

# CMPT 419/983: Theoretical Foundations of Reinforcement Learning

## Lecture 2

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# Recap

- **Input:**  $K$  arms (possible actions),  $T$  rounds.  $\mu_a := \mathbb{E}_{r \sim \nu_a}[r]$  is the (unknown) expected reward obtained by choosing action  $a$ .
- **Protocol:** In each round  $t \in [T]$ , the bandit algorithm chooses action  $a_t \in [K]$  and observes reward  $R_t \sim \nu_{a_t}$ .
- **Objective:** Minimize  $\text{Regret}(T) := \sum_{t=1}^T [\mu^* - \mathbb{E}[R_t]] = \sum_{a=1}^K \Delta_a \mathbb{E}[N_a(T)]$ .
- **Assumption:**  $\eta_t := R_t - \mu_{a_t}$  is 1 sub-Gaussian i.e. for all  $\lambda \in \mathbb{R}$ ,  $\mathbb{E}[\exp(\lambda \eta_t)] \leq \exp\left(\frac{\lambda^2}{2}\right)$ .
- **Concentration for sub-Gaussian r.v.:** If  $X$  is centered and  $\sigma$  sub-Gaussian, then for any  $\epsilon \geq 0$ ,  $\Pr[X \geq \epsilon] \leq \exp\left(-\frac{\epsilon^2}{2\sigma^2}\right)$ . For  $n$  i.i.d r.v's  $X_i$  s.t.  $\mathbb{E}[X_i] = \mu$ , if  $\hat{\mu} := \frac{1}{n} \sum_{i=1}^n X_i$  and  $X_i - \mu$  is  $\sigma$  sub-Gaussian, then  $\Pr[|\hat{\mu} - \mu| \geq \epsilon] \leq \exp\left(-\frac{n\epsilon^2}{2\sigma^2}\right)$
- **Explore-then-Commit (ETC):** Under a sub-Gaussian assumption, ETC results in  $O(\sqrt{KT})$  regret when exploring for  $m = O\left(\frac{1}{\Delta^2}\right)$  rounds, while it can only result in  $O(T^{2/3})$  regret when  $m$  is set independent of  $\Delta$ .

# $\epsilon$ -greedy Algorithm

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**Algorithm**  $\epsilon$ -greedy (EG)

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1: Input:  $\{\epsilon_t\}_{t=1}^T$ 
2: for  $t = 1 \rightarrow K$  do
3:   Select arm  $a_t = t$  and observe  $R_t$ 
4: end for
5: Calculate empirical mean reward for arm  $a \in [K]$  as  $\hat{\mu}_a(K) := \frac{\sum_{t=1}^K R_t \mathcal{I}\{a_t=a\}}{N_a(K)}$ 
6: for  $t = K + 1 \rightarrow T$  do
7:   Select arm  $\begin{cases} a_t = \arg \max_{a \in [K]} \hat{\mu}_a(t-1) \text{ w.p. } 1 - \epsilon_t \\ a_t \sim \mathcal{U}\{1, 2, \dots, K\} \text{ w.p. } \epsilon_t \end{cases}$ 
8:   Observe reward  $R_t$  and update for  $a \in [K]$ :
      
$$N_a(t) = N_a(t-1) + \mathcal{I}\{a_t = a\} \quad ; \quad \hat{\mu}_a(t) = \frac{N_a(t-1) \hat{\mu}_a(t-1) + R_t \mathcal{I}\{a_t = a\}}{N_a(t)}$$

9: end for
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- EG with  $\epsilon_t = \epsilon$  can result in linear regret.
- For  $K = 2$ , EG with  $\epsilon_t = O\left(\frac{1}{\Delta^2 t}\right)$  incurs  $O\left(\frac{\log(T)}{\Delta^2}\right)$  regret.

Prove in Assignment 1!

# Upper Confidence Bound (UCB) Algorithm

- Based on the principle of *optimism in the face of uncertainty*.

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**Algorithm** Upper Confidence Bound

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- 1: **Input:**  $\delta$
- 2: For each arm  $a \in [K]$ , initialize  $U_a(0, \delta) := \infty$ .
- 3: **for**  $t = 1 \rightarrow T$  **do**
- 4:   Select arm  $a_t = \arg \max_{a \in [K]} U_a(t-1, \delta)$  (*Choose the lower-indexed arm in case of a tie*)
- 5:   Observe reward  $R_t$  and update for  $a \in [K]$ :

$$N_a(t) = N_a(t-1) + \mathcal{I}\{a_t = a\} \quad ; \quad \hat{\mu}_a(t) = \frac{N_a(t-1) \hat{\mu}_a(t-1) + R_t \mathcal{I}\{a_t = a\}}{N_a(t)}$$

$$U_a(t, \delta) = \hat{\mu}_a(t) + \sqrt{\frac{2 \log(1/\delta)}{N_a(t)}}$$

- 6: **end for**
- 

- Intuitively, UCB pulls a “promising” arm (with higher empirical mean  $\hat{\mu}_a$ ) or one that has not been explored enough (with lower  $N_a(t)$ ).

