

# CMPT 210: Probability and Computing

## Lecture 2

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## Counting Sets - Example

**Q:** Suppose we want to buy 10 donuts. There are 5 donut varieties – chocolate, lemon-filled, sugar, glazed, plain. What is the number of ways to select the 10 donuts?

Let  $A$  be the set of ways to select the 10 donuts. Each element of  $A$  is a potential selection. For example, 4 chocolate, 3 lemon, 0 sugar, 2 glazed and 1 plain.

Let's map each way to a string as follows:

0000	000		00	0
				
chocolate	lemon	sugar	glazed	plain

Lets fix the ordering – chocolate, lemon, sugar, glazed and plain, and abstract this out further to get the sequence: 00001000110010. Hence, each way of choosing donuts is mapped to a binary sequence of length 14 with exactly 4 ones. Now, let  $B$  be all 14-bit sequences with exactly 4 ones. An element of  $B$  is 11110000000000.

**Q:** The above sequence corresponds to what donut order?

For every way to select donuts, we have an equivalent sequence in  $B$ . And every sequence in  $B$  implies a unique way to select donuts. Hence, the mapping from  $A \rightarrow B$  is a bijective function.

## Counting Sets - using a bijection

Hence,  $|A| = |B|$ , meaning that we have reduced the problem of counting the number of ways to select donuts to counting the number of 14-bit sequences with exactly 4 ones.

**General result:** The number of ways to choose  $n$  elements with  $k$  available varieties is equal to the number of  $n + k - 1$ -bit binary sequences with exactly  $k - 1$  ones.

**Q:** There are 2 donut varieties – chocolate and lemon-filled. Suppose we want to buy only 2 donuts. Use the above result to count the number of ways in which we can select the donuts? What are these ways?

**Q:** In the above example, I want at least one chocolate donut. What are the types of acceptable 3-bit sequences with this criterion? How many ways can we do this?

## Counting Sequences - using the product rule

**Q:** Suppose the university offers Math courses (denoted by the set  $M$ ), CS courses (set  $C$ ) and Statistics courses (set  $S$ ). We need to pick one course from each subject – Math, CS and Statistics. What is the number of ways we can select we can select the 3 courses?

The above problem is equivalent to counting the number of sequences of the form  $(m, c, s)$  that maps to choose the Math course  $m$ , CS course  $c$  and Stats course  $s$ .

Recall that the product of sets  $M \times C \times S$  is a set consisting of all sequences where the first component is drawn from  $M$ , the second component is drawn from  $C$  and the third from  $S$ , i.e.  $M \times C \times S = \{(m, c, s) | m \in M, c \in C, s \in S\}$ . Hence, counting the number of sequences is equivalent to computing  $|M \times C \times S|$ .

**Product Rule:**  $|M \times C \times S| = |M| \times |C| \times |S|$ .

Using the above equivalence, the number of sequences and hence, the number of ways to select the 3 courses is  $|M| \times |C| \times |S|$ .

## Counting Sequences - Example

Q: What is the number of length  $n$ -passwords that can be generated if each character in the password is allowed to be lower-case letter?

## Counting Sets - using the sum rule

**Q:** Let  $R$  be the set of rainy days,  $S$  be the set of snowy days and  $H$  be the set of really hot days in 2023. A bad day can be either rainy, snowy or really hot. What is the number of good days?

Let  $B$  be the set of bad days.  $B = R \cup S \cup H$ , and we want to estimate  $|\bar{B}|$ .  $|D| = 365$ .

$$|\bar{B}| = |D| - |B| = 365 - |B| = 365 - |R \cup S \cup H|.$$

Since the sets  $R$ ,  $S$  and  $H$  are disjoint,  $|R \cup S \cup H| = |R| + |S| + |H|$ , and hence the number of good days  $= 365 - |R| - |S| - |H|$ .

**Sum rule:** If  $A_1, A_2 \dots A_m$  are disjoint sets, then,  $|A_1 \cup A_2 \cup \dots \cup A_m| = \sum_{i=1}^m |A_i|$ .

## Counting Sets - Example

**Q:** What is the number of passwords that can be generated if the (i) first character is only allowed to be a lower-case letter, (ii) each subsequent character in the password is allowed to be lower-case letter or digit (0 – 9) and (iii) the length of the password is required to be between 6-8 characters?

