CMPT 210: Probability and Computation

Lecture 6

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Recap - Axioms of Probability

Sample (outcome) space $\mathcal{S}$: Nonempty (countable) set of possible outcomes. Example: When we threw one dice, the sample space is $\{1, 2, 3, 4, 5, 6\}$.

Outcome $\omega \in \mathcal{S}$: Possible "thing" that can happen. Example: When we threw one dice, a possible outcome is $\omega = 1$.

Event E: Any subset of the sample space. Example: When we threw one dice, a possible event is $E = \{6\}$ (first example) or $E = \{3, 6\}$ (second example).

Probability function on a sample space $\mathcal{S}$ is a total function $\Pr : \mathcal{S} \to [0, 1]$. For any $\omega \in \mathcal{S}$,

$$0 \leq Pr[\omega] \leq 1 \quad ; \quad \sum_{\omega \in \mathcal{S}} \Pr[\omega] = 1 \quad ; \quad \Pr[E] = \sum_{\omega \in E} \Pr[\omega]$$

Recap - Probability rules

Union: For mutually exclusive events $E_1, E_2, \ldots, E_n$,
$\Pr[E_1 \cup E_2 \cup \ldots E_n] = \Pr[E_1] + \Pr[E_2] + \ldots + \Pr[E_n]$.

Complement rule: $\Pr[E] = 1 - \Pr[E^c]$

Inclusion-Exclusion rule: For any two events E, F, $\Pr[E \cup F] = \Pr[E] + \Pr[F] - \Pr[E \cap F]$.

Union Bound: For any events $E_1, E_2, E_3, \ldots E_n$, $\Pr[E_1 \cup E_2 \cup E_3 \ldots \cup E_n] \leq \sum_{i=1}^{n} \Pr[E_i]$.

Uniform probability space: A probability space is said to be uniform if $\Pr[\omega]$ is the same for every outcome $\omega \in \mathcal{S}$. In this case, $\Pr[E] = \frac{|E|}{|\mathcal{S}|}$.

Probability - Examples

Q: Let us consider random permutations of the letters (i) ABBA (ii) ABBA'. What is the probability that the third letter is B?

Conditional Probability

Conditioning is revising probabilities based on partial information (an event).

For example, suppose we throw a "standard" dice, what is the probability of getting a 6 if we are told that the outcome is even?

Compute Pr(we get a 6|the outcome is even) (Probability of getting a 6 *given* that the outcome is even)/(Probability of getting a 6 *conditioned on the event* that the outcome is even).

Sample space: $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$, Event: $E = \{6\}$. Additional information: Event $F = \{2, 4, 6\}$ has happened. With conditioning, F can be interpreted as the new *sample space*.

Since each outcome in $F = \{2, 4, 6\}$ is still equally likely, the new probability space is still a uniform probability space. Hence, conditioned on the event that the outcome is even, $\sum_{\omega \in F} \Pr[\omega] = 1$ and $\Pr[\{\text{even number}\}] = \frac{1}{3}$ and $\Pr[\{\text{odd number}\}] = 0$.

Q: What is the probability of getting either a 3 or 6 if we are told that the outcome is even?

Conditional Probability

For two events E and F, we wish to compute the probability of event E conditioned on F i.e. event F has happened/constrained to happen.

By conditioning on F, the only outcomes we care about are in F i.e. for $\omega \notin F$, $\Pr[\omega] = 0$. Since we want to compute the probability that event E happens, we care about the outcomes that are in E. Hence, the outcomes we care about lie in both E *and* F, meaning that $\omega \in E \cap F$.

Hence, $\Pr[E|F] \propto \sum_{\omega \in (E \cap F)} \Pr[\omega]$. For some constant $c > 0$, $\Pr[E|F] = c \sum_{\omega \in (E \cap F)} \Pr[\omega]$.

We know that $\Pr[F|F] = 1$ (probability of event F given that F has happened). Hence, $\Pr[F|F] = 1 = c \sum_{\omega \in F} \Pr[\omega] \implies c = \frac{1}{\sum_{\omega \in \{F\}} \Pr[\omega]}$.

Substituting the value of c,

$$\Pr[E|F] = \frac{\sum_{\omega \in (E \cap F)} \Pr[\omega]}{\sum_{\omega \in F} \Pr[\omega]} = \frac{\Pr[E \cap F]}{\Pr[F]},$$

where $\Pr[F] \neq 0$.

Back to throwing dice

Suppose we throw a "standard" dice, what is the probability of getting a 6 if we are told that the outcome is even?

Sample space: $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$, Event: $E = \{6\}$. We are conditioning on $F = \{2, 4, 6\}$.

Hence, $\Pr(\text{we get a 6}|\text{the outcome is even}) = \Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]}$.

