CMPT 210: Probability and Computation

Lecture 15

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Collect your Midterm Exams

Logistics

- Collect your Midterm exams from TASC-1 9203 on Tuesdays between 10.30 am - 12 pm.
- Assignment 3 is out: `https://vaswanis.github.io/210-S22/A3.pdf`
  Due Friday 15 July in class.
- For A3, you can use your late-submission and submit on Tuesday 19 July in class.
- Solutions will be released on 19 July after class, meaning that no submissions will be allowed after that.
- If you have used your late-submission, and submit late again, you will lose 50% of the mark.
- If you have questions about either assignment or the marking, post it on Piazza: `https://piazza.com/sfu.ca/summer2022/cmpt210/home`

Recap

Expectation/mean of a random variable R is denoted by $\mathbb{E}[R]$ and "summarizes" its distribution. Formally, $\mathbb{E}[R] := \sum_{\omega \in \mathcal{S}} \Pr[\omega]\, R[\omega]$

Alternate definition of expectation: $\mathbb{E}[R] = \sum_{x \in \text{Range}(R)} x \Pr[R = x]$.

Linearity of Expectation: For n random variables $R_1, R_2, \ldots, R_n$ and constants $a_1, a_2, \ldots, a_n$, $\mathbb{E}\left[\sum_{i=1}^{n} a_i R_i\right] = \sum_{i=1}^{n} a_i \,\mathbb{E}[R_i]$.

Conditional Expectation: For random variable R, the expected value of R conditioned on an event A is given by:

$$\mathbb{E}[R|A] = \sum_{x \in \text{Range}(R)} x \Pr[R = x|A]$$

Law of Total Expectation: If R is a random variable $\mathcal{S} \to V$ and events $A_1, A_2, \ldots A_n$ form a partition of the sample space, then,

$$\mathbb{E}[R] = \sum_{i} \mathbb{E}[R|A_i] \Pr[A_i]$$

Independence of random variables

We define two random variables R_1 and R_2 to be independent if for *all* $x_1 \in \text{Range}(R_1)$ and $x_2 \in \text{Range}(R_2)$, events $[R_1 = x_1]$ and $[R_2 = x_2]$ are independent. More formally,

$$\Pr[(R_1 = x_1) \cap (R_2 = x_2)] = \Pr[(R_1 = x_1)]\Pr[(R_2 = x_2)]$$

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Q: Suppose we toss three independent, unbiased coins. Let C be r.v. equal to the number of heads that appear and M be the r.v. that is equal to 1 if all the coins match. Are random variables C and M independent?

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$\text{Range}(C) = \{0, 1, 2, 3\}$ and $\text{Range}(M) = \{0, 1\}$. $\Pr[C = 3] = \frac{1}{8}$ and $\Pr[M = 1] = \frac{1}{4}$. $\Pr[(C = 3) \cap (M = 1)] = \frac{1}{8} \neq \frac{1}{32} = \Pr[C = 3] \Pr[M = 1]$. Hence, C and M are not independent.

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$\Pr[H_1 = 0 \cap M_1 = 1] = \Pr[\{TTT\}] = \frac{1}{8} = \Pr[H_1 = 0]\,\Pr[M_1 = 1]$.

$\Pr[H_1 = 1 \cap M_1 = 1] = \Pr[\{HHH\}] = \frac{1}{8} = \Pr[H_1 = 1]\,\Pr[M_1 = 1]$.

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Hence, H_1 and M are independent.

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Hence, H_1 and M are independent.

Similar to events, random variables $R_1, R_2, \ldots, R_n$ are mutually independent if for all $x_1, x_2, \ldots, x_n$, events $[R_1 = x_1], [R_2 = x_2], \ldots [R_n = x_n]$ are mutually independent.

Independence - Examples

Q: Suppose that the successive daily changes of the price of a given stock are assumed to be independent and identically distributed random variables – for each day i, the PDF is:

$$\begin{aligned}
\Pr[\text{Daily change on day } i] &= -3 && \text{(With } p = 0.1,\text{)} \\
&= -2 && \text{(With } p = 0.1\text{)} \\
&= -1 && \text{(With } p = 0.2\text{)} \\
&= 0 && \text{(With } p = 0.3\text{)} \\
&= 1 && \text{(With } p = 0.2\text{)} \\
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\end{aligned}$$

If E is the event that the stocks price will increase successively by 1, 2, and 0 points in the next three days, compute $\mathbb{E}[\mathcal{I}_E]$.

