CMPT 210: Probability and Computing

Lecture 20

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Recap

- **Pairwise Independence**: Random variables $R_1, R_2, R_3, \ldots R_n$ are pairwise independent if for any pair R_i and R_j, for $x \in \text{Range}(R_i)$ and $y \in \text{Range}(R_j)$, $\Pr[(R_i = x) \cap (R_j = y)] = \Pr[R_i = x] \Pr[R_j = y]$.
- **Variance**: Standard way to measure the deviation from the mean. For r.v. X, $\text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_{x \in \text{Range}(X)} (x - \mu)^2 \Pr[X = x]$, where $\mu := \mathbb{E}[X]$.
- **Alternate Definition**: $\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.
- If $X \sim \text{Ber}(p)$, $\text{Var}[X] = p(1-p) \leq \frac{1}{4}$.

Back to throwing dice

Q: For a standard dice, if X is the r.v. equal to the number that comes up, compute $\text{Var}[X]$.

Recall that, for a standard dice, $X \sim \text{Uniform}(\{1,2,3,4,5,6\})$ and hence,

$$\mathbb{E}[X^2] = \sum_{x \in \{1,2,3,4,5,6\}} x^2 \Pr[X = x] = \frac{1}{6}\left[1^2 + 2^2 + \ldots + 6^2\right] = \frac{91}{6}$$

$$(\mathbb{E}[X])^2 = \left(\sum_{x \in \{1,2,3,4,5,6\}} x \Pr[X = x]\right)^2 = \left(\frac{1}{6}\left[1 + 2 + \ldots + 6\right]\right)^2 = \frac{49}{4}$$

$$\implies \text{Var}[X] = \frac{91}{6} - \frac{49}{4} \approx 2.917$$

Q: If $X \sim \text{Uniform}(\{v_1, v_2, \ldots v_n\})$, compute $\text{Var}[X]$.

$$\mathbb{E}[X] = \sum_{i=1}^{n} v_i \Pr[X = v_i] = \frac{1}{n}\left[v_1 + v_2 + \ldots v_n\right] \quad ; \quad \mathbb{E}[X^2] = \frac{1}{n}\left[v_1^2 + v_2^2 + \ldots v_n^2\right].$$

$$\implies \text{Var}[X] = \frac{\left[v_1^2 + v_2^2 + \ldots v_n^2\right]}{n} - \left(\frac{\left[v_1 + v_2 + \ldots v_n\right]}{n}\right)^2$$

Variance - Examples

Q: Calculate $\text{Var}[W]$, $\text{Var}[Y]$ and $\text{Var}[Z]$ whose PDF's are given as:

$$\begin{aligned} W &= 0 && \text{(with } p = 1) \\ Y &= -1 && \text{(with } p = 1/2) \\ &= +1 && \text{(with } p = 1/2) \\ Z &= -1000 && \text{(with } p = 1/2) \\ &= +1000 && \text{(with } p = 1/2) \end{aligned}$$

Recall that $\mathbb{E}[W] = \mathbb{E}[Y] = \mathbb{E}[Z] = 0$.

$\text{Var}[W] = \mathbb{E}[W^2] - (\mathbb{E}[W])^2 = \mathbb{E}[W^2] = \sum_{w \in \text{Range}(W)} w^2 \Pr[W = w] = 0^2(1) = 0$. The variance of W is zero because it can only take one value and the r.v. does not "vary".
$\text{Var}[Y] = \mathbb{E}[Y^2] = \sum_{y \in \text{Range}(Y)} y^2 \Pr[Y = y] = (-1)^2(1/2) + (1)^2(1/2) = 1$.

$\text{Var}[Z] = \mathbb{E}[Z^2] = \sum_{z \in \text{Range}(Z)} z^2 \Pr[Z = z] = (-1000)^2(1/2) + (1000)^2(1/2) = 10^6$.

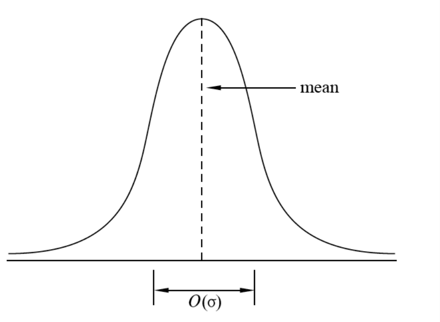
- Hence, the variance can be used to distinguish between r.v.'s that have the same mean.

