CMPT 210: Probability and Computation

Lecture 7

Sharan Vaswani

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Recap - Conditional Probability

For events E and F, we wish to compute $\Pr[E|F]$, the probability of event E conditioned on F.

Approach 1: With conditioning, F can be interpreted as the *new sample space* such that for $\omega \notin F$, $\Pr[\omega|F] = 0$.

Example: For computing Pr(we get a 6|the outcome is even), the new sample space is $F = \{2, 4, 6\}$ and the resulting probability space is uniform. $\Pr[\{\text{even number}\}] = \frac{1}{3}$ and $\Pr[\{\text{odd number}\}] = 0$.

Approach 2: $\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]}$.

Example: $E \cap F = \{6\}$. $\Pr[E \cap F] = \frac{1}{6}$. $\Pr[F] = \Pr[\{2\}] + \Pr[\{4\}] + \Pr[\{6\}] = \frac{1}{2}$. Hence, $\frac{\Pr[E \cap F]}{\Pr[F]} = \frac{1/6}{1/2} = \frac{1}{3}$.

Conditional Probability - Examples

Q: The organization that Jones works for is running a father–son dinner for those employees having at least one son. Each of these employees is invited to attend along with his youngest son. If Jones is known to have two children, what is the conditional probability that they are both boys given that he is invited to the dinner?

Sample space is the pair of genders of Jones' younger and older child. Hence, $\mathcal{S} = \{(b,b),(b,g),(g,b),(g,g)\}$.

The event that we care about is Jones has both boys. Hence, $E = \{(b,b)\}$

Additional information that we are conditioning on is that Jones is invited to the dinner meaning that he has at least one son. Hence, $F = \{(b,b),(b,g),(g,b)\}$.

Hence, $E \cap F = \{(b,b)\}$, $\Pr[E \cap F] = \frac{|E \cap F|}{|\mathcal{S}|} = \frac{1}{4}$. $\Pr[F] = \frac{|F|}{|\mathcal{S}|} = \frac{3}{4}$.

$\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]} = \frac{1/4}{3/4} = \frac{1}{3}$.

Conditional Probability - Examples

Q: Ms. Perez figures that there is a 30 percent chance that her company will set up a branch office in Phoenix. If it does, she is 60 percent certain that she will be made manager of this new operation. What is the probability that Perez will be an office manager in the Phoenix branch?

E = Perez will be a branch office manager; F = her company will set up a branch office in Phoenix; $E \cap F$ = Perez will be an office manager in the Phoenix branch.

From the question, we know that $\Pr[F] = 0.3$, $\Pr[E|F] = 0.6$. Hence, $\Pr[E \cap F] = \Pr[E] \Pr[E|F] = 0.3 \times 0.6 = 0.18$.

Conditional Probability Examples

Q: Suppose we have a bowl containing 6 white and 5 black balls. We randomly draw a ball. What is the probability that we draw a black ball

Q: We randomly draw two balls, one after the other (without putting the first back). What is the probability that we (i) draw a black ball followed by a white ball (ii) draw a white ball followed by a black ball (iii) we get one black ball and one white ball (iv) both black (v) both white?

B1 = Draw black first, W1 = Draw white first. B2 = Black second, W2 = White second.

(i) $\Pr[B1] = \frac{5}{11}$. $\Pr[W2|B1] = \frac{6}{10}$. Hence, $\Pr[B1 \cap W2] = \Pr[B1]\Pr[W2|B1] = \frac{30}{110}$.

(ii) $\Pr[W1] = \frac{6}{11}$. $\Pr[B2|W1] = \frac{5}{10}$. Hence, $\Pr[W1 \cap B2] = \Pr[W1]\Pr[B2|W1] = \frac{30}{110}$.

(iii) $G = (B1 \cap W2) \cup (W1 \cap B2)$. Events $B1 \cap W2$ and $B2 \cap W1$ are mutually exclusive. By the union rule for mutually exclusive events, $\Pr[G] = \Pr[B1 \cap W2] + \Pr[W1 \cap B2] = \frac{60}{110}$.

(iv) $\Pr[B1 \cap B2] = \Pr[B1]\Pr[B2|B1] = \frac{20}{110}$.

(v) $\Pr[W1 \cap W2] = \Pr[W1]\Pr[W2|W1] = \frac{20}{110}$.

