CMPT 210: Probability and Computing

Lecture 19

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March 21, 2024

Recap

Variance: Standard way to measure the deviation from the mean. For r.v. X, $\mathrm{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_{x \in \mathrm{Range}(X)} (x - \mu)^2 \Pr[X = x]$, where $\mu := \mathbb{E}[X]$.

Alternate Definition: $\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.

If $X \sim \mathrm{Ber}(p)$, $\mathrm{Var}[X] = p(1 - p)$.

If $X \sim \mathrm{Uniform}(\{v_1, v_2, \ldots v_n\})$, $\mathrm{Var}[X] = \frac{[v_1^2 + v_2^2 + \ldots v_n^2]}{n} - \left(\frac{[v_1 + v_2 + \ldots v_n]}{n}\right)^2$.

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Recall that for a coin s.t. $\Pr[\text{heads}] = p$, R is the r.v. equal to the number of coin tosses we need to get the first heads. Let A be the event that we get a heads in the first toss. Using the law of total expectation,

$$\mathbb{E}[R^2] = \mathbb{E}[R^2|A] \Pr[A] + \mathbb{E}[R^2|A^c] \Pr[A^c]$$

$\mathbb{E}[R^2|A] = 1$ ($R^2 = 1$ if we get a heads in the first coin toss) and $\Pr[A] = p$. Hence,

$$\mathbb{E}[R^2] = (1)\,(p) + \mathbb{E}[R^2|A^c]\,(1-p) \quad ; \quad \mathbb{E}[R^2|A^c] = \sum_{k=1} k^2 \Pr[R = k|A^c]$$

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Note that $\Pr[R = k|A^c] = \Pr[R = k|\text{ first toss is a tails}] = (1-p)^{k-2}\,p = \Pr[R = k-1]$

$$\implies \mathbb{E}[R^2|A^c] = \sum_{k=1} k^2 \Pr[R = k-1] = \sum_{t=0} (t+1)^2 \Pr[R = t] \qquad (t := k-1)$$

Variance - Examples

Continuing from the previous slide,

$$\mathbb{E}[R^2|A^c] = \sum_{t=0}(t+1)^2 \Pr[R=t] = \sum_{t=0} t^2 \Pr[R=t] + 2\sum_{t=0} t \Pr[R=t] + \sum_{t=0} \Pr[R=t]$$

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Putting everything together,

$$\mathbb{E}[R^2] = (1)\,(p) + (\mathbb{E}[R^2] + 2\mathbb{E}[R] + 1])\,(1-p) \implies p\,\mathbb{E}[R^2] = p + 2(1-p)\mathbb{E}[R] + (1-p)\mathbb{E}[1]$$
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$$\implies \mathrm{Var}[R] = \mathbb{E}[R^2] - (\mathbb{E}[R])^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

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Standard Deviation

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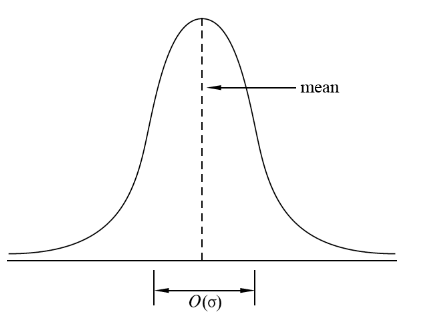

Standard deviation for a "bell"-shaped distribution indicates how wide the "main part" of the distribution is.

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&= (a^2\mathbb{E}[R^2] + 2ab\mathbb{E}[R] + b^2) - (a^2(\mathbb{E}[R])^2 + 2ab\mathbb{E}[R] + b^2) \\
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Similarly, for the standard deviation,

$$\sigma_{aR+b} = \sqrt{\text{Var}[aR+b]} = \sqrt{a^2\text{Var}[R]} = |a|\,\sigma_R$$

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Note the difference from the property of expectation,

$$\mathbb{E}[aR + b] = a\mathbb{E}[R] + b$$

Properties of Variance

Recall that for r.v's R and S, $\mathbb{E}[R+S] = \mathbb{E}[R] + \mathbb{E}[S]$. In general, such a property is not true for the variance, i.e. variance of a sum is not necessarily equal to the sum of the variances.

