CMPT 419/983: Theoretical Foundations of Reinforcement Learning

Lecture 4

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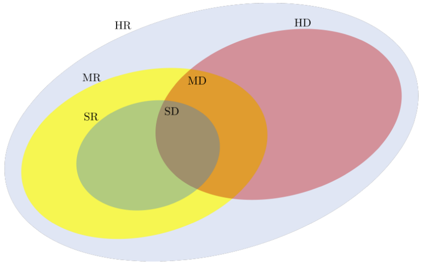
Recap

- Given an MDP $M = (\mathcal{S}, \mathcal{A}, \mathcal{P}, r, s_0)$, interacting with M using a fixed policy π results in a stochastic process $(S_0, A_0, S_1, \ldots)$ over the state-action space and a corresponding reward process $(R_0, R_1, \ldots,) = (r(S_0, A_0), r(S_1, A_1), \ldots)$.
- **Objective**: Find policy $\pi \in \Pi_{\text{HR}}$ that maximizes the *value function* $v^\pi(s_0) := \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R_t | S_0 = s_0\right]$.
- For each $s \in \mathcal{S}$, for a given policy $\pi = (\pi_0, \pi_1, \ldots) \in \Pi_{\text{HR}}$, there exists a policy $\pi' = (\pi'_0, \pi'_1, \ldots) \in \Pi_{\text{MR}}$ with the same value, conditioned on $S_0 = s_0$.
- Hence, considering the class Π_{MR} is sufficient when searching for the optimal policy.



Infinite-horizon Discounted Setting

Claim: For $\pi \in \Pi_{\text{MR}}$, if we define

$$\mathbf{r}_\pi \in \mathbb{R}^S \quad \text{s.t.} \quad \mathbf{r}_\pi(s) := \sum_{a \in \mathcal{A}} r(s,a)\, \pi[a|s]\,,$$

$$\mathbf{P}_\pi \in \mathbb{R}^{S \times S} \quad \text{s.t.} \quad \mathbf{P}_\pi[s,s'] = \Pr^\pi(s \to s') := \sum_{a \in \mathcal{A}} \Pr[s'|s,a]\, \pi(a|s)\,,$$

then, $v^\pi \in \mathbb{R}^S$ can be expressed as:

$$v^\pi = \sum_{t=0}^{\infty} \gamma^t \left[\prod_{j=0}^{t-1} \mathbf{P}_{\pi_j}\right] \mathbf{r}_{\pi_t}\,.$$

Furthermore, for a policy $\pi \in \Pi_{\text{SR}}$, $v^\pi = \mathbf{r}_\pi + \gamma\, \mathbf{P}_\pi\, v^\pi$. Examining each component,

$$v^\pi(s) = \mathbf{r}_\pi(s) + \gamma \sum_{s'} \mathbf{P}_\pi[s,s']\, v^\pi(s') = \sum_{a \in \mathcal{A}} r(s,a)\, \pi[a|s] + \gamma \sum_{s' \in \mathcal{S}} \sum_{a \in \mathcal{A}} \mathcal{P}[s'|s,a]\, \pi[a|s]\, v^\pi(s')$$

This is the **Bellman equation** for a fixed policy $\pi \in \Pi_{\text{SR}}$.

Infinite-horizon Discounted Setting

Proof: Starting from the definition of $v^\pi(s_0)$,

$$v^\pi(s_0) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R_t | S_0 = s_0\right] = \sum_{t=0}^{\infty} \gamma^t \sum_{s\in\mathcal{S}} \sum_{a\in\mathcal{A}} r(s,a)\, \Pr[S_t = s, A_t = a | S_0 = s_0]$$

Let us evaluate the first three terms in this sum,

For $t = 0:\ \sum_{s\in\mathcal{S}} \sum_{a\in\mathcal{A}} r(s,a)\, \Pr[S_0 = s, A_0 = a | S_0 = s_0] = \sum_{a\in\mathcal{A}} r(s_0, a)\, \pi_0(a|s_0) = \mathbf{r}_{\pi_{\mathbf{0}}}(s_0)$

