CMPT 210: Probability and Computing

Lecture 23

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April 4, 2024

Comparing the Bounds

For r.v's $T_1, T_2, \ldots T_n$, if $T_i \in \{0, 1\}$ and $\Pr[T_i = 1] = p_i$. Define $T := \sum_{i=1}^{n} T_i$. By linearity of expectation, $\mathbb{E}[T] = \sum_{i=1}^{n} p_i$. For $c \geq 1$,

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$$\implies \Pr[T - \mathbb{E}[T] \geq (c-1)\mathbb{E}[T]] \leq \frac{\mathrm{Var}[T]}{(c-1)^2\,(\mathbb{E}[T])^2)} \qquad (x = (c-1)\,\mathbb{E}[T])$$

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If the T_i's are pairwise independent, by linearity of variance, $\mathrm{Var}[T] = \sum_{i=1}^{n} p_i\,(1-p_i)$. Hence, $\Pr[T \geq c\mathbb{E}[T]] \leq \frac{\sum_{i=1}^{n} p_i\,(1-p_i)}{(c-1)^2\left(\sum_{i=1}^{n} p_i\right)^2}$. If for all i, $p_i = 1/2$, then, $\Pr[T \geq c\mathbb{E}[T]] \leq \frac{1}{(c-1)^2\,n}$.

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Chernoff Bound: If T_i' are mutually independent, then,
$\Pr[T \geq c\mathbb{E}[T]] \leq \exp(-\beta(c)\,\mathbb{E}[T]) = \exp\left(-(c\ln(c) - c + 1)\,\left(\sum_{i=1}^{n} p_i\right)\right)$. If for all i, $p_i = 1/2$, $\Pr[T \geq c\mathbb{E}[T]] \leq \exp\left(-\frac{n\,(c\ln(c) - c + 1)}{2}\right)$.

Chernoff Bound – Lottery Game

Q: Pick-4 is a lottery game in which you pay \$1 to pick a 4-digit number between 0000 and 9999. If your number comes up in a random drawing, then you win \$5,000. Your chance of winning is 1 in 10000. If 10 million people play, then the expected number of winners is 1000. When there are 1000 winners, the lottery keeps \$5 million of the \$10 million paid for tickets. The lottery operator's nightmare is that the number of winners is much greater – especially at the point where more than 2000 win and the lottery must pay out more than it received. What is the probability that will happen? (Assume that the players' picks and the winning number are random, independent and uniform)

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Let T_i be an indicator for the event that player i wins. Then $T := \sum_{i=1}^{n} T_i$ is the total number of winners. Using the independence assumptions, we can conclude that T_i are independent, as required by the Chernoff bound.

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We wish to compute $\Pr[T \geq 2000] = \Pr[T \geq 2\mathbb{E}[T]]$. Hence $c = 2$ and $\beta(c) \approx 0.386$. By the Chernoff bound,

$$\Pr[T \geq 2\mathbb{E}[T]] \leq \exp(-\beta(c)\mathbb{E}[T]) = \exp(-(0.386)\,1000) < \exp(-386) \approx 10^{-168}$$

Questions?

Chernoff Bound – Proof

Chernoff Bound: Let $T_1, T_2, \ldots, T_n$ be mutually independent r.v's such that $0 \leq T_i \leq 1$ for all i. If $T := \sum_{i=1}^{n} T_i$, for all $c \geq 1$ and $\beta(c) := c\ln(c) - c + 1$,

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$$\mathbb{E}[c^{T_i}] = \sum_{v \in \mathrm{Range}(T_i)} c^v \Pr[T_i = v] \leq \sum_{v \in \mathrm{Range}(T_i)} (1 + (c-1)v) \Pr[T_i = v]$$

(Since $T_i \in [0,1]$ and $c^v \leq 1 + (c-1)v$ for all $v \in [0,1]$.)

