CMPT 210: Probability and Computing

Lecture 14

Sharan Vaswani

March 5, 2024

Recap

Probability density function (PDF): Let R be a r.v. with codomain V. The probability density function of R is the function $\text{PDF}_R : V \to [0, 1]$, such that $\text{PDF}_R[x] = \Pr[R = x]$ if $x \in \text{Range(R)}$ and equal to zero if $x \notin \text{Range(R)}$.

Cumulative distribution function (CDF): The cumulative distribution function of R is the function $\text{CDF}_R : \mathbb{R} \to [0, 1]$, such that $\text{CDF}_R[x] = \Pr[R \leq x]$.

Importantly, neither PDF_R nor CDF_R involves the sample space of an experiment.

Example: If we flip three coins, and C counts the number of heads, then
$\text{PDF}_C[0] = \Pr[C = 0] = \frac{1}{8}$, and
$\text{CDF}_C[2.3] = \Pr[C \leq 2.3] = \Pr[C = 0] + \Pr[C = 1] + \Pr[C = 2] = \frac{7}{8}$.

Recap

A **distribution** can be specified by its probability density function (PDF) (denoted by f).

Bernoulli Distribution: $f_p(0) = 1 - p$, $f_p(1) = p$. *Example*: When tossing a coin such that $\Pr[\text{heads}] = p$, random variable R is equal to 1 if we get a heads (and equal to 0 otherwise). In this case, R follows the Bernoulli distribution i.e. $R \sim \text{Ber}(p)$.

Uniform Distribution: If $R : \mathcal{S} \to V$, then for all $v \in V$, $f(v) = 1/|V|$. *Example*: When throwing an n-sided die, random variable R is the number that comes up on the die. $V = \{1, 2, \ldots, n\}$. In this case, R follows the Uniform distribution i.e. $R \sim \text{Uniform}(1, n)$.

Binomial Distribution: $f_{n,p}(k) = \binom{n}{k} p^k (1-p)^{n-k}$. *Example*: When tossing n independent coins such that $\Pr[\text{heads}] = p$, random variable R is the number of heads in n coin tosses. In this case, R follows the Binomial distribution i.e. $R \sim \text{Bin}(n, p)$.

Geometric Distribution: $f_p(k) = (1-p)^{k-1} p$. *Example*: When repeatedly tossing a coin such that $\Pr[\text{heads}] = p$, random variable R is the number of tosses needed to get the first heads. In this case, R follows the Geometric distribution i.e. $R \sim \text{Geo}(p)$.

Distributions - Examples

Q: It is known that disks produced by a certain company will be defective with probability 0.01 independently of each other. The company sells the disks in packages of 10 and offers a money-back guarantee that at most 1 of the 10 disks is defective (the package can be returned if there is more than 1 defective disk). What proportion of packages is returned? If someone buys three packages, what is the probability that exactly one of them will be returned?

Let X be the random variable corresponding to the number of defective disks in a package. Let E be the event that the package is returned. We wish to compute $\Pr[E] = \Pr[X > 1]$. X follows the Binomial distribution $\mathrm{Bin}(10, 0.01)$. Hence,

$$\begin{aligned}\Pr[E] = \Pr[X > 1] = 1 - \Pr[X \leq 1] &= 1 - \Pr[X = 0] - \Pr[X = 1] \\ &= 1 - \binom{10}{0}(0.99)^{10} - \binom{10}{1}(0.99)^{9}(0.01)^{1} \approx 0.05\end{aligned}$$

Distributions - Examples

Q: It is known that disks produced by a certain company will be defective with probability 0.01 independently of each other. The company sells the disks in packages of 10 and offers a money-back guarantee that at most 1 of the 10 disks is defective (the package can be returned if there is more than 1 defective disk). If someone buys three packages, what is the probability that exactly one of them will be returned?

Let F be the event that someone bought 3 packages and exactly one of them is returned.

Answer 1: Let E_i be the event that package i is returned. From the previous question, we know that $\Pr[E_i] = \Pr[\text{Package } i \text{ has more than 1 defective disk}] \approx 0.05$.

$$F = (E_1 \cap E_2^c \cap E_3^c) \cup (E_1^c \cap E_2^c \cap E_3) \cup (E_1^c \cap E_2 \cap E_3^c)$$

$$\Pr[F] = \Pr[E_1](1 - \Pr[E_2])(1 - \Pr[E_3]) + (1 - \Pr[E_1])(1 - \Pr[E_2]) \Pr[E_3] + \dots$$

$$\Pr[F] \approx 3 \times (0.05)(0.95)(0.95) \approx 0.15.$$

Answer 2: Let Y be the random variable corresponding to the number of packages returned. Y follows the Binomial distribution $\mathrm{Bin}(3, 0.05)$ and we wish to compute $\Pr[F] = \Pr[Y = 1] \approx \binom{3}{1}(0.05)^1(0.95)^2 \approx 0.15.$

Distributions - Examples

Q: You are randomly and independently throwing darts. The probability that you hit the bullseye in throw i is p. Once you hit the bullseye you win and can go collect your reward. (a) What is the probability that you win after exactly k throws? (b) What is the probability you win in less than k throws?

