

CMPT 409/981: Optimization for Machine Learning

Lecture 6

Sharan Vaswani

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Recap

Gradient Descent: $w_{k+1} = w_k - \eta \nabla f(w_k)$.

Nesterov Acceleration: $w_{k+1} = [w_k + \beta_k(w_k - w_{k-1})] - \eta \nabla f(w_k + \beta_k(w_k - w_{k-1}))$.

Nesterov acceleration can be interpreted as doing GD on “extrapolated” points where β_k can be interpreted as the “momentum” in the previous direction ($w_k - w_{k-1}$).

Function class	L -smooth	L -smooth + convex	L -smooth + μ -strongly convex
Gradient Descent	$\Theta(1/\epsilon)$	$O(1/\epsilon)$	$O(\exp(-T/\kappa))$
Nesterov Acceleration	-	$\Theta(1/\sqrt{\epsilon})$	$\Theta(\exp(-T/\sqrt{\kappa}))$

Table 1: Optimization Zoo

For all cases, $\eta = \frac{1}{L}$ for both GD and Nesterov acceleration, and we can use Armijo line-search to estimate L and set the step-size.

Gradient Descent is adaptive to strong-convexity, however, Nesterov acceleration requires knowledge of μ to set β_k .

Heavy-Ball Momentum

Heavy-Ball/Polyak Momentum: $w_{k+1} = w_k - \eta \nabla f(w_k) + \beta_k(w_k - w_{k-1})$.

Nesterov Acceleration: $v_k = w_k + \beta_k(w_k - w_{k-1})$; $w_{k+1} = v_k - \eta \nabla f(v_k)$ i.e. extrapolate and compute the gradient at the extrapolated point v_k .

Polyak Momentum: $v_k = w_k + \beta_k(w_k - w_{k-1})$; $w_{k+1} = v_k - \eta \nabla f(w_k)$ i.e. compute the gradient at w_k and then extrapolate.

Unlike GD, Nesterov acceleration and Polyak momentum are not “descent” methods i.e. it is not guaranteed that $f(w_{k+1}) \leq f(w_k)$ for all k .

In order to minimize quadratics: $f(w) = \frac{1}{2}w^T A w - b w + c$ where A is symmetric, positive semi-definite, or equivalently solve linear systems of the form: $A w = b$, using Polyak momentum with *optimal* values of (η, β) is equivalent to Conjugate Gradient.

Brief History: For L -smooth + μ -strongly convex functions,

- *Quadratics*: HB momentum with a specific (η, β) can achieve the accelerated rate and obtain a dependence on $\sqrt{\kappa}$ (only an asymptotic rate). [Polyak, 1964]
- *General smooth, SC functions*: Using Polyak's (η, β) parameters can result in cycling and HB momentum is not guaranteed to converge. [Lessard et al, 2014]
- *General smooth, SC functions*: Using a different (η, β) , HB momentum can converge and match the GD rate (no acceleration). [Ghadimi et al, 2014]
- *General smooth, SC functions + Lipschitz-continuity of Hessian*: Using a different (η, β) , HB momentum matches the GD rate at the beginning, but achieves the accelerated rate after $O(\kappa)$ iterations. [Wang et al, 2022]

Heavy-Ball Momentum

Let us focus on minimizing quadratics: $f(w) = \frac{1}{2}w^\top Aw - bw + c$, where A is a symmetric positive definite matrix.

Claim: For L -smooth, μ -strongly convex quadratics, HB momentum with $\eta = \frac{4}{(\sqrt{L} + \sqrt{\mu})^2}$ and $\beta = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$ achieves the following convergence rate:

$$\|w_T - w^*\| \leq \sqrt{2} \left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1} + \epsilon_T \right)^T \|w_0 - w^*\|$$

where $\epsilon_T \geq 0$ and $\lim_{T \rightarrow \infty} \epsilon_T = 0$.

HB momentum can also achieve a slightly-worse, but still accelerated non-asymptotic rate [Wang et al, 2021].

$$\|w_T - w^*\| \leq 4\sqrt{\kappa} \left(1 - \frac{1}{2\sqrt{\kappa}} \right)^T \|w_0 - w^*\|$$

Questions?

Minimizing strongly-convex quadratics with GD

As a warm-up, let us first prove the optimal GD rate for smooth, strongly-convex quadratics.

