

CMPT 409/981: Optimization for Machine Learning

Lecture 18

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Adam: $w_{k+1} = \Pi_{\mathcal{C}}^k[w_k - \eta_k A_k^{-1} m_k]$; $m_k = \beta_1 m_{k-1} + (1 - \beta_1) \nabla f_k(w_k)$. $A_k = G_k^{\frac{1}{2}}$,
 $G_k = (1 - \beta_2) \sum_{i=1}^k \beta_2^{k-i} [\nabla f_i(w_i) \nabla f_i(w_i)^\top]$ and $m_k = (1 - \beta_1) \sum_{i=1}^k \beta_1^{k-i} [\nabla f_i(w_i)]$.

Adam does not guarantee that $A_k \succeq A_{k-1}$ for all k . There are simple counter-examples that exploit this and can result in the non-convergence of Adam.

AMSGrad – fixing the convergence of Adam

AMSGrad [RKK19] fixes the non-convergence of Adam by making a small modification (in red) to Adam. It has the following update – for $\beta_1, \beta_2 \in (0, 1)$,

$$\begin{aligned} G_k &= \beta_2 G_{k-1} + (1 - \beta_2) \text{diag} [\nabla f_k(w_k) \nabla f_k(w_k)^\top] \quad ; \quad A_k = \max\{G_k^{\frac{1}{2}}, A_{k-1}\} \\ w_{k+1} &= \Pi_C^k[w_k - \eta_k A_k^{-1} m_k]; \quad ; \quad m_k = \beta_1 m_{k-1} + (1 - \beta_1) \nabla f_k(w_k) \\ \Pi_C^k[v_{k+1}] &:= \arg \min_{w \in C} \frac{1}{2} \|w - v_{k+1}\|_{A_k}^2, \end{aligned}$$

where, for diagonal matrices A and B , $C = \max\{A, B\} \implies \forall i \in [d], C_{i,i} = \max\{A_{i,i}, B_{i,i}\}$.

The AMSGrad update ensures that $A_k \succeq A_{k-1}$ and is guaranteed to converge.

Convergence of AMSGrad

For a sequence of convex, G -Lipschitz functions,

- [RKK19] prove an $O(D^2 Gd \sqrt{T})$ regret bound for AMSGrad. The proof requires $\eta_k = O(1/\sqrt{k})$ and $\beta_1^k = O(\exp(-k))$ (decreasing step-size and momentum).
- [AMMC20] prove the same regret guarantee with a decreasing step-size, but constant β_1 .

Since AMSGrad is typically used with a constant step-size and momentum term, [VLK⁺20] analyze the convergence of this variant for smooth, convex functions. For this analysis, we will consider the stochastic optimization setting and make the following simplifying assumptions:

- **Bounded eigenvalues:** The eigenvalues of A_k are bounded for all iterations, i.e. for all k , there exists constants $a_{\min}, a_{\max} > 0$ such that $a_{\min}I_d \preceq A_k \preceq a_{\max}I_d$. This condition can be algorithmically ensured for the diagonal preconditioner.
- **Near-interpolation:** There exists a $\zeta < \infty$ such that $\zeta^2 := \mathbb{E}_i[f_i(w^*) - f_i^*]$ is small.
- **Bounded iterates:** The domain is unconstrained i.e. $\mathcal{C} = \mathbb{R}^d$ but the iterates remain bounded in a set of diameter D , i.e. for all k , $\|w_k - w^*\|^2 \leq D^2$.

Minimizing convex, smooth functions using AMSGrad

Let us prove the convergence of AMSGrad when minimizing a finite-sum of convex, L -smooth functions. As a warm-up, let us first analyze the case where $\beta_1 = 0$.

