CMPT 210: Probability and Computing

Lecture 8

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Recap - Conditional Probability

- **Conditioning** is revising probabilities based on partial information (an event).
- For events E and F, we wish to compute $\Pr[E|F]$, the probability of event E conditioned on event F.
- **Approach 1**: With conditioning, F can be interpreted as the *new sample space* such that for $\omega \notin F$, $\Pr[\omega|F] = 0$.
- *Example*: For computing $\Pr[\text{we get a } 6|\text{the outcome is even}]$, the new sample space is $F = \{2, 4, 6\}$ and the resulting probability space is uniform. $\Pr[\{\text{even number}\}] = \frac{1}{3}$ and $\Pr[\{\text{odd number}\}] = 0$.

Conditional Probability

Conditional Probability Rule: For two events E and F, $\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]}$, where $\Pr[F] \neq 0$.

Proof: By conditioning on F, the only outcomes we care about are in F i.e. for $\omega \notin F$, $\Pr[\omega|F] = 0$.

Since we want to compute the probability that event E happens, we care about the outcomes that are in E. Hence, the outcomes we care about lie in both E *and* F, meaning that $\omega \in E \cap F$.
$\implies \Pr[E|F] \propto \sum_{\omega \in (E \cap F)} \Pr[\omega]$. By definition of proportionality, for some constant $c > 0$, $\Pr[E|F] = c \sum_{\omega \in (E \cap F)} \Pr[\omega]$.

We know that $\Pr[F|F] = 1$ (probability of event F given that F has happened). Hence, $\Pr[F|F] = 1 = c \sum_{\omega \in F} \Pr[\omega] \implies c = \frac{1}{\sum_{\omega \in F} \Pr[\omega]}$.

Substituting the value of c,

$$\Pr[E|F] = \frac{\sum_{\omega \in (E \cap F)} \Pr[\omega]}{\sum_{\omega \in F} \Pr[\omega]} = \frac{\Pr[E \cap F]}{\Pr[F]}, \text{where } \Pr[F] \neq 0.$$

This formula gives an alternate way to compute conditional probabilities.

Back to throwing dice

Q: Suppose we throw a "standard" dice, what is the probability of getting a 6 if we are told that the outcome is even?

Sample space: $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$ and $\Pr[1] = \Pr[2] = \ldots = \Pr[6] = \frac{1}{6}$.
Event: $E = \{6\}$. We are conditioning on $F = \{2, 4, 6\}$.
$\Pr[\text{we get a 6}|\text{the outcome is even}] = \Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]}$.
$E \cap F = \{6\}$. $\Pr[E \cap F] = \frac{1}{6}$. $\Pr[F] = \Pr[2] + \Pr[4] + \Pr[6] = \frac{1}{2}$.
Hence, $\frac{\Pr[E \cap F]}{\Pr[F]} = \frac{1/6}{1/2} = \frac{1}{3}$.

Q: What is the probability of getting either a 3 or 6 if we are told that the outcome is even?

Q: Suppose we throw two standard dice one after the other. What is the probability that the sum of the dice is 6 given that the first dice came up 4?

Conditional Probability - Examples

Q: Suppose we select a card at random from a standard deck of 52 cards. What is the probability of getting:

- A spade conditioned on the event that I picked the red color
- A spade facecard conditioned on the event that I picked the black color
- A black card conditioned on the event that I picked a spade facecard
- The queen of hearts given that I picked a queen
- An ace given that I picked a spade

Conditional Probability - Generalization to multiple events

Multiplication Rule: For events E_1, E_2, E_3, $\Pr[E_1 \cap E_2 \cap E_3] = \Pr[E_1] \Pr[E_2|E_1] \Pr[E_3|E_1 \cap E_2]$.

Proof:

$$\Pr[E_1] \Pr[E_2|E_1] \Pr[E_3|E_1 \cap E_2] = \Pr[E_1] \frac{\Pr[E_2 \cap E_1]}{\Pr[E_1]} \frac{\Pr[E_1 \cap E_2 \cap E_3]}{\Pr[E_1 \cap E_2]} = \Pr[E_1 \cap E_2 \cap E_3]$$

We can order the events to compute $\Pr[E_1 \cap E_2 \cap E_3]$ more easily. For example,

$$\Pr[E_1 \cap E_2 \cap E_3] = \Pr[E_2] \Pr[E_3|E_2] \Pr[E_1|E_2 \cap E_3]$$

Can extend this to n events i.e. in general,

$$\Pr[E_1 \cap E_2 \ldots \cap E_n] = \Pr[E_1] \Pr[E_2|E_1] \Pr[E_3|E_1 \cap E_2] \ldots \Pr[E_n|E_1 \cap E_2 \cap \ldots E_{n-1}]$$

Conditional Probability - Examples

Q: The organization that Jones works for is running a father–son dinner for those employees having at least one son. Each of these employees is invited to attend along with his youngest son. If Jones is known to have two children, what is the conditional probability that they are both boys given that he is invited to the dinner? Assume that the sample space S is given by $S = \{(b, b), (b, g), (g, b), (g, g)\}$ and all outcomes are equally likely. For instance, (b, g) means that the younger child is a boy and the older child is a girl.

