CMPT 210: Probability and Computing

Lecture 3

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Recap - Counting

Product Rule: For sets A_1, $A_2 \ldots, A_m$, $|A_1 \times A_2 \times \ldots \times A_m| = \prod_{i=1}^{m} |A_i|$ (E.g: Selecting one course each from every subject.)

Sum rule: If $A_1, A_2 \ldots A_m$ are disjoint sets, then, $|A_1 \cup A_2 \cup \ldots \cup A_m| = \sum_{i=1}^{m} |A_i|$ (E.g Number of rainy, snowy or hot days in the year).

Generalized product rule: If S is the set of length k sequences such that the first entry can be selected in n_1 ways, after the first entry is chosen, the second one can be chosen in n_2 ways, and so on, then $|S| = n_1 \times n_2 \times \ldots n_k$. (E.g Number of ways n people can be arranged in a line $= n!$)

Division rule: $f : A \to B$ is a k-to-1 function, then, $|A| = k|B|$. (E.g. For arranging people around a round table, f : seatings $\to$ arrangements is an n-to-1 function).

Counting subsets (Combinations)

Q: How many size-k subsets of a size-n set are there?

Example: How many ways can we select 5 books from 100?

Let us form a permutation of the n elements, and pick the first k elements to form the subset. Every size k subset can be generated this way. There are $n!$ total such permutations.

The order of the first k elements in the permutation does not matter in forming the subset, and neither does the order of the remaining $n - k$ elements.

The first k elements can be ordered in $k!$ ways and the remaining $n - k$ elements can be ordered in $(n - k)!$ ways. Using the product rule, $k! \times (n - k)!$ permutations map to the same size k subset.

Hence, the function f : permutations $\to$ size k subsets is a $k! \times (n - k)!$-to-1 function. By the division rule, $|\text{permutations}| = k! \times (n - k)! \, |\text{size } k \text{ subsets}|$. Hence, the total number of size k subsets $= \frac{n!}{k! \times (n-k)!}$.

n **choose** $k = \binom{n}{k} := \frac{n!}{k! \times (n-k)!}$.

Counting subsets (Combinations)

Q: Prove that $\binom{n}{k} = \binom{n}{n-k}$ - both algebraically (using the formula for $\binom{n}{k}$) and combinatorially (without using the formula)

Q: Which is bigger? $\binom{8}{4}$ vs $\binom{8}{5}$?

Counting subsets – Example

Q: How many m-bit binary sequences contain exactly k ones?

Consider set $A = \{1, \ldots, m\}$ and selecting S, a subset of size k. For example, say $m = 10, k = 3$ and $S = \{3, 7, 10\}$. S records the positions of the 1's, and can mapped to the sequence 001 0001 001. Similarly, every m-bit sequence with exactly k ones can be mapped to a subset S of size k. Hence, there is a bijection:
f : m-bit sequence with exactly k ones $\to$ subsets of size k from size m-set, and
$|m\text{-bit sequence with exactly } k \text{ ones}| = |\text{subsets of size } k| = \binom{m}{k}$.

Q: Suppose we want to buy 10 donuts. There are 5 donut varieties – chocolate, lemon-filled, sugar, glazed, plain. What is the number of ways to select the 10 donuts?

Recall that the number of ways of selecting 10 donuts with 5 varieties = number of 14-bit sequences with exactly 4 ones $= \binom{14}{4} = 1001$.

Q: What is the number of ways of choosing n things with k varieties?

Counting subsets – Example

Q: What is the number of n-bit binary sequences with at least k ones?

Q: What is the number of n-bit binary sequences with less than k ones?

Q: What is the total number of n-bit binary sequences?

Binomial Theorem

For all $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Example: If $a = b = 1$, then $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ (result from previous slide).

If $n = 2$, then $(a+b)^2 = \binom{2}{0} a^2 + \binom{2}{1} ab + \binom{2}{2} b^2 = a^2 + 2ab + b^2$.

Q: What is the coefficient of the terms with ab^3 and a^2b^3 in $(a+b)^4$? .

Q: For $a, b > 0$, what is the coefficient of $a^{2n-7}b^7$ and $a^{2n-8}b^8$ in $(a+b)^{2n} + (a-b)^{2n}$?