Claim: For minimizing a finite-sum of convex, L -smooth functions, assuming that for all $k \in [T]$, $\|w_k - w^*\|^2 \leq D^2$, $a_{\min} I_d \preceq A_k \preceq a_{\max} I_d$, T iterations of the AMSGrad update with $\eta = \frac{a_{\min}}{2L}$, $\beta_1 = 0$ returns an iterate $\bar{w} = \sum_{k=1}^T w_k / T$ such that,

$$\mathbb{E}[f(\bar{w}_T) - f(w^*)] \leq \frac{D^2 2dL a_{\max}}{a_{\min} T} + \zeta^2 \quad \text{where} \quad \zeta^2 := \mathbb{E}_i[f_i(w^*) - f_i^*].$$

Proof: Define $P_k := \frac{A_k}{\eta}$. Starting from the update, $v_{k+1} = w_k - P_k^{-1} \nabla f_{ik}(w_k)$ and using the same steps as the AdaGrad proof,

$$\begin{aligned} v_{k+1} - w^* &= w_k - P_k^{-1} \nabla f_{ik}(w_k) - w^* \implies P_k[v_{k+1} - w^*] = P_k[w_k - w^*] - \nabla f_{ik}(w_k) \\ \implies [v_{k+1} - w^*]^\top P_k[v_{k+1} - w^*] &= [w_k - w^* - P_k^{-1} \nabla f_{ik}(w_k)]^\top [P_k[w_k - w^*] - \nabla f_{ik}(w_k)] \\ \|v_{k+1} - w^*\|_{P_k}^2 &= \|w_k - w^*\|_{P_k}^2 - 2\langle \nabla f_{ik}(w_k), w_k - w^* \rangle + [P_k^{-1} \nabla f_{ik}(w_k)]^\top [\nabla f_{ik}(w_k)] \\ \implies \|v_{k+1} - w^*\|_{P_k}^2 &= \|w_k - w^*\|_{P_k}^2 - 2\langle \nabla f_{ik}(w_k), w_k - w^* \rangle + \|\nabla f_{ik}(w_k)\|_{P_k^{-1}}^2 \end{aligned}$$

Minimizing convex, smooth functions using AMSGrad

Recall that $\|v_{k+1} - w^*\|_{P_k}^2 = \|w_k - w^*\|_{P_k}^2 - 2\langle \nabla f_{ik}(w_k), w_k - w^* \rangle + \|\nabla f_{ik}(w_k)\|_{P_k^{-1}}^2$. Since $\mathcal{C} = \mathbb{R}^d$, $w_{k+1} = v_{k+1}$,

$$\implies \|w_{k+1} - w^*\|_{P_k}^2 = \|w_k - w^*\|_{P_k}^2 - 2\langle \nabla f_{ik}(w_k), w_k - w^* \rangle + \|\nabla f_{ik}(w_k)\|_{P_k^{-1}}^2$$

$$f_{ik}(w_k) - f_{ik}(w^*) \leq \frac{\|w_k - w^*\|_{P_k}^2 - \|w_{k+1} - w^*\|_{P_k}^2}{2} + \frac{1}{2} \|\nabla f_{ik}(w_k)\|_{P_k^{-1}}^2 \quad (\text{Convexity of } f_{ik})$$

$$\implies \mathbb{E}[f(w_k) - f(w^*)] \leq \mathbb{E} \left[\frac{\|w_k - w^*\|_{P_k}^2 - \|w_{k+1} - w^*\|_{P_k}^2}{2} \right] + \frac{1}{2} \mathbb{E} \left[\|\nabla f_{ik}(w_k)\|_{P_k^{-1}}^2 \right]$$

$$\mathbb{E} \|\nabla f_{ik}(w_k)\|_{P_k^{-1}}^2 \leq \frac{\eta}{a_{\min}} \mathbb{E} \left[\|\nabla f_{ik}(w_k)\|^2 \right] \leq \frac{2L\eta}{a_{\min}} \mathbb{E} [f_{ik}(w_k) - f_{ik}^*] \leq \frac{2L\eta}{a_{\min}} \mathbb{E} [f(w_k) - f(w^*)] + \frac{2L\eta\zeta^2}{a_{\min}}$$

$$\implies \mathbb{E}[f(w_k) - f(w^*)] \leq \mathbb{E} \left[\frac{\|w_k - w^*\|_{P_k}^2 - \|w_{k+1} - w^*\|_{P_k}^2}{2} \right] + \frac{L\eta}{a_{\min}} \mathbb{E} [f(w_k) - f(w^*)] + \frac{L\eta\zeta^2}{a_{\min}}$$