# UCB – Regret Analysis

**Claim:** UCB with  $\delta = \frac{1}{T^2}$  achieves the following problem-dependent bound on the regret,

$$\text{Regret}(\text{UCB}, T) \leq 2 \sum_{a=1}^K \Delta_a + \sum_{a \in [K] | \Delta_a > 0} \frac{16 \log(T)}{\Delta_a}$$

*Proof:* Without loss of generality, assume that arm 1 is the best arm. Using the regret decomposition, we know that  $\text{Regret}(\text{UCB}, T) = \sum_a \Delta_a \mathbb{E}[N_a(T)]$ . Define a threshold  $\tau_a$  and  $\hat{\mu}_{a, \tau_a}$  as the mean for arm  $a$  after pulling it for the first  $\tau_a$  times. Define a “good” event  $G_a$  for each  $a \neq 1$ .

$$G_a = \left\{ \mu_1 < \min_{t \in [T]} U_1(t, \delta) \right\} \cap \left\{ \hat{\mu}_{a, \tau_a} + \sqrt{\frac{2 \log(1/\delta)}{\tau_a}} < \mu_1 \right\}$$

Consider two cases when bounding  $\mathbb{E}[N_a(T)]$ . Using the law of total expectation,

$$\begin{aligned} \mathbb{E}[N_a(T)] &= \mathbb{E}[N_a(T) | G_a] \Pr[G_a] + \mathbb{E}[N_a(T) | G_a^c] \Pr[G_a^c] \\ &\leq \underbrace{\mathbb{E}[N_a(T) | G_a]}_{\text{Term (i)}} + T \underbrace{\Pr[G_a^c]}_{\text{Term (ii)}} \quad (N_a(T) \leq T \text{ for all } a, \Pr[G_a] \leq 1) \end{aligned}$$

## UCB – Regret Analysis

Recall that  $G_a = \{\mu_1 < \min_{t \in [T]} U_1(t, \delta)\} \cap \left\{ \hat{\mu}_{a, \tau_a} + \sqrt{\frac{2 \log(1/\delta)}{\tau_a}} < \mu_1 \right\}$ . We will show (by contradiction) that Term (i)  $= \mathbb{E}[N_a(T) | G_a] \leq \tau_a$ .

Suppose  $\mathbb{E}[N_a(T) | G_a] > \tau_a$ , then there is a round  $t$  s.t.  $N_a(t-1) = \tau_a$ ,  $a_t = a$ . Since  $a_t = \arg \max_a U_a(t-1, \delta)$ , it follows that  $U_a(t-1, \delta) > U_1(t-1, \delta)$ . However, we know that,

$$\begin{aligned} U_a(t-1, \delta) &= \hat{\mu}_a(t-1) + \sqrt{\frac{2 \log(1/\delta)}{N_a(t-1)}} = \hat{\mu}_a(t-1) + \sqrt{\frac{2 \log(1/\delta)}{\tau_a}} \\ &\hspace{15em} \text{(By assumption, } N_a(t-1) = \tau_a) \\ &= \hat{\mu}_{a, \tau_a} + \sqrt{\frac{2 \log(1/\delta)}{\tau_a}} \hspace{10em} \text{(Since arm } a \text{ has been pulled } \tau_a \text{ times)} \\ &\leq \mu_1 < U_1(t-1, \delta), \hspace{10em} \text{(Since we are conditioning on } G_a) \end{aligned}$$

which is a contradiction. Hence,  $\mathbb{E}[N_a(T) | G_a] \leq \tau_a$ .

## UCB – Regret Analysis

$$\text{Bounding Term (ii)} = \Pr[G_a^c] \leq \Pr[\mu_1 \geq \min_{t \in [T]} U_1(t, \delta)] + \Pr\left[\hat{\mu}_{a, \tau_a} + \sqrt{\frac{2 \log(1/\delta)}{\tau_a}} \geq \mu_1\right].$$

$$\begin{aligned}\left\{\mu_1 \geq \min_{t \in [T]} U_1(t, \delta)\right\} &= \left\{\mu_1 \geq \min_{t \in [T]} \left\{\hat{\mu}_1(t) + \sqrt{\frac{2 \log(1/\delta)}{N_1(t)}}\right\}\right\} \\ &= \left\{\mu_1 \geq \min_{s \in [T]} \left\{\hat{\mu}_{1,s} + \sqrt{\frac{2 \log(1/\delta)}{s}}\right\}\right\} \\ &= \bigcup_{s=1}^T \left\{\mu_1 \geq \hat{\mu}_{1,s} + \sqrt{\frac{2 \log(1/\delta)}{s}}\right\}\end{aligned}$$

$$\Rightarrow \Pr\left[\mu_1 \geq \min_{t \in [T]} U_1(t, \delta)\right] \leq \sum_{s=1}^T \Pr\left[\mu_1 \geq \hat{\mu}_{1,s} + \sqrt{\frac{2 \log(1/\delta)}{s}}\right] \quad (\text{Union Bound})$$

$$\leq \sum_{s=1}^T \delta = \delta T \quad (\text{Using concentration for sub-Gaussian r.v's})$$

# UCB – Regret Analysis

Recall that Term (ii) =  $\Pr[G_a^c] \leq \delta T + \Pr\left[\hat{\mu}_{a,\tau_a} + \sqrt{\frac{2\log(1/\delta)}{\tau_a}} \geq \mu_1\right]$ . Assume that  $\tau_a$  is chosen such that  $\Delta_a - \sqrt{\frac{2\log(1/\delta)}{\tau_a}} \geq \frac{\Delta_a}{2}$ .

$$\begin{aligned}\Pr\left[\hat{\mu}_{a,\tau_a} + \sqrt{\frac{2\log(1/\delta)}{\tau_a}} \geq \mu_1\right] &= \Pr\left[\hat{\mu}_{a,\tau_a} - \mu_a + \sqrt{\frac{2\log(1/\delta)}{\tau_a}} \geq \Delta_a\right] \leq \Pr\left[\hat{\mu}_{a,\tau_a} - \mu_a \geq \frac{\Delta_a}{2}\right] \\ &\leq \exp\left(-\frac{\tau_a \Delta_a^2}{8}\right)\end{aligned}$$

