CMPT 210: Probability and Computing

Lecture 20

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Recap

Variance: Standard way to measure the deviation from the mean. For r.v. X, $\text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_{x \in \text{Range}(X)} (x - \mu)^2 \Pr[X = x]$, where $\mu := \mathbb{E}[X]$.

Alternate Definition: $\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.

Standard Deviation: For r.v. X, the standard deviation of X is defined as $\sigma_X := \sqrt{\text{Var}[X]} = \sqrt{\mathbb{E}[X^2] - (\mathbb{E}[X])^2}$.

For constants a, b and r.v. R, $\text{Var}[aR + b] = a^2 \text{Var}[R]$.

Pairwise Independence: Random variables $R_1, R_2, R_3, \ldots R_n$ are pairwise independent if for any pair R_i and R_j, for $x \in \text{Range}(R_i)$ and $y \in \text{Range}(R_j)$, $\Pr[(R_i = x) \cap (R_j = y)] = \Pr[R_i = x] \Pr[R_j = y]$.

Linearity of variance for pairwise independent r.v's: If $R_1, \ldots, R_n$ are pairwise independent, $\text{Var}[R_1 + R_2 + \ldots R_n] = \sum_{i=1}^{n} \text{Var}[R_i]$.

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\mathrm{Cov}[R,S] &= \mathbb{E}[(R-\mathbb{E}[R])(S-\mathbb{E}[S])] \\
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&= \mathbb{E}[RS] - \mathbb{E}[R\,\mathbb{E}[S]] - \mathbb{E}[S\,\mathbb{E}[R]] + \mathbb{E}[R]\,\mathbb{E}[S] \\
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The covariance between two r.v's is symmetric i.e. $\mathrm{Cov}[R, S] = \mathrm{Cov}[S, R]$.

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Recall from Lecture 19, Slide 6, where we showed that,

$$\operatorname{Var}[R+S] = \operatorname{Var}[R] + \operatorname{Var}[S] + 2(\mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]) = \operatorname{Var}[R] + \operatorname{Var}[S] + 2\operatorname{Cov}[R,S].$$

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Generalization to multiple random variables $R_1, R_2, \ldots R_n$ (Recall from Lecture 19, Slide 7):

$$\mathrm{Var}\left[\sum_{i=1}^{n} R_i\right] = \sum_{i=1}^{n} \mathrm{Var}[R_i] + 2 \sum_{1 \le i < j \le n} \mathrm{Cov}[R_i, R_j]$$

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We know that $\text{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$. Note that $X = \mathcal{I}_A$ and $Y = \mathcal{I}_B$. We can conclude that $XY = \mathcal{I}_{A \cap B}$ since $XY = 1$ iff both events A and B happen.

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$$\implies \mathbb{E}[X] = \Pr[A]\ ;\ \mathbb{E}[Y] = \Pr[B];\ \mathbb{E}[XY] = \Pr[A \cap B]$$

$$\implies \text{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] = \Pr[A \cap B] - \Pr[A]\Pr[B]$$

If $\text{Cov}[X, Y] > 0 \implies \Pr[A \cap B] > \Pr[A]\Pr[B]$. Hence,

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} > \frac{\Pr[A]\Pr[B]}{\Pr[B]} = \Pr[A]$$

If $Cov[X, Y] > 0$, it implies that $\Pr[A|B] > \Pr[A]$ and hence, the probability that event A happens increases if B is going to happen/has happened. Similarly, if $Cov[X, Y] < 0$, $\Pr[A|B] < \Pr[A]$. In this case, if B happens, then the probability of event A decreases.

Correlation

The correlation between two r.v's R_1 and R_2 is defined as:

$$\mathrm{Corr}[R_1, R_2] = \frac{\mathrm{Cov}[R_1, R_2]}{\sqrt{\mathrm{Var}[R_1]\,\mathrm{Var}[R_2]}}$$

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If $R_1 = R_2 = R$, then, $\text{Corr}[R, R] = \frac{\text{Cov}[R,R]}{\sqrt{\text{Var}[R]\,\text{Var}[R]}} = \frac{\text{Var}[R]}{\text{Var}[R]} = 1$.