The event that we care about is Jones has both boys. Hence, $E = \{(b, b)\}$.

Additional information that we are conditioning on is that Jones is invited to the dinner meaning that he has at least one son. Hence, $F = \{(b, b), (b, g), (g, b)\}$.

Hence, $E \cap F = \{(b, b)\}$, $\Pr[E \cap F] = \frac{|E \cap F|}{|\mathcal{S}|} = \frac{1}{4}$. $\Pr[F] = \frac{|F|}{|\mathcal{S}|} = \frac{3}{4}$.

$\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]} = \frac{1/4}{3/4} = \frac{1}{3}$.

Conditional Probability - Examples

Q: Ms. Perez figures that there is a 30 percent chance that her company will set up a branch office in Phoenix. If it does, she is 60 percent certain that she will be made manager of this new operation. What is the probability that there will be a branch in Phoenix and Perez will be its office manager?

E = Perez will be a branch office manager; F = her company will set up a branch office in Phoenix; $E \cap F$ = Perez will be an office manager in the Phoenix branch.

From the question, we know that $\Pr[F] = 0.3$, $\Pr[E|F] = 0.6$. Hence,
$\Pr[E \cap F] = \Pr[E]\Pr[E|F] = 0.3 \times 0.6 = 0.18$.

Conditional Probability Examples

Q: Suppose we have a bowl containing 6 white and 5 black balls. We randomly draw a ball. What is the probability that we draw a black ball?

Q: We randomly draw two balls, one after the other (without putting the first back). What is the probability that we (i) draw a black ball followed by a white ball (ii) draw a white ball followed by a black ball (iii) we get one black ball and one white ball (iv) both black (v) both white?

B1 = Draw black first, W1 = Draw white first. B2 = Black second, W2 = White second.

(i) $\Pr[B1] = \frac{5}{11}$. $\Pr[W2|B1] = \frac{6}{10}$. Hence, $\Pr[B1 \cap W2] = \Pr[B1]\,\Pr[W2|B1] = \frac{30}{110}$.

(ii) $\Pr[W1] = \frac{6}{11}$. $\Pr[B2|W1] = \frac{5}{10}$. Hence, $\Pr[W1 \cap B2] = \Pr[W1]\,\Pr[B2|W1] = \frac{30}{110}$.

(iii) $G = (B1 \cap W2) \cup (W1 \cap B2)$. Events $B1 \cap W2$ and $B2 \cap W1$ are mutually exclusive. By the union rule for mutually exclusive events, $\Pr[G] = \Pr[B1 \cap W2] + \Pr[W1 \cap B2] = \frac{60}{110}$.

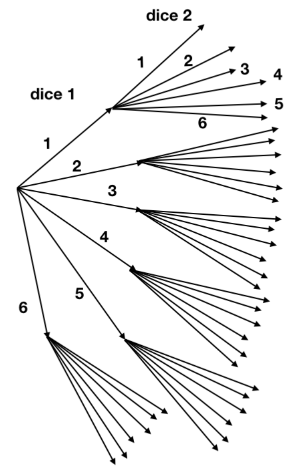
(iv) $\Pr[B1 \cap B2] = \Pr[B1]\,\Pr[B2|B1] = \frac{20}{110}$.

(v) $\Pr[W1 \cap W2] = \Pr[W1]\,\Pr[W2|W1] = \frac{30}{110}$.

Questions?

Back to throwing dice - Tree Diagram

Q: Suppose we throw two standard dice one after the other. What is the probability that we get two 6's in a row?



Identify Outcomes: Each leaf is an outcome and $\mathcal{S} = \{(1,1),(1,2),(1,3),\ldots(6,6)\}$.

Identify Event: $E = \{(6,6)\}$.

Compute probabilities: $\Pr[\text{Dice 1 is 6}] = \frac{1}{6}$.
$\Pr[(6,3)] = \Pr[\text{Dice 2 is 3} \cap \text{ Dice 1 is 6}] = \Pr[\text{Dice 2 is 3} \mid \text{Dice 1 is 6}] \Pr[\text{Dice 1 is 6}] = \frac{1}{6}\frac{1}{6} = \frac{1}{36}$.

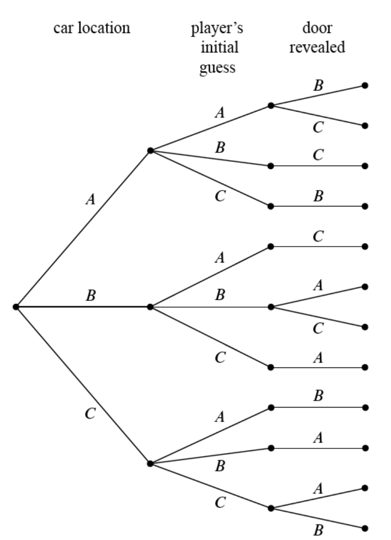
$\Pr[E] = \Pr[\text{dice 1 is 6 } \cap \text{dice 2 is 6 }] = \frac{1}{36}$.