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If E is the event that the stocks price will increase successively by 1, 2, and 0 points in the next three days, compute $\mathbb{E}[\mathcal{I}_E]$.

If X_i is the r.v. corresponding to the price increase on day i, we wish to compute $\mathbb{E}[\mathcal{I}_E] = \Pr[E] = \Pr[X_1 = 1 \cap X_2 = 2 \cap X_3 = 0]$. X_1, X_2 and X_3 are mutually independent and hence, $\Pr[X_1 = 1 \cap X_2 = 2 \cap X_3 = 0] = \Pr[X_1 = 1] \Pr[X_2 = 2] \Pr[X_3 = 0] = 0.006$.

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Yes! Recall the derivation of the linearity of expectation. We never assumed that R_1 and R_2 are independent for the proof and the linearity of expectation holds regardless of whether the random variables are independent.

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$$
\begin{aligned}
\mathbb{E}[R_1 R_2] &= \sum_{x \in \mathrm{Range}(R_1 R_2)} x \, \Pr[R_1 R_2 = x] = \sum_{r_1 \in \mathrm{Range}(R_1), r_2 \in \mathrm{Range}(R_2)} r_1 r_2 \, \Pr[R_1 = r_1 \cap R_2 = r_2] \\
&= \sum_{r_1 \in \mathrm{Range}(R_1)} \sum_{r_2 \in \mathrm{Range}(R_2)} r_1 r_2 \, \Pr[R_1 = r_1 \cap R_2 = r_2] \\
&= \sum_{r_1 \in \mathrm{Range}(R_1)} \sum_{r_2 \in \mathrm{Range}(R_2)} r_1 r_2 \, \Pr[R_1 = r_1] \Pr[R_2 = r_2] \\
&= \sum_{r_1 \in \mathrm{Range}(R_1)} r_1 \Pr[R_1 = r_1] \sum_{r_2 \in \mathrm{Range}(R_2)} r_2 \Pr[R_2 = r_2] = \mathbb{E}[R_1]\mathbb{E}[R_2]
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Independence of random variables

Alternate definition of independence – two random variables R_1 and R_2 are independent if for *all* $x_1 \in \text{Range}(R_1)$ and $x_2 \in \text{Range}(R_2)$,

$$\Pr[(R_1 = x_1)|(R_2 = x_2)] = \Pr[(R_1 = x_1)]$$
$$\Pr[(R_2 = x_2)|(R_1 = x_1)] = \Pr[(R_2 = x_2)]$$

Intuitively, this means that conditioning on the value of R_2 does not change the probability of the event $R_1 = x_1$, and vice-versa.

Expectation - Examples

Q: Suppose there is a dinner party where n people check in their coats. The coats are mixed up during dinner, so that afterward each person receives a random coat. In particular, each person gets their own coat with probability $\frac{1}{n}$. What is the expected number of people who get their own coat?

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Let G be the number of people that get their own coat. We wish to compute $\mathbb{E}[G]$. Define G_i to be the indicator r.v. that person i gets their own coat. Observe that $G = G_1 + G_2 + \ldots + G_n$ and by linearity of expectation $\mathbb{E}[G] = \mathbb{E}[G_1] + \mathbb{E}[G_2] + \ldots + \mathbb{E}[G_n]$. For each i, $\mathbb{E}[G_i] = \Pr[G_i] = \frac{1}{n}$. Hence, $\mathbb{E}[G] = 1$ meaning that on average one person will correctly receive their coat.

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No. Since if $G_1 = G_2 = \ldots G_{n-1} = 1$, then,
$\Pr[G_n = 1 | (G_1 = 1 \cap G_2 = 1 \cap \ldots \cap G_{n-1} = 1)] = 1 \neq \frac{1}{n} = \Pr[G_n = 1]$. Notice that we have used the linearity of expectation for the G_i's even though these r.v. are not mutually independent.

