CMPT 210: Probability and Computation

Lecture 4

Sharan Vaswani

May 20, 2022

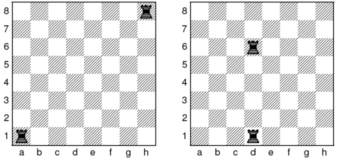
Counting Practice

Q: How many ways can I select 5 toppings for my pizza if there are 14 available toppings? What is the total number of different pizzas I can make?

Q: How many different solutions over $\mathbb{N}$ are there to the following equation: $x_1 + x_2 + x_3 = 100$

Counting Practice

Q: In how many ways can we place (i) two identical black rooks (ii) a black rook and a white rook such that they do not share the same row or column?



(a) valid (b) invalid

Figure 15.2 Two ways to place 2 rooks (♖) on a chessboard. The configuration in (b) is invalid because the rooks are in the same column.

Questions?

Introduction to Probability - Throwing dice

Suppose we throw a standard dice. What is the probability that the number that comes up is 6?

What are the possible things that can happen? The dice comes up one of the numbers in $1, 2, 3, 4, 5, 6$.

What are the things that we care about? Getting a 6.

In how many ways can this happen? Just one.

Probability of getting a $6 = \frac{\text{Number of ways in which the thing we care about happens}}{\text{Total number of ways in which something can happen}} = \frac{1}{6}$.

Suppose we throw a standard dice. What is the probability that we get either a 3 or a 6?

What are the possible *outcomes* that can happen? The dice comes up one of the numbers in $1, 2, 3, 4, 5, 6$.

What is the *event* that we care about? Getting either a 3 or 6.

In how many ways can this *event* happen? Two (the dice comes 3 or 6).

Probability of getting either a 3 or a 6 $= \frac{\text{Number of ways in which the event we care about happens}}{\text{Total number of outcomes}} = \frac{2}{6}$.

Introduction to Probability - Throwing dice

Suppose we throw two standard die one after the other. What is the probability that we get two 6's in a row?

What are the possible outcomes that can happen? The first dice comes up one of the numbers in $1, 2, 3, 4, 5, 6$, the second dice comes up one of the numbers in $1, 2, 3, 4, 5, 6$.

If we consider both die together, what are the possible outcomes – first dice is 1, second dice is 1; first is 1, second is 2, and so on. Let us write this compactly. The space of outcomes is $(1, 1), (1, 2), (1, 3), \ldots, (6, 6)$.

What is the size of this *outcome space*? 36 (By the product rule)

What is the event that we care about? Getting $(6, 6)$.

In how many ways can this happen? One (both die need to come up 6).

Probability of getting two 6's in a row $= \frac{\text{Number of ways in which the event we care about happens}}{|\text{outcome space}|} = \frac{1}{36}$.

Questions?

Probability Basics

Sample (outcome) space $\mathcal{S}$: Nonempty (countable) set of possible outcomes. Example: When we threw one dice, the sample space is $\{1, 2, 3, 4, 5, 6\}$. When we threw two die, the sample space is $\{(1, 1), (1, 2), (1, 3), \ldots\} = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$ (using the relation between sets and sequences).

The sample space does not necessarily need to be numbers. For example, if we are randomly choosing colors from the rainbow, then $\mathcal{S} = \{\text{violet, indigo, blue, green, yellow, orange, red}\}$.

Outcome $\omega \in \mathcal{S}$: Possible "things" that can happen. Example: When we threw one dice, a possible outcome is $\omega = 1$. For the rainbow example, the color "red" is a possible outcome.

Event E: Any subset of the sample space. Example: When we threw one dice, a possible event is $E = \{6\}$ (first example) or $E = \{3, 6\}$ (second example). When we threw two die, a possible event is $E = \{(6, 6)\}$.

Union of events

Since the event E is a set, all the set theory we learned is useful!

