CMPT 210: Probability and Computing

Lecture 14

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Recap

- **Random variable**: A random "variable" R on a probability space is a total function whose domain is the sample space $\mathcal{S}$. The codomain is denoted by V (usually a subset of the real numbers), meaning that $R : \mathcal{S} \to V$. A r.v partitions the sample space into several blocks.

Example: Suppose we toss three independent, unbiased coins. In this case, $\mathcal{S} = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. C is a random variable equal to the number of heads that appear such that $C : \mathcal{S} \to \{0, 1, 2, 3\}$. $C(HHT) = 2$.

- For r.v. R, for all $i \in \text{Range}(R)$, the event $[R = i] = \{\omega \in \mathcal{S} | R(\omega) = i\}$. For any r.v. R, $\sum_{i \in \text{Range(R)}} \Pr[R = i] = 1$.

Example: $[C = 2] = \{HHT, HTH, THH\}$ and $\Pr[C = 2] = \frac{3}{8}$.
$\sum_{i \in \text{Range(C)}} \Pr[C = i] = \Pr[C = 0] + \Pr[C = 1] + \Pr[C = 2] + \Pr[C = 3] = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$.

Recap

- **Indicator Random Variable**: An indicator random variable corresponding to an event E is denoted as $\mathcal{I}_E$ and is defined such that for $\omega \in E$, $\mathcal{I}_E[\omega] = 1$ and for $\omega \notin E$, $\mathcal{I}_E[\omega] = 0$.

Example: When throwing two dice, if E is the event that both throws of the dice result in a prime number, then $\mathcal{I}_E((2,4)) = 0$ and $\mathcal{I}_E((2,3)) = 1$.

- **Probability density function (PDF)**: Let R be a r.v. with codomain V. The probability density function of R is the function $\text{PDF}_R : V \to [0,1]$, such that $\text{PDF}_R[x] = \Pr[R = x]$ if $x \in \text{Range(R)}$ and equal to zero if $x \notin \text{Range(R)}$.

- **Cumulative distribution function (CDF)**: The cumulative distribution function of R is the function $\text{CDF}_R : \mathbb{R} \to [0,1]$, such that $\text{CDF}_R[x] = \Pr[R \leq x]$.

Importantly, neither PDF_R nor CDF_R involves the sample space of an experiment.

Example: If we flip three coins, and C counts the number of heads, then
$\text{PDF}_C[0] = \Pr[C = 0] = \frac{1}{8}$, and
$\text{CDF}_C[2.3] = \Pr[C \leq 2.3] = \Pr[C = 0] + \Pr[C = 1] + \Pr[C = 2] = \frac{7}{8}$.

Distributions

Many random variables turn out to have the same PDF and CDF. In other words, even though R and T might be different random variables on different probability spaces, it is often the case that $\text{PDF}_R = \text{PDF}_T$. Hence, by studying the properties of such PDFs, we can study different random variables and experiments.

- **Distribution** over a random variable can be fully specified using the cumulative distribution function (CDF) (usually denoted by F). The corresponding probability density function (PDF) is denoted by f.
- **Common Discrete Distributions** in Computer Science:
  - Bernoulli Distribution
  - Uniform Distribution
  - Binomial Distribution
  - Geometric Distribution

Bernoulli Distribution

Canonical Example: We toss a biased coin such that the probability of getting a heads is p. Let R be the random variable such that $R = 1$ when the coin comes up heads and $R = 0$ if the coin comes up tails. R follows the Bernoulli distribution.

PDF$_R$ for Bernoulli distribution: $f \colon \{0, 1\} \to [0, 1]$ meaning that Bernoulli random variables take values in $\{0, 1\}$. It can be fully specified by the "probability of success" (of an experiment) p (probability of getting a heads in the example). Formally, PDF$_R$ is given by:

$$f(1) = p \quad ; \quad f(0) = q := 1 - p.$$

In the example, $\Pr[R = 1] = f(1) = p = \Pr[\text{event that we get a heads}]$.