For $t = 1:\ \gamma \sum_{s\in\mathcal{S}} \sum_{a\in\mathcal{A}} r(s,a)\, \Pr[A_1 = a | S_1 = s, S_0 = s_0]\, \Pr[S_1 = s | S_0 = s_0]$

$$= \gamma \sum_{s\in\mathcal{S}} \mathbf{r}_{\pi_{\mathbf{1}}}(s)\, \Pr[S_1 = s | S_0 = s_0] = \gamma \sum_{s\in\mathcal{S}} \mathbf{r}_{\pi_{\mathbf{1}}}(s) \sum_{a\in\mathcal{A}} \mathcal{P}[s|s_0, a]\, \pi_0(a|s_0) = \gamma \sum_{s\in\mathcal{S}} \mathbf{r}_{\pi_{\mathbf{1}}}(s)\, \mathbf{P}_{\pi_{\mathbf{0}}}[s_0, s]$$

For $t = 2:\ \gamma^2 \sum_{s\in\mathcal{S}} \mathbf{r}_{\pi_{\mathbf{2}}}(s)\, \Pr[S_2 = s | S_0 = s_0] = \sum_{s\in\mathcal{S}} \mathbf{r}_{\pi_{\mathbf{2}}}(s) \sum_{\mathbf{s_1}\in\mathcal{S}} \mathbf{P}_{\pi_{\mathbf{1}}}[s_1, s]\, \mathbf{P}_{\pi_{\mathbf{0}}}[s_0, s_1]$

For a general $t:\ \gamma^t \sum_{s\in\mathcal{S}} \mathbf{r}_{\pi_t}(s) \sum_{s_{t-\mathbf{1}}\in\mathcal{S}} \dots \sum_{\mathbf{s_1}\in\mathcal{S}} \mathbf{P}_{\pi_{t-\mathbf{1}}}[s_{t-1}, s]\, \mathbf{P}_{\pi_{t-\mathbf{2}}}[s_{t-2}, s_{t-1}] \cdots \mathbf{P}_{\pi_{\mathbf{0}}}[s_0, s_1]$

Infinite-horizon Discounted Setting

Recall that, $v^\pi(s_0) = \sum_{t=0}^{\infty} \gamma^t \sum_{s \in \mathcal{S}} \sum_{a \in \mathcal{A}} r(s,a) \Pr[S_t = s, A_t = a | S_0 = s_0]$, and that term t in the above sum is equal to $\gamma^t \sum_{s \in \mathcal{S}} \mathbf{r}_{\pi_t}(s) \sum_{s_{t-\mathbf{1}} \in \mathcal{S}} \cdots \sum_{s_\mathbf{1} \in \mathcal{S}} \mathbf{P}_{\pi_{t-\mathbf{1}}}[s_{t-1}, s] \, \mathbf{P}_{\pi_{t-\mathbf{2}}}[s_{t-2}, s_{t-1}] \cdots \mathbf{P}_{\pi_\mathbf{0}}[s_0, s_1]$. Hence,

$$v^\pi(s_0) = \sum_{t=0}^{\infty} \gamma^t \sum_{s \in \mathcal{S}} \mathbf{r}_{\pi_t}(s) \sum_{s_{t-\mathbf{1}} \in \mathcal{S}} \cdots \sum_{s_\mathbf{1} \in \mathcal{S}} \mathbf{P}_{\pi_{t-\mathbf{1}}}[s_{t-1}, s] \, \mathbf{P}_{\pi_{t-\mathbf{2}}}[s_{t-2}, s_{t-1}] \cdots \mathbf{P}_{\pi_\mathbf{0}}[s_0, s_1]$$

$$\implies v^\pi = \sum_{t=0}^{\infty} \gamma^t \left[\prod_{j=0}^{t-1} \mathbf{P}_{\pi_j} \right] \mathbf{r}_{\pi_t} \qquad (v^\pi(s_0) \text{ is the } s_0 \text{ component of the vector } v^\pi)$$

