CMPT 210: Probability and Computing

Lecture 12

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Recap - (Basic) Frievald's Algorithm

- **Q**: For $n \times n$ matrices A, B and D, is $D = AB$?
- Last class, we proved that:

	Yes	No
$D = AB$	1	0
$D \neq AB$	$< \frac{1}{2}$	$\geq \frac{1}{2}$

Table 1: Probabilities for Basic Frievalds Algorithm

Frievald's Algorithm

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If $m = 20$, then Frievald's algorithm will make mistake with probability $1/2^{20} \approx 10^{-6}$.

Computational Complexity: $O(mn^2)$

Probability Amplification

Consider a randomized algorithm $\mathcal{A}$ that is supposed to solve a binary decision problem i.e. it is supposed to answer either Yes or No. It has a one-sided error – (i) if the true answer is Yes, then the algorithm $\mathcal{A}$ correctly outputs Yes with probability 1, but (ii) if the true answer is No, the algorithm $\mathcal{A}$ incorrectly outputs Yes with probability $\leq \frac{1}{2}$.

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Let us define a new algorithm $\mathcal{B}$ that runs algorithm $\mathcal{A}$ m times, and if *any* run of $\mathcal{A}$ outputs No, algorithm $\mathcal{B}$ outputs No. If *all* runs of $\mathcal{A}$ output Yes, algorithm $\mathcal{B}$ outputs Yes.

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Q: What is the probability that algorithm $\mathcal{B}$ correctly outputs Yes if the true answer is Yes, and correctly outputs No if the true answer is No?

Probability Amplification - Analysis

If A_i denotes run i of Algorithm $\mathcal{A}$, then

$$\begin{aligned}
&\Pr[\mathcal{B} \text{ outputs Yes} \mid \text{true answer is Yes}] \\
&= \Pr[\mathcal{A}_1 \text{ outputs Yes } \cap \mathcal{A}_2 \text{ outputs Yes } \cap \ldots \cap \mathcal{A}_m \text{ outputs Yes} \mid \text{true answer is Yes}] \\
&= \prod_{i=1}^{m} \Pr[\mathcal{A}_i \text{ outputs Yes} \mid \text{true answer is Yes}] = 1 \qquad \text{(Independence of runs)}
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&= 1 - \prod_{i=1}^{m} \Pr[\mathcal{A}_i \text{ outputs Yes} \mid \text{true answer is No }] \geq 1 - \frac{1}{2^m}.
\end{aligned}$$

When the true answer is Yes, both $\mathcal{B}$ and $\mathcal{A}$ correctly output Yes. When the true answer is No, $\mathcal{A}$ incorrectly outputs Yes with probability $< \frac{1}{2}$, but $\mathcal{B}$ incorrectly outputs Yes with probability $< \frac{1}{2^m} << \frac{1}{2}$. By repeating the experiment, we have "amplified" the probability of success.

Questions?

Random Variables

Definition: A random "variable" R on a probability space is a total function whose domain is the sample space $\mathcal{S}$. The codomain is usually a subset of the real numbers.

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Example: Suppose we toss three independent, unbiased coins. Let C be the number of heads that appear.

$\mathcal{S} = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

C is a total function that maps each outcome in $\mathcal{S}$ to a number as follows: $C(HHH) = 3$, $C(HHT) = C(HTH) = C(THH) = 2$, $C(HTT) = C(THT) = C(TTH) = 1$, $C(TTT) = 0$.

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Example: I toss a coin, and define the random variable R which is equal to 1 when I get a heads, and equal to 0 when I get a tails.

Bernoulli random variables: Random variables with the codomain $\{0, 1\}$ are called Bernoulli random variables. E.g. R is a Bernoulli r.v.

Back to throwing dice

Q: Suppose we throw two standard dice one after the other. Let us define R to be the random variable equal to the sum of the dice. What is the domain, range of R?

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Q: In the above example, what is $2 \times M((1, 4, 6))$? Is M an invertible function?

Random Variables and Events

Indicator Random Variable: An indicator random variable maps every outcome to either 0 or 1.

Example: Suppose we throw two standard dice, and define M to be the random variable that is 1 iff both throws of the dice produce a prime number, else it is 0.

$M : \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\} \to \{0, 1\}$. $M((2, 3)) = 1$, $M((3, 6)) = 0$.

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The indicator random variable corresponding to an event E is denoted as $\mathcal{I}_E$, meaning that for $\omega \in E$, $\mathcal{I}_E[\omega] = 1$ and for $\omega \notin E$, $\mathcal{I}_E[\omega] = 0$. In the above example, $M = \mathcal{I}_E$ and since $(2,4) \notin E$, $M((2,4)) = 0$ and since $(3,5) \in E$, $M((3,5)) = 1$.

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In general, a random variable that takes on several values partitions $\mathcal{S}$ into several blocks.