Minimizing convex, smooth functions using AMSGrad

Recall that $\mathbb{E}[f(w_k) - f(w^*)] \leq \mathbb{E} \left[\frac{\|w_k - w^*\|_{P_k}^2 - \|w_{k+1} - w^*\|_{P_k}^2}{2} \right] + \frac{L\eta}{a_{\min}} \mathbb{E}[f(w_k) - f(w^*)] + \frac{L\eta\zeta^2}{a_{\min}}.$

Setting $\eta = \frac{a_{\min}}{2L}$ and rearranging,

$$\mathbb{E}[f(w_k) - f(w^*)] \leq \mathbb{E} \left[\|w_k - w^*\|_{P_k}^2 - \|w_{k+1} - w^*\|_{P_k}^2 \right] + \zeta^2$$

Taking expectation w.r.t the randomness in iterations $k = 1$ to T and summing,

$$\sum_{k=1}^T \mathbb{E}[f(w_k) - f(w^*)] \leq \sum_{k=1}^T \mathbb{E} \left[\|w_k - w^*\|_{P_k}^2 - \|w_{k+1} - w^*\|_{P_k}^2 \right] + \zeta^2 T$$

Dividing by T , using Jensen's inequality on the LHS and the definition of \bar{w}_T

$$\mathbb{E}[f(\bar{w}_T) - f(w^*)] \leq \frac{\sum_{k=1}^T \mathbb{E} \left[\|w_k - w^*\|_{P_k}^2 - \|w_{k+1} - w^*\|_{P_k}^2 \right]}{T} + \zeta^2$$

Minimizing convex, smooth functions using AMSGrad

Recall that $\mathbb{E}[f(\bar{w}_T) - f(w^*)] \leq \frac{\sum_{k=1}^T \mathbb{E}[\|w_k - w^*\|_{P_k}^2 - \|w_{k+1} - w^*\|_{P_k}^2]}{T} + \zeta^2$.

$$\begin{aligned}
 & \sum_{k=1}^T \left[\|w_k - w^*\|_{P_k}^2 - \|w_{k+1} - w^*\|_{P_k}^2 \right] \\
 &= \sum_{k=2}^T [(w_k - w^*)^\top [P_k - P_{k-1}] (w_k - w^*)] + \|w_1 - w^*\|_{P_1}^2 - \|w_{T+1} - w^*\|_{P_T}^2 \\
 &\leq \sum_{k=2}^T \|w_k - w^*\|^2 \lambda_{\max}[P_k - P_{k-1}] + \|w_1 - w^*\|_{P_1}^2 \leq \sum_{k=2}^T D^2 \lambda_{\max}[P_k - P_{k-1}] + \|w_1 - w^*\|_{P_1}^2 \\
 &\quad \text{(Since } A_{k-1} \preceq A_k, P_{k-1} \preceq P_k, \lambda_{\max}[P_k - P_{k-1}] \geq 0 \text{ and } \|w_k - w^*\|^2 \leq D) \\
 &\sum_{k=1}^T \left[\|w_k - w^*\|_{P_k}^2 - \|w_{k+1} - w^*\|_{P_k}^2 \right] \leq D^2 \sum_{k=2}^T \text{Tr}[P_k - P_{k-1}] + \|w_1 - w^*\|_{P_1}^2 \leq D^2 \text{Tr}[P_T] \\
 &\quad \text{(By linearity of trace, and bounding } \|w_1 - w^*\|_{P_1}^2 \leq D^2 \text{Tr}[P_1])
 \end{aligned}$$