(Using concentration for sub-Gaussian r.v's)

Putting everything together,

$$\begin{aligned}\Rightarrow \Pr[G_a^c] &\leq \delta T + \exp\left(-\frac{\tau_a \Delta_a^2}{8}\right) \\ \Rightarrow \mathbb{E}[N_a(T)] &\leq \tau_a + T \left[ \delta T + \exp\left(-\frac{\tau_a \Delta_a^2}{8}\right) \right]\end{aligned}$$



## UCB – Regret Analysis

Recall that  $\mathbb{E}[N_a(T)] \leq \tau_a + T \left[ \delta T + \exp\left(-\frac{\tau_a \Delta_a^2}{8}\right) \right]$ .

$$\mathbb{E}[N_a(T)] \leq \frac{8 \log(1/\delta)}{\Delta_a^2} + T [\delta T + \delta] \quad (\text{Setting } \tau_a = \frac{8 \log(1/\delta)}{\Delta_a^2})$$

$$\leq \frac{8 \log(1/\delta)}{\Delta_a^2} + 2\delta T^2$$

$$= \frac{16 \log(T)}{\Delta_a^2} + 2 \quad (\text{Setting } \delta = 1/T^2)$$

$$\implies \text{Regret}(\text{UCB}, T) = \sum_a \Delta_a \mathbb{E}[N_a(T)] = 2 \sum_{a=1}^K \Delta_a + \sum_{a=2}^K \frac{16 \log(T)}{\Delta_a} \quad \square$$

# UCB – Regret Analysis

**Claim:** For  $\Delta \leq 1$ , UCB with  $\delta = \frac{1}{T^2}$  achieves the following worst-case regret,

$$\text{Regret}(\text{UCB}, T) \leq 2K + 8\sqrt{K T \log(T)}$$

*Proof:* Define  $C > 0$  to be a constant to be tuned later. From the regret decomposition result,

$$\begin{aligned} \text{Regret}(\text{UCB}, T) &= \sum_{a=1}^K \Delta_a \mathbb{E}[N_a(T)] = \sum_{a|\Delta_a < C} \Delta_a \mathbb{E}[N_a(T)] + \sum_{a|\Delta_a \geq C} \Delta_a \mathbb{E}[N_a(T)] \\ &\leq CT + \sum_{a|\Delta_a \geq C} \Delta_a \mathbb{E}[N_a(T)] && \text{(Since } \sum_{a=1}^K N_a(T) = T \text{)} \\ &\leq CT + \sum_{a|\Delta_a \geq C} \left[ \frac{16 \log(T)}{\Delta_a} + 2\Delta_a \right] && \text{(From the previous slide)} \\ &\leq CT + \left[ \frac{16K \log(T)}{C} + \sum_{a|\Delta_a \geq C} 2\Delta_a \right] && \text{(Setting } C = \sqrt{\frac{16K \log(T)}{T}} \text{)} \end{aligned}$$

$$\implies \text{Regret}(\text{UCB}, T) \leq 8\sqrt{K T \log(T)} + 2K\Delta_a \leq 2K + 8\sqrt{K T \log(T)}$$

# UCB vs ETC

- Similar to best-tuned ETC, UCB results in an  $\tilde{O}(\sqrt{KT})$  problem-independent regret.
- Unlike best-tuned ETC, UCB does not need to know the gaps  $\Delta$  to set algorithm parameters, but does require knowledge of the horizon  $T$ .

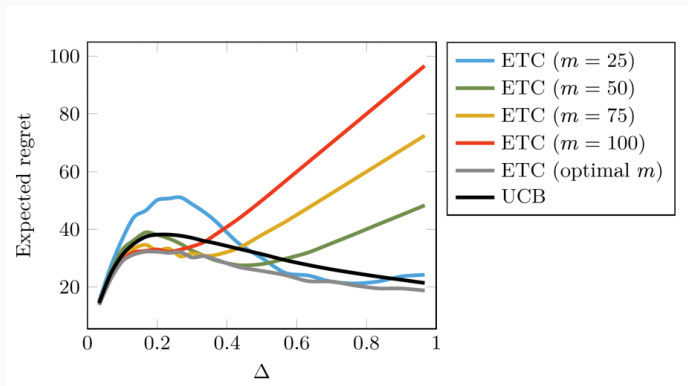


Figure 1: For  $K = 2$ ,  $T = 1000$ , Gaussian rewards, comparing UCB and ETC( $m$ ) as a function of the gap  $\Delta$ .

# Improvements to UCB

- **Problem:** UCB requires knowledge of  $T$  and hence, the number of rounds needs to be fixed.
- *Sol:* Define UCB as  $\hat{\mu}_a(t) + \sqrt{\frac{2 \log(f(t))}{N_a(t)}}$  where  $f(t) := 1 + t \log^2(t)$ . No dependence on  $T$ , but results in the same  $O(\sqrt{KT \log(T)})$  worst-case regret. (see [LS20, Chapter 8])
- **Lower-Bound:** For a fixed  $T$  and for every bandit algorithm, there exists a stochastic bandit problem with rewards in  $[0, 1]$  such that  $\text{Regret}(T) = \Omega(\sqrt{KT})$ . (see [LS20, Chapter 15]).
- **Problem:** UCB is sub-optimal by a  $\sqrt{\log(T)}$  factor compared to the lower-bound. Is it possible to develop an algorithm that does not incur this log factor?
- *Sol:* [Lat18, MG17] propose modifications of UCB that achieve  $O(\sqrt{KT})$  regret.