Let  $L = \{a, b, \dots, z\}$  and  $D = \{0, 1, 2, \dots\}$ . Using the equivalence between sequences and products of sets, the set of passwords of length 6 is given by  $P_6 = L \times (L \cup D)^5$ . Using the product rule,  $|P_6| = |L| \times (|L \cup D|)^5 = |L| \times (|L| + |D|)^5$ .

Since the total set of passwords are  $P = P_6 \cup P_7 \cup P_8$ , and a password can be either of length 6, 7 or 8, sets  $P_6$ ,  $P_7$  and  $P_8$  are disjoint. Using the sum rule,  $|P| = |P_6| + |P_7| + |P_8| = |L| \times [(|L| + |D|)^5(1 + (|L| + |D|) + (|L| + |D|)^2)] = 26 \times 36^5 \times [1 + 36 + 1296]$ .

Questions?



## Counting sets - using the generalized product rule

**Q:** Suppose we have  $p$  prizes to be handed amongst the set  $A$  of  $n$  students. What are the number of ways in which we can distribute the prizes?

**Q:** Suppose we have  $p$  prizes to be handed amongst the set  $A$  of  $n$  students. What are the number of ways in which we can distribute the prizes such that each prize goes to a different student i.e. no student receives more than one prize?

Consider sequences of length  $p$ . The first entry can be chosen in  $n$  ways (the first prize can be given to one of the  $n$  students). After the first entry is chosen, since the same student cannot receive the prize, the second entry can be chosen in  $n - 1$  ways, and so on. Hence, the total number of ways to distribute the prizes  $= n \times (n - 1) \times \dots \times (n - (p - 1))$ .

**Generalized product rule:** If  $S$  is the set of length  $k$  sequences such that the first entry can be selected in  $n_1$  ways, after the first entry is chosen, the second one can be chosen in  $n_2$  ways, and so on, then  $|S| = n_1 \times n_2 \times \dots \times n_k$ . If  $n_1 = n_2 = \dots = n_k$ , we recover the product rule.

## Counting Sets - Example

**Q:** A dollar bill is defective if some digit appears more than once in the 8-digit serial number. What is the fraction of non-defective bills?

In order to compute the fraction of non-defective bills, we need to compute the quantity

$$\frac{|\text{serial numbers with all different digits}|}{|\text{possible serial numbers}|}.$$

For computing  $|\text{possible serial numbers}|$ , each digit can be one of 10 numbers. Hence, using the product rule,  $|\text{possible serial numbers}| = 10 \times 10 \dots = 10^8$ .

For computing  $|\text{serial numbers with all different digits}|$ , the first digit can be one of 10 numbers. Once the first digit is chosen, the second one can be chosen in 9 ways, and so on. By the generalized product rule,  $|\text{serial numbers with all different digits}| = 10 \times 9 \times \dots \times 3 = 1,814,400$ .

$$\text{Fraction of non-defective bills} = \frac{1,814,400}{10^8} = 1.8144\%.$$

# Permutations

A permutation of a set  $S$  is a sequence of length  $|S|$  that contains every element of  $S$  exactly once. Permutations of  $\{a, b, c\}$  are  $(a, b, c), (a, c, b), (b, c, a), (b, a, c), (c, a, b), (c, b, a)$ .

**Q:** Given a set of size  $n$ , what is the total number of permutations?

Considering sequences of length  $n$ , the first entry can be chosen in  $n$  ways. Since each element can be chosen only once, the second entry can be chosen in  $n - 1$  ways, and so on.

By the generalized product rule, the number of permutations  $= n \times (n - 1) \times \dots \times 1$ .

**Factorial:**  $n! := n \times (n - 1) \times \dots \times 1$ . By convention:  $0! = 1$ .

How big is  $n!$ ? **Stirling approximation:**  $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$ .

**Q:** Which is bigger?  $n!$  vs  $n(n - 1)(n + 2)(n - 3)!$ ?

**Q:** In how many ways can we arrange  $n$  people in a line?

**$k$ -to-1 function:** Maps exactly  $k$  elements of the domain to every element of the codomain.

If  $f : A \rightarrow B$  is a  $k$ -to-1 function, then,  $|A| = k|B|$ .

