CMPT 210: Probability and Computation

Lecture 12

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Recap

Random variable: A random "variable" R on a probability space is a total function whose domain is the sample space $\mathcal{S}$. The codomain is denoted by V (is usually a subset of the real numbers), meaning that $R : \mathcal{S} \to V$.

Example: Suppose we toss three independent, unbiased coins. In this case, $\mathcal{S} = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. C is a random variable equal to the number of heads that appear such that $C(HHT) = 2$.

Indicator Random Variables: An indicator random variable corresponding to an event E is denoted as $\mathcal{I}_E$ and is defined such that for $\omega \in E$, $\mathcal{I}_E[\omega] = 1$ and for $\omega \notin E$, $\mathcal{I}_E[\omega] = 0$.

Example: When throwing two dice, if E is the event that both throws of the dice result in a prime number, then $\mathcal{I}_E((2,4)) = 0$ and $\mathcal{I}_E((2,3)) = 1$.

In general, a random variable that takes on several values partitions $\mathcal{S}$ into several blocks where each block is a subset of $\mathcal{S}$ and is therefore an event.

Example: When tossing three coins, $\Pr[C = 2] = \Pr[\{HHT, HTH, THH\}] = \frac{3}{8}$.

Recap

Probability density function (PDF): Let R be a random variable with codomain V. The probability density function of R is the function $\text{PDF}_R : V \to [0,1]$, such that $\text{PDF}_R[x] = \Pr[R = x]$ if $x \in \text{Range(R)}$ and equal to zero if $x \notin \text{Range(R)}$.

$\sum_{x \in V} \text{PDF}_R[x] = \sum_{x \in \text{Range(R)}} \Pr[R = x] = 1$.

Example: When tossing three coins, $\text{PDF}_C[2] = \Pr[C = 2] = \frac{3}{8}$.

Cumulative distribution function (CDF): The cumulative distribution function of R is the function $\text{CDF}_R : \mathbb{R} \to [0,1]$, such that $\text{CDF}_R[x] = \Pr[R \leq x]$.

Example: When tossing three coins,
$\text{CDF}_C[2.3] = \Pr[C \leq 2.3] = \Pr[C = 0] + \Pr[C = 1] + \Pr[C = 2] = \frac{7}{8}$.

Importantly, neither PDF_R nor CDF_R involves the sample space of an experiment.

Distributions

Many random variables turn out to have the same PDF and CDF. In other words, even though R and T might be different random variables on different probability spaces, it is often the case that $\text{PDF}_R = \text{PDF}_T$. Hence, by studying the properties of such PDFs, we can study different random variables and experiments.

Distribution over a random variable can be fully specified using the cumulative distribution function (CDF) (usually denoted by F). The corresponding probability density function (PDF) is denoted by f.

Common (Discrete) Distributions in Computer Science:

- Bernoulli Distribution
- Uniform Distribution
- Binomial Distribution
- Geometric Distribution

Bernoulli Distribution

We toss a biased coin such that the probability of getting a heads is p. Let R be the random variable such that $R = 0$ when the coin comes up heads and $R = 1$ if the coin comes up tails. R follows the Bernoulli distribution.

The Bernoulli distribution has the PDF $f \colon \{0, 1\} \to [0, 1]$ meaning that Bernoulli random variables take values in $\{0, 1\}$. It can be fully specified by specifying the "probability of success" (of an experiment) p (probability of getting a heads in the example). Formally, PDF_R is given by:

$$f(0) = p \quad ; \quad f(1) = q := 1 - p.$$

In the example, $\Pr[R = 0] = f(0) = p = \Pr[\text{event that we get a heads}]$.

The corresponding CDF_R is given by $F : \mathbb{R} \to [0, 1]$:

$$\begin{aligned} F(x) &= 0 && \text{(for } x < 0\text{)} \\ &= p && \text{(for } 0 \leq x < 1\text{)} \\ &= 1 && \text{(for } x \geq 1\text{)} \end{aligned}$$

Uniform Distribution

We roll a standard die. Let R be the random variable equal to the number that shows up on the die. R follows the uniform distribution.

A random variable R that takes on each possible value in its codomain V with the same probability is said to be uniform. The uniform distribution can be fully specified by $|V|$ and has PDF $f : V \to [0, 1]$ such that:

$$f(v) = {}^{1}/_{|V|}. \qquad \text{(for all } v \in V\text{)}$$

In the example, $f(1) = f(2) = \ldots = f(6) = \frac{1}{6}$.

For n elements in V arranged in increasing order – $(v_1, v_2, \ldots, v_n)$, the CDF is:

$$\begin{aligned} F(x) &= 0 && \text{(for } x < v_1\text{)} \\ &= {}^{k}/_{n} && \text{(for } v_k \leq x < v_{k+1}\text{)} \\ &= 1 && \text{(for } x \geq v_n\text{)} \end{aligned}$$

Q: If X has a Bernoulli distribution, when is X also uniform?