# Stochastic Linear Bandits

# Stochastic Linear Bandits

- MAB treat each arm (e.g. drug choice) independently. But the arms (and their rewards) can be dependent. E.g., drugs with similar chemical composition can have similar side-effects.
- Stochastic Linear Bandits can model linear dependence between different arms. For this, we require *feature vectors*  $X_a \in \mathbb{R}^d$  for each arm  $a \in [K]$ .
- **Reward Model:** For an unknown vector  $\theta^* \in \mathbb{R}^d$ , the mean reward for arm  $a$  is given as:  $\mu_a = \langle X_a, \theta^* \rangle$ . Hence, arms with similar feature vectors will have similar mean rewards.
- Similar to the MAB setting, on pulling arm  $a_t$  at round  $t$ , we observe the reward  $R_t = \mu_{a_t} + \eta_t = \langle X_{a_t}, \theta^* \rangle + \eta_t$ . We will assume that  $\eta_t$  is conditionally 1 sub-Gaussian noise, i.e. if  $\mathcal{H}_{t-1} := \{X_1, R_1, \dots, X_t\}$  as the *history* of interactions until round  $t$ , then for all  $\lambda \in \mathbb{R}$ ,  $\mathbb{E}[\exp(\lambda \eta_t) | \mathcal{H}_{t-1}] \leq \exp(\lambda^2/2)$ .
- $\text{Regret}(T) := \sum_{t=1}^T [\max_{a \in [K]} \langle X_a, \theta^* \rangle - \mathbb{E}[R_t]] = T \max_{a \in [K]} \langle X_a, \theta^* \rangle - \sum_{t=1}^T \mathbb{E}[R_t]$ .
- In the special case, when all the arms are independent, i.e.  $d = K$  and  $\forall a \in [K]$ ,  $X_a = e_a$  where  $\forall i \in [d]$ ,  $i \neq a$ ,  $e_a[i] = 0$  and  $e_a[a] = 1$ . Hence,  $\mu_a = \theta_a^*$  and the linear bandit setup strictly generalizes MAB.

## Stochastic Linear Bandits – Estimating $\hat{\mu}_a(t)$

At round  $t$ , we have collected the following data:  $\{X_s, R_s\}_{s=1}^t$ . **Q:** How do we estimate  $\hat{\mu}_a(t)$ ?

By solving regularized ridge regression, i.e. for a regularization parameter  $\lambda \geq 0$ ,

$$\hat{\theta}_t := \arg \min_{\theta} \left\{ \frac{1}{2} \sum_{s=1}^t [\langle X_s, \theta \rangle - R_s]^2 + \frac{\lambda}{2} \|\theta\|^2 \right\}$$

Setting the derivative to zero to solve the above minimization problem,

$$\begin{aligned} \sum_{s=1}^t \left[ X_s \left[ \langle X_s, \hat{\theta}_t \rangle - R_s \right] \right] + \lambda \hat{\theta}_t &= 0 \\ \Rightarrow \underbrace{\left[ \sum_{s=1}^t X_s X_s^T + \lambda I_d \right]}_{V_t \in \mathbb{R}^{d \times d}} \hat{\theta}_t &= \underbrace{\sum_{s=1}^t X_s R_s}_{b_t \in \mathbb{R}^{d \times 1}} \Rightarrow V_t \hat{\theta}_t = b_t \Rightarrow \hat{\theta}_t = V_t^{-1} b_t \end{aligned}$$

Hence, the empirical mean for each arm after  $t$  rounds:  $\hat{\mu}_a = \langle X_a, \hat{\theta}_t \rangle = X_a^T V_t^{-1} b$

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**Algorithm** Linear Upper Confidence Bound

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- 1: **Input:**  $\{\beta_t\}_{t=1}^T$ ,  $V_0 = \lambda I_d \in \mathbb{R}^{d \times d}$
- 2: For each arm  $a \in [K]$ , initialize  $U_a(0, \delta) := \infty$ .
- 3: **for**  $t = 1 \rightarrow T$  **do**
- 4:   Select arm  $a_t = \arg \max_{a \in [K]} U_a(t-1, \delta)$  *(Choose the lower-indexed arm in case of a tie)*
- 5:   Observe reward  $R_t$  and update:

$$V_t = V_{t-1} + X_t X_t^T \quad ; \quad b_t = b_{t-1} + R_t X_t \quad ; \quad \hat{\theta}_t = V_t^{-1} b_t$$
$$U_a(t) = \langle X_a, \hat{\theta}_t \rangle + \sqrt{\beta_t} \|X_a\|_{V_t^{-1}} \quad \quad \quad (\text{where } \|x\|_A := \sqrt{x^T A x})$$

6: **end for**

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In the special case, when all the arms are independent, Linear UCB with  $\beta_t = \beta = 2 \log(1/\delta)$  is equivalent to UCB, and hence, Linear UCB strictly generalizes UCB.

Prove this in Assignment 1!



**Claim:**  $U_a(t) = \langle X_a, \hat{\theta}_t \rangle + \sqrt{\beta_t} \|X_a\|_{V_t^{-1}} = \max_{\theta \in \mathcal{C}_t} \langle \theta, X_a \rangle$  where  $\mathcal{C}_t = \left\{ \theta \mid \left\| \theta - \hat{\theta}_t \right\|_{V_t}^2 \leq \beta_t \right\}$ .