For a policy $\pi \in \Pi_{\mathsf{SR}}$, $\mathbf{P}_{\pi_t} = \mathbf{P}_\pi$ and $\mathbf{r}_{\pi_t} = \mathbf{r}_\pi$ for all t. Hence,

$$v^\pi = \sum_{t=0}^{\infty} \gamma^t \, [\mathbf{P}_\pi]^t \, \mathbf{r}_\pi = \mathbf{r}_\pi + \gamma \, \mathbf{P}_\pi \, \mathbf{r}_\pi + \gamma^2 \, [\mathbf{P}_\pi]^2 \, \mathbf{r}_\pi + \dots$$

$$= \mathbf{r}_\pi + \gamma \, \mathbf{P}_\pi \, \left[\mathbf{r}_\pi + \gamma \, \mathbf{P}_\pi \, \mathbf{r}_\pi + \gamma^2 \, [\mathbf{P}_\pi]^2 \, \mathbf{r}_\pi + \dots \right] = \mathbf{r}_\pi + \gamma \, \mathbf{P}_\pi \, v^\pi$$

$$\implies v^\pi = \mathbf{r}_\pi + \gamma \, \mathbf{P}_\pi \, v^\pi \quad \square$$

Infinite-horizon Discounted Setting

For $\pi \in \Pi_{\text{SR}}$, we have seen that $v^\pi = \mathbf{r}_\pi + \gamma\, \mathbf{P}_\pi\, v^\pi$. This corresponds to a system of linear equations, and can be solved in closed form. Since $\gamma < 1$, and $\mathbf{P}_\pi$ is a stochastic matrix (i.e. its elements correspond to probabilities, and sums and columns add up to one), the eigenvalues of $I_S - \gamma \mathbf{P}_\pi$ are strictly positive and hence it is invertible.

$$v^\pi = \mathbf{r}_\pi + \gamma\, \mathbf{P}_\pi\, v^\pi \implies (I_S - \gamma \mathbf{P}_\pi)\, v^\pi = \mathbf{r}_\pi \implies v^\pi = (I_S - \gamma \mathbf{P}_\pi)^{-1}\, \mathbf{r}_\pi\,.$$

- By the Neumann series, $(I - A)^{-1} = \sum_{t=0}^{\infty} A^t$. Hence, $(I_S - \gamma \mathbf{P}_\pi)^{-1}\, \mathbf{r}_\pi = \sum_{t=0}^{\infty} \gamma^t\, [\mathbf{P}_\pi]^t\, \mathbf{r}_\pi$ which recovers the expression for v^π from the previous slide.
- Q: For a vector $x \geq 0$, prove that $(I_S - \gamma \mathbf{P}_\pi)^{-1}\, x \geq x \geq 0$ Ans: Use the Neumann series
- Q: For vectors $u \geq v$, prove that $(I_S - \gamma \mathbf{P}_\pi)^{-1}\, u \geq (I_S - \gamma \mathbf{P}_\pi)^{-1}\, v$ Ans: $x = u - v$ above.

Bellman policy evaluation operator for π: $\mathcal{T}_\pi : \mathbb{R}^S \to \mathbb{R}^S$ s.t. for vector $u \in \mathbb{R}^S$
$\mathcal{T}_\pi u = \mathbf{r}_\pi + \gamma \mathbf{P}_\pi u$ and $(\mathcal{T}_\pi u)(s) = \mathbf{r}_\pi(s) + \gamma \sum_{s'} \mathbf{P}_\pi[s, s']\, u(s')$.