Suppose E, F are two events in $\mathcal{S}$. Define the union $E \cup F$ to consist of outcomes that are either in E or F (this is just the definition of the union of two sets). Formally,

$$E \cup F = \{\omega | \omega \in E \text{ OR } \omega \in F\}$$

.

Another way to interpret this is to say event $E \cup F$ occurs if either event E or event F occurs.

Example: We considered the case where we threw down one dice and cared about getting *either* 3 or 6. In this case, event G "happens" if we get either 3 or 6. Formally, $E = \{3\}$, $F = \{6\}$, $G = E \cup F = \{3, 6\}$. And G occurs when the number that shows up is either 3 or 6.

Can define union between more than two events in the same way we defined union between more than two sets. $G = E_1 \cup E_2 \cup \dots E_n$. G happens when *at least one* of the events E_i happen.

Intersection of events

Suppose E, F are two events in $\mathcal{S}$. Define the intersection $E \cap F$ to consist of outcomes that are in both E and F (this is just the definition of the intersection of two sets). Formally,

$$E \cup F = \{\omega | \omega \in E \text{ AND } \omega \in F\}$$

.

Another way to interpret this is to say event $E \cap F$ occurs if both events E and F occur.

Example: We threw down two die and cared about getting 6 in the first throw *and* 6 in the second throw. In this case, $E = \{(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$, $F = \{(1,6), (2,6), (3,6), (4,6), (5,6), (6,6)\}$, $G = E \cap F = \{(6,6)\}$. G happens when both E and F happen i.e. the first dice has a 6 and the second dice has 6.

Can define intersection between more than two events in the same way we defined intersection between more than two sets. $G = E_1 \cap E_2 \cap \ldots E_n$. G happens when *all* of the events E_i happen.

Mutually exclusive and complement events

Mutually exclusive events: If E and F are two events such that $E \cap F = \{\}$, then events E and F are mutually exclusive.

Example: We threw one dice and want to get both 3 *and* 6. This is not possible. Formally, $E = \{3\}$, $F = \{6\}$ and $E \cap F = \{\}$, hence, events E and F are mutually exclusive.

Complement of an event: If E is an event, then its complement E^c is defined such that $E \cap E^c = \{\}$ and $E \cup E^c = \mathcal{S}$. Event E^c will occur if and only if event E does not occur.

Example: We threw one dice and want to get a 6 i.e. we define $E = \{6\}$. $E^c = \{1, 2, 3, 4, 5\}$. Clearly, $E \cap E^c = \{\}$, implying that an event and its complement are mutually exclusive.

Two complement events are mutually exclusive, but two mutually exclusive events need not be the complement. Example: E and F are are mutually exclusive, but not complements.

Subset: If $E \subset F$, then if E happens F will happen. Example: When we throw one dice, if $E = \{3\}$ and $F = \{1, 2, 3\}$ i.e. E is the event that we get 3 and F is the event that we can either $1, 2, 3$. Clearly, if E happens, F will happen.

Axioms of Probability

Probability function on a sample space $\mathcal{S}$ is a total function $\Pr : \mathcal{S} \to [0,1]$.

For any $\omega \in \mathcal{S}$,

$$0 \leq Pr[\omega] \leq 1 \quad \sum_{\omega \in \mathcal{S}} \Pr[\omega] = 1$$

Probability space: The outcome space $\mathcal{S}$ together with the probability function.

Recall that we can define functions on sets. In this case, for an event E, $\Pr[E] = \sum_{\omega \in E} \Pr[\omega]$.

For mutually exclusive events $E_1, E_2, \ldots, E_n$ (meaning that the sets $E_1, E_2, \ldots, E_n$ are disjoint),

$$\Pr[E_1 \cup E_2 \cup \ldots E_n] = \sum_{\omega \in \{\boldsymbol{E_1} \cup \boldsymbol{E_2} \cup \ldots E_n\}} \Pr[\omega]$$

Since E_i's are disjoint, any ω can only be in one of $E_1, E_2, \ldots E_n$

$$= \sum_{\omega \in \boldsymbol{E_1}} \Pr[\omega] + \sum_{\omega \in \boldsymbol{E_2}} \Pr[\omega] + \ldots + \sum_{\omega \in E_n} \Pr[\omega] = \Pr[E_1] + \Pr[E_2] + \ldots + \Pr[E_n].$$

Questions?

