CMPT 210: Probability and Computation

Lecture 12

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Logistics

Late submission for Assignment 2 is Tuesday, 21 June.

Solutions will be released after the Tuesday class, and no submissions are allowed after that.

Midterm is on Friday, 24 June. It will be 50 minutes with material from Lectures 1 - 12.

Please go through the slides and the relevant sections of (Meyer, Lehman, Leighton) to prepare.

The midterm will be "easy" – if your concepts are clear, you should be able to get full marks.

If you have questions about any of the material, ask them on Piazza. Or come to office hours.

Recap

Random variable: A random "variable" R on a probability space is a total function whose domain is the sample space $\mathcal{S}$. The codomain is denoted by V (usually a subset of the real numbers), meaning that $R : \mathcal{S} \to V$.

Example: Suppose we toss three independent, unbiased coins. In this case, $\mathcal{S} = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. C is a random variable equal to the number of heads that appear such that $C(HHT) = 2$.

Indicator Random Variables: An indicator random variable corresponding to an event E is denoted as $\mathcal{I}_E$ and is defined such that for $\omega \in E$, $\mathcal{I}_E[\omega] = 1$ and for $\omega \notin E$, $\mathcal{I}_E[\omega] = 0$.

Example: When throwing two dice, if E is the event that both throws of the dice result in a prime number, then $\mathcal{I}_E((2,4)) = 0$ and $\mathcal{I}_E((2,3)) = 1$.

In general, a random variable that takes on several values partitions $\mathcal{S}$ into several blocks where each block is a subset of $\mathcal{S}$ and is therefore an event.

Example: When tossing three coins, $\Pr[C = 2] = \Pr[\{HHT, HTH, THH\}] = \frac{3}{8}$.

Recap

Probability density function (PDF): Let R be a random variable with codomain V. The probability density function of R is the function $\text{PDF}_R : V \to [0,1]$, such that $\text{PDF}_R[x] = \Pr[R = x]$ if $x \in \text{Range(R)}$ and equal to zero if $x \notin \text{Range(R)}$.

$\sum_{x \in V} \text{PDF}_R[x] = \sum_{x \in \text{Range(R)}} \Pr[R = x] = 1$.

Example: When tossing three coins, $\text{PDF}_C[2] = \Pr[C = 2] = \frac{3}{8}$.

Cumulative distribution function (CDF): The cumulative distribution function of R is the function $\text{CDF}_R : \mathbb{R} \to [0,1]$, such that $\text{CDF}_R[x] = \Pr[R \leq x]$.

Example: When tossing three coins,
$\text{CDF}_C[2.3] = \Pr[C \leq 2.3] = \Pr[C = 0] + \Pr[C = 1] + \Pr[C = 2] = \frac{7}{8}$.

Importantly, neither PDF_R nor CDF_R involves the sample space of an experiment.

Distributions

Many random variables turn out to have the same PDF and CDF. In other words, even though R and T might be different random variables on different probability spaces, it is often the case that $\text{PDF}_R = \text{PDF}_T$. Hence, by studying the properties of such PDFs, we can study different random variables and experiments.

Distribution over a random variable can be fully specified using the cumulative distribution function (CDF) (usually denoted by F). The corresponding probability density function (PDF) is denoted by f.

Common (Discrete) Distributions in Computer Science:

- Bernoulli Distribution
- Uniform Distribution
- Binomial Distribution
- Geometric Distribution

Bernoulli Distribution

We toss a biased coin such that the probability of getting a heads is p. Let R be the random variable such that $R = 0$ when the coin comes up heads and $R = 1$ if the coin comes up tails. R follows the Bernoulli distribution.

The Bernoulli distribution has the PDF $f\colon \{0,1\} \to [0,1]$ meaning that Bernoulli random variables take values in $\{0,1\}$. It can be fully specified by the "probability of success" (of an experiment) p (probability of getting a heads in the example). Formally, PDF_R is given by:

$$f(0) = p \quad ; \quad f(1) = q := 1 - p.$$

In the example, $\Pr[R = 0] = f(0) = p = \Pr[\text{event that we get a heads}]$.

The corresponding CDF_R is given by $F : \mathbb{R} \to [0,1]$:

$$\begin{aligned} F(x) &= 0 && \text{(for } x < 0\text{)} \\ &= p && \text{(for } 0 \leq x < 1\text{)} \\ &= 1 && \text{(for } x \geq 1\text{)} \end{aligned}$$

Uniform Distribution

We roll a standard die. Let R be the random variable equal to the number that shows up on the die. R follows the uniform distribution.

A random variable R that takes on each possible value in its codomain V with the same probability is said to be uniform. The uniform distribution can be fully specified by V and has PDF $f : V \to [0,1]$ such that:

$$f(v) = {}^1\!/\!_{|V|}. \qquad \text{(for all } v \in V\text{)}$$

In the example, $f(1) = f(2) = \ldots = f(6) = \frac{1}{6}$.