Bellman Optimality Operator

Define the **Bellman optimality operator** $\mathcal{T} : \mathbb{R}^S \to \mathbb{R}^S$. For a vector $u \in R^S$,

$$(\mathcal{T}u)(s) = \max_{a \in \mathcal{A}} \left\{ r(s,a) + \gamma \sum_{s'} \mathcal{P}(s'|s,a) u(s') \right\}$$

Consider $w := \max_{\pi \in \Pi_{\mathbf{SD}}} \{ \mathbf{r}_\pi + \gamma \mathbf{P}_\pi u \}$,

$$w(s) = \max_{\pi \in \Pi_{\mathbf{SD}}} \left\{ \mathbf{r}_\pi(s) + \gamma \sum_{s'} \mathbf{P}_\pi[s,s'] u(s') \right\}$$

$$= \max_{\substack{\pi(\cdot|s) \\ \exists a^* \text{ s.t } \pi(a^*|s)=1}} \left\{ \sum_a \pi(a|s) \left[r(s,a) + \gamma \sum_{s'} \mathcal{P}(s'|s,a) u(s') \right] \right\}$$

(Optimization over degenerate distributions)

$$= \max_a \left\{ r(s,a) + \gamma \sum_{s'} \mathcal{P}(s'|s,a) u(s') \right\} = (\mathcal{T}u)(s)$$

$$\implies \mathcal{T}u = \max_{\pi \in \Pi_{\mathbf{SD}}} \{ \mathbf{r}_\pi + \gamma \mathbf{P}_\pi u \}$$

Bellman Optimality Operator

Claim: $\mathcal{T}$ is a contraction mapping with modulus γ, i.e. for any 2 vectors $u, w \in \mathbb{R}^S$ $\|\mathcal{T}u - \mathcal{T}w\|_\infty \leq \gamma \|u - w\|_\infty$.

Proof: For a fixed s, without loss of generality, consider the case when $(\mathcal{T}w)(s) \geq (\mathcal{T}u)(s)$. By the definition of $\mathcal{T}$, if $a^*(s) = \arg\max \{r(s,a) + \gamma \sum_{s'} \mathcal{P}(s'|s,a)w(s')\}$, then,

$$(\mathcal{T}w)(s) = r(s, a^*(s)) + \gamma \sum_{s'} \mathcal{P}(s'|s, a^*(s))w(s')$$

$$r(s, a^*(s)) + \gamma \sum_{s'} \mathcal{P}(s'|s, a^*(s))u(s') \leq \max_a \{r(s,a) + \gamma \sum_{s'} \mathcal{P}(s'|s,a)u(s')\} = (\mathcal{T}u)(s)$$

$$\begin{aligned} \implies (\mathcal{T}w)(s) - (\mathcal{T}u)(s) &\leq \gamma \sum_{s'} \mathcal{P}(s'|s, a^*(s)) \left[w(s') - u(s')\right] \\ &\leq \gamma \left\|\mathcal{P}(\cdot|s, a^*(s))\right\|_1 \|w - u\|_\infty = \gamma \|w - u\|_\infty \end{aligned}$$

Similarly, $(\mathcal{T}w)(s) - (\mathcal{T}u)(s) \leq \gamma \|w - u\|_\infty$. Since this result is true for an arbitrary s,

$$\|\mathcal{T}u - \mathcal{T}w\|_\infty \leq \gamma \|u - w\|_\infty \quad \square$$

Banach's Fixed Point Theorem

Fact: Under certain technical assumptions, if L is a contraction mapping, then,

- There exists a unique fixed point u^* such that $Lu^* = u^*$.
- For any vector u_0, $u_{n+1} = Lu_n = L^{n+1} u_0$ converges to u^* i.e $\|u_n - u^*\|_\infty \to 0$ as $n \to \infty$.

Since the Bellman optimality operator, $\mathcal{T}$ is a contraction mapping, using Banach's Fixed Point Theorem above, there exists a fixed point $u^* \in \mathbb{R}^S$ s.t. $\mathcal{T} u^* = u^*$.