Claim: For L -smooth, μ -strongly convex quadratics, GD with $\eta = \frac{2}{\mu+L}$ achieves the following convergence rate:

$$\|w_T - w^*\| \leq \left(\frac{\kappa - 1}{\kappa + 1} \right)^T \|w_0 - w^*\|$$

Proof: For quadratics, $\nabla f(w) = Aw - b$,

$$w_{k+1} = w_k - \eta \nabla f(w_k) = w_k - \eta[Aw_k - b]$$

$$\implies \|w_{k+1} - w^*\| = \|w_k - w^* - \eta[Aw_k - b]\|$$

$$= \|w_k - w^* - \eta[Aw_k - Aw^*]\| \quad (\text{Since } \nabla f(w^*) = 0 \implies Aw^* = b)$$

$$\implies \|w_{k+1} - w^*\| = \|(I_d - \eta A)(w_k - w^*)\| \leq \|I_d - \eta A\|_2 \|w_k - w^*\|$$

(By definition of the matrix norm: for matrix B , $\|B\|_2 = \max \left\{ \frac{\|Bv\|_2}{\|v\|_2} \right\}$ for all vectors $v \neq 0$, and)

We have thus reduced the problem to bounding $\|I_d - \eta A\|_2$.

Minimizing strongly-convex quadratics with GD

Recall that $\|w_{k+1} - w^*\| = \|I_d - \eta A\|_2 \|w_k - w^*\|$. Since f is L -smooth and μ -strongly convex, $\mu I_d \preceq \nabla^2 f(w) = A \preceq L I_d$.

If $A = U \Lambda U^\top$ is the eigen-decomposition of A , and $\lambda_1, \lambda_2, \dots, \lambda_d$ are the eigenvalues of A , then, $I_d - \eta A = U S U^\top$ where $S_{i,i} = 1 - \eta \lambda_i$.

Since U is an orthonormal matrix, $\|I_d - \eta A\| = \|S\|$. By definition of the matrix norm, for symmetric matrices,

$$\|B\|_2 = \rho(B) := \max\{|\lambda_1[B]|, |\lambda_2[B]|, \dots, |\lambda_d[B]|\}$$

where $\rho(B)$ is the spectral radius of B .

Hence,

$$\|I_d - \eta A\| = \|S\| = \rho(S) = \max\{|\lambda_1[S]|, |\lambda_2[S]|, \dots, |\lambda_d[S]|\} = \max_{\lambda \in [\mu, L]} \{ |1 - \eta \lambda| \}$$

$$\|I_d - \eta A\| = \max\{|1 - \eta \mu|, |1 - \eta L|\} \quad (\text{Since } 1 - \eta \lambda \text{ is linear in } \lambda)$$

Minimizing strongly-convex quadratics with GD

Recall that $\|w_{k+1} - w^*\| = \|I_d - \eta A\| \|w_k - w^*\|$ and $\|I_d - \eta A\| = \max\{|1 - \eta\mu|, |1 - \eta L|\}$.

Let us choose a step-size $\eta \in \left[\frac{1}{L}, \frac{1}{\mu}\right]$. Hence,

$$\|I_d - \eta A\| \leq \max\{1 - \eta\mu, \eta L - 1\} = \frac{L - \mu}{L + \mu}$$

(By setting $\eta = \frac{2}{\mu + L}$, we minimize $\max\{1 - \eta\mu, \eta L - 1\}$)

Putting everything together,

$$\|w_{k+1} - w^*\| \leq \frac{L - \mu}{L + \mu} \|w_k - w^*\| = \frac{\kappa - 1}{\kappa + 1} \|w_k - w^*\|$$

Recurring from $k = 0$ to $T - 1$,

$$\|w_T - w^*\| \leq \left(\frac{\kappa - 1}{\kappa + 1}\right)^T \|w_0 - w^*\|.$$

Questions?