Expectation - Examples

For a random variable $X : \mathcal{S} \to V$ and a function $g : V \to \mathbb{R}$, we define $\mathbb{E}[g(X)]$ as follows:

$$\mathbb{E}[g(X)] := \sum_{x \in \mathrm{Range}(X)} g(x) \Pr[X = x]$$

If $g(x) = x$ for all $x \in \mathrm{Range}(X)$, then $\mathbb{E}[g(X)] = \mathbb{E}[X]$.

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For a standard dice, $X \sim \mathrm{Uniform}(\{1, 2, 3, 4, 5, 6\})$ and hence,

$$\mathbb{E}[X^2] = \sum_{x \in \{1,2,3,4,5,6\}} x^2 \Pr[X = x] = \frac{1}{6}\left[1^2 + 2^2 + \ldots + 6^2\right] = \frac{91}{6}$$

$$(\mathbb{E}[X])^2 = \left(\sum_{x \in \{1,2,3,4,5,6\}} x \Pr[X = x] = \frac{1}{6}\left[1 + 2 + \ldots + 6\right]\right) = \frac{49}{4}$$

Questions?

Joint distribution

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If X and Y are independent random variables, $\text{PDF}_{X,Y}[x, y] = \text{PDF}_X[x]\,\text{PDF}_Y[y]$.

If $\text{Range}[X] = \{x_1, x_2, \dots x_n\}$, $\text{Range}[Y] = \{y_1, y_2, \dots y_n\}$, then for $x \in \text{Range}(X)$, $[X = x] = [X = x \cap y = y_1] \cup [X = x \cap y = y_2] \cup \ldots \cup [X = x \cap y = y_n] \implies \Pr[X = x] = \Pr[X = x \cap y = y_1] + \Pr[X = x \cap y = y_2] + \ldots + \Pr[X = x \cap y = y_n]$.

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$$\implies \mathrm{PDF}_X[x] = \sum_i \mathrm{PDF}_{X,Y}[x, y_i]$$

Hence, we can obtain the distribution for each r.v. from the joint distribution by "marginalizing" over the other r.v's.

Joint distribution - Examples

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$\mathrm{PDF}_{X,Y}[0,0] = \frac{\binom{5}{3}}{\binom{12}{3}} = 10/220$, $\mathrm{PDF}_{X,Y}[1,2] = \frac{\binom{3}{1}\binom{4}{2}\binom{5}{2}}{\binom{12}{3}} = 18/220$.

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Table 4.1 $P\{X = i, Y = j\}$.

i \ j	**0**	**1**	**2**	**3**	**Row Sum** $= P\{X = i\}$
0	$\frac{10}{220}$	$\frac{40}{220}$	$\frac{30}{220}$	$\frac{4}{220}$	$\frac{84}{220}$
1	$\frac{30}{220}$	$\frac{60}{220}$	$\frac{18}{220}$	0	$\frac{108}{220}$
2	$\frac{15}{220}$	$\frac{12}{220}$	0	0	$\frac{27}{220}$
3	$\frac{1}{220}$	0	0	0	$\frac{1}{220}$
Column Sums = $P\{Y = j\}$	$\frac{56}{220}$	$\frac{112}{220}$	$\frac{48}{220}$	$\frac{4}{220}$	

Questions?

Deviation from the Mean

We have developed tools to calculate the mean of random variables. Getting a handle on the expectation is useful because it tell us what would happen on average.

Summarizing the PDF using the mean is typically not enough. We also want to know how "spread" the distribution is.

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Example: Consider three random variables W, Y and Z whose PDF's can be given as:

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\begin{aligned}
W &= 0 && \text{(with } p = 1) \\
Y &= -1 && \text{(with } p = 1/2) \\
&= +1 && \text{(with } p = 1/2) \\
Z &= -1000 && \text{(with } p = 1/2) \\
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\end{aligned}$$

Though $\mathbb{E}[W] = \mathbb{E}[Y] = \mathbb{E}[Z] = 0$, these distributions are quite different. Z can take values really far away from its expected value, while W can take only one value equal to the mean.

Hence, we want to understand how much does a random variable "deviate" from its mean.

Variance

Standard way to measure the deviation from the mean is to calculate the *variance*. For r.v. X,

$$\mathrm{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_{x \in \mathrm{Range}(X)} (x - \mu)^2 \Pr[X = x] \qquad (\text{where } \mu := \mathbb{E}[X])$$

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Q: If $X \sim \mathrm{Ber}(p)$, compute $\mathrm{Var}[X]$.