Minimizing convex, smooth functions using AMSGrad

Recall that $\mathbb{E}[f(\bar{w}_T) - f(w^*)] \leq \frac{D^2 \text{Tr}[P_T]}{T} + \zeta^2$.

$$\begin{aligned} D^2 \text{Tr}[P_T] &\leq \frac{D^2}{\eta} \text{Tr}[A_T] = \frac{D^2 2L \text{Tr}[A_T]}{a_{\min}} \leq \frac{D^2 2L d \lambda_{\max}[A_T]}{a_{\min}} \leq \frac{D^2 2L d a_{\max}}{a_{\min}} \\ \implies \mathbb{E}[f(\bar{w}_T) - f(w^*)] &\leq \frac{D^2 2dL a_{\max}}{a_{\min} T} + \zeta^2 \end{aligned}$$

When minimizing smooth, convex functions, AMSGrad with a constant step-size *without momentum* will converge to a neighbourhood of the solution at an $O(1/T)$ rate. Similar to SGD, this neighbourhood depends on ζ , the extent to which interpolation is violated.

Next, we will consider the $\beta_1 \neq 0$ case and prove a similar convergence result for constant step-size AMSGrad.

Questions?

Minimizing convex, smooth functions using AMSGrad

Claim: For minimizing a finite-sum of convex, L -smooth functions, assuming that for all $k \in [T]$, $\|w_k - w^*\|^2 \leq D^2$, $a_{\min} l_d \preceq A_k \preceq a_{\max} l_d$, T iterations of the AMSGrad update with $\eta = \frac{1-\beta}{1+\beta} \frac{a_{\min}}{2L}$, $\beta_1 = \beta \in (0, 1)$ returns an iterate $\bar{w} = \sum_{k=1}^T w_k / T$ such that,

$$\mathbb{E}[f(\bar{w}_T) - f(w^*)] \leq \left(\frac{1+\beta}{1-\beta} \right)^2 \frac{D^2 2dL a_{\max}}{a_{\min} T} + \zeta^2 \quad \text{where} \quad \zeta^2 := \mathbb{E}_i[f_i(w^*) - f_i^*].$$

Proof: Proceeding similar to the case for $\beta_1 = 0$, define $P_k := \frac{A_k}{\eta}$ and $\beta := \beta_1$. Starting from the update, $v_{k+1} = w_k - P_k^{-1} m_k$ where $m_k = \beta m_{k-1} + (1-\beta) \nabla f_{ik}(w_k)$.

$$v_{k+1} - w^* = w_k - P_k^{-1} m_k - w^* \implies P_k[v_{k+1} - w^*] = P_k[w_k - w^*] - m_k$$

$$[v_{k+1} - w^*]^\top P_k[v_{k+1} - w^*] = [w_k - w^* - P_k^{-1} m_k]^\top [P_k[w_k - w^*] - m_k]$$

$$\|v_{k+1} - w^*\|_{P_k}^2 = \|w_k - w^*\|_{P_k}^2 - 2\langle m_k, w_k - w^* \rangle + [P_k^{-1} m_k]^\top [m_k]$$

$$\|w_{k+1} - w^*\|_{P_k}^2 = \|w_k - w^*\|_{P_k}^2 - 2(1-\beta) \langle w_k - w^*, \nabla f_{ik}(w_k) \rangle - 2\beta \langle w_k - w^*, m_{k-1} \rangle + \|m_k\|_{P_k^{-1}}^2.$$

(Since $\mathcal{C} = \mathbb{R}^d$, $w_{k+1} = v_{k+1}$)

Minimizing convex, smooth functions using AMSGrad

$$\|w_{k+1} - w^*\|_{P_k}^2 = \|w_k - w^*\|_{P_k}^2 - 2(1-\beta) \langle w_k - w^*, \nabla f_{ik}(w_k) \rangle - 2\beta \langle w_k - w^*, m_{k-1} \rangle + \|m_k\|_{P_k^{-1}}^2.$$

