3e} + \frac{\Delta t}{2\eta_D}p - (\epsilon_{3e})_{tr} = 0$. It can be shown that solving the first two equations exactly will result in the third equation being satisfied automatically. Adding the equations 9.34 and 9.35 will result in $\epsilon_{1e} + \epsilon_{2e} + \frac{\Delta t}{2\eta_D}(\tau_1 + \tau_2 + 2p) - [(\epsilon_{1e})_{tr} + (\epsilon_{2e})_{tr}] = 0$. This, in conjunction with the assumption of incompressibility and plane stress condition results in $\epsilon_{3e} + \frac{\Delta t}{2\eta_D}p - (\epsilon_{3e})_{tr} = 0$, which is the third equation.

The equations 9.34 and 9.35 are solved iteratively using Newton-Raphson technique.

$$r_A^{(k+1)} = r_A^{(k)} + \frac{\partial r_A}{\partial \epsilon_{Be}}\Delta\epsilon_{Be} = 0. \tag{9.36}$$

This in turn requires the evaluation of $K_{AB} = \dfrac{\partial r_A}{\partial \epsilon_{Be}}$. This can be evaluated as

$$K_{AB} = \frac{\partial r_A}{\partial \epsilon_{Be}} = \delta_{AB} + \frac{\Delta t}{2\eta_D}\left(\frac{\partial \tau_A}{\partial \epsilon_{Be}} + \frac{\partial p}{\partial \epsilon_{Be}}\right). \tag{9.37}$$

The pressure derivative can be computed from equation 9.33 as

$$\frac{\partial p}{\partial \epsilon_{Ae}} = 2\lambda_{Ae}^2\frac{\partial p}{\partial \lambda_{Ae}^2}. \tag{9.38}$$

$$\frac{\partial p}{\partial \lambda_{Ae}^2} = 2\lambda_{Ce}^2 \left[\frac{\partial^2 \psi}{\partial I_1 \partial I_1} \frac{\partial I_1}{\partial \lambda_{Ae}^2} + \frac{\partial^2 \psi}{\partial I_2 \partial I_1} \frac{\partial I_2}{\partial \lambda_{Ae}^2} + \frac{\partial \psi}{\partial I_2} + (I_1 - \lambda_C^2) \left(\frac{\partial^2 \psi}{\partial I_1 \partial I_2} \frac{\partial I_1}{\partial \lambda_{Ae}^2} + \frac{\partial^2 \psi}{\partial I_2 \partial I_2} \frac{\partial I_2}{\partial \lambda_{Ae}^2} \right) \right] -$$
$$2 \left[\frac{\partial \psi}{\partial I_1} + \frac{\partial \psi}{\partial I_2} (I_1 - \lambda_C^2) \right] \lambda_{Ae}^{-4} \lambda_{Be}^{-2} - \frac{2}{3} \left[\frac{\partial \psi}{\partial I_1} \frac{\partial I_1}{\partial \lambda_{Ae}^2} + I_1 \left(\frac{\partial^2 \psi}{\partial I_1 \partial I_1} \frac{\partial I_1}{\partial \lambda_{Ae}^2} + \frac{\partial^2 \psi}{\partial I_1 \partial I_2} \frac{\partial I_2}{\partial \lambda_{Ae}^2} \right) \right] -$$
$$\frac{2}{3} \left[(I_1 - \lambda_A^2)\lambda_A^2 + (I_1 - \lambda_B^2)\lambda_B^2 + (I_1 - \lambda_C^2)\lambda_C^2 \right] \left(\frac{\partial^2 \psi}{\partial I_1 \partial I_2} \frac{\partial I_1}{\partial \lambda_{Ae}^2} + \frac{\partial^2 \psi}{\partial I_2 \partial I_2} \frac{\partial I_2}{\partial \lambda_{Ae}^2} \right) -$$
$$\frac{2}{3} \frac{\partial \psi}{\partial I_2} \left[I_1 - \lambda_{Ae}^2 - \lambda_{Ae}^{-2} \lambda_{Be}^{-2} + \lambda_{Be}^2 (1 - \lambda_{Ae}^{-4} \lambda_{Be}^{-2}) + \lambda_{Ce}^2 - (I_1 - \lambda_{Ce}^2) \lambda_{Ae}^{-4} \lambda_{Be}^{-2} \right]. \quad (9.39)$$

The computation of derivative $\frac{\partial \tau_A}{\partial \epsilon_{Be}}$ can be carried out as shown in the next section.