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Intuitively, the variance measures the weighted (by the probability) average of how far the random variable is from the mean μ.

Q: If $X \sim \mathrm{Ber}(p)$, compute $\mathrm{Var}[X]$. Since X is a Bernoulli random variable, $X = 1$ with probability p and $X = 0$ with probability $1 - p$. Recall that $\mathbb{E}[X] = \mu = (0)(1-p) + (1)(p) = p$.

$$\mathrm{Var}[X] = \sum_{x \in \{0,1\}} (x - p)^2 \Pr[X = x] = (0 - p)^2 \Pr[X = 0] + (1 - p)^2 \Pr[X = 1]$$
$$= p^2(1-p) + (1-p)^2 p = p(1-p)[p + 1 - p] = p(1-p).$$

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For a Bernoulli r.v. X, $\mathrm{Var}[X] = p(1 - p) \leq \frac{1}{4}$. Hence, the variance is maximum when $p = 1/2$ (equal probability of getting heads/tails).

Variance

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$$
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\mathrm{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] &= \sum_{x \in \mathrm{Range}(X)} (x - \mu)^2 \Pr[X = x] \\
&= \sum_{x \in \mathrm{Range}(X)} (x^2 - 2\mu x + \mu^2) \Pr[X = x] \\
&= \sum_{x \in \mathrm{Range}(X)} (x^2 \Pr[X = x]) - (2\mu x \Pr[X = x]) + (\mu^2) \Pr[X = x] \\
&= \sum_{x \in \mathrm{Range}(X)} x^2 \Pr[X = x] - 2\mu \sum_{x \in \mathrm{Range}(X)} x \Pr[X = x] + \mu^2 \sum_{x \in \mathrm{Range}(X)} \Pr[X = x] \\
&\qquad \text{(Since } \mu \text{ is a constant does not depend on the } x \text{ in the sum.)} \\
&= \mathbb{E}[X^2] - 2\mu \mathbb{E}[X] + \mu^2 \sum_{x \in \mathrm{Range}(X)} \Pr[X = x] \qquad \text{(Definition of } \mathbb{E}[X] \text{ and } \mathbb{E}[X^2]\text{)} \\
&= \mathbb{E}[X^2] - 2\mu^2 + \mu^2 \qquad \text{(Definition of } \mu\text{)} \\
\implies \mathrm{Var}[X] = \mathbb{E}[X^2] - \mu^2 &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2.
\end{aligned}
$$

Back to throwing dice

Q: For a standard dice, if X is the r.v. corresponding to the number that comes up on the dice, compute $\text{Var}[X]$

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Recall that, for a standard dice, $X \sim \text{Uniform}(\{1,2,3,4,5,6\})$ and hence,

$$\mathbb{E}[X^2] = \sum_{x \in \{1,2,3,4,5,6\}} x^2 \Pr[X = x] = \frac{1}{6}\left[1^2 + 2^2 + \ldots + 6^2\right] = \frac{91}{6}$$

$$(\mathbb{E}[X])^2 = \left(\sum_{x \in \{1,2,3,4,5,6\}} x \Pr[X = x] = \frac{1}{6}\left[1 + 2 + \ldots + 6\right] \right) = \frac{49}{4}$$

$$\implies \text{Var}[X] = \frac{91}{6} - \frac{49}{4} \approx 2.917$$

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In general, if $X \sim \text{Uniform}(\{v_1, v_2, \ldots v_n\})$,

$$\text{Var}[X] = \frac{[v_1^2 + v_2^2 + \ldots v_n^2]}{n} - \left(\frac{[v_1 + v_2 + \ldots v_n]}{n}\right)^2$$

Variance - Examples

Q: Calculate $\text{Var}[W]$, $\text{Var}[Y]$ and $\text{Var}[Z]$ whose PDF's are given as:

$$\begin{aligned}
W &= 0 && \text{(with } p = 1\text{)} \\
Y &= -1 && \text{(with } p = 1/2\text{)} \\
&= +1 && \text{(with } p = 1/2\text{)} \\
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Recall that $\mathbb{E}[W] = \mathbb{E}[Y] = \mathbb{E}[Z] = 0$.