Standard Deviation

Standard Deviation: For r.v. X, the standard deviation in X is defined as:

$$\sigma_X := \sqrt{\mathrm{Var}[X]} = \sqrt{\mathbb{E}[X^2] - (\mathbb{E}[X])^2}$$

- Standard deviation has the same units as expectation.



- Standard deviation for a "bell"-shaped distribution indicates how wide the "main part" of the distribution is.

Q: If $R \sim \text{Geo}(p)$, calculate $\text{Var}[R]$.

$$\text{Var}[R] = \mathbb{E}[R^2] - (\mathbb{E}[R])^2 = \mathbb{E}[R^2] - \frac{1}{p^2}$$

Recall that for a coin s.t. $\Pr[\text{heads}] = p$, R is the r.v. equal to the number of coin tosses we need to get the first heads. Let A be the event that we get a heads in the first toss. Using the law of total expectation,

$$\mathbb{E}[R^2] = \mathbb{E}[R^2|A] \Pr[A] + \mathbb{E}[R^2|A^c] \Pr[A^c]$$

$\mathbb{E}[R^2|A] = 1$ ($R^2 = 1$ if we get a heads in the first coin toss) and $\Pr[A] = p$. Hence,

$$\mathbb{E}[R^2] = (1)\,(p) + \mathbb{E}[R^2|A^c]\,(1-p) \quad ; \quad \mathbb{E}[R^2|A^c] = \sum_{k=1} k^2 \Pr[R = k|A^c]$$

Note that $\Pr[R = k|A^c] = \Pr[R = k|\text{ first toss is a tails}] = (1-p)^{k-2}\, p = \Pr[R = k-1]$

$$\implies \mathbb{E}[R^2|A^c] = \sum_{k=1} k^2 \Pr[R = k-1] = \sum_{t=0} (t+1)^2 \Pr[R = t] \qquad (t := k-1)$$

Variance - Examples

Continuing from the previous slide,

$$\begin{aligned}\mathbb{E}[R^2|A^c] &= \sum_{t=0}(t+1)^2 \Pr[R=t] = \sum_{t=0} t^2 \Pr[R=t] + 2\sum_{t=0} t \Pr[R=t] + \sum_{t=0} \Pr[R=t] \\ &= \sum_{t=1} t^2 \Pr[R=t] + 2\sum_{t=1} t \Pr[R=t] + \sum_{t=1} \Pr[R=t] = \mathbb{E}[R^2] + 2\mathbb{E}[R] + 1\end{aligned}$$

Putting everything together,

$$\begin{aligned}\mathbb{E}[R^2] &= (1)\,(p) + (\mathbb{E}[R^2] + 2\mathbb{E}[R] + 1])\,(1-p) \implies p\,\mathbb{E}[R^2] = p + 2(1-p)\mathbb{E}[R] + (1-p)\mathbb{E}[1] \\ &\implies p\,\mathbb{E}[R^2] = p + \frac{2(1-p)}{p} + (1-p) \qquad (\mathbb{E}[R] = \tfrac{1}{p},\ \mathbb{E}[1] = 1) \\ &\implies \mathbb{E}[R^2] = \frac{2(1-p)}{p^2} + \frac{1}{p} \implies \mathbb{E}[R^2] = \frac{2-p}{p^2} \\ &\implies \mathrm{Var}[R] = \mathbb{E}[R^2] - (\mathbb{E}[R])^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}\end{aligned}$$

Properties of Variance

Q: For constants a, b and r.v. R, prove that $\text{Var}[a\,R + b] = a^2\text{Var}[R]$.