Conditional Probability Examples

Q: Two teams A and B are asked to separately design a new product within a month. From past experience we know that, (a) The probability that team A is successful is 2/3, (b) The probability that team B is successful is 1/2, (c) The probability that at least one team is successful is 3/4. Assuming that exactly one successful design is produced, what is the probability that it was designed by team B.

Let SS be the event that both teams are successful, SF be the event that team A succeeded but team B failed, FS be the event that team B succeeded but team A failed and FF be the event that both teams failed. Hence, $\mathcal{S} = \{(s,s),(s,f),(f,s),(f,f)\}$.

Since exactly one successful design is produced, we know that exactly one of the teams succeeded. Hence, we wish to compute $\Pr[\{FS\}|\{FS \cup SF\}]$.

$\Pr[SS \cup SF] = \frac{2}{3}$, $\Pr[SS \cup FS] = \frac{1}{2}$, $\Pr[SS \cup SF \cup SS] = 3/4$. Since these are mutually exclusive,

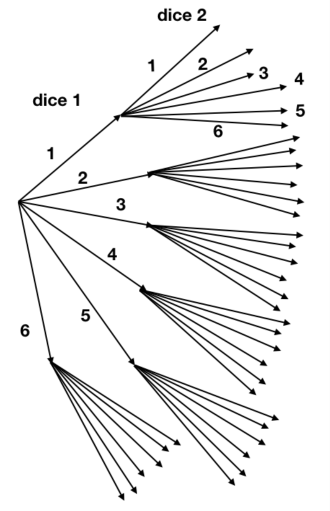
$$\Pr[SS] + \Pr[SF] = \frac{2}{3} \quad \Pr[SS] + \Pr[FS] = \frac{1}{2} \quad \Pr[SS] + \Pr[SF] + \Pr[FS] = \frac{3}{4}$$

Solving these, $\Pr[\{FS\}|\{FS \cup SF\}] = \frac{3/4-2/3}{(3/4-2/3)+(3/4-1/2)} = \frac{1/12}{1/12+1/4} = \frac{1}{4}$.

Questions?

Back to throwing dice - Tree Diagram

Suppose we throw two standard dice one after the other. What is the probability that we get two 6's in a row?



Identify Outcomes: Each leaf is an outcome and $\mathcal{S} = \{(1,1),(1,2),(1,3),\ldots(6,6)\}$.

Identify Event: $E = \{(6,6)\}$.

Compute probabilities: $\Pr[\text{Dice 1 is 6}] = \frac{1}{6}$.
$\Pr[(3,6)] = \Pr[\text{Dice 2 is 3} \; \cap \; \text{Dice 1 is 6}] =$
$\Pr[\text{Dice 2 is 3} \mid \text{Dice 1 is 6}] \; \Pr[\text{Dice 1 is 6}] = \frac{1}{6}\frac{1}{6} = \frac{1}{36}$.

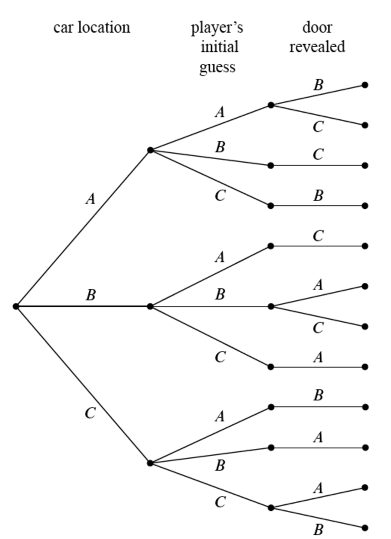
$\Pr[E] = \Pr[\text{dice 1 is 6} \; \cap \text{dice 2 is 6}\;] = \frac{1}{36}$.

Monty Hall Problem

Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car, behind the others, goats. You pick a door, say A, and the host, who knows what's behind the doors, opens another door, say C, which has a goat. He says to you, "Do you want to pick door B?" Is it to your advantage to switch your choice of doors?

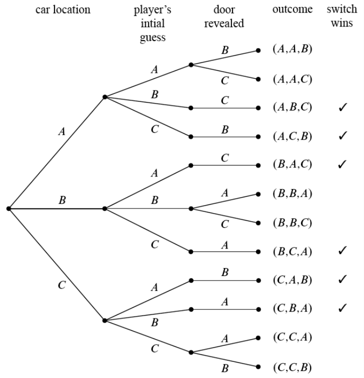
- The car is equally likely to be hidden behind each of the three doors.
- The player is equally likely to pick each of the three doors, regardless of the car's location.
- After the player picks a door, the host must open a different door with a goat behind it and offer the player the choice of staying with the original door or switching.
- If the host has a choice of which door to open, then he is equally likely to select each of them.