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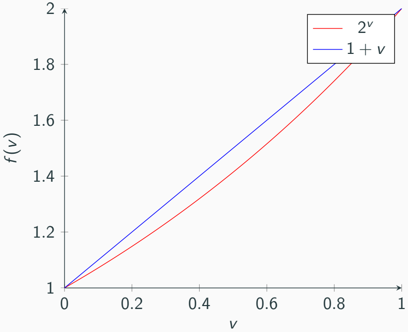
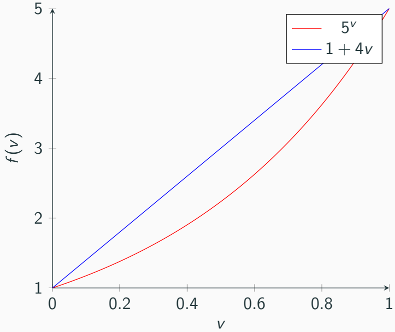
For $c = 2$ and $c = 5$,





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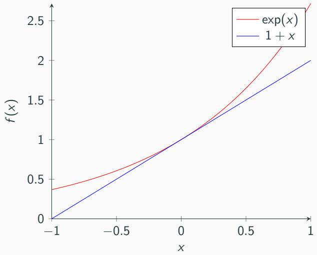

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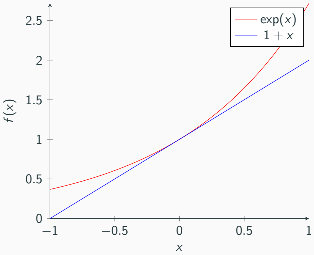

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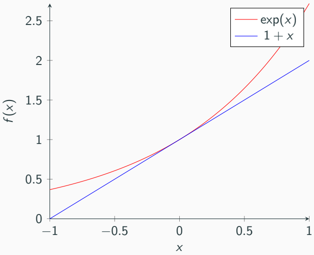
&= 1 + (c-1)\,\mathbb{E}[T_i] \leq \exp((c-1)\mathbb{E}[T_i]) \qquad \text{(Since } 1 + x \leq \exp(x) \text{ for all } x\text{)} \\
\implies \mathbb{E}[c^{T_i}] &\leq \exp((c-1)\,\mathbb{E}[T_i])
\end{aligned}
$$



Hence we have proved the Chernoff Bound!

Questions?

Randomized Load Balancing

Fussbook is a new social networking site oriented toward unpleasant people. Like all major web services, Fussbook has a load balancing problem: it receives lots of forum posts that computer servers have to process. If any server is assigned more work than it can complete in a given interval, then it is overloaded and system performance suffers. That would be bad because Fussbook users are not a tolerant bunch.

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Fussbook receives 24000 forum posts in every 10-minute interval. Each post is assigned to one of several servers for processing, and each server works sequentially through its assigned tasks. It takes a server an average of $1/4$ second to process a post. No post takes more than 1 second.

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This implies that a server could be overloaded when it is assigned more than 600 units of work in a 10-minute interval. On average, for $24000 \times \frac{1}{4} = 6000$ units of work in a 10-minute interval, Fussbook requires at least 10 servers to ensure that no server is overloaded (with perfect load-balancing).

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Q: There might be random fluctuations in the load or the load-balancing is not perfect. How many servers does Fussbook need to ensure that their servers are not overloaded with high-probability?

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Let m be the number of servers that Fussbook needs to use. Recall that a server may be overloaded if the load it is assigned exceeds 600 units. Let us first look at server 1 and define T to be the r.v. corresponding to the number of units of work assigned to the first server.

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Let T_i be the number of seconds server 1 spends on processing post i. $T_i = 0$ if the task is assigned to a different (not the first server). The maximum amount of time spent on post i is 1-second. Hence, $T_i \in [0, 1]$.

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Since there are $n := 24000$ posts in every 10-minute interval, the load (amount of units) assigned to the first server is equal to $T = \sum_{i=1}^{n} T_i$. Server 1 may be overloaded if $T \geq 600$, and hence we want to upper-bound the probability $\Pr[T \geq 600]$.

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Since the assignment of a post to a server is independent of the time required to process the post, the T_i r.v's are mutually independent. Hence, we can use the Chernoff bound.

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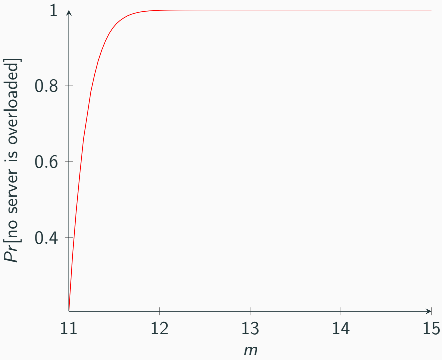
Randomized Load Balancing

Plotting Pr[no server is overloaded] as a function of m.



Hence, as $m \geq 12$, the probability that no server gets overloaded tends to 1 and hence none of the Fussbook servers crash!

Questions?