Proof:

$$\begin{aligned}
\text{Var}[aR + b] &= \mathbb{E}[(aR + b)^2] - (\mathbb{E}[aR + b])^2 = \mathbb{E}[a^2R^2 + 2abR + b^2] - (\mathbb{E}[aR] + \mathbb{E}[b])^2 \\
&= (a^2\mathbb{E}[R^2] + 2ab\mathbb{E}[R] + b^2) - (a\mathbb{E}[R] + b)^2 \\
&= (a^2\mathbb{E}[R^2] + 2ab\mathbb{E}[R] + b^2) - (a^2(\mathbb{E}[R])^2 + 2ab\mathbb{E}[R] + b^2) \\
&= a^2\left[\mathbb{E}[R^2] - (\mathbb{E}[R])^2\right]
\end{aligned}$$

$$\implies \text{Var}[aR + b] = a^2\text{Var}[R]$$

- Similarly, for the standard deviation,

$$\sigma_{aR+b} = \sqrt{\text{Var}[aR + b]} = \sqrt{a^2\text{Var}[R]} = |a|\,\sigma_R$$

- Note the difference from the property of expectation,

$$\mathbb{E}[aR + b] = a\mathbb{E}[R] + b$$

Properties of Variance

Q: If the r.v's R_1 and R_2 are *independent* r.v., prove that $\text{Var}[R_1 + R_2] = \text{Var}[R_1] + \text{Var}[R_2]$.

- In order to prove this result, we need an additional definition: for 2 r.v's R_1 and R_2,

$$\mathbb{E}[R_1\, R_2] := \sum_{z \in \text{Range}(R_1) \times \text{Range}(R_2)} z \Pr[R_1\, R_2 = z] = \sum_{x \in \text{Range}(R_1)} \sum_{y \in \text{Range}(R_2)} x\, y \Pr[R_1 = x \cap R_2 = y]$$

Proof: We will first prove that for two independent r.v's R_1 and R_2, $\mathbb{E}[R_1\, R_2] = \mathbb{E}[R_1]\, \mathbb{E}[R_2]$.

$$\begin{aligned}
\mathbb{E}[R_1 R_2] &= \sum_{x \in \text{Range}(R_1)} \sum_{y \in \text{Range}(R_2)} x\, y \Pr[R_1 = x \cap R_2 = y] \\
&= \sum_{x \in \text{Range}(R_1)} \sum_{y \in \text{Range}(R_2)} xy \Pr[R_1 = x] \Pr[R_2 = y] && \text{(Independence)} \\
&= \sum_{x \in \text{Range}(R_1)} x \Pr[R_1 = x] \sum_{y \in \text{Range}(R_2)} y \Pr[R_2 = y] \\
&= \mathbb{E}[R_1]\, \mathbb{E}[R_2]
\end{aligned}$$

Properties of Variance

Continuing the proof from the previous slide and using the definition of $\mathrm{Var}[R_1 + R_2]$,

$$\begin{aligned}
\mathrm{Var}[R_1 + R_2] &= \mathbb{E}[(R_1 + R_2)^2] - (\mathbb{E}[R_1 + R_2])^2 \\
&= \mathbb{E}[R_1^2 + R_2^2 + 2R_1\,R_2] - (\mathbb{E}[R_1] + \mathbb{E}[R_2])^2 \\
&= \mathbb{E}[R_1^2 + R_2^2 + 2R_1\,R_2] - [(\mathbb{E}[R_1])^2 + (\mathbb{E}[R_2])^2 + 2\mathbb{E}[R_1]\,\mathbb{E}[R_2]] \\
&\hfill \text{(Expanding the terms)} \\
&= [\mathbb{E}[R_1^2] - (\mathbb{E}[R_1])^2] + [\mathbb{E}[R_2^2] - (\mathbb{E}[R_2])^2] + 2(\mathbb{E}[R_1\,R_2] - \mathbb{E}[R_1]\,\mathbb{E}[R_2]) \\
&\hfill \text{(Linearity of expectation)} \\
&= \mathrm{Var}[R_1] + \mathrm{Var}[R_2] + 2(\mathbb{E}[R_1\,R_2] - \mathbb{E}[R_1]\,\mathbb{E}[R_2]) \quad \text{(Definition of variance)}
\end{aligned}$$

From the previous slide, if R_1 and R_2 are independent, then, $\mathbb{E}[R_1\,R_2] = \mathbb{E}[R_1]\,\mathbb{E}[R_2]$,

$$\implies \mathrm{Var}[R_1 + R_2] = \mathrm{Var}[R_1] + \mathrm{Var}[R_2]$$

Properties of Variance

Q: For pairwise independent random variables $R_1, R_2, R_3, \ldots R_n$, $\text{Var}[\sum_{i=1}^n R_i] = \sum_{i=1}^n \text{Var}[R_i]$.