9.4 TANGENT COMPUTATION

9.4.1 For elastic branch

The computation of tangent first involves the computation of $\mathfrak{C} = 2\frac{\partial \boldsymbol{S}}{\partial \boldsymbol{C}}$. The derivative can be computed by noting that (**Bonet2001**)

$$\dot{\boldsymbol{S}} = \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{C}} : \dot{\boldsymbol{C}}. \quad (9.40)$$

Since $\boldsymbol{C} = \sum_{i=1}^2 \lambda_i^2 \boldsymbol{N}_i \otimes \boldsymbol{N}_i$, $\dot{\boldsymbol{C}} = \sum_{i=1}^2 \left[\frac{\partial \lambda_i^2}{\partial t} \boldsymbol{N}_i \otimes \boldsymbol{N}_i + \lambda_i^2 \dot{\boldsymbol{N}}_i \otimes \boldsymbol{N}_i + \lambda_i^2 \boldsymbol{N}_i \otimes \dot{\boldsymbol{N}}_i \right]$. $\dot{\boldsymbol{N}}_i = \sum_{j=1}^2 W_{ij} \boldsymbol{N}_j$, where $W_{ij} = -W_{ji}$ are the components of a skew symmetric tensor. Hence

$$\dot{\boldsymbol{C}} = \sum_{i=1}^{2} \frac{\partial \lambda_i^2}{\partial t} \boldsymbol{N}_i \otimes \boldsymbol{N}_i + \sum_{i,j=1, i \neq j}^{2} W_{ij} (\lambda_i^2 - \lambda_j^2) \boldsymbol{N}_i \otimes \boldsymbol{N}_j. \quad (9.41)$$

As a consequence of isotropy, $\boldsymbol{S}$ and $\boldsymbol{C}$ share eigen vectors. Hence, following the same procedure,

$$\dot{\boldsymbol{S}} = \sum_{i,j=1}^{2} 2 \frac{\partial^2 \psi}{\partial \lambda_i^2 \partial \lambda_j^2} \frac{\partial \lambda_j^2}{\partial t} \boldsymbol{N}_i \otimes \boldsymbol{N}_i + \sum_{i,j=1, i \neq j}^{2} W_{ij} (S_i - S_j) \boldsymbol{N}_i \otimes \boldsymbol{N}_j. \quad (9.42)$$

The tangent can hence be written as

$$\mathfrak{C} = \sum_{i,j=1}^{2} 4 \frac{\partial^2 \psi}{\partial \lambda_i^2 \partial \lambda_j^2} \boldsymbol{N}_i \otimes \boldsymbol{N}_i \otimes \boldsymbol{N}_j \otimes \boldsymbol{N}_j + \sum_{i,j=1, i \neq j}^{2} \frac{S_i - S_j}{\lambda_i^2 - \lambda_j^2} (\boldsymbol{N}_i \otimes \boldsymbol{N}_j \otimes \boldsymbol{N}_i \otimes \boldsymbol{N}_j + \boldsymbol{N}_i \otimes \boldsymbol{N}_j \otimes \boldsymbol{N}_j \otimes \boldsymbol{N}_i). \quad (9.43)$$

Its push forward to the spatial configuration can be seen to be

$$\mathfrak{c} = \sum_{i,j=1}^{2} (C_{ij} - 2\sigma_i \delta_{ij}) \boldsymbol{n}_i \otimes \boldsymbol{n}_i \otimes \boldsymbol{n}_j \otimes \boldsymbol{n}_j$$
$$+ \sum_{i,j=1, i \neq j}^{2} \frac{\sigma_i \lambda_j^2 - \sigma_j \lambda_i^2}{\lambda_i^2 - \lambda_j^2} (\boldsymbol{n}_i \otimes \boldsymbol{n}_j \otimes \boldsymbol{n}_i \otimes \boldsymbol{n}_j + \boldsymbol{n}_i \otimes \boldsymbol{n}_j \otimes \boldsymbol{n}_j \otimes \boldsymbol{n}_i), \quad (9.44)$$