$\mathcal{C}_t$  is an ellipsoid centered at  $\hat{\theta}_t$  with the principle axes being the eigenvectors of  $V_t$  and the corresponding lengths being the reciprocal of the eigenvalues. As  $t$  increases, the eigenvalues of matrix  $V_t$  increases and the volume of the ellipsoid decreases.

Prove this in Assignment 1! For the subsequent proof, we will use this equivalence.

**Claim:** Assuming (i)  $\|\theta^*\| \leq 1$ , (ii)  $\|X_a\| \leq 1$  for all  $a$  and (iii)  $R_t \in [0, 1]$ , UCB with  $\beta_t = \sqrt{d \log \left( \frac{\lambda d + t}{\lambda} \right)} + 2 \log(1/\delta) + \sqrt{\lambda}$  achieves the following worst-case bound on the regret,

$$\text{Regret}(\text{LinUCB}, T) \leq O\left(d\sqrt{T} \log(T)\right)$$

## Linear UCB – Regret Analysis

*Proof:* Define a “good” event  $G := \{\forall t \in [T] | \theta^* \in \mathcal{C}_t := \left\{ \theta \mid \left\| \theta - \hat{\theta}_t \right\|_{V_t}^2 \leq \beta_t \right\}\}$ , and denote the instantaneous expected regret at round  $t$  as  $r_t = \max_a \langle X_a, \theta^* \rangle - \langle X_t, \theta^* \rangle$ . Using the law of total expectation,

$$\begin{aligned} \text{Regret}(\text{LinUCB}, T) &= \mathbb{E}[\text{Regret}(\text{LinUCB}, T) | G] \Pr[G] + \mathbb{E}[\text{Regret}(T) | G^c] \Pr[G^c] \\ &\leq \mathbb{E}[\text{Regret}(\text{LinUCB}, T) | G] + T \Pr[G^c] \\ &\quad (\text{Regret}(\text{LinUCB}, T) \leq T \text{ and } \Pr[G] \leq 1) \\ &= \sum_{t=1}^T \mathbb{E}[r_t | G] + T \Pr[G^c] \leq \sqrt{T \sum_{t=1}^T [\mathbb{E}[r_t | G]]^2} + T \Pr[G^c] \\ &\quad (\text{Cauchy Schwarz inequality: } \langle x, y \rangle \leq \|x\| \|y\| \text{ with } x, y \in \mathbb{R}^T \text{ and } x[t] = 1, y[t] = r_t) \end{aligned}$$

# Linear UCB – Regret Analysis

Recall that  $\text{Regret}(\text{LinUCB}, T) \leq \sqrt{T \sum_{t=1}^T [\mathbb{E}[r_t|G]]^2} + T \Pr[G^c]$ . Let us first bound  $\mathbb{E}[r_t|G]$ . If event  $G$  happens, then  $\theta^* \in \mathcal{C}_t$ . Hence, for all  $a \in [K]$ ,

$$\langle \theta^*, X_a \rangle \leq \max_{\theta \in \mathcal{C}_t} \langle \theta, X_a \rangle = U_a(t) \leq U_{a_t}(t)$$

(Using the equivalence on Slide 15 and the algorithm)

$$\implies \max_{a \in [K]} \langle \theta^*, X_a \rangle \leq U_{a_t}(t) = \max_{\theta \in \mathcal{C}_t} \langle \theta, X_t \rangle = \langle \tilde{\theta}_t, X_t \rangle \quad (\tilde{\theta}_t := \arg \max_{\theta \in \mathcal{C}_t} \langle \theta, X_t \rangle)$$

$$\implies \mathbb{E}[r_t|G] = \mathbb{E}[\max_a \langle X_a, \theta^* \rangle - \langle X_t, \theta^* \rangle | G] \leq \langle \tilde{\theta}_t - \theta^*, X_t \rangle$$

$$\leq \left\| \tilde{\theta}_t - \theta^* \right\|_{V_t} \|X_t\|_{V_t^{-1}}$$

(Cauchy Schwarz inequality with  $x, y \in \mathbb{R}^d$  and  $x = V_t(\tilde{\theta}_t - \theta^*)$ ,  $y = V_t^{-1}X_t$ )

$$\leq \left[ \left\| \tilde{\theta}_t - \hat{\theta}_t \right\|_{V_t} + \left\| \theta^* - \hat{\theta}_t \right\|_{V_t} \right] \|X_t\|_{V_t^{-1}} \quad (\text{Triangle inequality})$$

$$\implies \mathbb{E}[r_t|G] \leq 2\sqrt{\beta_t} \|X_t\|_{V_t^{-1}} \quad (\text{Since } \theta^*, \tilde{\theta}_t \in \mathcal{C}_t)$$

# Linear UCB – Regret Analysis

Putting everything together,

$$\begin{aligned}\text{Regret}(\text{LinUCB}, T) &\leq \sqrt{T \sum_{t=1}^T [\mathbb{E}[r_t|G]]^2} + T \Pr[G^c] \leq 2 \sqrt{T \sum_{t=1}^T \beta_t \|X_t\|_{V_t^{-1}}^2} + T \Pr[G^c] \\ &\leq 2 \sqrt{T \beta_T \sum_{t=1}^T \|X_t\|_{V_t^{-1}}^2} + T \Pr[G^c] \quad (\text{Since } \beta_t \leq \beta_T \text{ for all } t \in [T])\end{aligned}$$