Proof: Following the same proof, we can show that for any pair of pairwise independent r.v's, R_i and R_j, $\mathbb{E}[R_i R_j] = \mathbb{E}[R_i]\,\mathbb{E}[R_j]$.

$$\begin{aligned}
\text{Var}[R_1 + R_2 + \ldots R_n] &= \mathbb{E}[(R_1 + R_2 + \ldots R_n)^2] - (\mathbb{E}[R_1 + R_2 + \ldots R_n])^2 \\
&= \sum_{i=1}^n [\mathbb{E}[R_i^2] - (\mathbb{E}[R_i])^2] + 2 \sum_{i,j | 1 \le i < j \le n} [\mathbb{E}[R_i R_j] - \mathbb{E}[R_i]\,\mathbb{E}[R_j]] \\
&\qquad \text{(Expanding the terms and using linearity of expectation)}
\end{aligned}$$

$$\implies \text{Var}[R_1 + R_2 + \ldots R_n] = \sum_{i=1}^n \text{Var}[R_i] \qquad \text{(Since the r.v's are pairwise independent)}$$

- In general, the pairwise independence of r.v.'s is a necessary condition for the linearity of variance. To see this, consider $R_1 = R_2 = R$ i.e. the two r.v's are not independent. In this case, $\text{Var}[R_1 + R_2] = \text{Var}[2R] = 4\text{Var}[R] \neq 2\text{Var}[R] = \text{Var}[R_1] + \text{Var}[R_2]$.

Variance - Examples

Q: If $R \sim \text{Bin}(n, p)$, calculate $\text{Var}[R]$.

Define R_i to be the indicator random variable that we get a heads in toss i of the coin. Recall that R is the random variable equal to the number of heads in n tosses.

Hence,

$$R = R_1 + R_2 + \ldots + R_n \implies \text{Var}[R] = \text{Var}[R_1 + R_2 + \ldots + R_n]$$

Since $R_1, R_2, \ldots, R_n$ are mutually independent indicator random variables and mutual independence implies pairwise independence,

$$\text{Var}[R] = \text{Var}[R_1] + \text{Var}[R_2] + \ldots + \text{Var}[R_n]$$

Since the variance of an indicator (Bernoulli) r.v. is $p(1-p)$,

$$\text{Var}[R] = n\,p\,(1-p).$$

Questions?

Matching Birthdays

Q: In a class of n students, what is the probability that two students share the same birthday? Assume that (i) each student is equally likely to be born on any day of the year, (ii) no leap years and (iii) student birthdays are independent of each other.

For $d := 365$ (since no leap years),

$$\Pr[\text{two students share the same birthday}] = 1 - \frac{d \times (d-1) \times (d-2) \times \ldots (d-(n-1))}{d^n}$$

Q: On average, how many pairs of students have matching birthdays?

Define M to be the number of pairs of students with matching birthdays. For a fixed ordering of the students, let $X_{i,j}$ be the indicator r.v. corresponding to the event $E_{i,j}$ that the birthdays of students i and j match. Hence,

$$M = \sum_{i,j \mid 1 \leq i < j \leq n} X_{i,j} \implies \mathbb{E}[M] = \mathbb{E}[\sum_{i,j \mid 1 \leq i < j \leq n} X_{i,j}] = \sum_{i,j \mid 1 \leq i < j \leq n} \mathbb{E}[X_{i,j}] = \sum_{i,j \mid 1 \leq i < j \leq n} \Pr[E_{i,j}]$$

(Linearity of expectation)

Matching Birthdays

For a pair of students i, j, let B_i be the r.v. equal to the day of student i's birthday. Range(B_i) = {1,2,. . . , d}. For all $k \in [d]$, $\Pr[B_i = k] = 1/d$ (each student is equally likely to be born on any day of the year).

$$E_{i,j} = (B_i = 1 \cap B_j = 1) \cup (B_i = 2 \cap B_j = 2) \cup \dots$$

$$\implies \Pr[E_{i,j}] = \sum_{k=1}^{d} \Pr[B_i = k \cap B_j = k] = \sum_{k=1}^{d} \Pr[B_i = k] \Pr[B_j = k] = \sum_{k=1}^{d} \frac{1}{d^2} = \frac{1}{d}$$

(student birthdays are independent of each other)

$$\implies \mathbb{E}[M] = \sum_{i,j|1 \le i < j \le n} \Pr[E_{i,j}] = \frac{1}{d} \sum_{i,j|1 \le i < j \le n} (1) = \frac{1}{d}[(n-1) + (n-2) + \ldots + 1] = \frac{n(n-1)}{2d}$$

Hence, in our class of 100 students, on average, there are $\frac{(100)(50)}{365} = 13.7$ students with matching birthdays.