To simplify the $\langle w_k - w^*, m_{k-1} \rangle$ term, we will prove the following lemma: for any set of vectors a, b, c, d , if $a = b + c$, then, $2\langle c, a - d \rangle = \|a - d\|^2 + \|a - b\|^2 - \|b - d\|^2$.

$$\|a - d\|^2 = \|b + c - d\|^2 = \|b - d\|^2 + 2\langle a - b, b - d \rangle + \|a - b\|^2 \quad (a = b + c, c = b - a)$$

$$\|a - d\|^2 = \|b - d\|^2 + 2\langle a - b, b - a + a - d \rangle + \|a - b\|^2 = \|b - d\|^2 + 2\langle c, a - d \rangle - \|a - b\|^2$$

$$\implies 2\langle c, a - d \rangle = \|a - d\|^2 + \|a - b\|^2 - \|b - d\|^2$$

$$-2\langle w_k - w^*, m_{k-1} \rangle = -2\langle w_k - w^*, P_{k-1}(w_{k-1} - w_k) \rangle = 2\langle P_{k-1}^{1/2}(w_k - w^*), P_{k-1}^{1/2}(w_k - w_{k-1}) \rangle$$

$$= 2\langle \underbrace{P_{k-1}^{1/2}(w_k - w^*)}_{=c}, \underbrace{P_{k-1}^{1/2}(w_k - w^*)}_{=a} - \underbrace{P_{k-1}^{1/2}(w_{k-1} - w^*)}_{=d} \rangle$$

$$\leq \|w_k - w_{k-1}\|_{P_{k-1}}^2 + \|w_k - w^*\|_{P_{k-1}}^2 - \|w_{k-1} - w^*\|_{P_{k-1}}^2$$

$$(\text{Lemma with } a = c = P_{k-1}^{1/2}(w_k - w^*), b = 0, d = P_{k-1}^{1/2}(w_{k-1} - w^*))$$

$$\implies -2\langle w_k - w^*, m_{k-1} \rangle \leq \|m_{k-1}\|_{P_{k-1}^{-1}}^2 + \|w_k - w^*\|_{P_k}^2 - \|w_{k-1} - w^*\|_{P_{k-1}}^2$$

$$(\text{Since } P_{k-1}(w_k - w_{k-1}) = m_{k-1} \text{ and } P_{k-1} \preceq P_k) \quad 10$$

Minimizing convex, smooth functions using AMSGrad

Putting everything together,

$$\begin{aligned}\|w_{k+1} - w^*\|_{P_k}^2 &= \|w_k - w^*\|_{P_k}^2 - 2(1 - \beta) \langle w_k - w^*, \nabla f_{ik}(w_k) \rangle - 2\beta \langle w_k - w^*, m_{k-1} \rangle + \|m_k\|_{P_k^{-1}}^2 \\ &\leq \|w_k - w^*\|_{P_k}^2 - 2(1 - \beta) \langle w_k - w^*, \nabla f_{ik}(w_k) \rangle \\ &\quad + \beta \left[\|m_{k-1}\|_{P_{k-1}^{-1}}^2 + \|w_k - w^*\|_{P_k}^2 - \|w_{k-1} - w^*\|_{P_{k-1}}^2 \right] + \|m_k\|_{P_k^{-1}}^2 \\ &\leq \|w_k - w^*\|_{P_k}^2 - 2(1 - \beta) [f_{ik}(w_k) - f_{ik}(w^*)] \quad (\text{By convexity}) \\ &\quad + \beta \left[\|m_{k-1}\|_{P_{k-1}^{-1}}^2 + \|w_k - w^*\|_{P_k}^2 - \|w_{k-1} - w^*\|_{P_{k-1}}^2 \right] + \|m_k\|_{P_k^{-1}}^2\end{aligned}$$

$$\begin{aligned}\implies & 2(1 - \beta) [f_{ik}(w_k) - f_{ik}(w^*)] \\ & \leq \underbrace{\left[\|w_k - w^*\|_{P_k}^2 - \|w_{k+1} - w^*\|_{P_k}^2 \right]}_{\text{Will telescope}} + \beta \underbrace{\left[\|w_k - w^*\|_{P_k}^2 - \|w_{k-1} - w^*\|_{P_{k-1}}^2 \right]}_{\text{Will telescope}} \\ & \quad + \underbrace{\left[\beta \|m_{k-1}\|_{P_{k-1}^{-1}}^2 + \|m_k\|_{P_k^{-1}}^2 \right]}_{\text{Will handle next}}\end{aligned}$$