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Pairwise Independence: Random variables $R_1, R_2, R_3, \ldots R_n$ are *pairwise* independent if for any pair R_i and R_j, for $x \in \text{Range}(R_i)$ and $y \in \text{Range}(R_j)$, events $\Pr[R_i = x]$ and $\Pr[R_j = y]$ are pairwise independent implying that $\Pr[(R_i = x) \cap (R_j = y)] = \Pr[R_i = x] \Pr[R_j = y]$.

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\textit{Proof}: \text{Var}[R_1 + R_2 + \ldots R_n] &= \mathbb{E}[(R_1 + R_2 + \ldots R_n)^2] - (\mathbb{E}[R_1 + R_2 + \ldots R_n])^2 \\
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Importantly, we do not require the r.v's to be mutually independent. Mutual independence $\implies$ pairwise independence, but pairwise independence $\not\Rightarrow$ mutual independence.

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Hence,

$$R = R_1 + R_2 + \ldots + R_n \implies \text{Var}[R] = \text{Var}[R_1 + R_2 + \ldots + R_n]$$

Since $R_1, R_2, \ldots, R_n$ are mutually independent indicator random variables,

$$\text{Var}[R] = \text{Var}[R_1] + \text{Var}[R_2] + \ldots + \text{Var}[R_n]$$

Since the variance of an indicator (Bernoulli) r.v. is $p(1 - p)$,

$$\text{Var}[R] = n\, p\, (1 - p).$$

Questions?

Matching Birthdays

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$$\Pr[\text{two students share the same birthday}] = 1 - \frac{d \times (d-1) \times (d-2) \times \ldots (d-(n-1))}{d^n}$$

Q: On average, how many pairs of students have matching birthdays?

Define M to be the number of pairs of students with matching birthdays. For a fixed ordering of the students, let $X_{i,j}$ be the indicator r.v. corresponding to the event $E_{i,j}$ that the birthdays of students i and j match. Hence,

$$M = \sum_{i,j \mid 1 \leq i < j \leq n} X_{i,j} \implies \mathbb{E}[M] = \mathbb{E}[\sum_{i,j \mid 1 \leq i < j \leq n} X_{i,j}] = \sum_{i,j \mid 1 \leq i < j \leq n} \mathbb{E}[X_{i,j}] = \sum_{i,j \mid 1 \leq i < j \leq n} \Pr[E_{i,j}]$$

(Linearity of expectation)

Matching Birthdays

For a pair of students i, j, let B_i be the r.v. equal to the day of student i's birthday. Range(B_i) = {1,2,. . . , d}. For all $k \in [d]$, $\Pr[B_i = k] = 1/d$ (each student is equally likely to be born on any day of the year).

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$$E_{i,j} = (B_i = 1 \cap B_j = 1) \cup (B_i = 2 \cap B_j = 2) \cup \dots$$

$$\implies \Pr[E_{i,j}] = \sum_{k=1}^{d} \Pr[B_i = k \cap B_j = k] = \sum_{k=1}^{d} \Pr[B_i = k] \Pr[B_j = k] = \sum_{k=1}^{d} \frac{1}{d^2} = \frac{1}{d}$$

(student birthdays are independent of each other)

$$\implies \mathbb{E}[M] = \sum_{i,j|1 \le i < j \le n} \Pr[E_{i,j}] = \frac{1}{d} \sum_{i,j|1 \le i < j \le n} (1) = \frac{1}{d}[(n-1) + (n-2) + \ldots + 1] = \frac{n(n-1)}{2d}$$

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Hence, in our class of 75 students, on average, there are $\frac{(21)\,(41)}{365} = 7.60$ students with matching birthdays.

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Yes, because for all i, j and i', j' (where $i \neq i'$), $\Pr[X_{i,j} = 1 | X_{i',j'} = 1] = \Pr[X_{i,j} = 1]$ because if students i' and j' have matching birthdays, it does not tell us anything about whether i and j have matching birthdays.