Matching Birthdays

Q: Are the $X_{i,j}$ r.v's mutually independent?

No, because if $X_{i,j} = 1$ and $X_{j,k} = 1$, then,
$\Pr[X_{i,k} = 1 | X_{j,k} = 1 \cap X_{i,j} = 1] = 1 \neq \frac{1}{d} = \Pr[X_{i,k} = 1]$.

Q: Are the $X_{i,j}$ pairwise independent?

Yes, because for all i, j and i', j' (where $i \neq i'$), $\Pr[X_{i,j} = 1 | X_{i',j'} = 1] = \Pr[X_{i,j} = 1]$ because if students i' and j' have matching birthdays, it does not tell us anything about whether i and j have matching birthdays.

Matching Birthdays

Q: If M is the random variable equal to the number of pairs of students with matching birthdays, calculate $\mathrm{Var}[M]$.

$$\mathrm{Var}[M] = \mathrm{Var}[\sum_{i,j \mid 1 \le i < j \le n} X_{i,j}]$$

Since $X_{i,j}$ are pairwise independent, the variance of the sum is equal to the sum of the variance.

$$\implies \mathrm{Var}[M] = \sum_{i,j \mid 1 \le i < j \le n} \mathrm{Var}[X_{i,j}] = \sum_{i,j \mid 1 \le i < j \le n} \frac{1}{d}\left(1 - \frac{1}{d}\right) = \frac{1}{d}\left(1 - \frac{1}{d}\right)\frac{n(n-1)}{2}$$

(Since $X_{i,j}$ is an indicator (Bernoulli) r.v. and $\Pr[X_{i,j} = 1] = \frac{1}{d}$)

Hence, in our class of 75 students, the standard deviation for the matching birthdays is equal to $\sqrt{\frac{(100)(50)}{365}\frac{364}{365}} \approx 3.7$.

Questions?

Covariance

Definition: For two random variables R and S, the covariance between R and S is defined as:

$$\mathrm{Cov}[R,S] := \mathbb{E}[(R-\mathbb{E}[R])\,(S-\mathbb{E}[S])] = \mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]$$

$$\begin{aligned}
\mathrm{Cov}[R,S] &= \mathbb{E}[(R-\mathbb{E}[R])\,(S-\mathbb{E}[S])] \\
&= \mathbb{E}\left[RS - R\,\mathbb{E}[S] - S\,\mathbb{E}[R] + \mathbb{E}[R]\,\mathbb{E}[S]\right] && \text{(Expanding the terms)} \\
&= \mathbb{E}[RS] - \mathbb{E}[R\,\mathbb{E}[S]] - \mathbb{E}[S\,\mathbb{E}[R]] + \mathbb{E}[R]\,\mathbb{E}[S] && \text{(Linearity of Expectation)} \\
\implies \mathrm{Cov}[R,S] &= \mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S] - \mathbb{E}[S]\,\mathbb{E}[R] + \mathbb{E}[R]\,\mathbb{E}[S] = \mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]
\end{aligned}$$

- Covariance generalizes the notion of variance to multiple random variables.

$$\mathrm{Cov}[R,R] = \mathbb{E}[R\,R] - \mathbb{E}[R]\,\mathbb{E}[R] = \mathrm{Var}[R]$$

- If R and S are independent r.v's, $\mathbb{E}[RS] = \mathbb{E}[R]\,\mathbb{E}[S]$ and $\mathrm{Cov}[R,S] = 0$.
- The covariance between two r.v's is symmetric i.e. $\mathrm{Cov}[R,S] = \mathrm{Cov}[S,R]$.

Covariance

- For two arbitrary (not necessarily independent) r.v's, R and S,

$$\mathrm{Var}[R+S] = \mathrm{Var}[R] + \mathrm{Var}[S] + 2\,\mathrm{Cov}[R,S]$$

- Recall from Slide 9, where we showed that,

$$\mathrm{Var}[R+S] = \mathrm{Var}[R] + \mathrm{Var}[S] + 2(\mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]) = \mathrm{Var}[R] + \mathrm{Var}[S] + 2\,\mathrm{Cov}[R,S].$$

If R and S are independent, $\mathrm{Cov}[R,S] = 0$ and we recover the formula for the sum of independent variables.

