CMPT 210: Probability and Computing

Lecture 17

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Recap

- **Expectation**/mean of a random variable R is denoted by $\mathbb{E}[R]$ and "summarizes" its distribution. Formally, $\mathbb{E}[R] := \sum_{\omega \in \mathcal{S}} \Pr[\omega]\, R[\omega]$
- **Alternate definition of expectation**: $\mathbb{E}[R] = \sum_{x \in \text{Range}(R)} x \Pr[R = x]$.
- **Linearity of Expectation**: For n random variables $R_1, R_2, \ldots, R_n$ and constants $a_1, a_2, \ldots, a_n$, $\mathbb{E}\left[\sum_{i=1}^{n} a_i R_i\right] = \sum_{i=1}^{n} a_i\, \mathbb{E}[R_i]$.
- **Expectation for Common Distributions**:
  - If $R \sim \text{Bernoulli}(p)$, $\mathbb{E}[R] = p$. *Example*: When tossing a coin, if R is the random variable equal to 1 if we get a heads.
  - If $R \sim \text{Uniform}(\{v_1, \ldots, v_n\})$, $\mathbb{E}[R] = \frac{v_1 + v_2 + \ldots + v_n}{n}$. *Example*: When throwing an n-sided dice with numbers $v_1, \ldots v_n$, if R is the random variable equal to the number.
  - If $R \sim \text{Bin}(n, p)$, $\mathbb{E}[R] = np$. *Example*: When tossing n independent coins, if R is the random variable equal to the number of heads.
  - If $R \sim \text{Geo}(p)$, $\mathbb{E}[R] = \frac{1}{p}$. *Example*: When tossing a coin repeatedly, if R is the random variable equal to the number of tosses required to get the first heads.

Conditional Expectation

- Similar to probabilities, expectations can be conditioned on some event.
- **Definition**: For random variable R, the expected value of R conditioned on an event A is:

$$\mathbb{E}[R|A] := \sum_{x \in \text{Range}(R)} x \Pr[R = x|A]$$

Q: If we throw a standard dice and define R to be the random variable equal to the number that comes up, what is the expected value of R given that the number is at most 4?

Let A be the event that the number is at most 4, i.e. $A = \{1, 2, 3, 4\}$.
$\Pr[R = 1|A] = \frac{\Pr[(R=1) \cap A]}{\Pr[A]} = \frac{\Pr[R=1]}{\Pr[A]} = \frac{1/6}{4/6} = 1/4$.
Similarly, $\Pr[R = 2|A] = \Pr[R = 3|A] = \Pr[R = 4|A] = \frac{1}{4}$ and $\Pr[R = 5|A] = \Pr[R = 6|A] = 0$.

$$\mathbb{E}[R|A] = \sum_{x \in \{1,2,3,4\}} x \Pr[R = x|A] = \frac{1}{4}[1 + 2 + 3 + 4] = \frac{5}{2}.$$

Q: What is the expected value of R given that the number is at least 4?

Law of Total Expectation

Law of Total Expectation: If R is a random variable $\mathcal{S} \to V$ and events $A_1, A_2, \ldots A_n$ form a partition of the sample space i.e. for all i, j, $A_i \cap A_j = \emptyset$ and $A_1 \cup A_2 \cup \ldots \cup A_n = \mathcal{S}$, then,

$$\mathbb{E}[R] = \sum_i \mathbb{E}[R|A_i] \Pr[A_i] \,.$$

Proof:

$$\mathbb{E}[R] = \sum_{x \in \text{Range}(R)} x \Pr[R = x] = \sum_{x \in \text{Range}(R)} x \sum_i \Pr[R = x|A_i] \Pr[A_i] \qquad \text{(Law of total probability)}$$

$$= \sum_i \Pr[A_i] \sum_{x \in \text{Range}(R)} x \Pr[R = x|A_i] \qquad \text{(Rearranging the summations)}$$

$$\implies \mathbb{E}[R] = \sum_i \Pr[A_i] \, \mathbb{E}[R|A_i].$$

Conditional Expectation - Examples

Q: Suppose that 49.6% of the people in the world are male and the rest female. If the expected height of a randomly chosen male is 5 feet 11 inches, while the expected height of a randomly chosen female is 5 feet 5 inches, what is the expected height of a randomly chosen person?