Monty Hall Problem

Q: Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car, behind the others, goats. You pick a door, say A, and the host, who knows what's behind the doors, opens another door, say C, which has a goat. He says to you, "Do you want to pick door B?" Is it to your advantage to switch your choice of doors?

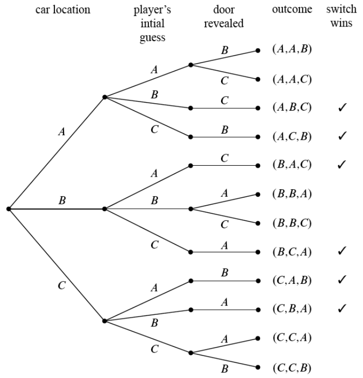
- The car is equally likely to be hidden behind each of the three doors.
- The player is equally likely to pick each of the three doors, regardless of the car's location.
- After the player picks a door, the host must open a different door with a goat behind it and offer the player the choice of staying with the original door or switching.
- If the host has a choice of which door to open, then he is equally likely to select each of them.

Tree Diagram for the Monty Hall Problem - Identify Outcomes



$\mathcal{S} = \{(A, A, B), (A, A, C), (A, B, C), (A, C, B), \ldots\}$.
$E_1 =$ Prize is behind door C $=$
$\{(C, A, B), (C, B, A), (C, C, A), (C, C, B)\}$

Tree Diagram for the Monty Hall Problem - Identify Event

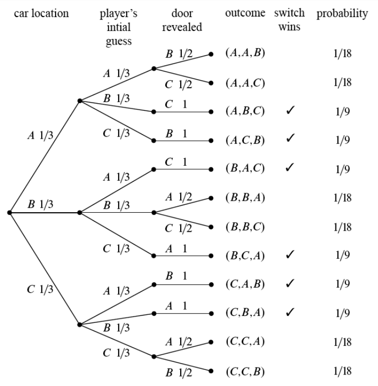


E = Switching wins =
$\{(A, B, C), (A, C, B), (B, A, C), (B, C, A), (C, A, B), (C, B, A)\}$

$\Pr[(A, A)] = \Pr[\text{Car is at A } \cap \text{ Player picks A }] =$
$\Pr[\text{Player picks A} \mid \text{Car is at A}] \Pr[\text{Car is at A}] = \frac{1}{3}\frac{1}{3} = \frac{1}{9}.$

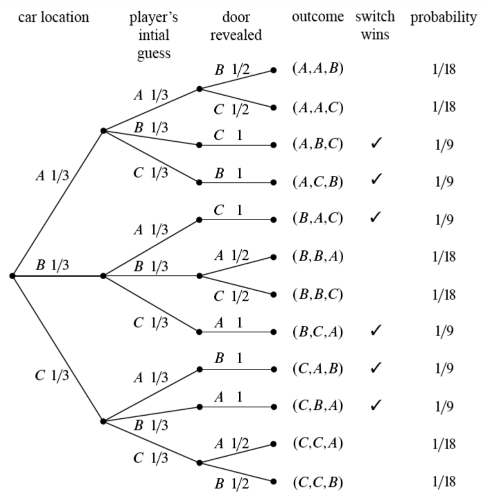
$\Pr[(A, A, B)] = \Pr[\text{Door B is revealed } \cap \text{ AA}] =$
$\Pr[\text{Door B is revealed} \mid \text{AA}] \Pr[\text{AA}] = \frac{1}{2}\frac{1}{9} = \frac{1}{18}.$

Tree Diagram for the Monty Hall Problem - Compute Probabilities



$$\Pr[E] = \Pr[(A,B,C)] + \Pr[(A,C,B)] + \Pr[(B,A,C)] + \Pr[(B,C,A)] + \Pr[(C,A,B)] + \Pr[(C,B,A)] = \frac{1}{9} \times 6 = \frac{2}{3}.$$

Monty Hall Problem and Conditional Probability



Q: What is the probability of winning by switching, if we pick door A and door B is opened.

$$\Pr[\text{win by switching}|\text{pick A and door B is opened}] = \frac{\Pr[\text{win by switching} \cap \text{pick A and door B is opened}]}{\Pr[\text{pick A and door B is opened}]}$$

$$= \frac{\Pr[(C,A,B)]}{\Pr[\{(A,A,B),(C,A,B)\}]} = \frac{1/9}{1/9+1/18} = \frac{2}{3}.$$

Q: Compute

$\Pr[\text{win by switching}|\text{pick A and door C is opened}]$?

Questions?