$\text{Var}[W] = \mathbb{E}[W^2] - \mathbb{E}[W^2] = \mathbb{E}[W^2] = \sum_{w \in \text{Range}(W)} w^2 \Pr[W = w] = 0^2(1) = 0$. The variance of W is zero because it can only take one value and the r.v. does not "vary".

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$\mathrm{Var}[Y] = \mathbb{E}[Y^2] = \sum_{y \in \mathrm{Range}(Y)} y^2 \Pr[Y = y] = (-1)^2(1/2) + (1)^2(1/2) = 1$.

$\mathrm{Var}[Z] = \mathbb{E}[Z^2] = \sum_{z \in \mathrm{Range}(Z)} z^2 \Pr[Z = z] = (-1000)^2(1/2) + (1000)^2(1/2) = 10^6$.

The variance of Z is the largest because it can take values that are far away from the mean. Hence, the variance can be used to distinguish between r.v.'s that have the same mean.

Variance - Examples

Q: Game A: We win \$2 with probability 2/3 and lose \$1 with probability 1/3. Game B: We win \$1002 with probability 2/3 and lose \$2001 with probability 1/3. Which game is better financially?

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Hence, on average, the two games have the same payoff. To get more information, let us analyze the variance.

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Intuitively, this means that the payoff in Game A is usually close to the expected value of \$1, but the payoff in Game B can deviate very far from this expected value.

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Intuitively, this means that the payoff in Game A is usually close to the expected value of \$1, but the payoff in Game B can deviate very far from this expected value. High variance is often associated with high risk. For example, in ten rounds of Game A, we expect to make \$10, but could conceivably lose \$10 instead (if we lose each game). On the other hand, in ten rounds of game B, we also expect to make \$10, but could actually lose more than \$20000!

Variance - Examples

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Similar to Slide 13 of Lecture 14, let A be the event that we get a heads in the first toss. Using the law of total expectation,

$$\mathbb{E}[R^2] = \mathbb{E}[R^2|A]\,\Pr[A] + \mathbb{E}[R^2|A^c]\,\Pr[A^c]$$

We know that, $\mathbb{E}[R^2|A] = 1$ ($R^2 = 1$ if we get a heads in the first coin toss). $\Pr[A] = p$. Hence,

$$\mathbb{E}[R^2] = (1)(p) + \mathbb{E}[R^2|A^c](1-p)$$

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$$\Pr[R = k| \text{ if first toss is a tails}] = \Pr[R = k-1]$$

$$\mathbb{E}[R^2|A^c] = \sum_{k=1} k^2 \Pr[R = k-1] = \sum_{t=0} (t+1)^2 \Pr[R = t] = \mathbb{E}[(R+1)^2] \qquad (t = k-1)$$

Variance - Examples

Putting everything together,

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$$\mathbb{E}[R^2] = (1)(p) + \mathbb{E}[R^2 + 2R + 1](1-p) \implies p\,\mathbb{E}[R^2] = p + 2(1-p)\mathbb{E}[R] + (1-p)\mathbb{E}[1]$$

$$\implies p\mathbb{E}[R^2] = p + 2(1-p)\frac{1}{p} + (1-p)$$

$$\implies \mathbb{E}[R^2] = \frac{2(1-p)}{p^2} + \frac{1}{p} \implies \mathbb{E}[R^2] = \frac{2-p}{p^2}$$

$$\implies \mathrm{Var}[R] = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

Questions?

Standard Deviation

In the gambling example on Slide 17, the random variable is in dollars, then the expectation is also in dollars, but the variance is in square dollars. To get the right units, we define the standard deviation. For r.v. X, the standard deviation in X is defined as:

$$\sigma_X := \sqrt{\mathsf{Var}[X]} = \sqrt{\mathbb{E}[X^2] - (\mathbb{E}[X])^2}$$

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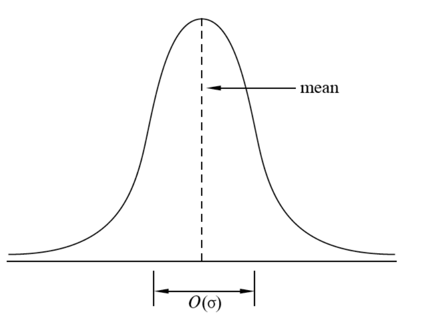

Standard deviation for a "bell"-shaped distribution indicates how wide the "main part" of the distribution is.