CDF$_R$ for Bernoulli distribution: $F : \mathbb{R} \to [0, 1]$:

$$\begin{aligned} F(x) &= 0 && \text{(for } x < 0\text{)} \\ &= 1 - p && \text{(for } 0 \leq x < 1\text{)} \\ &= 1 && \text{(for } x \geq 1\text{)} \end{aligned}$$

Uniform Distribution

Canonical Example: We roll a standard die. Let R be the random variable equal to the number that shows up on the die. R follows the uniform distribution.

A random variable R that takes on each possible value in its codomain V with the same probability is said to be uniform.

PDF$_R$ for Uniform distribution: $f : V \to [0, 1]$ such that for all $v \in V$, $f(v) = {}^1/_{|V|}$. In the example, $f(1) = f(2) = \ldots = f(6) = \frac{1}{6}$.

CDF$_R$ for Uniform distribution: For n elements in V arranged in increasing order – $(v_1, v_2, \ldots, v_n)$, the CDF is:

$$\begin{aligned} F(x) &= 0 && \text{(for } x < v_1\text{)} \\ &= {}^k/_n && \text{(for } v_k \leq x < v_{k+1}\text{)} \\ &= 1 && \text{(for } x \geq v_n\text{)} \end{aligned}$$

Q: If X has a Bernoulli distribution, when is X also uniform?

Binomial Distribution

Canonical Example: We toss n biased coins independently. The probability of getting a heads for each coin is p. Let R be the random variable equal to the number of heads in the n coin tosses. R follows the Binomial distribution.

PDF$_R$ for Binomial distribution: $f : \{0, 1, 2, \ldots, n\} \to [0, 1]$. For $k \in \{0, 1, \ldots, n\}$, $f(k) = \binom{n}{k} p^k (1-p)^{n-k}$.

Proof: Let E_k be the event we get k heads. Let A_i be the event we get a heads in toss i.

$$E_k = (A_1 \cap A_2 \ldots A_k \cap A_{k+1}^c \cap A_{k+2}^c \cap \ldots \cap A_n^c) \cup (A_1^c \cap A_2 \ldots A_k \cap A_{k+1} \cap A_{k+2}^c \cap \ldots \cap A_n^c) \cup \ldots$$

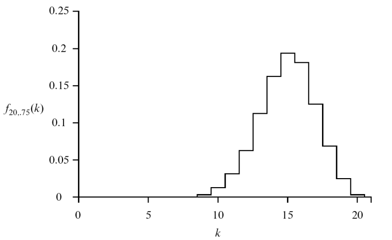
$$\begin{aligned}
\Pr[E_k] &= \Pr[(A_1 \cap A_2 \ldots A_k \cap A_{k+1}^c \cap A_{k+2}^c \cap \ldots \cap A_n^c)] + \Pr[A_1^c \cap A_2 \ldots A_k \cap A_{k+1} \cap \ldots \cap] + \ldots \\
&= \Pr[A_1] \Pr[A_2] \Pr[A_k] \Pr[A_{k+1}^c] \Pr[A_{k+2}^c] \ldots \Pr[A_n^c] + \ldots \qquad \text{(Independence of tosses)} \\
&= p^k (1-p)^{n-k} + p^k (1-p)^{n-k} + \ldots
\end{aligned}$$

$$\implies \Pr[E_k] = \binom{n}{k} p^k (1-p)^{n-k}$$

(Number of terms = number of ways to choose the k tosses that result in heads = $\binom{n}{k}$)

Binomial Distribution

For the Binomial distribution, $\text{PDF}_R(k) = \binom{n}{k} p^k (1-p)^{n-k}$.



Q: Prove that $\sum_{k \in \text{Range(R)}} \text{PDF}_R[k] = 1$.
By the Binomial Theorem, $\sum_{k \in \text{Range(R)}} \text{PDF}_R[k] = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + 1 - p)^n = 1$.

CDF_R for Binomial distribution: $F : \mathbb{R} \to [0,1]$:

$$
\begin{aligned}
F(x) &= 0 && \text{(for } x < 0\text{)} \\
&= \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} && \text{(for } k \le x < k+1\text{)} \\
&= 1. && \text{(for } x \ge n\text{)}
\end{aligned}
$$

Geometric Distribution

Canonical Example: We toss a biased coin independently multiple times. The probability of getting a heads is p. Let R be the random variable equal to the number of tosses needed to get the first heads. R follows the geometric distribution.