Define H to be the random variable equal to the height (in feet) of a randomly chosen person. Define M to be the event that the person is male and F the event that the person is female. We wish to compute $\mathbb{E}[H]$ and we know that $\mathbb{E}[H|M] = 5 + \frac{11}{12}$ and $\mathbb{E}[H|F] = 5 + \frac{5}{12}$.
$\Pr[M] = 0.496$ and $\Pr[F] = 1 - 0.496 = 0.504$.
Hence, $\mathbb{E}[H] = \mathbb{E}[H|M] \Pr[M] + \mathbb{E}[H|F] \Pr[F] = \frac{71}{12}(0.496) + \frac{65}{12}(0.504)$.

Questions?

Independence of random variables

Definition: Random variables R_1 and R_2 are independent iff for *all* $x_1 \in \text{Range}(R_1)$ and $x_2 \in \text{Range}(R_2)$, events $[R_1 = x_1]$ and $[R_2 = x_2]$ are independent. Formally, we require,

$$\Pr[(R_1 = x_1) \cap (R_2 = x_2)] = \Pr[(R_1 = x_1)] \Pr[(R_2 = x_2)]$$

Q: Suppose we toss three independent, unbiased coins. Let C be r.v. equal to the number of heads that appear and M be the r.v. that is equal to 1 if all the coins match (else it is 0). Are random variables C and M independent?

$\mathcal{S} = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
$\text{Range}(C) = \{0, 1, 2, 3\}$ and $\text{Range}(M) = \{0, 1\}$. $\Pr[C = 3] = \frac{1}{8}$ and $\Pr[M = 1] = \frac{1}{4}$.
$\Pr[(C = 3) \cap (M = 1)] = \frac{1}{8} \neq \frac{1}{32} = \Pr[C = 3] \Pr[M = 1]$. Hence, C and M are not independent.

Independence - Examples

Q: Suppose we toss three independent, unbiased coins. If H_1 is the indicator r.v. equal to one if the first toss is a heads (else it is 0) and M be the r.v. that is equal to 1 if all the coins match (else it is 0), are H_1 and M independent?

The sample space is: $\mathcal{S} = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
$\Pr[H_1 = 1] = \Pr[H_1 = 0] = \frac{1}{2}$, $\Pr[M = 1] = \frac{1}{4}$, $\Pr[M = 0] = \frac{3}{4}$.
$\Pr[H_1 = 0 \cap M = 1] = \Pr[\{TTT\}] = \frac{1}{8} = \Pr[H_1 = 0]\Pr[M = 1]$.
$\Pr[H_1 = 1 \cap M = 1] = \Pr[\{HHH\}] = \frac{1}{8} = \Pr[H_1 = 1]\Pr[M = 1]$.
$\Pr[H_1 = 0 \cap M = 0] = \Pr[\{THH, THT, TTH\}] = \frac{3}{8} = \Pr[H_1 = 0]\Pr[M = 0]$.
$\Pr[H_1 = 1 \cap M = 0] = \Pr[\{HHT, HTH, HTT\}] = \frac{3}{8} = \Pr[H_1 = 1]\Pr[M = 0]$.
Hence, H_1 and M are independent.

Independence of random variables

Alternate definition of independence: Random variables R_1 and R_2 are independent iff for *all* $x_1 \in \text{Range}(R_1)$ and $x_2 \in \text{Range}(R_2)$,

$$\Pr[(R_1 = x_1)|(R_2 = x_2)] = \Pr[(R_1 = x_1)]$$
$$\Pr[(R_2 = x_2)|(R_1 = x_1)] = \Pr[(R_2 = x_2)]$$

Pairwise Independence: Similar to events, r.v's $R_1, R_2, R_3, \ldots R_n$ are *pairwise* independent iff for *any* pair R_i and R_j, for *all* $x \in \text{Range}(R_i)$ and $y \in \text{Range}(R_j)$, events $\Pr[R_i = x]$ and $\Pr[R_j = y]$ are pairwise independent implying that

$$\Pr[(R_i = x) \cap (R_j = y)] = \Pr[R_i = x] \Pr[R_j = y]$$

Alternatively, R_i and R_j are pairwise independent iff for *all* $x \in \text{Range}(R_i)$ and $y \in \text{Range}(R_j)$,

$$\Pr[(R_i = x)|(R_j = y)] = \Pr[(R_i = x)]$$
$$\Pr[(R_j = y)|(R_i = x)] = \Pr[(R_j = y)]$$

Independence of random variables

- Similar to events, random variables $R_1, R_2, \ldots, R_n$ are mutually independent if for *all* $x_1, x_2, \ldots, x_n$, events $[R_1 = x_1], [R_2 = x_2], \ldots [R_n = x_n]$ are mutually independent.