For n elements in V arranged in increasing order – $(v_1, v_2, \ldots, v_n)$, the CDF is:

$$\begin{aligned} F(x) &= 0 && \text{(for } x < v_1\text{)} \\ &= {}^k\!/\!_n && \text{(for } v_k \leq x < v_{k+1}\text{)} \\ &= 1 && \text{(for } x \geq v_n\text{)} \end{aligned}$$

Q: If X has a Bernoulli distribution, when is X also uniform? Ans: When $p = 1/2$

Questions?

Binomial Distribution

We toss n biased coins independently. The probability of getting a heads for each coin is p. Let R be the random variable equal to the number of heads in the n coin tosses. R follows the Binomial distribution.

$V = \{0, 1, 2, \ldots, n\}$. Hence PDF_R is a function $f : \{0, 1, 2, \ldots, n\} \to [0, 1]$.

Let E_k be the event we get k heads in n tosses. Let A_i be the event we get a heads in toss i.

$$E_k = (A_1 \cap A_2 \ldots A_k \cap A_{k+1}^c \cap A_{k+2}^c \cap \ldots \cap A_n^c) \cup (A_1^c \cap A_2 \ldots A_k \cap A_{k+1} \cap A_{k+2}^c \cap \ldots \cap A_n^c) \cup \ldots$$

$$\begin{aligned}
\Pr[E_k] &= \Pr[(A_1 \cap A_2 \ldots A_k \cap A_{k+1}^c \cap A_{k+2}^c \cap \ldots \cap A_n^c)] + \Pr[A_1^c \cap A_2 \ldots A_k \cap A_{k+1} \cap \ldots \cap] + \ldots \\
&= \Pr[A_1] \Pr[A_2] \Pr[A_k] \Pr[A_{k+1}^c] \Pr[A_{k+2}^c] \ldots \Pr[A_n^c] + \ldots = p^k (1-p)^{n-k} + p^k (1-p)^{n-k} + \ldots
\end{aligned}$$

$$\implies \Pr[E_k] = \binom{n}{k} p^k (1-p)^{n-k}$$

Sanity check: Since $\text{PDF}_R[k] = \Pr[E_k]$ and $V = \{0, 1, 2, \ldots, n\}$,

$$\sum_{i \in V} \text{PDF}_R[i] = \sum_{i=0}^{n} \Pr[E_i] = \sum_{i=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + 1 - p)^n = 1. \quad \text{(Binomial Theorem)}$$

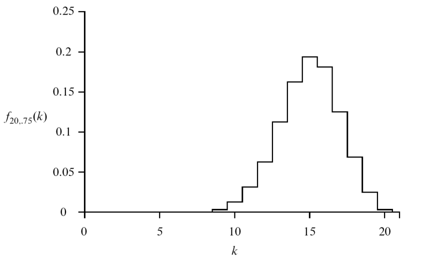
Binomial Distribution

The binomial distribution can be fully specified by n, p and has PDF $f : \{0, 1, \ldots n\} \to [0, 1]$:

$$f(k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

The corresponding CDF is given by $F : \mathbb{R} \to [0, 1]$:

$$\begin{aligned} F(x) &= 0 && (\text{for } x < 0) \\ &= \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} && (\text{for } k \leq x < k+1) \\ &= 1. && (\text{for } x \geq n) \end{aligned}$$



Q: If X has a Bernoulli distribution with parameter p, does it also follow the Binomial distribution? With what parameters? Ans: Yes. With $n = 1$ and $p = p$.

Geometric Distribution

We toss a biased coin independently multiple times. The probability of getting a heads is p. Let R be the random variable equal to the number of tosses needed to get the first heads. R follows the geometric distribution.

$V = \{1, 2, \ldots,\}$. Hence PDF_R is a function $f : \{1, 2, \ldots\} \to [0, 1]$.

Let E_k be the event that we need k tosses to get the first heads. Let A_i be the event that we get a heads in toss i.

$$\begin{aligned}
E_k &= A_1^c \cap A_2^c \cap \ldots \cap A_k \\
\Pr[E_k] &= \Pr[A_1^c \cap A_2^c \cap \ldots \cap A_k] = \Pr[A_1^c]\Pr[A_2^c]\ldots\Pr[A_k] \\
\implies \Pr[E_k] &= (1-p)^{k-1} p
\end{aligned}$$

Sanity check: Since $\text{PDF}_R[k] = \Pr[E_k]$ and $V = \{1, 2, \ldots,\}$,

$$\sum_{i \in V} \text{PDF}_R[i] = \sum_{i=1}^{\infty} \Pr[E_i] = \sum_{i=1}^{\infty} (1-p)^{i-1} p = \frac{p}{1-(1-p)} = 1. \quad \text{(Sum of geometric series)}$$

Geometric Distribution

The geometric distribution can be fully specified by p and has PDF $f : \{1, 2, \ldots\} \to [0, 1]$:

$$f(k) = (1-p)^{k-1}p.$$

The corresponding CDF is given by $F : \mathbb{R} \to [0, 1]$:

$$\begin{aligned} F(x) &= 0 && \text{(for } x < 1\text{)} \\ &= \sum_{i=0}^{k}(1-p)^{i-1}p && \text{(for } k \leq x < k+1\text{)} \end{aligned}$$