where $C_{ij} = \dfrac{\partial^2 \psi}{\partial \ln \lambda_i \partial \ln \lambda_j} = \dfrac{\partial \tau_i}{\partial \epsilon_j}$. The components of the above fourth order tensor can be stored in a matrix as

$$[\mathfrak{c}] = \begin{bmatrix} 1111 & 1122 & 1112 \\ 2211 & 2222 & 2212 \\ 1211 & 1222 & 1212 \end{bmatrix}_{\boldsymbol{n}_1,\boldsymbol{n}_2} . \tag{9.45}$$

The components of the tangent can be converted to Cartesian basis by using the transformation (**Reese1995**)

$$[\mathfrak{c}]_{(\boldsymbol{e}_1,\boldsymbol{e}_2)} = [P][\mathfrak{c}]_{(\boldsymbol{n}_1,\boldsymbol{n}_2)}[P]^T, \tag{9.46}$$

where

$$[P] = \begin{bmatrix} Q_{11}^2 & Q_{12}^2 & 2Q_{11}Q_{12} \\ Q_{21}^2 & Q_{22}^2 & 2Q_{21}Q_{22} \\ Q_{11}Q_{21} & Q_{12}Q_{22} & Q_{11}Q_{22} + Q_{12}Q_{21} \end{bmatrix}. \tag{9.47}$$

Here, Q_{ij}s are the elements of $[Q]$ matrix which is the transpose of $[\tilde{Q}]$, $[Q] = [\tilde{Q}]^T$. The columns of $[\tilde{Q}]$ matrix are the components of eigen vectors of $\boldsymbol{b}$ in cartesian basis.

The tangent to be supplied to abaqus ($\mathcal{C}^{(JK)}$) corresponds to the Jaumann rate of the Kirchhoff stress (**Nguyen2016**) written as

$$\overset{\nabla(JK)}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} + \boldsymbol{\tau}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{\tau} = \mathcal{C}^{(JK)} : \boldsymbol{d}. \tag{9.48}$$

$\mathcal{C}^{(JK)}$ is related to $\mathfrak{c}$ as

$$\mathcal{C}^{(JK)}_{ijkl} = \mathfrak{c}_{ijkl} + \frac{1}{2}(\sigma_{ij}\delta_{kl} + \sigma_{kl}\delta_{ij} + \sigma_{il}\delta_{jk} + \sigma_{jk}\delta_{il}). \tag{9.49}$$

The computation of $\mathfrak{c}$ requires the computation of $\dfrac{\partial \tau_i}{\partial \epsilon_j}$, which can be carried out as follows

$$\frac{\partial \tau_i}{\partial \epsilon_j} = 2\lambda_j^2 \frac{\partial \tau_i}{\partial \lambda_j^2}, \quad \text{i,j=1,2.} \tag{9.50}$$

$$\frac{\partial \tau_i}{\partial \lambda_j^2} = 2\left[\frac{\partial^2 \psi}{\partial \lambda_j^2 \partial I_1}\lambda_i^2 \frac{\partial I_1}{\partial \lambda_i^2} + \frac{\partial \psi}{\partial I_1}\frac{\partial}{\partial \lambda_j^2}\left(\lambda_i^2 \frac{\partial I_1}{\partial \lambda_i^2}\right) + \frac{\partial^2 \psi}{\partial \lambda_j^2 \partial I_2}\lambda_i^2 \frac{\partial I_2}{\partial \lambda_i^2} + \frac{\partial \psi}{\partial I_2}\frac{\partial}{\partial \lambda_j^2}\left(\lambda_i^2 \frac{\partial I_2}{\partial \lambda_i^2}\right)\right]. \tag{9.51}$$