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Each block is a subset of the sample space and is therefore an event. For example, $[C = 2]$ is the event that the number of heads is two and consists of the outcomes $\{HHT, HTH, THH\}$.

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Since it is an event, we can compute its probability i.e. $\Pr[C = 2] = \Pr[\{HHT, HTH, THH\}] = \Pr[\{HHT\}] + \Pr[\{HTH\}] + \Pr[\{THH\}]$. Since this is a uniform probability space, $\Pr[\omega] = \frac{1}{8}$ for $\omega \in \mathcal{S}$ and hence $\Pr[C = 2] = \frac{3}{8}$.

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Q: What is $\Pr[C = 0]$, $\Pr[C = 1]$ and $\Pr[C = 3]$?

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Q: What is $\Pr[C = 0]$, $\Pr[C = 1]$ and $\Pr[C = 3]$?

Q: What is $\sum_{i=0}^{3} \Pr[C = i]$?

Since a random variable R is a total function that maps every outcome in $\mathcal{S}$ to some value in the codomain, $\sum_{i \in \text{Range of R}} \Pr[R = i] = \sum_{i \in \text{Range of R}} \sum_{\omega \text{ s.t. } R(\omega)=i} \Pr[\omega] = \sum_{\omega \in \mathcal{S}} \Pr[\omega] = 1$.

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Q: If M is the indicator random variable equal to 1 iff both throws of the dice produces a prime number, what is $\Pr[M = 1]$?

Distribution Functions

Probability density function (PDF): Let R be a random variable with codomain V. The probability density function of R is the function $\text{PDF}_R : V \to [0,1]$, such that $\text{PDF}_R[x] = \Pr[R = x]$ if $x \in \text{Range(R)}$ and equal to zero if $x \notin \text{Range(R)}$.

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Cumulative distribution function (CDF): If the codomain is a subset of the real numbers, then the cumulative distribution function is the function $\text{CDF}_R : \mathbb{R} \to [0,1]$, such that $\text{CDF}_R[x] = \Pr[R \leq x]$.

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Example: If we flip three coins, and C counts the number of heads, then
$\text{PDF}_C[0] = \Pr[C = 0] = \frac{1}{8}$, and
$\text{CDF}_C[2.3] = \Pr[C \leq 2.3] = \Pr[C = 0] + \Pr[C = 1] + \Pr[C = 2] = \frac{7}{8}$.

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For a general random variable R, as $x \to \infty$, $\text{CDF}_R[x] \to 1$ and $x \to -\infty$, $\text{CDF}_R[x] \to 0$.

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Recall that $T : \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\} \to V$ where $V = \{2, 3, 4, \ldots 12\}$.

$\text{PDF}_T : V \to [0, 1]$ and $\text{CDF}_T : \mathbb{R} \to [0, 1]$.

For example, $\text{PDF}_T[4] = \Pr[T = 4] = \frac{3}{36}$ and $\text{PDF}_T[12] = \Pr[T = 12] = \frac{1}{36}$.

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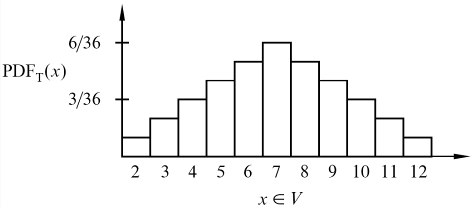

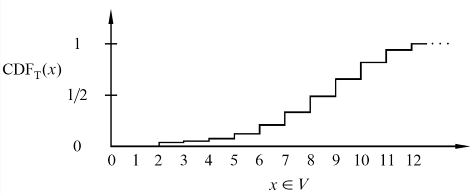

Questions?

Distributions

Many random variables turn out to have the same PDF and CDF. In other words, even though R and T might be different random variables on different probability spaces, it is often the case that $\text{PDF}_R = \text{PDF}_T$. Hence, by studying the properties of such PDFs, we can study different random variables and experiments.

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Common Discrete Distributions in Computer Science:

- Bernoulli Distribution
- Uniform Distribution
- Binomial Distribution
- Geometric Distribution

Bernoulli Distribution

Canonical Example: We toss a biased coin such that the probability of getting a heads is p. Let R be the random variable such that $R = 1$ when the coin comes up heads and $R = 0$ if the coin comes up tails. R follows the Bernoulli distribution.

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PDF$_R$ for Bernoulli distribution: $f\colon \{0, 1\} \to [0, 1]$ meaning that Bernoulli random variables take values in $\{0, 1\}$. It can be fully specified by the "probability of success" (of an experiment) p (probability of getting a heads in the example). Formally, PDF$_R$ is given by:

$$f(1) = p \quad ; \quad f(0) = q := 1 - p.$$

In the example, $\Pr[R = 1] = f(1) = p = \Pr[\text{event that we get a heads}]$.