We will prove the following results: (i)  $\sum_{t=1}^T \|X_t\|_{V_t^{-1}}^2 \leq 2d \log\left(\frac{\lambda d + T}{\lambda d}\right)$  and (ii)

$$\sqrt{\beta_t} = \sqrt{d \log\left(\frac{\lambda d + t}{\lambda}\right) + 2 \log(T) + \sqrt{\lambda}}, \Pr[G^c] \leq \frac{1}{T}.$$

Given these results,

$$\text{Regret}(\text{LinUCB}, T) \leq 2 \sqrt{2d T \beta_T \log\left(\frac{\lambda d + T}{\lambda d}\right)} + 1 = O\left(d\sqrt{T} \log(T)\right) \quad \square$$

## Linear UCB – Regret Analysis

**Claim:** If  $\|X_a\| \leq 1$  for all  $a$ ,  $\sum_{t=1}^T \|X_t\|_{V_t^{-1}}^2 \leq 2d \log\left(\frac{\lambda d + T}{\lambda d}\right)$ .

*Proof:*

$$\begin{aligned} V_t &= V_{t-1} + X_t X_t^\top = V_{t-1}^{1/2} \left[ I_d + V_{t-1}^{-1/2} X_t X_t^\top V_{t-1}^{-1/2} \right] V_{t-1}^{1/2} \\ \implies \det[V_t] &= \det[V_{t-1}^{1/2}] \det \left[ I_d + V_{t-1}^{-1/2} X_t X_t^\top V_{t-1}^{-1/2} \right] \det[V_{t-1}^{1/2}] \\ &\hspace{25em} (\det[XY] = \det[X] \det[Y]) \\ &= \det[V_{t-1}] \det \left[ I_d + V_{t-1}^{-1/2} X_t [V_{t-1}^{-1/2} X_t]^\top \right] \quad (\det[X^{1/2}] = \sqrt{\det[X]}) \\ &= \det[V_{t-1}] \left( 1 + \left\| V_{t-1}^{-1/2} X_t \right\|^2 \right) = \det[V_{t-1}] \left( 1 + \|X_t\|_{V_t^{-1}}^2 \right) \\ &\hspace{2em} (\text{Matrix Determinant Lemma: } \det[I_d + x x^\top] = 1 + x^\top x = 1 + \|x\|^2) \\ \implies \ln \left( 1 + \|X_t\|_{V_t^{-1}}^2 \right) &= \ln \left( \frac{\det[V_t]}{\det[V_{t-1}]} \right) \end{aligned}$$

# Linear UCB – Regret Analysis

Recall that  $\ln \left( 1 + \|X_t\|_{V_t^{-1}}^2 \right) = \ln \left( \frac{\det[V_t]}{\det[V_{t-1}]} \right)$ .

Hence,  $\sum_{t=1}^T \ln \left( 1 + \|X_t\|_{V_t^{-1}}^2 \right) = \ln \left( \frac{\det[V_T]}{\det[V_0]} \right)$ . For any  $x \geq 0$ ,  $x \leq 2 \ln(1 + x)$ . Hence,  $\sum_{t=1}^T \|X_t\|_{V_t^{-1}}^2 \leq 2 \sum_{t=1}^T \ln(1 + \|X_t\|_{V_t^{-1}}^2)$ , implying,

$$\sum_{t=1}^T \|X_t\|_{V_t^{-1}}^2 \leq 2 \sum_{t=1}^T \ln(1 + \|X_t\|_{V_t^{-1}}^2) = 2 \ln \left( \frac{\det[V_T]}{\det[V_0]} \right) = 2 \ln \left( \frac{\det[V_T]}{\det[V_0]} \right)$$

$$\det[V_T] \leq \left( \frac{\text{Tr}[V_T]}{d} \right)^d \quad (\det[A] = \prod \lambda_i = \left( (\prod \lambda_i)^{1/d} \right)^d \leq \left( \frac{\sum \lambda_i}{d} \right)^d)$$

$$= \left( \frac{\text{Tr}[V_0] + \sum_{t=1}^T X_t X_t^T}{d} \right)^d \leq \left( \frac{\text{Tr}[V_0] + T}{d} \right)^d = \left( \frac{d \lambda + T}{d} \right)^d$$

(Since  $\|X_t\| \leq 1$ )

$$\Rightarrow \sum_{t=1}^T \|X_t\|_{V_t^{-1}}^2 \leq 2 \ln \left( \left( \frac{(d \lambda + T)/d}{(\det[V_0])^{1/d}} \right)^d \right) = 2d \log \left( \frac{\lambda d + T}{\lambda d} \right) \quad \square$$

## Digression – (Super)-Martingales

**Martingale:** Sequence of random variables for which, at a particular time, the conditional expectation of the next value in the sequence is equal to the present value, regardless of all prior values.

A sequence of random variables –  $M_1, M_2, \dots$  is a discrete-time martingale if for all  $t$ ,

$$\mathbb{E}[|M_t|] \leq \infty \quad ; \quad \mathbb{E}[M_t | M_1, M_2, \dots, M_{t-1}] = M_{t-1}$$

*Example 1:* An unbiased random walk

*Example 2:* Gambler's fortune: Suppose  $M_t$  is a gambler's fortune after  $t$  tosses of a fair coin, where the gambler wins \$1 if the coin comes up heads and loses \$1 if it comes up tails.