Tree Diagram for the Monty Hall Problem - Identify Outcomes



$\mathcal{S} = \{(A, A, B), (A, A, C), (A, B, C), (A, C, B), \ldots\}$.
$E_1 =$ Prize is behind door C $=$
$\{(C, A, B), (C, B, A), (C, C, A), (C, C, B)\}$

Tree Diagram for the Monty Hall Problem - Identify Event

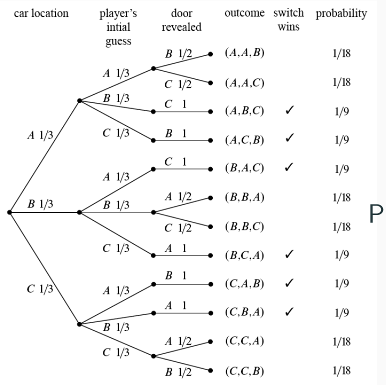


E = Switching wins =
$\{(A,B,C),(A,C,B),(B,A,C),(B,C,A),(C,A,B),(C,B,A)\}$

$\Pr[(A,A)] = \Pr[\text{Car is at A} \cap \text{Player picks A}] =$
$\Pr[\text{Player picks A} \mid \text{Car is at A}] \Pr[\text{Car is at A}] = \frac{1}{3}\frac{1}{3} = \frac{1}{9}.$

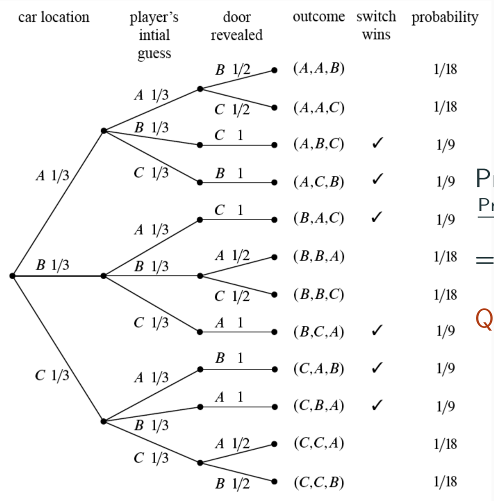
$\Pr[(A,A,B)] = \Pr[\text{Door B is revealed} \cap \text{AA}] =$
$\Pr[\text{Door B is revealed} \mid \text{AA}] \Pr[\text{AA}] = \frac{1}{2}\frac{1}{9} = \frac{1}{18}.$

Tree Diagram for the Monty Hall Problem - Compute Probabilities



$$\Pr[E] = \Pr[(A, B, C)] + \Pr[(A, B, C)] + \Pr[(A, B, C)] + \Pr[(A, B, C)] + \ldots = \frac{1}{9} + \frac{1}{9} + \ldots = \frac{2}{3}.$$

Monty Hall Problem and Conditional Probability



$$\Pr[\text{win by switching}|\text{pick A and door B is opened}] = \frac{\Pr[\text{win by switching} \cap \text{pick A and door B is opened}]}{\Pr[\text{pick A and door B is opened}]}$$

$$= \frac{\Pr[(C,A,B)]}{\Pr[\{(A,A,B),(C,A,B)\}]} = \frac{1/9}{1/9+1/18} = \frac{2}{3}.$$

Q: $\Pr[\text{win by switching}|\text{pick A and door C is opened}]$?

Questions?

Conditional Probability - Examples

In a best-of-three series, the local hockey team wins the first game with probability $\frac{1}{2}$. In subsequent games, their probability of winning is determined by the outcome of the previous game. If the team won the previous game, then they are invigorated by victory and win the current game with probability $\frac{2}{3}$. If they lost the previous game, then they are demoralized by defeat and win the current game with probability only $\frac{1}{3}$. What is the probability that the local team wins the series, given that they win the first game?