Claim: For $u_0 \in \mathbb{R}^S$, $\|u^* - \mathcal{T}^n u_0\|_\infty \leq \gamma^n \|u^* - u_0\|_\infty$ i.e. $u_n := \mathcal{T}^n u_0$ converges to u^* at a linear rate.

Q: *Proof?* Ans: For any $s \leq n$,

$$\|u^* - u_s\|_\infty = \|\mathcal{T} u^* - \mathcal{T} u_{s-1}\|_\infty \leq \gamma \|u^* - u_{s-1}\|_\infty$$
$$\implies \|u^* - u_n\|_\infty \leq \gamma \|u^* - u_{n-1}\|_\infty \leq \gamma^n \|u^* - u_0\|_\infty \quad \square$$

Similarly, $\mathcal{T}_\pi$ is a γ-contraction, and converges to a unique fixed point equal to v^π at a linear rate. Prove in Assignment 2!

Fundamental Theorem

Claim: There exists a policy $\pi^* \in \Pi_{\text{SD}}$ s.t. $v^{\pi^*}(s) = \max_{\pi \in \Pi_{\mathbf{HR}}} v^{\pi}(s)$ for all $s \in \mathcal{S}$.

- Hence, for MDPs, it is sufficient to only consider the class of stationary, deterministic policies in order to compute the optimal policy.

Proof: We know the following:

(a) From Slide 19 in Lecture 3, $\max_{\pi \in \Pi_{\mathbf{HR}}} v^{\pi}(s) = \max_{\pi \in \Pi_{\mathbf{MR}}} v^{\pi}(s)$.

(b) If v^* is the fixed point of $\mathcal{T}$ and $\pi^* \in \Pi_{\text{SD}}$ is the *greedy* policy s.t. $\pi^*(s) = \arg\max_a \{r(s,a) + \gamma \sum_{s'} \mathcal{P}(s'|s,a)\, v^*(s')\}$, then,

$$v^* = \mathcal{T} v^* = \max_{\pi \in \Pi_{\mathbf{SD}}} \{\mathbf{r}_\pi + \gamma \mathbf{P}_\pi v^*\} = \mathcal{T}_{\pi^*} v^* = \mathbf{r}_{\pi^*} + \gamma \mathbf{P}_{\pi^*} v^*$$

(c) $\max_{\pi \in \Pi_{\mathbf{SD}}} \{\mathbf{r}_\pi + \gamma \mathbf{P}_\pi v^*\} = \max_{\pi \in \Pi_{\mathbf{SR}}} \{\mathbf{r}_\pi + \gamma \mathbf{P}_\pi v^*\}$ i.e. randomized policies cannot increase the value. (Prove in Assignment 2!)

We will prove that for a v s.t. $v = \mathcal{T} v$, $v = \max_{\pi \in \Pi_{\mathbf{HR}}} v^{\pi}$. Together with (b), this implies that $v^* = \max_{\pi \in \Pi_{\mathbf{HR}}} v^{\pi}$ and that this value function corresponds to the policy $\pi^* \in \Pi_{\text{SD}}$.

Fundamental Theorem

We will now prove that:

(i) If $v \geq \mathcal{T} v$, then $v \geq \max_{\pi \in \Pi_{\mathbf{HR}}} v^{\pi}$.
(ii) If $v \leq \mathcal{T} v$, then $v \leq \max_{\pi \in \Pi_{\mathbf{HR}}} v^{\pi}$.

Hence, if $v = \mathcal{T} v$, then $v = \max_{\pi \in \Pi_{\mathbf{HR}}} v^{\pi}$.