**Super-Martingale:** A sequence of random variables –  $M_1, M_2, \dots$  is a discrete-time super-martingale if for all  $t$ ,

$$\mathbb{E}[|M_t|] \leq \infty \quad ; \quad \mathbb{E}[M_t | M_1, M_2, \dots, M_{t-1}] \leq M_{t-1}$$

# Linear UCB – Regret Analysis

**Claim:** If (i)  $\|\theta^*\| \leq 1$  and (ii)  $\|X_a\| \leq 1$  for all  $a$ , for  $\sqrt{\beta_t} = \sqrt{d \log\left(\frac{\lambda d + t}{\lambda}\right) + 2 \log(T)} + \sqrt{\lambda}$  and  $G := \{\forall t \in [T] \mid \theta^* \in \mathcal{C}_t := \left\{ \theta \mid \left\| \theta - \hat{\theta}_t \right\|_{V_t}^2 \leq \beta_t \right\}\}$ ,  $\Pr[G^c] \leq \frac{1}{T}$ .

*Proof:* Define  $S_t := \sum_{s=1}^t \eta_s X_s$  and  $K_t := \sum_{s=1}^t X_s X_s^\top$ . We will prove the claim in 4 steps:

- (i)  $\left\| \theta - \hat{\theta}_t \right\|_{V_t} \leq \|S_t\|_{V_t^{-1}} + \sqrt{\lambda}$ .
- (ii)  $M_t(z) = \exp\left(\langle z, S_t \rangle - \frac{1}{2} \|z\|_{K_t}^2\right)$  is a non-negative super-martingale with  $M_0(z) = 1$ .
- (iii) Use the fact that a mixture of super-martingales given by  $\bar{M}_t = \int_z M_t(z) h(z) dz$  is also a non-negative super-martingale for any probability density function  $h(z)$ .
- (iv) Use the maximal inequality for super-martingales to bound  $\Pr\left[\sup_{t \in [T]} \log(\bar{M}_t(z)) \geq \log(1/\delta)\right]$  and hence bound  $\left\| \theta - \hat{\theta}_t \right\|_{V_t}$ .



# Linear UCB – Regret Analysis

**Part (i):** If  $S_t := \sum_{s=1}^t \eta_s X_s$  and  $K_t := \sum_{s=1}^t X_s X_s^\top$ , then  $\|\theta^* - \hat{\theta}_t\|_{V_t} \leq \|S_t\|_{V_t^{-1}} + \sqrt{\lambda}$ .

*Proof:*

$$b_t = \sum_{s=1}^t X_s R_s = \sum_{s=1}^t X_s [\langle X_s \theta^* + \eta_s \rangle] = \sum_{s=1}^t X_s^\top X_s \theta^* + \sum_{s=1}^t X_s \eta_s = S_t + \sum_{s=1}^t X_s^\top X_s \theta^*.$$

$$\Rightarrow \hat{\theta}_t = V_t^{-1} b_t = V_t^{-1} S_t + V_t^{-1} \left[ \sum_{s=1}^t X_s^\top X_s \right] \theta^*$$

$$\begin{aligned} \|\theta^* - \hat{\theta}_t\|_{V_t} &= \|V_t^{-1} S_t + V_t^{-1} (K_t - I_d) \theta^*\|_{V_t} \leq \|S_t\|_{V_t^{-1}} + \sqrt{\theta^{*\top} (K_t - I_d) V_t (K_t - I_d) \theta^*} \\ &= \|S_t\|_{V_t^{-1}} + \sqrt{\theta^{*\top} (I_d - V_t^{-1} K_t) \theta^*} \end{aligned}$$

$$\Rightarrow \|\theta^* - \hat{\theta}_t\|_{V_t} \leq \|S_t\|_{V_t^{-1}} + \sqrt{\lambda} \|\theta^*\| \leq \|S_t\|_{V_t^{-1}} + \sqrt{\lambda} \quad \square$$

## Linear UCB – Regret Analysis

**Part (ii):** If  $S_t := \sum_{s=1}^t \eta_s X_s$  and  $K_t := \sum_{s=1}^t X_s X_s^\top$ ,  $M_t(z) = \exp \left( \langle z, S_t \rangle - \frac{1}{2} \|z\|_{K_t}^2 \right)$  is a non-negative super-martingale with  $M_0(z) = 1$ .

*Proof:* It is clear that  $M_t(z) = \exp \left( \langle z, S_t \rangle - \frac{1}{2} \|z\|_{K_t}^2 \right)$  is non-negative and  $M_0(z) = 1$ . By our assumption on the noise,  $\mathbb{E}[\exp(\lambda \eta_t) | \mathcal{H}_{t-1}] \leq \exp \left( \frac{\lambda^2}{2} \right)$ . Setting  $\lambda = \langle z, X_t \rangle$ , implies that

$$\mathbb{E}[\exp(\langle z, X_t \rangle \eta_t) | \mathcal{H}_{t-1}] \leq \exp \left( \frac{\|z\|_{X_t X_t^\top}^2}{2} \right).$$

$$\begin{aligned} \mathbb{E}[M_t(z) | \mathcal{H}_{t-1}] &= \mathbb{E} \left[ \exp \left( \langle z, S_{t-1} + \eta_t X_t \rangle - \frac{1}{2} \|z\|_{G_{t-1} + X_t X_t^\top}^2 \right) | \mathcal{H}_{t-1} \right] \\ &= \mathbb{E} \left[ \exp \left( \langle z, \eta_t X_t \rangle - \frac{1}{2} \|z\|_{X_t X_t^\top}^2 \right) | \mathcal{H}_{t-1} \right] \mathbb{E} \left[ \exp \left( \langle z, S_{t-1} \rangle - \frac{1}{2} \|z\|_{V_{t-1}}^2 \right) | \mathcal{H}_{t-1} \right] \\ &= M_{t-1}(z) \mathbb{E} \left[ \exp \left( \langle z, \eta_t X_t \rangle - \frac{1}{2} \|z\|_{X_t X_t^\top}^2 \right) | \mathcal{H}_{t-1} \right] \end{aligned}$$