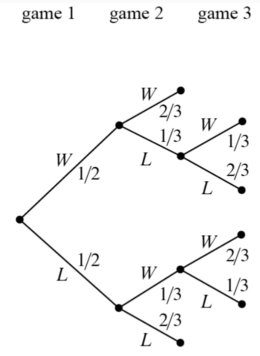
Conditional Probability - Examples

Sample space: $\mathcal{S} = \{(W,W),(W,L,W),(W,L,L),(L,W,W),(L,W,L),(L,L)\}$.

Events: $T = \{(W,W),(W,L,W),(L,W,W)\}$, $F = \{(W,W),(W,L,W),(W,L,L)\}$.

$$
\begin{aligned}
\Pr[T|F] &= \frac{\Pr[T \cap F]}{\Pr[F]} \\
&= \frac{\Pr[\{(W,W),(W,L,W)\}]}{\Pr[\{(W,W),(W,L,W),(W,L,L)\}]} \\
&= \frac{\Pr[\{(W,W)\} + \Pr[\{(W,L,W)\}]}{\Pr[\{(W,W)\} + \Pr[\{(W,L,W)\} + \Pr[\{(W,L,L)\}]]} \\
&= \frac{1/3 + 1/18}{1/3 + 1/18 + 1/9} = \frac{7}{9}
\end{aligned}
$$

Conditional Probability - Examples



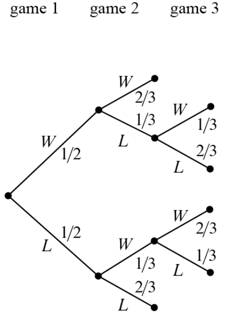
outcome	event A: win the series	event B: win game 1	outcome probability
WW	✓	✓	1/3
WLW	✓	✓	1/18
WLL		✓	1/9
LWW	✓		1/9
LWL			1/18
LL			1/3

Q: What is the probability that the team wins the series if they lose Game 1?

Q: What is the probability that the team wins the series?

Q: What is the probability that the series goes to Game 3?

Conditional Probability - Examples



outcome	event A: win the series	event B: win game 1	outcome probability
WW	✓	✓	1/3
WLW	✓	✓	1/18
WLL		✓	1/9
LWW	✓		1/9
LWL			1/18
LL			1/3

Q: What is the probability that the team won their first game given that they won the series?

Recall that $T = \{(W,W),(W,L,W),(L,W,W)\}$, $F = \{(W,W),(W,L,W),(W,L,L)\}$. We wish to compute $\Pr[F|T] = \frac{\Pr[F \cap T]}{\Pr[T]} = \frac{\Pr[\{(W,W),(W,L,W)\}]}{\Pr[\{(W,W),(W,L,W),(L,W,W)\}]} = \frac{1/3+1/18}{1/3+1/18+1/9} = \frac{7}{9}$.

Bayes Rule

For events E and F if $\Pr[E] \neq 0$, and $\Pr[F] \neq 0$, then,

$$\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]} \quad ; \quad \Pr[F|E] = \frac{\Pr[F \cap E]}{\Pr[E]}$$

$$\implies \Pr[E \cap F] = \Pr[E|F]\Pr[F] \quad ; \quad \Pr[F \cap E] = \Pr[F|E]\Pr[E]$$

$$\implies \Pr[E|F]\Pr[F] = \Pr[F|E]\Pr[E]$$

$$\implies \Pr[F|E] = \frac{\Pr[E|F]\Pr[F]}{\Pr[E]} \tag{Bayes Rule}$$

Allows us to compute $\Pr[F|E]$ using $\Pr[E|F]$. We will see an application of the Bayes rule to machine learning.

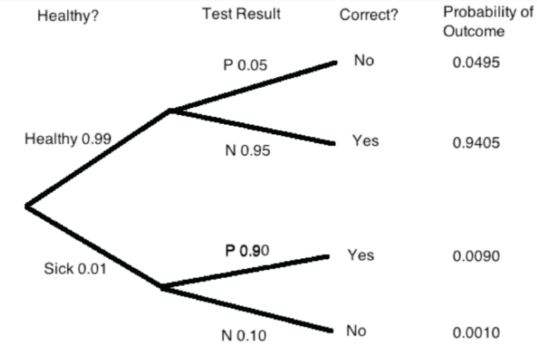
Conditional Probability - Examples

A test for detecting cancer has the following accuracy – (i) If a person has cancer, there is a 10% chance that the test will say that the person does not have it. This is called a "false negative" and (ii) If a person does not have cancer, there is a 5% chance that the test will say that the person does have it. This is called a "false positive". For patients that have no family history of cancer, the incidence of cancer is 1%. Person X does not have any family history of cancer, but is detected to have cancer. What is the probability that the Person X does have cancer?

