

CMPT 210: Probability and Computing

Lecture 22

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Tail inequalities bound the probability that the r.v. takes a value much different from its mean.

Markov's Theorem: If X is a non-negative random variable, then for all $x > 0$,
 $\Pr[X \geq x] \leq \frac{\mathbb{E}[X]}{x}$.

Chebyshev's Theorem: For a r.v. X and all $x > 0$, $\Pr[|X - \mathbb{E}[X]| \geq x] \leq \frac{\text{Var}[X]}{x^2}$.

Pairwise Independent Sampling

Claim: Let G_1, G_2, \dots, G_n be pairwise independent random variables with the same mean μ and standard deviation σ . Define $S_n := \sum_{i=1}^n G_i$, then,

$$\Pr \left[\left| \frac{S_n}{n} - \mu \right| \geq \epsilon \right] \leq \frac{1}{n} \left(\frac{\sigma}{\epsilon} \right)^2.$$

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Proof: Let us compute $\mathbb{E}[S_n/n]$ and $\text{Var}[S_n/n]$.

$$\mathbb{E}[S_n] = \mathbb{E} \left[\sum_{i=1}^n G_i \right] = \sum_{i=1}^n \mathbb{E}[G_i] = n\mu \implies \mathbb{E}[S_n/n] = \frac{1}{n} \mathbb{E}[S_n] = \mu$$

(Using linearity of expectation)

$$\text{Var}[S_n] = \text{Var} \left[\sum_{i=1}^n G_i \right] = \sum_{i=1}^n \text{Var}[G_i] = n\sigma^2$$

(Using linearity of variance for pairwise independent r.v's)

$$\implies \text{Var}[S_n/n] = \frac{1}{n^2} \text{Var}[S_n] = \frac{\sigma^2}{n}$$

Pairwise Independent Sampling

Using Chebyshev's Theorem,

$$\Pr \left[\left| \frac{S_n}{n} - \mathbb{E} \left[\frac{S_n}{n} \right] \right| \geq \epsilon \right] = \Pr \left[\left| \frac{S_n}{n} - \mu \right| \geq \epsilon \right] \leq \frac{\text{Var}[S_n/n]}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}$$

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Weak Law of Large Numbers: Let G_1, G_2, \dots, G_n be pairwise independent variables with the same mean μ and (finite) standard deviation σ . Define $X_n := \frac{\sum_{i=1}^n G_i}{n}$, then for every $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \Pr[|X_n - \mu| \leq \epsilon] = 1.$$

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Proof: Follows from the theorem on pairwise independent sampling since $\lim_{n \rightarrow \infty} \Pr[|X_n - \mu| \leq \epsilon] = \lim_{n \rightarrow \infty} \left[1 - \frac{\sigma^2}{n\epsilon^2} \right] = 1.$

Questions?

Sums of Random Variables

If we know that the r.v X is (i) non-negative and (ii) $\mathbb{E}[X]$, we can use Markov's Theorem to bound the probability of deviation from the mean.

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Chernoff Bound: Let T_1, T_2, \dots, T_n be mutually independent r.v's such that $0 \leq T_i \leq 1$ for all i . If $T := \sum_{i=1}^n T_i$, for all $c \geq 1$ and $\beta(c) := c \ln(c) - c + 1$,

$$\Pr[T \geq c\mathbb{E}[T]] \leq \exp(-\beta(c)\mathbb{E}[T])$$

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If $T_i \sim \text{Ber}(p)$ and are mutually independent, then $T_i \in \{0, 1\}$ and we can use the Chernoff bound to bound the deviation from the mean for $T \sim \text{Bin}(n, p)$. In general, if $T_i \in [0, 1]$, the Chernoff Bound can be used even if the T_i 's have different distributions!

Chernoff Bound – Binomial Distribution

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Let T_i be the r.v. for the event that coin i comes up heads, and let T denote the total number of heads. Hence, $T = \sum_{i=1}^{1000} T_i$. For all i , $T_i \in \{0, 1\}$ and are mutually independent r.v.'s. Hence, we can use the Chernoff Bound.