Minimizing convex, smooth functions using AMSGrad

Let us focus on bounding the $\beta \|m_{k-1}\|_{P_{k-1}^{-1}}^2 + \|m_k\|_{P_k^{-1}}^2$ term.

$$\begin{aligned} & \beta \|m_{k-1}\|_{P_{k-1}^{-1}}^2 + \|m_k\|_{P_k^{-1}}^2 \\ &= \beta \|m_{k-1}\|_{P_{k-1}^{-1}}^2 + (1 + \delta) \|m_k\|_{P_k^{-1}}^2 - \delta \|m_k\|_{P_k^{-1}}^2 && \text{(For some } \delta > 0\text{)} \\ &= \beta \|m_{k-1}\|_{P_{k-1}^{-1}}^2 + (1 + \delta) \|\beta m_{k-1} + (1 - \beta) \nabla f_{ik}(w_k)\|_{P_k^{-1}}^2 - \delta \|m_k\|_{P_k^{-1}}^2 \\ &\leq \beta \|m_{k-1}\|_{P_{k-1}^{-1}}^2 + (1 + \delta) \left[(1 + \epsilon) \beta^2 \|m_{k-1}\|_{P_k^{-1}}^2 + (1 + 1/\epsilon) (1 - \beta)^2 \|\nabla f_{ik}(w_k)\|_{P_k^{-1}}^2 \right] - \delta \|m_k\|_{P_k^{-1}}^2 \\ &\quad \text{(By Young's inequality: for some } \epsilon > 0, (a + b)^2 = a^2 + 2ab + b^2 \leq a^2(1 + \epsilon) + b^2(1 + 1/\epsilon)\text{)} \\ &= \left[(\beta + (1 + \delta)(1 + \epsilon) \beta^2) \|m_{k-1}\|_{P_{k-1}^{-1}}^2 - \delta \|m_k\|_{P_k^{-1}}^2 \right] + (1 + \delta)(1 + 1/\epsilon) (1 - \beta)^2 \|\nabla f_{ik}(w_k)\|_{P_k^{-1}}^2 \\ &\quad \text{(Since } P_{k-1} \preceq P_k, P_{k-1}^{-1} \succeq P_k^{-1}\text{)} \end{aligned}$$

We want $\beta + (1 + \delta)(1 + \epsilon) \beta^2 = \delta$. Hence, $\delta = \frac{\beta + \beta^2(1 + \epsilon)}{1 - (1 + \epsilon)\beta^2}$. Since $\delta > 0 \implies \beta < \frac{1}{\sqrt{1 + \epsilon}}$,

$$\beta \|m_{k-1}\|_{P_{k-1}^{-1}}^2 + \|m_k\|_{P_k^{-1}}^2 \leq \delta \left[\|m_{k-1}\|_{P_{k-1}^{-1}}^2 - \|m_k\|_{P_k^{-1}}^2 \right] + (1 + \delta)(1 + 1/\epsilon) (1 - \beta)^2 \|\nabla f_{ik}(w_k)\|_{P_k^{-1}}^2$$