Minimizing strongly-convex quadratics with HB momentum

Update: $w_{k+1} = w_k - \eta \nabla f(w_k) + \beta(w_k - w_{k-1})$

Claim: For L -smooth, μ -strongly convex quadratics, HB momentum with $\eta = \frac{4}{(\sqrt{L} + \sqrt{\mu})^2}$ and $\beta = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$ achieves the following convergence rate:

$$\|w_T - w^*\| \leq \sqrt{2} \left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1} + \epsilon_T \right)^T \|w_0 - w^*\|, \text{ where, } \lim_{T \rightarrow \infty} \epsilon_T \rightarrow 0.$$

Proof:

$$\begin{aligned} \begin{bmatrix} w_{k+1} - w^* \\ w_k - w^* \end{bmatrix} &= \begin{bmatrix} w_k - w^* - \eta \nabla f(w_k) + \beta(w_k - w_{k-1}) \\ w_k - w^* \end{bmatrix} \\ &= \begin{bmatrix} w_k - w^* - \eta A(w_k - w^*) + \beta(w_k - w^*) - \beta(w_{k-1} - w^*) \\ w_k - w^* \end{bmatrix} \end{aligned}$$

(Since $\nabla f(w) = Aw$, $Aw^* = b$)

$$\Rightarrow \begin{bmatrix} w_{k+1} - w^* \\ w_k - w^* \end{bmatrix} = \begin{bmatrix} (1 + \beta)I_d - \eta A & -\beta I_d \\ I_d & 0 \end{bmatrix} \begin{bmatrix} w_k - w^* \\ w_{k-1} - w^* \end{bmatrix}$$

If $\beta = 0$, we can recover the same equation as GD.

Minimizing strongly-convex quadratics with HB momentum

$$\underbrace{\begin{bmatrix} w_{k+1} - w^* \\ w_k - w^* \end{bmatrix}}_{:= \Delta_{k+1} \in \mathbb{R}^{2d}} = \underbrace{\begin{bmatrix} (1 + \beta)I_d - \eta A & -\beta I_d \\ I_d & 0 \end{bmatrix}}_{:= \mathcal{H} \in \mathbb{R}^{2d \times 2d}} \underbrace{\begin{bmatrix} w_k - w^* \\ w_{k-1} - w^* \end{bmatrix}}_{:= \Delta_k \in \mathbb{R}^{2d}} \implies \Delta_{k+1} = \mathcal{H} \Delta_k$$

Recurring from $k = 0$ to $T - 1$, and taking norm,

$$\|\Delta_T\| = \|\mathcal{H}^T \Delta_0\| \leq \|\mathcal{H}^T\| \left\| \begin{bmatrix} w_0 - w^* \\ w_{-1} - w^* \end{bmatrix} \right\| \quad (\text{By definition of the matrix norm})$$

Define $w_{-1} = w_0$ and lower-bounding the LHS,

$$\|w_T - w^*\| \leq \sqrt{2} \|\mathcal{H}^T\| \|w_0 - w^*\|$$

Hence, we have reduced the problem to bounding $\|\mathcal{H}^T\|$.

Minimizing strongly-convex quadratics with HB momentum

Recall that for symmetric matrices, $\|B\|_2 = \rho(B)$. Unfortunately, this relation is not true for general asymmetric matrices, and $\|B\| \geq \rho(B)$.

Gelfand's Formula: For a matrix $B \in \mathbb{R}^{d \times d}$ such that $\rho(B) := \max_{i \in [d]} |\lambda_i|$, then there exists a sequence $\epsilon_k \geq 0$ such that $\lim_{k \rightarrow \infty} \epsilon_k = 0$ and,

$$\|B^k\| \leq (\rho(B) + \epsilon_k)^k.$$

Using this formula with our bound,

$$\|w_T - w^*\| \leq (\rho(\mathcal{H}) + \epsilon_T)^T \|w_0 - w^*\|$$

Hence, we have reduced the problem to bounding $\rho(\mathcal{H})$.

Minimizing strongly-convex quadratics with HB momentum

Similar to the GD case, let $A = U\Lambda U^\top$ be the eigen-decomposition of A , then, $(1 + \beta)I_d - \eta A = USU^\top$ where $S_{i,i} = 1 + \beta - \eta\lambda_i$. Hence,

$$\mathcal{H} = \begin{bmatrix} U^\top & 0 \\ 0 & U^\top \end{bmatrix} \underbrace{\begin{bmatrix} (1 + \beta)I_d - \eta\Lambda & -\beta I_d \\ I_d & 0 \end{bmatrix}}_{:=H} \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}$$

Since U is orthonormal, $\rho(\mathcal{H}) = \rho(H)$. Hence we have reduced the problem to bounding $\rho(H)$.