Counting Practice

Q: A standard dice (with numbers $\{1, 2, 3, 4, 5, 6\}$) is rolled 6 times in succession.

- How many rolls will have no 6?
- How many rolls will have each number once?
- How many rolls will have 6 come up exactly once?
- How many rolls will have 6 come up exactly k times (for $k \leq 6$)?

Counting Practice

Q: How many 5 digit numbers are there which contain at least one zero? Note that a number is different from a string, i.e. 01234 is not a 5-digit number and is hence not allowed.

Counting Practice

Q: How many non-negative integer solutions ($x_1, x_2, x_3 \geq 0$) are there to the following equation:

$$x_1 + x_2 + x_3 = 40$$

Questions?

Generalization to Multinomials

We saw how to split a set into two subsets - one that contains some elements, while the other does not. Can generalize the arguments to split a set into more than two subsets.

A $(k_1, k_2, \ldots, k_m)$-split of set A is a sequence of sets $(A_1, A_2, \ldots A_m)$ s.t. sets A_i form a partition ($A_1 \cup A_2 \cup \ldots = A$ and for $i \neq j$, $A_i \cap A_j = \emptyset$) and $|A_i| = k_i$.

An example of a $(2, 1, 3)$-split of $A = \{1, 2, 3, 4, 5, 6\}$ is $(\{2, 4\}, \{1\}, \{3, 5, 6\})$. Here, $m = 3$, $A_1 = \{2, 4\}$, $A_2 = \{1\}$, $A_3 = \{3, 5, 6\}$ s.t. $|A_1| = 2$, $|A_2| = 1$, $|A_3| = 3$, $A_1 \cup A_2 \cup A_3 = A$ and for $i \neq j$, $A_i \cap A_j = \emptyset$.

Example: Consider strings of length 6 of a's, b's and c's such that number of a's $= 2$; number of b's $= 1$ and number of c's $= 3$. Possible strings: abaccc, ccbaac, bacacc, cbacac.

Each possible string, e.g. bacacc can be written as a $(2, 1, 3)$-split of $A = \{1, 2, 3, 4, 5, 6\}$ as $(\{2, 4\}, \{1\}, \{3, 5, 6\})$ where A_1 records the positions of a, A_2 records the positions of b and A_3 records the positions of c.

Generalization to Multinomials

Q: Show that the number of ways to obtain an $(k_1, k_2, \ldots, k_m)$ split of A with $|A| = n$ is $\binom{n}{k_1, k_2, \ldots k_m} = \frac{n!}{k_1!\, k_2! \ldots k_m!}$ where $\sum_i k_i = n$.

Can map any permutation $(a_1, a_2, \ldots a_n)$ into a split by selecting the first k_1 elements to form set A_1, next k_2 to form set A_2 and so on. For the same split, the order of the elements in each subset does not matter. Hence f : number of permutations $\rightarrow$ number of splits is a $k_1!\, k_2!\, \ldots k_m!$-to-1 function.

Hence, $|\text{number of splits}| = \frac{|\text{number of permutations}|}{k_1!\, k_2! \ldots k_m!} = \frac{n!}{k_1!\, k_2! \ldots k_m!}$.

Generalization to Multinomials - Example

Q: Count the number of permutations of the letters in the word BOOKKEEPER.

We want to count sequences of the form $(1E, 1P, 2E, 1B, 1K, 1R, 2O, 1K) = EPEEBKROOK$. There is a bijection between such sequences and $(1, 2, 2, 3, 1, 1)$ split of $A = \{1, 2, \ldots, 10\}$ where A_1 is the set of positions of B's, A_2 is the set of positions of O's, A_3 is set of positions of K and so on.

For example, the above sequence maps to the following split:
({5}, {8,9}, { 6, 10}, { 1,3,4 }, { 2 }, { 7 })

Hence, the total number of sequences that can be formed from the letters in BOOKKEEPER = number of $(1, 2, 2, 3, 1, 1)$ splits of $A = [10] = \{1, 2, \ldots, 10\} = \frac{10!}{1!\,2!\,2!\,3!\,1!\,1!}$.

Q: Count the number of permutations of the letters in the word (i) ABBA (ii) A_1BBA_2 and (iii) $A_1B_1B_2A_2$?