Minimizing convex, smooth functions using AMSGrad

Putting everything together and taking expectation w.r.t randomness at iteration k ,

$$\begin{aligned}
 & 2(1 - \beta) \mathbb{E}[f(w_k) - f(w^*)] \\
 & \leq \mathbb{E} \left[\|w_k - w^*\|_{P_k}^2 - \|w_{k+1} - w^*\|_{P_k}^2 \right] + \beta \mathbb{E} \left[\|w_k - w^*\|_{P_k}^2 - \|w_{k-1} - w^*\|_{P_{k-1}}^2 \right] \\
 & + \delta \mathbb{E} \left[\|m_{k-1}\|_{P_{k-1}^{-1}}^2 - \|m_k\|_{P_k^{-1}}^2 \right] + (1 + \delta)(1 + 1/\epsilon) (1 - \beta)^2 \mathbb{E} \|\nabla f_{ik}(w_k)\|_{P_k^{-1}}^2
 \end{aligned}$$

Bounding $\mathbb{E} \|\nabla f_{ik}(w_k)\|_{P_k^{-1}}^2$ using the smoothness of f_{ik} ,

$$\begin{aligned}
 \mathbb{E} \|\nabla f_{ik}(w_k)\|_{P_k^{-1}}^2 & \leq \frac{\eta}{a_{\min}} \mathbb{E} \left[\|\nabla f_{ik}(w_k)\|^2 \right] \leq \frac{2L\eta}{a_{\min}} \mathbb{E} [f_{ik}(w_k) - f_{ik}^*] \leq \frac{2L\eta}{a_{\min}} \mathbb{E} [f(w_k) - f(w^*)] + \frac{2L\eta\zeta^2}{a_{\min}} \\
 & \left[\underbrace{2(1 - \beta) - (1 + \delta)(1 + 1/\epsilon) (1 - \beta)^2}_{:=\alpha} \frac{2L\eta}{a_{\min}} \right] \mathbb{E}[f(w_k) - f(w^*)] \\
 & \leq \mathbb{E} \left[\|w_k - w^*\|_{P_k}^2 - \|w_{k+1} - w^*\|_{P_k}^2 \right] + \beta \mathbb{E} \left[\|w_k - w^*\|_{P_k}^2 - \|w_{k-1} - w^*\|_{P_{k-1}}^2 \right] \\
 & + \delta \mathbb{E} \left[\|m_{k-1}\|_{P_{k-1}^{-1}}^2 - \|m_k\|_{P_k^{-1}}^2 \right] + (1 + \delta)(1 + 1/\epsilon) (1 - \beta)^2 \frac{2L\eta\zeta^2}{a_{\min}}
 \end{aligned}$$

Minimizing convex, smooth functions using AMSGrad

Taking expectation w.r.t randomness from iterations $k = 1$ to T and summing,

$$\begin{aligned}
 & \alpha \sum_{k=1}^T \mathbb{E}[f(w_k) - f(w^*)] \\
 & \leq \underbrace{\mathbb{E} \sum_{k=1}^T \left[\|w_k - w^*\|_{P_k}^2 - \|w_{k+1} - w^*\|_{P_k}^2 \right]}_{:= T_1} + \underbrace{\beta \mathbb{E} \sum_{k=1}^T \left[\|w_k - w^*\|_{P_k}^2 - \|w_{k-1} - w^*\|_{P_{k-1}}^2 \right]}_{:= T_2} \\
 & \quad + \underbrace{\delta \mathbb{E} \sum_{k=1}^T \left[\|m_{k-1}\|_{P_{k-1}^{-1}}^2 - \|m_k\|_{P_k^{-1}}^2 \right]}_{:= T_3} + (1 + \delta)(1 + 1/\epsilon)(1 - \beta)^2 \frac{2L\eta\zeta^2 T}{a_{\min}}
 \end{aligned}$$

As before, $T_1 \leq \frac{D^2}{\eta} \text{Tr}[A_T] \leq \frac{D^2 d a_{\max}}{\eta}$. $T_2 \leq \frac{1}{\eta} \|w_T - w^*\|_{A_T}^2 \leq \frac{D^2 d a_{\max}}{\eta}$. $T_3 \leq \frac{1}{\eta} \|m_0\|_{A_0}^2 = 0$.