PDF$_R$ for Geometric distribution: $f : \{1, 2, \ldots\} \to [0, 1]$. For $k \in \{1, 2, \ldots, \infty\}$, $f(k) = (1-p)^{k-1}\, p$.

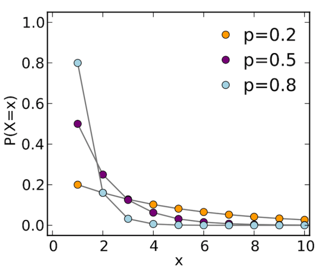
Proof: Let E_k be the event that we need k tosses to get the first heads. Let A_i be the event that we get a heads in toss i.

$$\begin{aligned} E_k &= A_1^c \cap A_2^c \cap \ldots \cap A_k \\ \Pr[E_k] &= \Pr[A_1^c \cap A_2^c \cap \ldots \cap A_k] = \Pr[A_1^c]\Pr[A_2^c] \ldots \Pr[A_k] \quad \text{(Independence of tosses)} \\ \implies \Pr[E_k] &= (1-p)^{k-1} p \end{aligned}$$

Q: Prove that $\sum_{k \in \text{Range(R)}} \text{PDF}_R[k] = 1$.

By the sum of geometric series, $\sum_{k \in \text{Range}(R)} \text{PDF}_R[k] = \sum_{k=1}^{\infty} (1-p)^{k-1} p = \frac{p}{1-(1-p)} = 1$.

Geometric Distribution



For the Geometric distribution, $\text{PDF}_R(k) = (1-p)^{k-1}p$.

CDF_R for Geometric distribution: $F : \mathbb{R} \to [0,1]$:

$$\begin{aligned} F(x) &= 0 && \text{(for } x < 1\text{)} \\ &= \sum_{i=1}^{k} (1-p)^{i-1}p && \text{(for } k \leq x < k+1\text{)} \end{aligned}$$

Questions?

Distributions - Examples

Q: It is known that disks produced by a certain company will be defective with probability 0.01 independently of each other. The company sells the disks in packages of 10 and offers a money-back guarantee that at most 1 of the 10 disks is defective (the package can be returned if there is more than 1 defective disk). What proportion of packages is returned? If someone buys three packages, what is the probability that exactly one of them will be returned?

Let X be the random variable corresponding to the number of defective disks in a package. Let E be the event that the package is returned. We wish to compute $\Pr[E] = \Pr[X > 1]$. X follows the Binomial distribution $\text{Bin}(10, 0.01)$. Hence,

$$\begin{aligned}\Pr[E] = \Pr[X > 1] = 1 - \Pr[X \leq 1] &= 1 - \Pr[X = 0] - \Pr[X = 1] \\ &= 1 - \binom{10}{0}(0.99)^{10} - \binom{10}{1}(0.99)^{9}(0.01)^{1} \approx 0.05\end{aligned}$$

Distributions - Examples

Q: It is known that disks produced by a certain company will be defective with probability 0.01 independently of each other. The company sells the disks in packages of 10 and offers a money-back guarantee that at most 1 of the 10 disks is defective (the package can be returned if there is more than 1 defective disk). If someone buys three packages, what is the probability that exactly one of them will be returned?

Let F be the event that someone bought 3 packages and exactly one of them is returned.

Answer 1: Let E_i be the event that package i is returned. From the previous question, we know that $\Pr[E_i] = \Pr[\text{Package } i \text{ has more than 1 defective disk}] \approx 0.05$.

$$F = (E_1 \cap E_2^c \cap E_3^c) \cup (E_1^c \cap E_2^c \cap E_3) \cup (E_1^c \cap E_2 \cap E_3^c)$$
$$\Pr[F] = \Pr[E_1](1 - \Pr[E_2])(1 - \Pr[E_3]) + (1 - \Pr[E_1])(1 - \Pr[E_2])\Pr[E_3] + \dots$$
$$\Pr[F] \approx 3 \times (0.05)(0.95)(0.95) \approx 0.15.$$

Answer 2: Let Y be the random variable corresponding to the number of packages returned. Y follows the Binomial distribution $\text{Bin}(3, 0.05)$ and we wish to compute $\Pr[F] = \Pr[Y = 1] \approx \binom{3}{1}(0.05)^1(0.95)^2 \approx 0.15.$

Q: You are randomly and independently throwing darts. The probability that you hit the bullseye in throw i is p. Once you hit the bullseye you win and can go collect your reward. (a) What is the probability that you win after exactly k throws? (b) What is the probability you win in less than k throws?