(a) The number of throws (T) to hit the bullseye and win follows a geometric distribution $\text{Geo}(p)$ and we wish to compute $\Pr[T = k]$. Using the PDF for the Geometric distribution, this is equal to $(1 - p)^{k-1} p$.

(b) **Answer 1**: If E is the event that we win in less than k throws,
$\Pr[E] = \Pr[T < k] = \sum_{i=1}^{k-1} \Pr[T = i] = p \sum_{i=1}^{k-1} (1 - p)^{i-1} = 1 - (1 - p)^{k-1}$.

Answer 2:
$\Pr[E] = 1 - \Pr[E^c] = 1 - \Pr[\text{do not hit the bullseye in } k - 1 \text{ throws}] = 1 - (1 - p)^{k-1}$.

Number Guessing Game

Q: We have two envelopes. Each contains a distinct number in $\{0, 1, 2, \ldots, 100\}$. To win the game, we must determine which envelope contains the larger number. We are allowed to peek at the number in one envelope selected at random. Can we devise a winning strategy?

Strategy 1: We pick an envelope at random and guess that it contains the larger number (without even peeking at the number).

Q: What is the probability that we win with this strategy? Ans: 0.5

Strategy 2: We peek at the number and if its below 50, we choose the other envelope.

But the numbers in the envelopes need not be random! The numbers are chosen "adversarially" in a way that will defeat our guessing strategy. For example, to "beat" Strategy 2, the two numbers can always be chosen to be below 50.

Q: Can we do better than 50% chance of winning?

Number Guessing Game

Suppose that we somehow knew a number x that was in between the numbers in the envelopes. If we peek in one envelope and see a number. If it is bigger than x, we know its the higher number and choose that envelope. If it is smaller than x, we know that is the smaller number and choose the other envelope.

Of course, we do not know such a number x. But we can guess it!

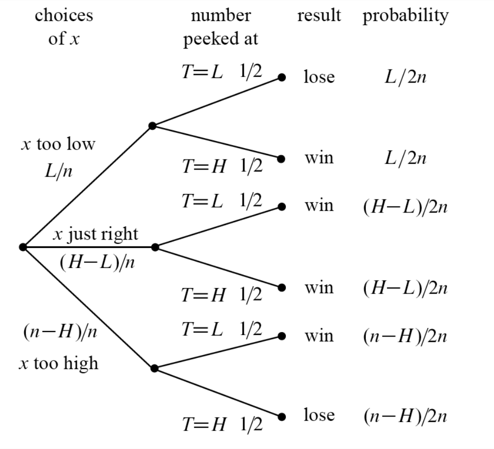
Strategy 3: Choose a random number x from $\{0.5, 1.5, 2.5, \ldots n - 1/2\}$ according to the uniform distribution i.e. $\Pr[x = 0.5] = \Pr[1.5] = \ldots = 1/n$. Then we peek at the number (denoted by T) in one envelope, and if $T > x$, we choose that envelope, else we choose the other envelope.

The advantage of such a randomized strategy is that the adversary cannot easily "adapt" to it.

Q: But does it have better than 50% chance of winning?

Number Guessing Game

Let the numbers in the two envelopes be L (lower number) and H (the higher number).



$$\begin{aligned}\Pr[\text{win}] &= \frac{L}{2n} + \frac{H-L}{2n} + \frac{H-L}{2n} + \frac{n-H}{2n} \\ &= \frac{1}{2} + \frac{H-L}{2n} \geq \frac{1}{2} + \frac{1}{2n} > \frac{1}{2}\end{aligned}$$

Hence our strategy has a greater than 50% chance of winning! If $n = 10$, $\Pr[\text{win}] \geq 0.55$, for $n = 100$, $\Pr[\text{win}] \geq 0.505$.

Q: For $n = 100$, if $L = 23$ and $H = 54$, compute $\Pr[\text{ guessing too low} \mid \text{we win }]$

Ans: $\Pr[\text{ guessing too low} \mid \text{we win }] = \frac{\Pr[\text{ we win} \cap \text{guessing too low }]}{\Pr[\text{ we win }]} = \frac{L/2n}{1/2 + (H-L)/2n} = \frac{L}{n+H-L} = \frac{23}{131}$.

Questions?