Let us first prove (i). Define an arbitrary $\pi' := \{\pi'_1, \pi'_2, \ldots, \} \in \Pi_{\mathrm{MR}}$. For an arbitrary i, define $\pi_i := \{\pi'_i, \pi'_i, \ldots\} \in \Pi_{\mathrm{SR}}$.

$$v \geq \mathcal{T} v = \max_{\pi \in \Pi_{\mathbf{SD}}} \{\mathbf{r}_\pi + \gamma \mathbf{P}_\pi v\} = \max_{\pi \in \Pi_{\mathbf{SR}}} \{\mathbf{r}_\pi + \gamma \mathbf{P}_\pi v\} \geq \mathbf{r}_{\pi_i} + \gamma \mathbf{P}_{\pi_i} v \tag{Using (c)}$$

$$\implies v \geq \mathbf{r}_{\pi_{\mathbf{0}}} + \gamma \mathbf{P}_{\pi_{\mathbf{0}}} v \geq \mathbf{r}_{\pi_{\mathbf{0}}} + \gamma \mathbf{P}_{\pi_{\mathbf{0}}} [\mathbf{r}_{\pi_{\mathbf{1}}} + \gamma \mathbf{P}_{\pi_{\mathbf{1}}} v] \implies v \geq \sum_{t=0}^{\infty} \gamma^t \left[\prod_{j=0}^{t-1} \mathbf{P}_{\pi_j}\right] \mathbf{r}_{\pi_t} \tag{Recursing}$$

$$\implies v \geq v^{\pi'} \implies v \geq \max_{\pi \in \Pi_{\mathbf{MR}}} v^{\pi} = \max_{\pi \in \Pi_{\mathbf{HR}}} v^{\pi} \quad \text{(Using def of } v^{\pi'} \text{ for } \pi' \in \Pi_{\mathrm{MR}}\text{, and then (a))}$$

Fundamental Theorem

Now let us prove (ii): if $v \le \mathcal{T} v$, then $v \le \max_{\pi \in \Pi_{\mathbf{HR}}} v^{\pi}$. For a specific $\pi \in \Pi_{\mathrm{SD}}$,

$$v \le \mathcal{T} v = \mathcal{T}_{\pi} v = \mathbf{r}_{\pi} + \gamma \mathbf{P}_{\pi} v \le \mathbf{r}_{\pi} + \gamma \mathbf{P}_{\pi} \left[\mathbf{r}_{\pi} + \gamma \mathbf{P}_{\pi} v \right] \implies v \le \sum_{t=0}^{\infty} \gamma^{t} \left[\mathbf{P}_{\pi} \right]^{t} \mathbf{r}_{\pi} \tag{Recursing}$$

$$\implies v \le v^{\pi} \le \max_{\pi \in \Pi_{\mathbf{SD}}} v^{\pi} \tag{By def of v^{π} for $\pi \in \Pi_{\mathrm{SD}}$}$$

$$= \max_{\pi \in \Pi_{\mathbf{SR}}} v^{\pi} \le \max_{\pi \in \Pi_{\mathbf{MR}}} v^{\pi} \implies v \le \max_{\pi \in \Pi_{\mathbf{HR}}} v^{\pi} \quad \square \tag{Using (c) and then (a)}$$

The fundamental theorem immediately suggests a way to calculate π^*:

- Starting from an arbitrary vector $v_0 \in \mathbb{R}^S$, iterate $v = \mathcal{T} v$ to converge to a fixed point v^*.
- Once we have computed v^*, compute the greedy policy in each state $s \in \mathcal{S}$: $\pi^*(s) = \arg\max_a \{ r(s,a) + \gamma \sum_{s'} \mathcal{P}(s'|s,a) \, v^*(s') \}$.

This is value iteration!