Back to throwing dice

Suppose we throw a standard dice. What is the probability that the number that comes up is 6?

$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$. Since the dice is "standard", each outcome is equally likely, i.e. $\Pr[\{1\}] = \Pr[\{2\}] = \ldots = \Pr[\{6\}]$.

Since $\Pr[\mathcal{S}] = 1 \implies \sum_{\omega \in \mathcal{S}} \Pr[\omega] = 1 \implies [\Pr[\{1\}] + \Pr[\{2\}] + \ldots \Pr[\{6\}]] = 1 \implies \Pr[\{6\}] = \frac{1}{6}$.

Back to throwing dice

Suppose we throw a standard dice. What is the probability that we get either a 3 or a 6?

$E = \{3\}$, $F = \{6\}$, $G = \{3, 6\}$. Since $E \cap F = \{\}$, E and F are mutually exclusive events, implying that $\Pr[G] = \Pr[E] + \Pr[F] = \Pr[\{3\}] + \Pr[\{6\}] = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

Hence, probability of getting either a 3 or a 6 is equal to $\frac{1}{3}$.

Q: Compute the probability of getting either 1, 2 or 3.

Q: Compute the probability of getting an even number.

Q: Compute the probability of getting either 1, 2, 3, 4, 5, 6

Probability Rules

Complement rule: $\Pr[E] = 1 - \Pr[E^c]$ Recall that $E \cap E^c = \{\}$ and $E \cup E^c = \mathcal{S}$. Since E and E^c are disjoint,

$$\Pr[E \cup E^c] = \Pr[E] + \Pr[E^c] \implies \Pr[S] = \Pr[E] + \Pr[E^c] \implies \Pr[E^c] \quad = 1 - \Pr[E].$$

Inclusion-Exclusion rule: For any two events E, F, $\Pr[E \cup F] = \Pr[E] + \Pr[F] - \Pr[E \cap F]$.

$$\Pr[E \cup F] = \sum_{\omega \in \{E \cup F\}} \Pr[\omega] = \sum_{\omega \in \{E - F\}} \Pr[\omega] + \sum_{\omega \in \{F - E\}} \Pr[\omega] + \sum_{\omega \in \{E \cap F\}} \Pr[\omega]$$

(Since these are disjoint)

$$= \left[\sum_{\omega \in \{E - F\}} \Pr[\omega] + \sum_{\omega \in \{E \cap F\}} \Pr[\omega]\right] + \left[\sum_{\omega \in \{F - E\}} \Pr[\omega] + \sum_{\omega \in \{E \cap F\}} \Pr[\omega]\right] - \sum_{\omega \in \{E \cap F\}} \Pr[\omega]$$

$$= \sum_{\omega \in E} \Pr[\omega] + \sum_{\omega \in F} \Pr[\omega] - \sum_{\omega \in \{E \cap F\}} \Pr[\omega] = \Pr[E] + \Pr[F] - \Pr[E \cap F]$$

$$\implies \Pr[E \cup F] = \Pr[E] + \Pr[F] - \Pr[E \cap F]$$

Probability Rules

Union Bound: For any events $E_1, E_2, E_3, \ldots E_n$,

$$\Pr[E_1 \cup E_2 \cup E_3 \ldots \cup E_n] \leq \sum_{i=1}^{n} \Pr[E_i] \qquad . \qquad \text{(Since probabilities are positive)}$$

Monotonicity rule: For events A and B, if $A \subset B$, then $\Pr[A] < \Pr[B]$.

Probability - Examples

Q: Suppose we select a card at random from a standard deck of 52 cards. What is the probability of getting:

- A spade
- A spade facecard
- A black card
- The queen of hearts
- An ace