(a) The number of throws (T) to hit the bullseye and win follows a geometric distribution Geo(p) and we wish to compute $\Pr[T = k]$. Using the PDF for the Geometric distribution, this is equal to $(1 - p)^{k-1} p$.

(b) **Answer 1**: If E is the event that we win in less than k throws,
$\Pr[E] = \Pr[T < k] = \sum_{i=1}^{k-1} \Pr[T = i] = p \sum_{i=1}^{k-1} (1 - p)^{i-1} = 1 - (1 - p)^{k-1}$.

Answer 2:
$\Pr[E] = 1 - \Pr[E^c] = 1 - \Pr[\text{do not hit the bullseye in } k - 1 \text{ throws}] = 1 - (1 - p)^{k-1}$.

Number Guessing Game

Q: We have two envelopes. Each contains a distinct number in $\{0, 1, 2, \ldots, 100\}$. To win the game, we must determine which envelope contains the larger number. We are allowed to peek at the number in one envelope selected at random. Can we devise a winning strategy?

Strategy 1: We pick an envelope at random and guess that it contains the larger number (without even peeking at the number).

Q: What is the probability that we win with this strategy?

Strategy 2: We peek at the number and if its below 50, we choose the other envelope.

But the numbers in the envelopes need not be random! The numbers are chosen "adversarially" in a way that will defeat our guessing strategy. For example, to "beat" Strategy 2, the two numbers can always be chosen to be below 50.

Q: Can we do better than 50% chance of winning?

Number Guessing Game

Suppose that we somehow knew a number x that was in between the numbers in the envelopes. If we peek in one envelope and see a number. If it is bigger than x, we know its the higher number and choose that envelope. If it is smaller than x, we know that is the smaller number and choose the other envelope.

Of course, we do not know such a number x. But we can guess it!

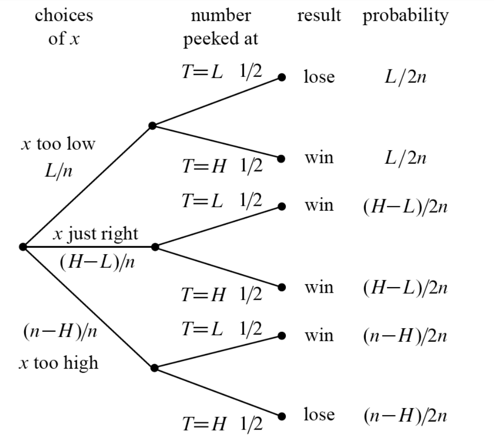
Strategy 3: Choose a random number x from $\{0.5, 1.5, 2.5, \ldots n - 1/2\}$ according to the uniform distribution i.e. $\Pr[x = 0.5] = \Pr[1.5] = \ldots = 1/n$. Then we peek at the number (denoted by T) in one envelope, and if $T > x$, we choose that envelope, else we choose the other envelope.

The advantage of such a randomized strategy is that the adversary cannot easily "adapt" to it.

Q: But does it have better than 50% chance of winning?

Number Guessing Game

Let the numbers in the two envelopes be L (lower number) and H (the higher number).



$$\begin{aligned}\Pr[\text{win}] &= \frac{L}{2n} + \frac{H-L}{2n} + \frac{H-L}{2n} + \frac{n-H}{2n} \\ &= \frac{1}{2} + \frac{H-L}{2n} \geq \frac{1}{2} + \frac{1}{2n} > \frac{1}{2}\end{aligned}$$

Hence our strategy has a greater than 50% chance of winning! If $n = 10$, $\Pr[\text{win}] \geq 0.55$, for $n = 100$, $\Pr[\text{win}] \geq 0.505$.

Q: For $n = 100$, if $L = 23$ and $H = 54$, compute $\Pr[\text{ guessing too low} \mid \text{we win }]$

Questions?