Conditional Probability - Examples

$\mathcal{S} = \{(Healthy, Positive), (Healthy, Positive), (Sick, Negative), (Sick, Negative)\}$.

A is the event that Person X has cancer. B is the event that the test is positive.



$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} = \frac{\{(S,P)\}}{\{(S,P),(H,P)\}} = \frac{0.0090}{0.0090+0.0495} \approx 15.4\%.$$

Law of Total Probability and Bayes rule

For events E and F,

$$\begin{aligned} E &= (E \cap F) \cup (E \cap F^c) \\ \implies \Pr[E] &= \Pr[(E \cap F) \cup (E \cap F^c)] = \Pr[E \cap F] + \Pr[E \cap F^c] \\ & \hspace{20em} \text{(By union-rule for disjoint events)} \\ \Pr[E] &= \Pr[E|F]\Pr[F] + \Pr[E|F^c]\Pr[F^c] \qquad \text{(By definition of conditional probability)} \end{aligned}$$

$$\Pr[F|E] = \frac{\Pr[F \cap E]}{\Pr[E]} = \frac{\Pr[E|F]\ \Pr[F]}{\Pr[E]} \qquad \text{(By definition of conditional probability)}$$

$$\Pr[F|E] = \frac{\Pr[E|F]\ \Pr[F]}{\Pr[E|F]\Pr[F] + \Pr[E|F^c]\Pr[F^c]} \qquad \text{(By law of total probability)}$$

Law of Total Probability and Bayes rule

For disjoint events E_1, E_2, E_3 such that $\Pr[E_1 \cup E_2 \cup E_3] = 1$ and $\Pr[E_1 \cap E_2 \cap E_3] = 0$ i.e. events E_1, E_2 and E_3 form a partition, for any event A,

$$A = (A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \qquad \text{(Since } \Pr[E_1 \cup E_2 \cup E_3] = 1\text{)}$$

$$\implies \Pr[A] = \Pr[A \cap E_1] + \Pr[A \cap E_2] + \Pr[A \cap E_2] \qquad \text{(By union-rule for disjoint events)}$$

$$\Pr[A] = \Pr[A|E_1]\Pr[E_1] + \Pr[A|E_2]\Pr[E_2] + \Pr[A|E_3]\Pr[E_3]$$

(By definition of conditional probability)

Similarly, we can obtain the Bayes rule for 3 events,

$$\Pr[E_1|A] = \frac{\Pr[A|E_1]\ \Pr[E_1]}{\Pr[A|E_1]\Pr[E_1] + \Pr[A|E_2]\Pr[E_2] + \Pr[A|E_3]\Pr[E_3]}$$

Total Probability - Examples

Q: We flip a fair coin. If heads comes up, then we roll one die and take the result. If tails comes up, then we roll two dice and take the sum of the two results. What is the probability that this process yields a 2?

C = event that the coin comes up heads. And T = event that the process results in 2. We wish to compute $\Pr[T]$. Since the coin is fair, $\Pr[C] = \frac{1}{2}$. $\Pr[T|C] = \frac{1}{6}$ (with one roll of the dice). $\Pr[C^c] = \frac{1}{2}$. $\Pr[T|C^c] = \frac{1}{36}$ (both dice need to be 1).

Using the law of total probability, $\Pr[E] = \Pr[E|C]\Pr[C] + \Pr[E|C^c]\Pr[C^c] = \frac{7}{72}$.

Q: What is the probability that this process yields a (i) 4, (ii) 6, (iii) 8?

Q: What is the probability that the first dice (in the two dice when we get a tails) is 4 given that the process yields a 6?

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Total Probability - Examples

Q: An insurance company believes that people can be divided into two classes — those that are accident prone and those that are not. Their statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability 0.4, whereas this probability decreases to 0.2 for a non-accident-prone person. If we assume that 30% of the population is accident prone, what is the probability that a new policy holder will have an accident within a year of purchasing a policy?

Let $A =$ event that a new policy holder will have an accident within a year of purchasing a policy. Let $B =$ event that the new policy holder is accident prone. We know that $\Pr[B] = 0.3$, $\Pr[A|B] = 0.4$, $\Pr[A|B^c] = 0.2$. By the law of total probability,
$\Pr[A] = \Pr[A|B]\Pr[B] + \Pr[A|B^c]\Pr[B^c] = (0.4)(0.3) + (0.2)(0.7) = 0.26$.