Questions?

Binomial Distribution

We toss n biased coins independently. The probability of getting a heads for each coin is p. Let R be the random variable equal to the number of heads in the n coin tosses. R follows the Binomial distribution.

$V = \{0, 1, 2, \ldots, n\}$. Hence PDF_R is a function $f : \{0, 1, 2, \ldots, n\} \to [0, 1]$.

Let E_k be the event we get k heads in n tosses. Let A_i be the event we get a heads in toss i.

$$E_k = (A_1 \cap A_2 \ldots A_k \cap A_{k+1}^c \cap A_{k+2}^c \cap \ldots \cap A_n^c) \cup (A_1^c \cap A_2 \ldots A_k \cap A_{k+1} \cap A_{k+2}^c \cap \ldots \cap A_n^c) \cup \ldots$$

$$\begin{aligned}\Pr[E_k] &= \Pr[(A_1 \cap A_2 \ldots A_k \cap A_{k+1}^c \cap A_{k+2}^c \cap \ldots \cap A_n^c)] + \Pr[A_1^c \cap A_2 \ldots A_k \cap A_{k+1} \cap \ldots \cap] + \ldots \\ &= \Pr[A_1] \Pr[A_2] \Pr[A_k] \Pr[A_{k+1}^c] \Pr[A_{k+2}^c] \ldots \Pr[A_n^c] + \ldots = p^k (1-p)^{n-k} + p^k (1-p)^{n-k} + \ldots\end{aligned}$$

$$\implies \Pr[E_k] = \binom{n}{k} p^k (1-p)^{n-k}$$

Sanity check: Since $\text{PDF}_R[k] = \Pr[E_k]$ and $V = \{0, 1, 2, \ldots, n\}$,

$$\sum_{i \in V} \text{PDF}_R[i] = \sum_{i=0}^{n} \Pr[E_i] = \sum_{i=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + 1 - p)^n = 1. \quad \text{(Binomial Theorem)}$$

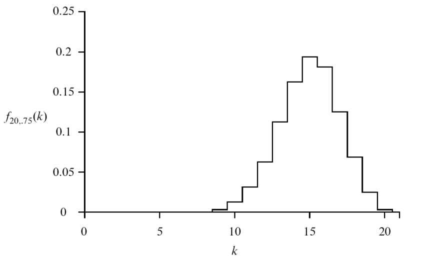
Binomial Distribution

The binomial distribution can be fully specified by n, p and has PDF $f : \{0, 1, \dots n\} \to [0, 1]$:

$$f(k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

The corresponding CDF is given by $F : \mathbb{R} \to [0, 1]$:

$$\begin{aligned} F(x) &= 0 && \text{(for } x < 0\text{)} \\ &= \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} && \text{(for } k \le x < k+1\text{)} \\ &= 1. && \text{(for } x \ge n\text{)} \end{aligned}$$



Q: If X has a Bernoulli distribution with parameter p, does it also follow the Binomial distribution? With what parameters?

Geometric Distribution

We toss a biased coin independently multiple times. The probability of getting a heads is p. Let R be the random variable equal to the number of tosses needed to get the first heads. R follows the geometric distribution.

$V = \{1, 2, \ldots, \}$. Hence PDF_R is a function $f : \mathbb{N} \to [0, 1]$.

Let E_k be the event that we need k tosses to get the first heads. Let A_i be the event that we get a heads in toss i.

$$\begin{aligned} E_k &= A_1^c \cap A_2^c \cap \ldots \cap A_k \\ \Pr[E_k] &= \Pr[A_1^c \cap A_2^c \cap \ldots \cap A_k] = \Pr[A_1^c] \Pr[A_2^c] \ldots \Pr[A_k] \\ \implies \Pr[E_k] &= (1-p)^{k-1} p \end{aligned}$$

Sanity check: Since $\text{PDF}_R[k] = \Pr[E_k]$ and $V = \{1, 2, \ldots, \}$,

$$\sum_{i \in V} \text{PDF}_R[i] = \sum_{i=1}^{\infty} \Pr[E_i] = \sum_{i=1}^{\infty} (1-p)^{i-1} p = \frac{p}{1 - (1-p)} = 1. \quad \text{(Sum of geometric series)}$$

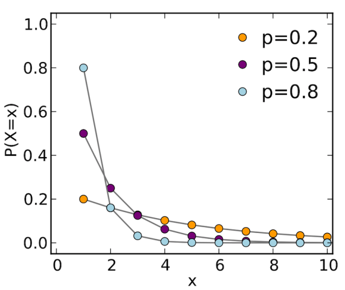
Geometric Distribution

The geometric distribution can be fully specified by p and has PDF $f : \{0, 1, \ldots n\} \to [0, 1]$:

$$f(k) = (1 - p)^{k-1} p.$$

The corresponding CDF is given by $F : \mathbb{R} \to [0, 1]$:

$$\begin{aligned} F(x) &= 0 && (\text{for } x < 1) \\ &= \sum_{i=0}^{k} (1 - p)^{i-1} p && (\text{for } k \leq x < k + 1) \end{aligned}$$



Q: We throw a standard dice multiple times until we get a 6. What is the probability that we get a 6 on the 4$^{\text{th}}$ trial?