$$\implies \alpha \sum_{k=1}^T \mathbb{E}[f(w_k) - f(w^*)] \leq \frac{D^2 d a_{\max} (1 + \beta)}{\eta} + (1 + \delta)(1 + 1/\epsilon)(1 - \beta)^2 \frac{2L\eta\zeta^2 T}{a_{\min}}$$

Minimizing convex, smooth functions using AMSGrad

Recall that $\alpha \sum_{k=1}^T \mathbb{E}[f(w_k) - f(w^*)] \leq \frac{D^2 d a_{\max} (1+\beta)}{\eta} + (1+\delta)(1+1/\epsilon) (1-\beta)^2 \frac{2L\eta \zeta^2 T}{a_{\min}}$. Here, $\delta = \frac{\beta + \beta^2(1+\epsilon)}{1 - (1+\epsilon)\beta^2}$, $\beta < \frac{1}{\sqrt{1+\epsilon}}$ and $\alpha = 2(1-\beta) - (1+\delta)(1+1/\epsilon) (1-\beta)^2 2L\eta/a_{\min}$. For $\epsilon > 0$, setting

$$\beta = \frac{1}{1+\epsilon} < \frac{1}{\sqrt{1+\epsilon}} \implies \delta = \frac{\beta + \beta^2 \frac{1}{\beta}}{1 - \frac{1}{\beta}\beta^2} = \frac{2\beta}{1-\beta}$$

$$\alpha = 2(1-\beta) - \left(1 + \frac{2\beta}{1-\beta}\right) (1+1/\epsilon) (1-\beta)^2 2L\eta/a_{\min} = 2(1-\beta) - (1+\beta) 2L\eta/a_{\min}$$

For $\alpha > 0$, we want that $\eta < \frac{1-\beta}{1+\beta} \frac{a_{\min}}{L}$. Setting $\eta = \frac{1-\beta}{1+\beta} \frac{a_{\min}}{2L}$, $\alpha = 1-\beta$. With these settings,

$$\implies \sum_{k=1}^T \mathbb{E}[f(w_k) - f(w^*)] \leq \frac{D^2 d a_{\max} (1+\beta)}{\alpha \eta} + \frac{(1-\beta) \zeta^2 T}{\alpha}$$

Dividing by T , using Jensen's inequality on the LHS and using the definition of \bar{w}_T ,

$$\mathbb{E}[f(\bar{w}) - f(w^*)] \leq \left(\frac{1+\beta}{1-\beta}\right)^2 \frac{D^2 2dL a_{\max}}{a_{\min}} + \zeta^2$$


Minimizing convex, smooth functions using AMSGrad

When minimizing smooth, convex functions, AMSGrad with a constant step-size will converge to a neighbourhood of the solution at an $O(1/T)$ rate. Similar to SGD, this neighbourhood depends on ζ , the extent to which interpolation is violated.

Unlike the guarantee for AdaGrad that holds for any η (Slide 5, Lecture 16), the above AMSGrad guarantee above requires knowledge of L to set the step-size. Moreover, it results in an $O(1/T + \zeta^2)$ bound as compared to the noise-adaptive $O(1/T + \zeta^2/\sqrt{T})$ bound for AdaGrad (using online-batch conversion with the regret guarantee).

Since Stochastic Heavy Ball (SHB) is a special case of AMSGrad with $A_k = I_d$, we can prove a similar $O(1/T + \zeta^2)$ rate of convergence (Prove in Assignment 4!).

Questions?

-  Ahmet Alacaoglu, Yura Malitsky, Panayotis Mertikopoulos, and Volkan Cevher, *A new regret analysis for adam-type algorithms*, International conference on machine learning, PMLR, 2020, pp. 202–210.
-  Sashank J Reddi, Satyen Kale, and Sanjiv Kumar, *On the convergence of adam and beyond*, arXiv preprint arXiv:1904.09237 (2019).
-  Sharan Vaswani, Issam H Laradji, Frederik Kunstner, Si Yi Meng, Mark Schmidt, and Simon Lacoste-Julien, *Adaptive gradient methods converge faster with over-parameterization (and you can do a line-search)*.