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&= (a^2\mathbb{E}[R^2] + 2ab\mathbb{E}[R] + b^2) - (a\mathbb{E}[R] + b)^2 \\
&= (a^2\mathbb{E}[R^2] + 2ab\mathbb{E}[R] + b^2) - (a^2(\mathbb{E}[R])^2 + 2ab\mathbb{E}[R] + b^2) \\
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Similarly, for the standard deviation,

$$\sigma_{aR+b} = \sqrt{\mathrm{Var}[aR+b]} = \sqrt{a^2\mathrm{Var}[R]} = a\,\sigma_R$$

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Similarly, for the standard deviation,

$$\sigma_{aR+b} = \sqrt{\text{Var}[aR + b]} = \sqrt{a^2\text{Var}[R]} = a\,\sigma_R$$

Note the difference from the property of expectation,

$$\mathbb{E}[aR + b] = a\mathbb{E}[R] + b$$

Variance

Recall that for r.v's R and S, $\mathbb{E}[R+S] = \mathbb{E}[R] + \mathbb{E}[S]$.

In general, such a property is not true for the variance, i.e. variance of a sum is not equal to the sum of the variances.

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$$\begin{aligned}
\mathrm{Var}[R+S] &= \mathbb{E}[(R+S)^2] - (\mathbb{E}[R+S])^2 = \mathbb{E}[R^2 + S^2 + 2RS] - (\mathbb{E}[R] + \mathbb{E}[S])^2 \\
&= \mathbb{E}[R^2 + S^2 + 2RS] - [(\mathbb{E}[R])^2 + (\mathbb{E}[S])^2 - 2\mathbb{E}[R]\,\mathbb{E}[S]] \\
&= [\mathbb{E}[R^2] - (\mathbb{E}[R])^2] + [\mathbb{E}[S^2] - (\mathbb{E}[S])^2] + 2(\mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]) \\
&= \mathrm{Var}[R] + \mathrm{Var}[S] + 2(\mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S])
\end{aligned}$$

Recall that if r.v. are independent, $\mathbb{E}[RS] = \mathbb{E}[R]\,\mathbb{E}[S]$,

$$\implies \mathrm{Var}[R+S] = \mathrm{Var}[R] + \mathrm{Var}[S]$$

Variance

Random variables $R_1, R_2, R_3, \ldots R_n$ are *pairwise* independent if for any pair R_i and R_j, for $x \in \text{Range}(R_i)$ and $y \in \text{Range}(R_j)$, events $\Pr[R_i = x]$ and $\Pr[R_j = y]$ are pairwise independent implying that $\Pr[(R_i = x) \cap (R_j = y)] = \Pr[R_i = x] \Pr[R_j = y]$.

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$$\begin{aligned}
\text{Var}[R_1 + R_2 + \ldots R_n] &= \mathbb{E}[(R_1 + R_2 + \ldots R_n)^2] - (\mathbb{E}[R_1 + R_2 + \ldots R_n])^2 \\
&= \sum_{i=1}^{n} [\mathbb{E}[R_i^2] - (\mathbb{E}[R_i])^2] + 2 \sum_{i,j \mid 1 \leq i < j \leq n} [\mathbb{E}[R_i R_j] - \mathbb{E}[R_i]\mathbb{E}[R_j]] \\
\text{Var}[R_1 + R_2 + \ldots R_n] &= \sum_{i=1}^{n} \text{Var}[R_i] \qquad \text{(Since the r.v's are pairwise independent)}
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\text{Var}[R_1 + R_2 + \ldots R_n] &= \sum_{i=1}^{n} \text{Var}[R_i] \qquad \text{(Since the r.v's are pairwise independent)}
\end{aligned}$$

Importantly, we do not require the r.v's to be mutually independent. Similar to events, mutual independence $\implies$ pairwise independence, but pairwise independence $\not\Rightarrow$ mutual independence.

Variance - Examples

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$$R = R_1 + R_2 + \ldots + R_n \implies \text{Var}[R] = \text{Var}[R_1 + R_2 + \ldots + R_n]$$

Since $R_1, R_2, \ldots, R_n$ are independent indicator random variables, and hence pairwise independent,

$$\text{Var}[R] = \text{Var}[R_1] + \text{Var}[R_2] + \ldots + \text{Var}[R_n]$$

Since the variance of an indicator (Bernoulli) r.v. is $p(1-p)$,

$$\text{Var}[R] = n\, p\, (1-p).$$

Back to throwing dice

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Q: We throw 10 independent dice and define R to be the random variable equal to the number of dice that have an even number. What is $\mathrm{Var}[R]$?