Sampling distributions

If we have a Bernoulli distribution $\text{Ber}(p)$, then $X \sim \text{Ber}(p)$ (read as X is "sampled from" or "distributed according to" a Bernoulli distribution with parameter p) means that the random variable X follows a Bernoulli distribution with parameter p.

Example: In Frievald's algorithm, we sampled each entry of x (the vector we multiplied the matrices by) according to a Bernoulli distribution with $p = 1/2$. Formally, for all i, $x_i \sim \text{Ber}(1/2)$.

If X is sampled from/distributed according to a uniform distribution with parameter n, then $X \sim \text{Uniform}(n)$. If the domain of PDF_X is $\{1, 2, \ldots, n\}$, then $\Pr[X = 1] = \Pr[X = 2] \ldots = \frac{1}{n}$.

Similarly, if $X \sim \text{Bin}(n, p)$, then $\Pr[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$ and if $X \sim \text{Geo}(p)$, then $\Pr[X = k] = (1-p)^{k-1} p$.

Questions?

Number Guessing Game

We saw an application of the Bernoulli distribution in Frievald's algorithm where we sampled each entry of x (the "probe" vector we multiplied the matrices by) according to a Bernoulli distribution with $p = 1/2$. Let us now study another randomized algorithm and use the uniform distribution.

We have two envelopes. Each contains a distinct number in $\{0, 1, 2, \ldots, 100\}$. To win the game, we must determine which envelope contains the larger number. We are allowed to peek at the number in one envelope selected at random. Can we devise a winning strategy?

Strategy 1: We pick an envelope at random and guess that it contains the larger number (without even peeking at the number). This strategy wins only 50% of the time.

Strategy 2: We peek at the number and if its below 50, we choose the other envelope.

But the numbers in the envelopes need not be random! The numbers are chosen "adversarially" in a way that will defeat our guessing strategy. For example, to "beat" Strategy 2, the two numbers can always be chosen to be below 50.

Q: Can we do better than 50% chance of winning?

Number Guessing Game

Suppose that we somehow knew a number x that was in between the numbers in the envelopes. If we peek in one envelope and see a number. If it is bigger than x, we know its the higher number and choose that envelope. If it is smaller than x, we know that is the smaller number and choose the other envelope.

Of course, we do not know such a number x. But we can guess it!

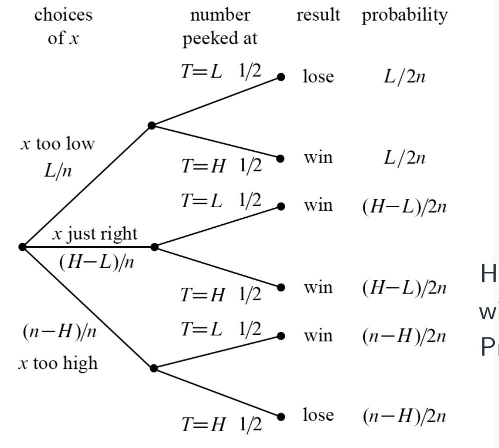
Strategy 3: Choose a random number x from $\{0.5, 1.5, 2.5, \ldots n - 1/2\}$ according to the uniform distribution i.e. $\Pr[x = 0.5] = \Pr[1.5] = \ldots = 1/n$. Then we peek at the number (denoted by T) in one envelope, and if $T > x$, we choose that envelope, else we choose the other envelope.

The advantage of such a randomized strategy is that the adversary cannot easily "adapt" to it.

Q: But does it have better than 50% chance of winning?

Number Guessing Game

Let the numbers in the two envelopes be L (lower number) and H (the higher number). Let us construct a tree diagram.



$$
\begin{aligned}
\Pr[\text{win}] &= \frac{L}{2n} + \frac{H-L}{2n} + \frac{H-L}{2n} + \frac{n-H}{2n} \\
&= \frac{1}{2} + \frac{H-L}{2n} \geq \frac{1}{2} + \frac{1}{2n} \geq \frac{1}{2}
\end{aligned}
$$

Hence our strategy has a greater than 50% chance of winning! If $n = 10$, the $\Pr[\text{win}] = 0.55$, if $n = 100$ then $\Pr[\text{win}] = 0.505$.

Questions?