$E \cap F = \{6\}$. $\Pr[\{E\}] = \frac{1}{6}$. $\Pr[\{F\}] = \Pr[\{2\}] + \Pr[\{4\}] + \Pr[\{6\}] = \frac{1}{2}$. Hence, $\frac{\Pr[E \cap F]}{\Pr[F]} = \frac{1/6}{1/2} = \frac{1}{3}$.

Q: What is the probability of getting either a 3 or 6 if we are told that the outcome is even?

Questions?

Conditional Probability - Examples

Q: Suppose we select a card at random from a standard deck of 52 cards. What is the probability of getting:

- A spade conditioned on the event that I picked the red color
- A spade facecard conditioned on the event that I picked the black color
- A black card conditioned on the event that I picked a spade facecard
- The queen of hearts given that I picked a queen
- An ace given that I picked a spade

Conditional Probability - Examples

Q: The organization that Jones works for is running a father–son dinner for those employees having at least one son. Each of these employees is invited to attend along with his youngest son. If Jones is known to have two children, what is the conditional probability that they are both boys given that he is invited to the dinner?

Sample space is the pair of genders of Jones' younger and older child. Hence, $\mathcal{S} = \{(b, b), (b, g), (g, b), (g, g)\}$.

The event that we care about is Jones has both boys. Hence, $E = \{(b, b)\}$

Additional information that we are conditioning on is that Jones is invited to the dinner meaning that he has at least one son. Hence, $F = \{(b, b), (b, g), (g, b)\}$.

Hence, $E \cap F = \{(b, b)\}$, $\Pr[E \cap F] = \frac{|E \cap F|}{|\mathcal{S}|} = \frac{1}{4}$. $\Pr[F] = \frac{|F|}{|\mathcal{S}|} = \frac{3}{4}$.

$\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]} = \frac{1/4}{3/4} = \frac{1}{3}$.

Conditional Probability - Examples

Q: Ms. Perez figures that there is a 30 percent chance that her company will set up a branch office in Phoenix. If it does, she is 60 percent certain that she will be made manager of this new operation. What is the probability that Perez will be an office manager in the Phoenix branch?

E = Perez will be a branch office manager; F = her company will set up a branch office in Phoenix; $E \cap F$ = Perez will be an office manager in the Phoenix branch.

From the question, we know that $\Pr[F] = 0.3$, $\Pr[E|F] = 0.6$. Hence, $\Pr[E \cap F] = \Pr[E]\Pr[E|F] = 0.3 \times 0.6 = 0.18$.

Conditional Probability Examples

Q: Suppose we have a bowl containing 6 white and 5 black balls. We randomly draw a ball. What is the probability that we draw a black ball

Q: We randomly draw two balls, one after the other (without putting the first back). What is the probability that we (i) draw a black ball followed by a white ball (ii) draw a white ball followed by a black ball (iii) we get one black ball and one white ball (iv) both black (v) both white?

B1 = Draw a black first, W1 = Draw a white first. B2 = Black second, W2 = White second.

(i) $\Pr[B1] = \frac{5}{11}$. $\Pr[W2|B1] = \frac{6}{10}$. Hence, $\Pr[B1 \cap W2] = \Pr[B1]\Pr[W2|B1] = \frac{30}{110}$.

(ii) $\Pr[W1] = \frac{6}{11}$. $\Pr[B2|W1] = \frac{5}{10}$. Hence, $\Pr[B1 \cap W2] = \Pr[W1]\Pr[B2|W1] = \frac{30}{110}$.

(iii) $G = (B1 \cap W2) \cup (W1 \cap B2)$. Events $B1 \cap W2$ and $B2 \cap W1$ are mutually exclusive. By the union rule for mutually exclusive events, $\Pr[G] = \Pr[B1 \cap W2] + \Pr[W1 \cap B2] = \frac{60}{110}$.

(iv) $\Pr[B1 \cap B2] = \Pr[B1]\Pr[B2|B1] = \frac{20}{110}$.

(v) $\Pr[W1 \cap W2] = \Pr[W1]\Pr[W2|W1] = \frac{20}{110}$.

Conditional Probability Examples

Q: Two teams A and B are asked to separately design a new product within a month. From past experience we know that, (a) The probability that team A is successful is 2/3, (b) The probability that team B is successful is 1/2, (c) (c) The probability that at least one team is successful is 3/4. Assuming that exactly one successful design is produced, what is the probability that it was designed by team B.

Let SS be the event that both teams are successful, SF be the event that team A succeeded but team B failed, FS be the event that team B succeeded but team A failed and FF be the event that both teams failed.

Since exactly one successful design is produced, we know that exactly one of the teams succeeded. Hence, we wish to compute $\Pr[\{FS\}|\{FS \cup SF\}]$.