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CDF$_R$ for Bernoulli distribution: $F : \mathbb{R} \to [0,1]$:

$$\begin{aligned} F(x) &= 0 && \text{(for } x < 0) \\ &= 1 - p && \text{(for } 0 \leq x < 1) \\ &= 1 && \text{(for } x \geq 1) \end{aligned}$$

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Canonical Example: We roll a standard die. Let R be the random variable equal to the number that shows up on the die. R follows the uniform distribution.

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$$\begin{aligned} F(x) &= 0 && \text{(for } x < v_1\text{)} \\ &= {}^k/_n && \text{(for } v_k \leq x < v_{k+1}\text{)} \\ &= 1 && \text{(for } x \geq v_n\text{)} \end{aligned}$$

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Q: If X has a Bernoulli distribution, when is X also uniform?

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$$E_k = (A_1 \cap A_2 \ldots A_k \cap A_{k+1}^c \cap A_{k+2}^c \cap \ldots \cap A_n^c) \cup (A_1^c \cap A_2 \ldots A_k \cap A_{k+1} \cap A_{k+2}^c \cap \ldots \cap A_n^c) \cup \ldots$$

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$$\Pr[E_k] = \Pr[(A_1 \cap A_2 \ldots A_k \cap A^c_{k+1} \cap A^c_{k+2} \cap \ldots \cap A^c_n)] + \Pr[A^c_1 \cap A_2 \ldots A_k \cap A_{k+1} \cap \ldots \cap] + \ldots$$

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$$= \Pr[A_1] \Pr[A_2] \Pr[A_k] \Pr[A_{k+1}^c] \Pr[A_{k+2}^c] \ldots \Pr[A_n^c] + \ldots \qquad \text{(Independence of tosses)}$$

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\Pr[E_k] &= \Pr[(A_1 \cap A_2 \ldots A_k \cap A_{k+1}^c \cap A_{k+2}^c \cap \ldots \cap A_n^c)] + \Pr[A_1^c \cap A_2 \ldots A_k \cap A_{k+1} \cap \ldots \cap] + \ldots \\
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(Number of terms = number of ways to choose the k tosses that result in heads = $\binom{n}{k}$)

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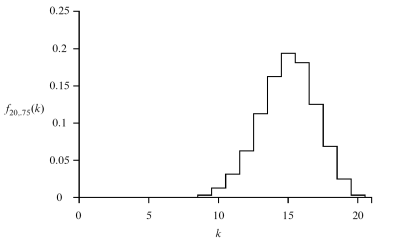

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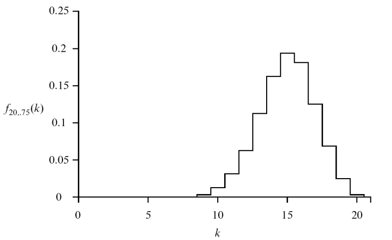

Q: Prove that $\sum_{k \in \text{Range(R)}} \text{PDF}_R[k] = 1$.

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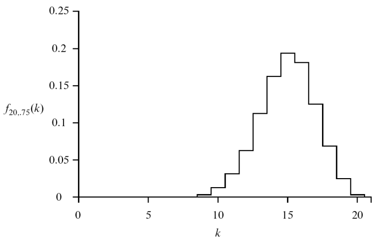

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By the Binomial Theorem, $\sum_{k \in \mathrm{Range(R)}} \mathrm{PDF}_R[k] = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + 1 - p)^n = 1$.

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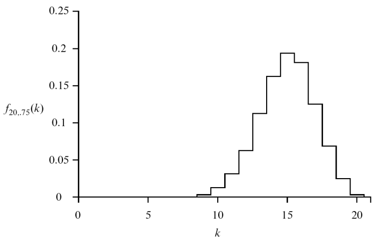

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Canonical Example: We toss a biased coin independently multiple times. The probability of getting a heads is p. Let R be the random variable equal to the number of tosses needed to get the first heads. R follows the geometric distribution.

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By the sum of geometric series, $\sum_{k \in \text{Range}(R)} \text{PDF}_R[k] = \sum_{k=1}^{\infty} (1 - p)^{k-1} p = \frac{p}{1-(1-p)} = 1$.

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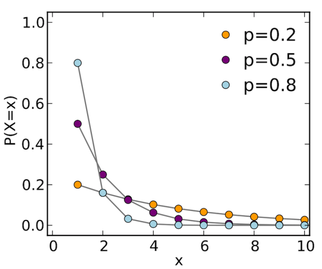

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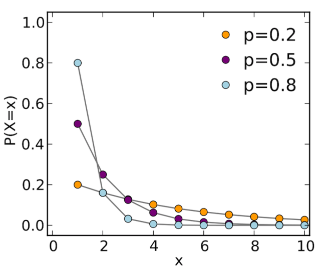

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$$F(x) = 0 \qquad \text{(for } x < 1\text{)}$$

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Questions?