**Example:**  $E$  is the set of ears in this room, and  $P$  is the set of people. Then  $f$  mapping the ears to people is a 2-to-1 function. Hence,  $|E| = 2|P|$ .

**Q:** If  $f : A \rightarrow B$  is a  $k$ -to-1 function, and  $g : B \rightarrow C$  is a  $m$ -to-1 function, then what is  $|A|/|C|$ ?

**Q:** If  $f : A \rightarrow B$  is a  $k$ -to-1 function, and  $g : C \rightarrow B$  is a  $m$ -to-1 function, then what is  $|A|/|C|$ ?

## Counting sets - Example

**Q:** In how many ways can we arrange  $n$  people around a round table? Two seatings are considered to be the same *arrangement* if each person sits with the same person on their left in both seatings.

Starting from the head of the table, and going clockwise, each seating has an equivalent sequence.  $|\text{seatings}| = \text{number of permutations} = n!$ .

$n$  different seatings can result in the same arrangement (by clockwise rotation).

Hence,  $f : \text{seatings} \rightarrow \text{arrangements}$  is an  $n$ -to-1 function. Hence, the  $|\text{seatings}| = n |\text{arrangements}|$ , meaning that the  $|\text{arrangements}| = (n - 1)!$ .

Questions?

# Counting subsets

**Q:** How many size- $k$  subsets of a size- $n$  set are there?

*Example:* How many ways can we select 5 books from 100?

Let us form a permutation of the  $n$  elements, and pick the first  $k$  elements to form the subset. Every size  $k$  subset can be generated this way. There are  $n!$  total such permutations.

The order of the first  $k$  elements in the permutation does not matter in forming the subset, and neither does the order of the remaining  $n - k$  elements.

The first  $k$  elements can be ordered in  $k!$  ways and the remaining  $n - k$  elements can be ordered in  $(n - k)!$  ways. Using the product rule,  $k! \times (n - k)!$  permutations map to the same size  $k$  subset.

Hence, the function  $f : \text{permutations} \rightarrow \text{size } k \text{ subsets}$  is a  $k! \times (n - k)!$ -to-1 function. By the division rule,  $|\text{permutations}| = k! \times (n - k)! |\text{size } k \text{ subsets}|$ . Hence, the total number of size  $k$  subsets  $= \frac{n!}{k! \times (n - k)!}$ .

$$n \text{ choose } k = \binom{n}{k} := \frac{n!}{k! \times (n - k)!}.$$

Q: Prove that  $\binom{n}{k} = \binom{n}{n-k}$  - both algebraically (using the formula for  $\binom{n}{k}$ ) and combinatorially (without using the formula)

Q: Which is bigger?  $\binom{8}{4}$  vs  $\binom{8}{5}$ ?



## Counting subsets - Example

**Q:** How many  $m$ -bit binary sequences contain exactly  $k$  ones?

Consider set  $A = \{a_1, \dots, a_m\}$  and selecting  $S$ , a subset of size  $k$ . For example, say  $m = 10, k = 3$  and  $S = \{a_3, a_7, a_{10}\}$ , then  $S$  records the positions of the 1's, and can be mapped to the sequence 001 0001 001. Similarly, every  $m$ -bit sequence with exactly  $k$  ones can be mapped to a subset of size  $k$ . Hence, there is a bijection:

$f : m\text{-bit sequence with exactly } k \text{ ones} \rightarrow \text{subsets of size } k \text{ from size } m\text{-set, and}$   
 $|m\text{-bit sequence with exactly } k \text{ ones}| = |\text{subsets of size } k| = \binom{m}{k}.$

**Q:** Suppose we want to buy 10 donuts. There are 5 donut varieties – chocolate, lemon-filled, sugar, glazed, plain. What is the number of ways to select the 10 donuts?

Recall that the number of ways of selecting 10 donuts with 5 varieties = number of 14-bit sequences with exactly 4 ones =  $\binom{14}{4} = 1001$ .

**Q:** What is the number of ways of choosing  $n$  things with  $k$  varieties?

Questions?