Mutual Independence of events: A set of events is said to be mutually independent if the probability of each event in the set is the same no matter which of the events has occurred. For events E_1, E_2 and E_3 to be mutually independent, all the following equalities should hold:

$$\Pr[E_1 \cap E_2] = \Pr[E_1]\Pr[E_2] \quad \Pr[E_1 \cap E_3] = \Pr[E_1]\Pr[E_3]$$
$$\Pr[E_2 \cap E_3] = \Pr[E_2]\Pr[E_3] \quad \Pr[E_1 \cap E_2 \cap E_3] = \Pr[E_1]\Pr[E_2]\Pr[E_3].$$

Alternatively, (i) $\forall i$ and $j \neq i$, $\Pr[E_i | E_j] = \Pr[E_i]$ and (ii) $\forall i$ and $j, k \neq i$, $\Pr[E_i | E_j \cap E_k] = \Pr[E_i]$.

- For 2 r.v's R_1 and R_2, mutual independence and pairwise independence is the same.
- For more than 2 r.v's $R_1, R_2, \ldots, R_n$, mutual independence implies pairwise independence.

Expectation/Independence

Q: If R_1 and R_2 are not independent, is $\mathbb{E}[R_1 + R_2] = \mathbb{E}[R_1] + \mathbb{E}[R_2]$?

Yes! Recall the derivation of the linearity of expectation. We never assumed that R_1 and R_2 are independent for the proof and the linearity of expectation holds regardless of whether the random variables are independent.

Q: If R_1 and R_2 are independent, is $\mathbb{E}[R_1 R_2] = \mathbb{E}[R_1]\,\mathbb{E}[R_2]$? Yes!

$$\mathbb{E}[R_1 R_2] = \sum_{x \in \text{Range}(R_1 R_2)} x \Pr[R_1 R_2 = x] = \sum_{r_1 \in \text{Range}(R_1), r_2 \in \text{Range}(R_2)} r_1 r_2 \Pr[R_1 = r_1 \cap R_2 = r_2] \qquad (x = r_1\, r_2)$$

$$= \sum_{r_1 \in \text{Range}(R_1)} \sum_{r_2 \in \text{Range}(R_2)} r_1 r_2 \Pr[R_1 = r_1 \cap R_2 = r_2] \qquad \text{(Splitting the sum)}$$

$$= \sum_{r_1 \in \text{Range}(R_1)} \sum_{r_2 \in \text{Range}(R_2)} r_1 r_2 \Pr[R_1 = r_1] \Pr[R_2 = r_2] \qquad \text{(Independence)}$$

$$= \sum_{r_1 \in \text{Range}(R_1)} r_1 \Pr[R_1 = r_1] \sum_{r_2 \in \text{Range}(R_2)} r_2 \Pr[R_2 = r_2] = \mathbb{E}[R_1]\mathbb{E}[R_2]$$

Expectation/Independence - Examples

Q: Suppose there is a dinner party where n people check in their coats. The coats are mixed up during dinner, so that afterward each person receives a random coat i.e. a person gets their own coat with probability $\frac{1}{n}$. What is the expected number of people who get their own coat?

Let G be the number of people that get their own coat. We wish to compute $\mathbb{E}[G]$. Define G_i to be the indicator r.v. that person i gets their own coat. Observe that $G = G_1 + G_2 + \ldots + G_n$ and by linearity of expectation $\mathbb{E}[G] = \mathbb{E}[G_1] + \mathbb{E}[G_2] + \ldots + \mathbb{E}[G_n]$. For each i, $\mathbb{E}[G_i] = \Pr[G_i] = \frac{1}{n}$. Hence, $\mathbb{E}[G] = 1$ meaning that on average one person will correctly receive their coat.

Q: If G_i is the indicator r.v. that person i gets their own coat, are the random variables $G_1, G_2, \ldots G_n$ mutually independent?

No. Since if $G_1 = G_2 = \ldots G_{n-1} = 1$, then,
$\Pr[G_n = 1 | (G_1 = 1 \cap G_2 = 1 \cap \ldots \cap G_{n-1} = 1)] = 1 \neq \frac{1}{n} = \Pr[G_n = 1]$. Conditioning on $(G_1, G_2, \ldots, G_{n-1})$ changes $\Pr[G_n]$, and hence the r.v's are not independent. Notice that we used the linearity of expectation even though these r.v's are not mutually independent.

Questions?