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If R_1 and R_2 are independent, $\operatorname{Cov}[R_1, R_2] = 0$ and $\operatorname{Corr}[R_1, R_2] = 0$.

If $R_1 = -R_2 = R$, then,

$$\begin{aligned}
\operatorname{Corr}[R, -R] &= \frac{\operatorname{Cov}[R, -R]}{\sqrt{\operatorname{Var}[R]\operatorname{Var}[-R]}} = \frac{\operatorname{Cov}[R, -R]}{\sqrt{\operatorname{Var}[R]\,(-1)^2\operatorname{Var}[R]}} = \frac{\operatorname{Cov}[R, -R]}{\operatorname{Var}[R]} \\
&= \frac{\mathbb{E}[-R^2] - \mathbb{E}[R]\,\mathbb{E}[-R]}{\operatorname{Var}[R]} = \frac{-\mathbb{E}[R^2] + \mathbb{E}[R]\,\mathbb{E}[R]}{\operatorname{Var}[R]} = \frac{-\operatorname{Var}[R]}{\operatorname{Var}[R]} = -1
\end{aligned}$$

Questions?

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$$\begin{aligned}
\textit{Proof}: \mathbb{E}[X] &= \sum_{x \in \text{Range}(X)} x \Pr[X = x] = \sum_{x | x \geq 300} x \Pr[X = x] + \sum_{x | 0 \leq x < 300} x \Pr[X = x] \\
&\geq \sum_{x | x \geq 300} (300) \Pr[X = x] + \sum_{x | 0 \leq x < 300} x \Pr[X = x] \\
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\end{aligned}$$

If $\Pr[X \geq 300] > \frac{1}{3}$, then, $\mathbb{E}[X] > (300)\,\frac{1}{3} + \sum_{x|0 \leq x < 300} x \Pr[X = x] > 100$ (since the second term is always non-negative). Hence, if $\Pr[X \geq 300] > \frac{1}{3}$, $\mathbb{E}[X] > 100$ which is a contradiction since $\mathbb{E}[X] = 99.99$.

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Markov's Theorem: If X is a non-negative random variable, then for all $x > 0$,

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$$\implies \Pr[X \geq x] \leq \frac{\mathbb{E}[X]}{x}.$$

Since the above theorem holds for all $x > 0$, we can set $x = c\mathbb{E}[X]$ for $c \geq 1$. In this case, $\Pr[X \geq c\mathbb{E}[X]] \leq \frac{1}{c}$. Hence, the probability that X is "far" from the mean in terms of the multiplicative factor c is upper-bounded by $\frac{1}{c}$.

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Recall that if G is the r.v. corresponding to the number of people that receive their own coat, then we used the linearity of expectation to derive that $\mathbb{E}[G] = 1$. Using Markov's Theorem,

$$\Pr[G \geq x] \leq \frac{\mathbb{E}[G]}{x} = \frac{1}{x}.$$

Hence, we can bound the probability that x people receive their own coat. For example, there is no better than 20% chance that more than 5 people get their own coat.

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Define $Y := X - 100$. $\mathbb{E}[Y] = \mathbb{E}[X] - 100 = 50$ and Y is non-negative.

$$\Pr[X \geq 200] = \Pr[Y + 100 \geq 200] = \Pr[Y \geq 100] \leq \frac{\mathbb{E}[Y]}{100} = \frac{50}{100} = \frac{1}{2}$$

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Hence, if we have additional information (in the form of a lower-bound that a r.v. can not be smaller than some constant $b > 0$), we can use Markov's Theorem on the shifted r.v. (Y in our example) and obtain a tighter bound on the probability of deviation.