Back to throwing dice

Q: We throw a standard dice, and define a random variable R which is equal to 1 if we get an even number and 0 otherwise.What is $\mathrm{Var}[R]$?

Q: We throw 10 independent dice and define R to be the random variable equal to the number of dice that have an even number. What is $\mathrm{Var}[R]$?

Q: We repeatedly and independently throw the dice until we get an even number. We define a random variable R equal to the number of throws we need to get an even number. What is $\mathrm{Var}[R]$?

Questions?

Matching Birthdays

Q: In a class of n students, what is the probability that two students share the same birthday? Assume that (i) each student is equally likely to be born on any day of the year, (ii) no leap years and (iii) student birthdays are independent of each other.

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For $d := 365$,

$$\Pr[\text{two students share the same birthday}] = 1 - \frac{d \times (d-1) \times (d-2) \times \ldots (d-(n-1))}{d^n}$$

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Q: On average, how many matched birthdays should we expect in the class? Define M to be the number of pairs of students with matching birthdays. For a fixed ordering of the students, let $X_{i,j}$ be the indicator r.v. corresponding to the event $E_{i,j}$ that the birthdays of students i and j match. Hence,

$$M = \sum_{i,j | 1 \leq i < j \leq n} X_{i,j} \implies \mathbb{E}[M] = \mathbb{E}[\sum_{i,j | 1 \leq i < j \leq n} X_{i,j}] = \sum_{i,j | 1 \leq i < j \leq n} \mathbb{E}[X_{i,j}] = \sum_{i,j | 1 \leq i < j \leq n} \Pr[E_{i,j}]$$

(Linearity of expectation)

Matching Birthdays

For a pair of students i, j, let B_i, B_j be the r.v. equal to the day of student i and j's birthday. Range(B_i) $= \{1{,}2{,}\ldots, 365\}$ and for all $k \in [365]$, $\Pr[B_i = k] = 1/d$.

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$$E_{i,j} = (B_i = 1 \cap B_j = 1) \cup (B_i = 2 \cap B_j = 2) \cup \ldots$$

$$\implies \Pr[E_{i,j}] = \sum_{k=1}^{d} \Pr[B_i = k \cap B_j = k] = \sum_{k=1}^{d} \Pr[B_i = k] \Pr[B_j = k] = \sum_{k=1}^{d} \frac{1}{d^2} = \frac{1}{d}$$

$$\implies \mathbb{E}[M] = \sum_{i,j|1\leq i<j\leq n} \Pr[E_{i,j}] = \frac{1}{d} \sum_{i,j|1\leq i<j\leq n} = \frac{1}{d}[(n-1) + (n-2) + \ldots + 1] = \frac{n\,(n-1)}{2d}$$

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Hence, in our class of 48 students, on average, there are $\frac{(24)\,(47)}{365} = 3.09$ students with matching birthdays.

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Yes, because for all i, j and i', j' (where $i \neq i'$), $\Pr[X_{i,j} = 1 | X_{i',j'} = 1] = \Pr[X_{i,j} = 1]$ because if students i' and j' have matching birthdays, it does not tell us anything about whether i and j have matching birthdays.

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$$\mathrm{Var}[M] = \mathrm{Var}[\sum_{i,j|1 \le i < j \le n} X_{i,j}]$$

Since $X_{i,j}$ are pair-wise independent, the variance of the sum is equal to the sum of the variance.

$$\implies \mathrm{Var}[M] = \sum_{i,j|1 \le i < j \le n} \mathrm{Var}[X_{i,j}] = \sum_{i,j|1 \le i < j \le n} \frac{1}{d}\left(1 - \frac{1}{d}\right) = \frac{1}{d}\left(1 - \frac{1}{d}\right)\frac{n\,(n-1)}{2}$$

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Hence, in our class of 48 students, the standard deviation for the matching birthdays is equal to $\sqrt{\frac{(24)(47)}{365}\frac{364}{365}} \approx 1.75$.

Questions?