$\Pr[SS \cup SF] = \frac{2}{3}$, $\Pr[SS \cup FS] = \frac{1}{2}$, $\Pr[SS \cup SF \cup SS] = 3/4$. Since these are mutually exclusive,

$$\Pr[SS] + \Pr[SF] = \frac{2}{3} \quad \Pr[SS] + \Pr[FS] = \frac{1}{2} \quad \Pr[SS] + \Pr[SF] + \Pr[FS] = \frac{3}{4}$$

Solving these, $\Pr[\{FS\}|\{FS \cup SF\}] = \frac{3/4-2/3}{(3/4-2/3)+(3/4-1/2)} = \frac{1/12}{1/12+1/4} = \frac{1}{4}$.

Questions?

Back to throwing dice

Suppose we throw two standard dice one after the other. What is the probability that we get two 6's in a row?

E = We get a 6 in the second throw. F = We get a 6 in the first throw. $E \cap F$ = we get two 6's in a row. Wish to compute $\Pr[E \cap F]$. $\Pr[E] = \Pr[F] = \frac{1}{6}$.

$\Pr[E|F] = \frac{\Pr[E \cap F]}{\Pr[F]} \implies \Pr[E \cap F] = \Pr[E|F]\,\Pr[F]$.

Since the two dice are *independent*, knowing that we got a 6 in the first throw does not change the probability that we will get a 6 in the second throw. Hence, $\Pr[E|F] = \Pr[E]$ (conditioning does not change the probability of the event).

Hence, $\Pr[E \cap F] = \Pr[E|F]\,\Pr[F] = \Pr[E]\,\Pr[F] = \frac{1}{6}\,\frac{1}{6} = \frac{1}{36}$.

Independent Events

Events E and F are said to be independent, if knowledge that F has occurred does not change the probability that E occurs. Formally,

$$\Pr[E \cap F] = \Pr[E] \Pr[F]$$

Q: I toss two independent, fair coins. What is the probability that I get the HT sequence?

Define E to be the event that I get a heads in the first toss, and F be the event that I get a tails in the second toss. Since the two coins are independent, events E and F are also independent. $\Pr[E \cap F] = \Pr[E] \Pr[F] = \frac{1}{2} \frac{1}{2} = \frac{1}{4}$.

Q: I randomly choose a number from $\{1, 2, \ldots, 10\}$. E is the event that the number I picked is a prime number. F is the event that the event I picked is an odd number. Are E and F independent?

$\Pr[E] = \frac{2}{5}$, $\Pr[F] = \frac{1}{2}$, $\Pr[E \cap F] = \frac{3}{10}$. $\Pr[E \cap F] \neq \Pr[E] \Pr[F]$. Another way: $\Pr[E|F] = \frac{3}{5}$ and $\Pr[E] = \frac{2}{5}$, and hence $\Pr[E|F] \neq \Pr[E]$. Conditioning on F tell us that prime number cannot be 2, so it changes the probability of E.

Independent Events - Example

Q: We have a machine that has 2 independent components. The machine breaks if each of its 2 components break. Suppose each component can break with probability p, what is the probability that the machine does not break?

Let E_1 = Event that the first component breaks, E_2 = Event that the second component breaks. M = Event that the machine breaks = $E_1 \cap E_2$.

$\Pr[M] = \Pr[E_1 \cap E_2]$. Since the two components are independent, E_1 and E_2 are independent, meaning that $Pr[E_1 \cap E_2] = Pr[E_1] \Pr[E_2] = p^2$.

Probability that the machine does not break = $\Pr[M^c] = 1 - \Pr[M] = 1 - p^2$.

Q: We have a new machine that breaks if either of its 2 components break. Suppose each component can break with probability p, what is the probability that the machine breaks?

For this machine, let M' be the event that it breaks. In this case, $\Pr[M] = \Pr[E_1 \cup E_2]$. Since E_1 and E_2 are mutually exclusive, by the union rule, $\Pr[E_1 \cup E_2] = \Pr[E_1] + \Pr[E_2] = 2p$.

Independent Events

Mutual Independence: A set of events is said to be mutually independent if the probability of each event in the set is the same no matter which of the events has occurred.

For any selection of two or more of the events, the probability that all the selected events occur equals the product of the probabilities of the selected events.

Example: For events E_1, E_2 and E_3 to be mutually independent, all the following equalities should hold:

$$\Pr[E_1 \cap E_2] = \Pr[E_1]\Pr[E_2] \quad \Pr[E_1 \cap E_3] = \Pr[E_1]\Pr[E_3]$$
$$\Pr[E_2 \cap E_3] = \Pr[E_2]\Pr[E_3] \quad \Pr[E_1 \cap E_2 \cap E_3] = \Pr[E_1]\Pr[E_2]\Pr[E_3].$$

Can generalize this concept to mutual independence between n events – E_1, E_2, ..., E_n are mutually independent, if for *every* subset of events $S = \{E_i, E_j, \ldots\}$, $\Pr[E_i \cap E_j \cap \ldots] = \Pr[E_i]\ \Pr[E_j] \ldots$.