- If $R = S$, $\mathrm{Var}[R+R] = \mathrm{Var}[R] + \mathrm{Var}[R] + 2\mathrm{Cov}[R,R] = \mathrm{Var}[R] + \mathrm{Var}[R] + 2\mathrm{Var}[R] = 4\mathrm{Var}[R]$ which is consistent with our previous formula that $\mathrm{Var}[2R] = 2^2\mathrm{Var}[R]$.

- Generalization to multiple random variables $R_1, R_2, \ldots R_n$ (Recall from Slide 10):

$$\mathrm{Var}\left[\sum_{i=1}^{n} R_i\right] = \sum_{i=1}^{n} \mathrm{Var}[R_i] + 2 \sum_{1 \le i < j \le n} \mathrm{Cov}[R_i, R_j]$$

Covariance - Example

Q: If X and Y are indicator r.v's for events A and B respectively, calculate the covariance between X and Y

We know that $\text{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$. Note that $X = \mathcal{I}_A$ and $Y = \mathcal{I}_B$. We can conclude that $XY = \mathcal{I}_{A\cap B}$ since $XY = 1$ iff both events A and B happen.

$$\implies \mathbb{E}[X] = \Pr[A]\ ; \mathbb{E}[Y] = \Pr[B]; \mathbb{E}[XY] = \Pr[A \cap B]$$

$$\implies \text{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] = \Pr[A \cap B] - \Pr[A]\Pr[B]$$

If $\text{Cov}[X, Y] > 0 \implies \Pr[A \cap B] > \Pr[A]\Pr[B]$. Hence,

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} > \frac{\Pr[A]\Pr[B]}{\Pr[B]} = \Pr[A]$$

If $Cov[X, Y] > 0$, it implies that $\Pr[A|B] > \Pr[A]$ and hence, the probability that event A happens increases if B is going to happen/has happened. Similarly, if $Cov[X, Y] < 0$, $\Pr[A|B] < \Pr[A]$. In this case, if B happens, then the probability of event A decreases.

Correlation

Definition: The correlation between two r.v's R_1 and R_2 is defined as:

$$\mathrm{Corr}[R_1, R_2] = \frac{\mathrm{Cov}[R_1, R_2]}{\sqrt{\mathrm{Var}[R_1]\,\mathrm{Var}[R_2]}}$$

$\mathrm{Corr}[R_1, R_2] \in [-1, 1]$ and indicates the strength of the relationship between R_1 and R_2.

- If $\mathrm{Corr}[R_1, R_2] > 0$, then R_1 and R_2 are said to be positively correlated, else if $\mathrm{Corr}[R_1, R_2] < 0$, the r.v's are negatively correlated.
- If $R_1 = R_2 = R$, then, $\mathrm{Corr}[R, R] = \frac{\mathrm{Cov}[R,R]}{\sqrt{\mathrm{Var}[R]\,\mathrm{Var}[R]}} = \frac{\mathrm{Var}[R]}{\mathrm{Var}[R]} = 1$.
- If R_1 and R_2 are independent, $\mathrm{Cov}[R_1, R_2] = 0$ and $\mathrm{Corr}[R_1, R_2] = 0$.
- If $R_1 = -R_2 = R$, then,

$$\mathrm{Corr}[R, -R] = \frac{\mathrm{Cov}[R, -R]}{\sqrt{\mathrm{Var}[R]\,\mathrm{Var}[-R]}} = \frac{\mathrm{Cov}[R, -R]}{\sqrt{\mathrm{Var}[R]\,(-1)^2\mathrm{Var}[R]}} = \frac{\mathrm{Cov}[R, -R]}{\mathrm{Var}[R]}$$

$$= \frac{\mathbb{E}[-R^2] - \mathbb{E}[R]\,\mathbb{E}[-R]}{\mathrm{Var}[R]} = \frac{-\mathbb{E}[R^2] + \mathbb{E}[R]\,\mathbb{E}[R]}{\mathrm{Var}[R]} = \frac{-\mathrm{Var}[R]}{\mathrm{Var}[R]} = -1$$