$$\implies \mathbb{E}[M_t(z) | \mathcal{H}_{t-1}] \leq M_{t-1}(z)$$

# Linear UCB – Regret Analysis

**Fact 1:** For a probability density  $h$ , if  $M_t(z)$  is a non-negative super-martingale with  $M_0(z) = 1$ , the “mixture”  $\bar{M}_t := \int_z M_t(z) h(z) dz$  is also a non-negative super-martingale with  $\bar{M}_0 = 1$ .

**Fact 2:** For a non-negative super-martingale  $\bar{M}_t$  s.t.  $\bar{M}_0 = 1$ , for any  $\epsilon > 0$ ,  $\Pr[\sup_{t \in [T]} \bar{M}_t \geq \epsilon] \leq \frac{1}{\epsilon}$ .

In order to construct  $\bar{M}_t$ , we will choose  $h = \mathcal{N}(0, H^{-1})$  and  $H = \lambda I_d$ .

$$\bar{M}_t = \int_z M_t(z) h(z) dz = \frac{1}{\sqrt{(2\pi)^d \det[H^{-1}]}} \int_z \exp \left( \langle z, S_t \rangle - \frac{1}{2} \|z\|_{K_t}^2 - \frac{1}{2} \|z\|_H^2 \right) dz$$

From **Fact 1**,  $\bar{M}_t$  is a non-negative super-martingale, and hence using **Fact 2** with  $\epsilon = 1/\delta$

$$\Pr \left[ \sup_{t \in [T]} \bar{M}_t \geq \epsilon \right] = \Pr \left[ \sup_{t \in [T]} \log(\bar{M}_t) \geq \log(\epsilon) \right] = \Pr \left[ \sup_{t \in [T]} \log(\bar{M}_t) \geq \log(1/\delta) \right] \leq \delta$$

In the last part of the proof, we will relate  $\bar{M}_t$  to  $\|S_t\|_{V_t^{-1}}$ .

# Linear UCB – Regret Analysis

$$\text{Recall that } \bar{M}_t = \int_z M_t(z) h(z) dz = \frac{1}{\sqrt{(2\pi)^d \det[H^{-1}]}} \int_z \exp \left( \langle z, S_t \rangle - \frac{1}{2} \|z\|_{K_t}^2 - \frac{1}{2} \|z\|_H^2 \right) dz.$$

$$\langle z, S_t \rangle - \frac{1}{2} \|z\|_{K_t}^2 - \frac{1}{2} \|z\|_H^2 = \frac{1}{2} \|S_t\|_{(K_t+H)}^2 - \frac{1}{2} \|z - (K_t + H)^{-1} S_t\|_{(K_t+H)}^2$$

$$\implies \int_z M_t(z) h(z) dz = \frac{\exp \left( \frac{1}{2} \|S_t\|_{V_t}^2 \right)}{\sqrt{(2\pi)^d \det[H^{-1}]}} \int_z \exp \left( -\frac{1}{2} \|z - V_t^{-1} S_t\|_{V_t}^2 \right) dz$$

$$\implies \bar{M}_t = \frac{\exp \left( \frac{1}{2} \|S_t\|_{V_t}^2 \right)}{\sqrt{(2\pi)^d \det[H^{-1}]}} \sqrt{(2\pi)^d \det[V_t^{-1}]} = \sqrt{\frac{\det[H]}{\det[V_t]}} \exp \left( \frac{1}{2} \|S_t\|_{V_t}^2 \right)$$

(Integral of a Gaussian density)

## Linear UCB – Regret Analysis




Putting everything together, we know that for all  $t \in [T]$ , w.p  $1 - \delta$ ,  $\log(\bar{M}_t) \leq \log(1/\delta)$ . Using the result from the previous slide, w.p  $1 - \delta$ , for all  $t \in [T]$

$$\begin{aligned} \frac{1}{2} \|S_t\|_{V_t}^2 + \frac{1}{2} \log \left( \frac{\det[H]}{\det[V_t]} \right) &\leq \log(1/\delta) \implies \|S_t\|_{V_t} \leq \sqrt{\log \left( \frac{\det[V_t]}{\lambda^d} \right) + 2 \log(1/\delta)} \\ &\implies \|S_t\|_{V_t} \leq \sqrt{d \log \left( \frac{\lambda d + t}{\lambda} \right) + 2 \log(1/\delta)} \end{aligned}$$

From **Part (i)**, we know that,

$$\left\| \theta^* - \hat{\theta}_t \right\|_{V_t} \leq \|S_t\|_{V_t^{-1}} + \sqrt{\lambda} \leq \underbrace{\sqrt{d \log \left( \frac{\lambda d + t}{\lambda} \right) + 2 \log(1/\delta)}}_{:= \sqrt{\beta_t}} + \sqrt{\lambda}$$

Hence, we have shown that w.p.  $1 - \frac{1}{T}$ ,  $\left\| \theta^* - \hat{\theta}_t \right\|_{V_t}^2 \leq \beta_t$ . □

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