Q: Suppose that a new policy holder has an accident within a year of purchasing his policy. What is the probability that he is accident prone?

Compute $\Pr[B|A] = \frac{\Pr[A|B]\ \Pr[B]}{\Pr[A]} = \frac{0.12}{0.26} = 0.4615$.

Total Probability - Examples

In answering a question on a multiple-choice test, a student either knows the answer or she guesses. Let p be the probability that she knows the answer and $1-p$ the probability that she guesses. Assume that a student who guesses at the answer will be correct with probability $\frac{1}{m}$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that she answered it correctly?

Let C be the event that the student answers the question correctly. Let K be the event that the student knows the answer. We wish to compute $\Pr[K|C]$.

We know that $\Pr[K] = p$ and $\Pr[C|K^c] = 1/m$, $\Pr[C|K] = 1$. Hence,
$\Pr[C] = \Pr[C|K]\Pr[K] + \Pr[C|K^c]\Pr[K^c] = (1)(p) + \frac{1}{m}(1-p)$.

$\Pr[K|C] = \frac{\Pr[C|K]\ \Pr[K]}{\Pr[C]} = \frac{mp}{1+(m-1)p}$.

At a certain stage of a criminal investigation, the inspector in charge is 60% convinced of the guilt of a certain suspect. Suppose now that a new piece of evidence that shows that the criminal has a certain characteristic (such as left-handedness, baldness, brown hair, etc.) is uncovered. If 20% of the general population possesses this characteristic, how certain of the guilt of the suspect should the inspector now be if it turns out that the suspect is among this group?

Let G be the event that the suspect is guilty. Let C be the event that the suspect possesses the characteristic found at the crime scene. We wish to compute $\Pr[G|C]$.

We know that $\Pr[G] = 0.6$, $\Pr[C|G] = 1$, $\Pr[C|G^c] = 0.2$.
$\Pr[C] = \Pr[C|G]\Pr[G] + \Pr[C|G^c]\Pr[G^c] = (1)(0.6) + (0.2)(0.4) = 0.68$

$\Pr[G|C] = \frac{\Pr[G]\,\Pr[C|G]}{\Pr[C]} = \frac{0.6}{0.68} = 0.882.$

Hence, the additional evidence has corroborated the inspector's theory and increased the probability of guilt.

Simpson's Paradox

In 1973, there was a lawsuit against a university with the claim that a male candidate is more likely to be admitted to the university than a female.

Let us consider a simplified case – there are two departments, EE and CS, and men and women apply to the program of their choice. Let us define the following events: A is the event that the candidate is admitted to the program of their choice, F_E is the event that the candidate is a woman applying to EE, F_C is the event that the candidate is a woman applying to CS. Similarly, we can define M_E and M_C. Assumption: Candidates are either men or women, and that no candidate is allowed to be part of both EE and CS.

Lawsuit claim: Male candidate is more likely to be admitted to the university than a female i.e. $\Pr[A|M_E \cup M_C] > \Pr[A|F_E \cup F_C]$.

University response: In any given department, a male applicant is less likely to be admitted than a female i.e. $\Pr[A|F_E] > \Pr[A|M_E]$ and $\Pr[A|F_C] > \Pr[A|M_C]$.

Simpson's Paradox: Both the above statements can be simultaneously true.

Simpson's Paradox

CS	2 men admitted out of 5 candidates	40%
	50 women admitted out of 100 candidates	50%
EE	70 men admitted out of 100 candidates	70%
	4 women admitted out of 5 candidates	80%
Overall	72 men admitted, 105 candidates	$\approx$ 69%
	54 women admitted, 105 candidates	$\approx$ 51%

In the above example, $\Pr[A|F_E] = 0.8 > 0.7 = \Pr[A|M_E]$ and $\Pr[A|F_C] = 0.5 > 0.4 = \Pr[A|M_C]$. $\Pr[A|F_E \cup F_C] \approx 0.51$. Similarly, $\Pr[A|M_E \cup M_C] \approx 0.69$.

In general, Simpson's Paradox occurs when multiple small groups of data all exhibit a similar trend, but that trend reverses when those groups are aggregated.

Questions?