The partial derivatives can be further evaluated as

$$\frac{\partial^2 \psi}{\partial \lambda_i^2 \partial I_j} = \frac{\partial^2 \psi}{\partial I_1 \partial I_j}\frac{\partial I_1}{\partial \lambda_i^2} + \frac{\partial^2 \psi}{\partial I_2 \partial I_j}\frac{\partial I_2}{\partial \lambda_i^2} \tag{9.52}$$

$$\frac{\partial}{\partial \lambda_A^2}\left(\lambda_A^2 \frac{\partial I_1}{\partial \lambda_A^2}\right) = 1 + 1/\lambda_A^4\lambda_B^2, \quad \frac{\partial}{\partial \lambda_B^2}\left(\lambda_A^2 \frac{\partial I_1}{\partial \lambda_A^2}\right) = 1/\lambda_A^2\lambda_B^4, \tag{9.53}$$

$$\frac{\partial}{\partial \lambda_A^2}\left(\lambda_A^2 \frac{\partial I_2}{\partial \lambda_A^2}\right) = \lambda_B^2 + 1/\lambda_A^4, \quad \frac{\partial}{\partial \lambda_B^2}\left(\lambda_A^2 \frac{\partial I_2}{\partial \lambda_A^2}\right) = \lambda_A^2. \tag{9.54}$$

9.4.2 For viscous branches

For the viscous branches, instead of computing $\mathfrak{C}_{IJKL} = 2\dfrac{\partial S_{IJ}}{\partial C_{KL}}$ and pushing it forward by $\boldsymbol{F}$ to obtain $\mathfrak{c}$, the following will be used (**Reese1998**).

In the elastic trial state, since the inelastic strain is held fixed, $\bar{\boldsymbol{F}}^n = \boldsymbol{F}_e^{tr}\boldsymbol{F}_i^{n-1} \implies \bar{\boldsymbol{C}}^n = (\boldsymbol{F}_i^{n-1})^T\boldsymbol{C}_e^{tr}\boldsymbol{F}_i^{n-1}$. Hence,

$$\frac{\partial S_{IJ}}{\partial C_{KL}} = \frac{\partial S_{IJ}}{\partial (C_e^{tr})_{\alpha\beta}}\frac{\partial (C_e^{tr})_{\alpha\beta}}{\partial C_{KL}} = \left((\boldsymbol{F}_i^{n-1})^{-1}\right)_{K\alpha}\left((\boldsymbol{F}_i^{n-1})^{-1}\right)_{L\beta}\frac{\partial S_{IJ}}{\partial (C_e^{tr})_{\alpha\beta}}. \tag{9.55}$$

where the symmetry of $\boldsymbol{C}_e^{tr}$ has been used. Since in viscous branches,

$$\boldsymbol{S} = \bar{\boldsymbol{F}}^{-1}\bar{\boldsymbol{\tau}}\bar{\boldsymbol{F}}^{-T} = (\boldsymbol{F}_i^{n-1})^{-1}.\underbrace{(\boldsymbol{F}_e^{tr})^{-1}\bar{\boldsymbol{\tau}}(\boldsymbol{F}_e^{tr})^{-T}}_{\tilde{\boldsymbol{S}}}.(\boldsymbol{F}_i^{n-1})^{-T}, \tag{9.56}$$

the stress derivative can be further refined as

$$\frac{\partial S_{IJ}}{\partial (C_e^{tr})_{\alpha\beta}} = \left((\boldsymbol{F}_i^{n-1})^{-1}\right)_{I\gamma}\left((\boldsymbol{F}_i^{n-1})^{-1}\right)_{J\delta}\frac{\partial \tilde{S}_{\gamma\delta}}{\partial (C_e^{tr})_{\alpha\beta}}. \tag{9.57}$$

Hence,

$$2\frac{\partial S_{IJ}}{\partial C_{KL}} = 2\left((\boldsymbol{F}_i^{n-1})^{-1}\right)_{I\gamma}\left((\boldsymbol{F}_i^{n-1})^{-1}\right)_{J\delta}\left((\boldsymbol{F}_i^{n-1})^{-1}\right)_{K\alpha}\left((\boldsymbol{F}_i^{n-1})^{-1}\right)_{L\beta}\frac{\partial \tilde{S}_{\gamma\delta}}{\partial (C_e^{tr})_{\alpha\beta}}. \tag{9.58}$$