Minimizing strongly-convex quadratics with HB momentum

Let P be a permutation matrix such that:

$$P_{i,j} = \begin{cases} 1 & i \text{ is odd, } j = i \\ 1 & i \text{ is even, } j = 2d + i \\ 0 & \text{otherwise} \end{cases} \quad B = P H P^\top = \begin{bmatrix} H_1 & 0 & \dots & 0 \\ 0 & H_2 & \dots & 0 \\ \vdots & \ddots & & \\ 0 & & 0 & H_d \end{bmatrix}$$

where,

$$H_i = \begin{bmatrix} (1 + \beta) - \eta\lambda_i & -\beta \\ 1 & 0 \end{bmatrix}$$

Note that $\rho(H) = \rho(B)$ (a permutation matrix does not change the eigenvalues). Since B is a block diagonal matrix, $\rho(B) = \max_i [\rho(H_i)]$. Hence we have reduced the problem to bounding $\rho(H_i)$.

Minimizing strongly-convex quadratics with HB momentum

For a fixed $i \in [2d]$, let us compute the eigenvalues of $H_i \in \mathbb{R}^{2 \times 2}$ by solving the characteristic polynomial: $\det(H_i - uI_2) = 0$ w.r.t u .

$$u^2 - (1 + \beta - \eta\lambda_i)u + \beta = 0 \implies u = \frac{1}{2} \left[(1 + \beta - \eta\lambda_i) \pm \sqrt{(1 + \beta - \eta\lambda_i)^2 - 4\beta} \right]$$

Let us set β such that, $(1 + \beta - \eta\lambda_i)^2 \leq 4\beta$. This ensures that the roots to the above equation are complex conjugates. Hence,

$$1 + \beta - \eta\lambda_i \geq -2\sqrt{\beta} \implies (\sqrt{\beta} + 1) \geq \sqrt{\eta\lambda_i} \implies \beta \geq (1 - \sqrt{\eta\lambda_i})^2$$

If we ensure that $\beta \geq (1 - \sqrt{\eta\lambda_i})^2$

$$\begin{aligned} u &= \frac{1}{2} \left[(1 + \beta - \eta\lambda_i) \pm i\sqrt{4\beta - (1 + \beta - \eta\lambda_i)^2} \right] \\ \implies |u|^2 &= \frac{1}{4} \left[(1 + \beta - \eta\lambda_i)^2 + 4\beta - (1 + \beta - \eta\lambda_i)^2 \right] = \beta \implies |u| = \sqrt{\beta}. \end{aligned}$$

Hence, if $\beta \geq (1 - \sqrt{\eta\lambda_i})^2$, $\rho(H_i) = \sqrt{\beta}$ and $\rho(B) = \max_i [\rho(H_i)] = \sqrt{\beta}$.

Minimizing strongly-convex quadratics with HB momentum

Using the result from the previous slide, if we ensure that for all i , $\beta \geq (1 - \sqrt{\eta\lambda_i})^2$, then, $\rho(B) = \sqrt{\beta}$. Hence, we want that,

$$\beta = \max_i \{(1 - \sqrt{\eta\lambda_i})^2\} = \max_{\lambda \in [\mu, L]} \{(1 - \sqrt{\eta\lambda})^2\} = \max\{(1 - \sqrt{\eta\mu})^2, (1 - \sqrt{\eta L})^2\}$$

Similar to GD, we equate the two terms in the max,

$$1 + \eta\mu - 2\sqrt{\eta\mu} = 1 + \eta L - 2\sqrt{\eta L} \implies \eta = \frac{4}{(\sqrt{L} + \sqrt{\mu})^2}.$$

With this value of η , $\rho(\mathcal{H}) = \rho(H) = \rho(B) \leq \sqrt{\beta} = \sqrt{\left(1 - \frac{2\sqrt{\mu}}{(\sqrt{L} + \sqrt{\mu})}\right)^2} = \frac{\sqrt{L} - \sqrt{\mu}}{\sqrt{L} + \sqrt{\mu}} = \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}$.

Putting everything together,

$$\|w_T - w^*\| \leq \sqrt{2} \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} + \epsilon_T \right)^T \|w_0 - w^*\|$$

Questions?