Value Iteration

Value Iteration

Algorithm Value Iteration

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Input: MDP $M = (\mathcal{S}, \mathcal{A}, \mathcal{P}, r, \rho)$, $v_0 = 0$.
for $k = 1 \to K$ do
    $\forall s \in \mathcal{S}$, $v_k(s) = \max_{a \in \mathcal{A}} \{r(s,a) + \gamma \sum_{s'} \mathcal{P}(s'|s,a) v_{k-1}(s')\} = (\mathcal{T} v_{k-1})(s)$
end for
$\forall s \in \mathcal{S}$, return $\hat{\pi}(s) = \arg\max_a \{r(s,a) + \gamma \sum_{s'} \mathcal{P}(s'|s,a)\, v_K(s')\}$
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Q: What is the computational complexity of VI? Ans: $O(S^2 AK)$

Claim: After $K \geq \frac{\log(1/\epsilon\,(1-\gamma))}{1-\gamma}$ iterations, value iteration returns a v_K s.t. $\|v_K - v^*\|_\infty \leq \epsilon$.

Proof: By using the contraction property of $\mathcal{T}$,

$$\|v_K - v^*\|_\infty \leq \gamma^K \|v_0 - v^*\|_\infty = \gamma^K \|v^*\|_\infty \leq \gamma^K \frac{1}{1-\gamma}$$

Setting $K \geq \frac{\log(1/\epsilon\,(1-\gamma))}{1-\gamma} \geq \frac{\log(1/\epsilon\,(1-\gamma))}{\log(1/\gamma)}$ ensures that $\|v_K - v^*\|_\infty \leq \epsilon$. ($\because 1 - \gamma \leq \log(1/\gamma)$)

Recall that the greedy step w.r.t v_K can also be written as: $\mathcal{T} v_K = \mathcal{T}_{\hat{\pi}} v_K$.

Value Iteration

- The previous result gives a bound on the quality of v_K.
- Since $\hat{\pi}$ is the policy returned by VI, we want a bound on $\left\| v^* - v^{\hat{\pi}} \right\|_\infty$.
- We will prove a general result bounding the error for the greedy policy inferred from v.

Claim: For an arbitrary $v \in \mathbb{R}^S$ if (i) π is the greedy policy w.r.t v, i.e. $\pi(s) = \arg\max_a \{r(s,a) + \gamma \sum_{s'} \mathcal{P}(s'|s,a)\, v(s')\}$, (ii) v^π is the value function corresponding to policy π i.e. $v^\pi = \mathcal{T}_\pi v^\pi = \mathbf{r}_\pi + \gamma \mathbf{P}_\pi v^\pi$, then,

$$v^\pi \geq v^* - \frac{2\gamma \left\| v - v^* \right\|_\infty}{1-\gamma} \mathbf{1}$$

- Hence, the error in $\left\| v - v^* \right\|_\infty$ "blows up" when inferring policy π.
- This result is sharp meaning that the constant $\frac{2\gamma}{1-\gamma}$ cannot be improved.
- Using this result, we conclude that VI requires $K \geq \frac{\log(2\gamma/\epsilon\,(1-\gamma)^2)}{1-\gamma}$ iterations to obtain a greedy policy $\hat{\pi}$ s.t. $v^* - v^{\hat{\pi}} \leq \epsilon \mathbf{1}$.

Value Iteration

- We have seen that VI requires $O\left(\frac{S^2 A \log(1/\epsilon)}{1-\gamma}\right)$ operations to produce an ϵ-optimal policy π that guarantees $v^\pi \geq v^* - \epsilon\mathbf{1}$.
- **Lower Bound**: For $\epsilon \in [0, \gamma/1-\gamma)$, any algorithm guaranteed to produce ϵ-optimal policies in an MDP with finite state-action spaces (with sizes S and A respectively) and bounded (in $[0,1]$) rewards requires $\Omega(S^2 A)$ operations (no dependence on ϵ) (see Csaba's notes, Lecture 3 for details).
- Is our VI analysis loose or is the $O(\log(1/\epsilon))$ dependence necessary?
- There exists a family of MDPs with deterministic transitions, three states, two actions and bounded (in $[0,1]$) rewards such that the worst-case iteration complexity of VI to find an *exactly* optimal policy is infinite. (see Csaba's notes, Lecture 4 for details).
- In the next class, we will study Policy Iteration (PI) which can converge to the optimal policy with finite operations.