The push-forward of above by $\boldsymbol{F}$ results in

$$\begin{aligned}\mathfrak{c}_{ijkl} &= 2(\det \boldsymbol{F})^{-1}F_{iI}F_{jJ}F_{kK}F_{lL}\frac{\partial S_{IJ}}{\partial C_{KL}}\\ &= 2(\boldsymbol{F}_e^{tr})_{i\gamma}(\boldsymbol{F}_e^{tr})_{j\delta}(\boldsymbol{F}_e^{tr})_{k\alpha}(\boldsymbol{F}_e^{tr})_{l\beta}\frac{\partial \tilde{S}_{\gamma\delta}}{\partial (C_e^{tr})_{\alpha\beta}}.\end{aligned} \tag{9.59}$$

$\tilde{\boldsymbol{S}}$, written in Eigen basis is

$$\tilde{\boldsymbol{S}} = \sum_{A=1}^{2}\frac{\tau_A}{(\lambda_{Ae})_{tr}^2}\tilde{\boldsymbol{N}}_A\otimes\tilde{\boldsymbol{N}}_A. \tag{9.60}$$

It is to be noted that the τ_A in the above equation is a function of ϵ_e. The development from here is similar to that used to arrive at the equation 9.44 except that C_{ij} in that equation will be replaced by C_{ij}^{alg}, which will be defined below. A crucial factor is that $\boldsymbol{b}_e$ and $(\boldsymbol{b}_e)_{tr}$ share the same eigen space as a consequence of isotropy.

$$\begin{aligned}\mathfrak{c} &= \sum_{i,j=1}^{2}(C_{ij}^{alg} - 2\sigma_i\delta_{ij})\boldsymbol{n}_i\otimes\boldsymbol{n}_i\otimes\boldsymbol{n}_j\otimes\boldsymbol{n}_j\\ &+ \sum_{i,j=1,i\neq j}^{2}\frac{\sigma_i(\lambda_j)_{tr}^2 - \sigma_j(\lambda_i)_{tr}^2}{(\lambda_i)_{tr}^2 - (\lambda_j)_{tr}^2}(\boldsymbol{n}_i\otimes\boldsymbol{n}_j\otimes\boldsymbol{n}_i\otimes\boldsymbol{n}_j + \boldsymbol{n}_i\otimes\boldsymbol{n}_j\otimes\boldsymbol{n}_j\otimes\boldsymbol{n}_i).\end{aligned} \tag{9.61}$$

In the above, $C_{AC}^{alg} = \dfrac{\partial \tau_A}{\partial (\epsilon_{Ce})_{tr}}$. Since τ_As are a function of ϵ_es, the derivative is computed using chain rule.

$$\frac{\partial \tau_A}{\partial (\epsilon_{Ce})_{tr}} = \frac{\partial \tau_A}{\partial \epsilon_{Be}}\frac{\partial \epsilon_{Be}}{\partial (\epsilon_{Ce})_{tr}}. \tag{9.62}$$

The derivative $\dfrac{\partial \epsilon_{Be}}{\partial (\epsilon_{Ce})_{tr}}$ can be computed by realizing that the equations $r_B = 0$ are satisfied at the end of Newton iterations. Hence, $\dfrac{\partial r_B}{\partial (\epsilon_{Ce})_{tr}} = 0$ as well. Hence,

$$\frac{\partial \epsilon_{Be}}{\partial (\epsilon_{Ce})_{tr}} = K_{BC}^{-1}, \tag{9.63}$$

where K_{BC} is defined in equation 9.37. The expression for dissipation becomes

$$\mathcal{D} = \frac{1}{\eta_D} \mathrm{dev}\{\boldsymbol{\tau}_{NEQ}\} : \mathrm{dev}\{\boldsymbol{\tau}_{NEQ}\} = (\boldsymbol{\tau} + p\boldsymbol{I}) : (\boldsymbol{\tau} + p\boldsymbol{I}) + p^2, \tag{9.64}$$

which can be seen to be positive since $\eta_D > 0$. In the above equation, $\boldsymbol{\tau}$ is as evaluated in the equation 9.31.