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$$\mathrm{Var}[M] = \mathrm{Var}[\sum_{i,j|1\leq i<j\leq n} X_{i,j}]$$

Since $X_{i,j}$ are pairwise independent, the variance of the sum is equal to the sum of the variance.

$$\implies \mathrm{Var}[M] = \sum_{i,j|1\leq i<j\leq n} \mathrm{Var}[X_{i,j}] = \sum_{i,j|1\leq i<j\leq n} \frac{1}{d}\left(1-\frac{1}{d}\right) = \frac{1}{d}\left(1-\frac{1}{d}\right)\frac{n(n-1)}{2}$$

(Since $X_{i,j}$ is an indicator (Bernoulli) r.v. and $\Pr[X_{i,j}=1]=\frac{1}{d}$)

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Hence, in our class of 75 students, the standard deviation for the matching birthdays is equal to $\sqrt{\frac{(37)(75)}{365}\frac{364}{365}} \approx 2.75$.

Questions?

Covariance

For two random variables R and S, the covariance between R and S is defined as:

$$\mathrm{Cov}[R, S] := \mathbb{E}[(R - \mathbb{E}[R])\,(S - \mathbb{E}[S])] = \mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]$$

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The covariance between two r.v's is symmetric i.e. $\mathrm{Cov}[R,S] = \mathrm{Cov}[S,R]$.

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For two arbitrary (not necessarily independent) r.v's, R and S,

$$\mathrm{Var}[R + S] = \mathrm{Var}[R] + \mathrm{Var}[S] + 2\,\mathrm{Cov}[R, S]$$

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Recall from Lecture 17, Slide 7, where we showed that,

$$\mathrm{Var}[R+S] = \mathrm{Var}[R] + \mathrm{Var}[S] + 2(\mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]) = \mathrm{Var}[R] + \mathrm{Var}[S] + 2\,\mathrm{Cov}[R,S].$$

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Generalization to multiple random variables $R_1, R_2, \ldots R_n$ (Recall from Lecture 17, Slide 8):

$$\mathrm{Var}\left[\sum_{i=1}^{n} R_i\right] = \sum_{i=1}^{n} \mathrm{Var}[R_i] + 2\sum_{1\leq i<j\leq n} \mathrm{Cov}[R_i, R_j]$$

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$$\implies \mathbb{E}[X] = \Pr[A]\ ; \mathbb{E}[Y] = \Pr[B]; \mathbb{E}[XY] = \Pr[A \cap B]$$

$$\implies \text{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] = \Pr[A \cap B] - \Pr[A]\Pr[B]$$

If $\text{Cov}[X, Y] > 0 \implies \Pr[A \cap B] > \Pr[A]\Pr[B]$. Hence,

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} > \frac{\Pr[A]\Pr[B]}{\Pr[B]} = \Pr[A]$$

If $Cov[X, Y] > 0$, it implies that $\Pr[A|B] > \Pr[A]$ and hence, the probability that event A happens increases if B is going to happen/has happened. Similarly, if $Cov[X, Y] < 0$, $\Pr[A|B] < \Pr[A]$. In this case, if B happens, then the probability of event A decreases.

Correlation

The correlation between two r.v's R_1 and R_2 is defined as:

$$\mathrm{Corr}[R_1, R_2] = \frac{\mathrm{Cov}[R_1, R_2]}{\sqrt{\mathrm{Var}[R_1]\,\mathrm{Var}[R_2]}}$$

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If R_1 and R_2 are independent, $\text{Cov}[R_1, R_2] = 0$ and $\text{Corr}[R_1, R_2] = 0$.

If $R_1 = -R_2 = R$, then,

$$\begin{aligned}\text{Corr}[R, -R] &= \frac{\text{Cov}[R, -R]}{\sqrt{\text{Var}[R]\,\text{Var}[-R]}} = \frac{\text{Cov}[R, -R]}{\sqrt{\text{Var}[R]\,(-1)^2\text{Var}[R]}} = \frac{\text{Cov}[R, -R]}{\text{Var}[R]} \\ &= \frac{\mathbb{E}[-R^2] - \mathbb{E}[R]\,\mathbb{E}[-R]}{\text{Var}[R]} = \frac{-\mathbb{E}[R^2] + \mathbb{E}[R]\,\mathbb{E}[R]}{\text{Var}[R]} = \frac{-\text{Var}[R]}{\text{Var}[R]} = -1\end{aligned}$$

Questions?