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We want to compute the probability that the number of heads is larger than the expectation by 20% meaning that $c = 1.2$ for the Chernoff Bound. Computing $\beta(c) = c \ln(c) - c + 1 \approx 0.0187$. Since the coin is fair, $\mathbb{E}[T] = 1000 \cdot \frac{1}{2} = 500$. Plugging into the Chernoff Bound,

$$\Pr[T \geq c\mathbb{E}[T]] \leq \exp(-\beta(c)\mathbb{E}[T]) \implies \Pr[T \geq 1.2\mathbb{E}[T]] \leq \exp(-(0.0187)(500)) \approx 0.0000834.$$

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Comparing this to using Chebyshev's inequality,

$$\begin{aligned} \Pr[T \geq c\mathbb{E}[T]] &= \Pr[T - \mathbb{E}[T] \geq (c - 1)\mathbb{E}[T]] \leq \Pr[|T - \mathbb{E}[T]| \geq (c - 1)\mathbb{E}[T]] \\ &\leq \frac{\text{Var}[T]}{(c - 1)^2 (\mathbb{E}[T])^2} = \frac{1000 \cdot \frac{1}{4}}{(1.2 - 1)^2 (500^2)} = \frac{250}{0.2^2 500^2} = \frac{250}{10000} = 0.025. \end{aligned}$$

Chernoff Bound – Lottery Game

Q: Pick-4 is a lottery game in which you pay \$1 to pick a 4-digit number between 0000 and 9999. If your number comes up in a random drawing, then you win \$5,000. Your chance of winning is 1 in 10000. If 10 million people play, then the expected number of winners is 1000. When there are 1000 winners, the lottery keeps \$5 million of the \$10 million paid for tickets. The lottery operator's nightmare is that the number of winners is much greater – especially at the point where more than 2000 win and the lottery must pay out more than it received. What is the probability that will happen? (Assume that the players' picks and the winning number are random, independent and uniform)

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We wish to compute $\Pr[T \geq 2000] = \Pr[T \geq 2\mathbb{E}[T]]$. Hence $c = 2$ and $\beta(c) \approx 0.386$. By the Chernoff bound,

$$\Pr[T \geq 2\mathbb{E}[T]] \leq \exp(-\beta(c)\mathbb{E}[T]) = \exp(-(0.386)1000) < \exp(-386) \approx 10^{-168}$$

Comparing the Bounds

For r.v's T_1, T_2, \dots, T_n , if $T_i \in \{0, 1\}$ and $\Pr[T_i = 1] = p_i$. Define $T := \sum_{i=1}^n T_i$. By linearity of expectation, $\mathbb{E}[T] = \sum_{i=1}^n p_i$. For $c \geq 1$,

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Markov's Theorem: $\Pr[T \geq c\mathbb{E}[T]] \leq \frac{1}{c}$. Does not require T_i 's to be independent.

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If the T_i 's are pairwise independent, by linearity of variance, $\text{Var}[T] = \sum_{i=1}^n p_i(1-p_i)$. Hence, $\Pr[T \geq c\mathbb{E}[T]] \leq \frac{\sum_{i=1}^n p_i(1-p_i)}{(c-1)^2 (\sum_{i=1}^n p_i)^2}$. If for all i , $p_i = 1/2$, then, $\Pr[T \geq c\mathbb{E}[T]] \leq \frac{1}{(c-1)^2}$.

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Chernoff Bound: If T_i ' are mutually independent, then,

$$\Pr[T \geq c\mathbb{E}[T]] \leq \exp(-\beta(c)\mathbb{E}[T]) = \exp(-(c \ln(c) - c + 1) (\sum_{i=1}^n p_i)).$$

If for all i , $p_i = 1/2$,

$$\Pr[T \geq c\mathbb{E}[T]] \leq \exp\left(-\frac{n(c \ln(c) - c + 1)}{2}\right).$$

Questions?