CMPT 210: Probability and Computing

Lecture 5

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Recap - Introduction to Probability

Sample (outcome) space $\mathcal{S}$: Nonempty (countable) set of possible outcomes. Example: When we threw one dice, the sample space is $\{1, 2, 3, 4, 5, 6\}$.

Outcome $\omega \in \mathcal{S}$: Possible "thing" that can happen. Example: When we threw one dice, a possible outcome is $\omega = 1$.

Event E: Any subset of the sample space. Example: When we threw one dice, a possible event is $E = \{6\}$ (first example) or $E = \{3, 6\}$ (second example).

An event E "happens" if the outcome ω (from some process) is in set E i.e. if $\omega \in E$.

Recap - Introduction to Probability

Union: For events E, F, $G = E \cup F = \{\omega | \omega \in E \text{ OR } \omega \in F\}$. G happens when *at least one* of the events E or F happen.

Intersection: For events E, F, $G = E \cap F = \{\omega | \omega \in E \text{ AND } \omega \in F\}$. G happens when *both* events E and F happen.

Mutually exclusive events: If $E \cap F = \{\}$, then events E and F are mutually exclusive.

Complement of an event: The complement of an event E is E^c such that $E \cap E^c = \{\}$ and $E \cup E^c = \mathcal{S}$. Event E^c will occur if and only if event E does not occur.

Subset: If $E \subset F$, then if E happens F will happen.

Axioms of Probability

Probability function on a sample space $\mathcal{S}$ is a total function $\Pr : \mathcal{S} \to [0, 1]$.

For any $\omega \in \mathcal{S}$, $0 \leq Pr[\omega] \leq 1$; $\sum_{\omega \in \mathcal{S}} \Pr[\omega] = 1$

Probability space: The outcome space $\mathcal{S}$ together with the probability function.

Recall that we can define functions on sets. In this case, for an event E, $\Pr[E] = \sum_{\omega \in E} \Pr[\omega]$.

Union: For mutually exclusive events $E_1, E_2, \ldots, E_n$ (sets $E_1, E_2, \ldots, E_n$ are disjoint), $\Pr[E_1 \cup E_2 \cup \ldots E_n] = \Pr[E_1] + \Pr[E_2] + \ldots + \Pr[E_n]$.

Proof :

$$\Pr[E_1 \cup E_2 \cup \ldots E_n] = \sum_{\omega \in \{E_1 \cup E_2 \cup \ldots E_n\}} \Pr[\omega]$$

Since E_i's are disjoint, any ω can only be in one of $E_1, E_2, \ldots E_n$

$$= \sum_{\omega \in E_1} \Pr[\omega] + \sum_{\omega \in E_2} \Pr[\omega] + \ldots + \sum_{\omega \in E_n} \Pr[\omega] = \Pr[E_1] + \Pr[E_2] + \ldots + \Pr[E_n].$$

Back to throwing dice

Q: Suppose we throw a standard dice. What is the probability that the number that comes up is 6?

$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$. Since the dice is "standard", each outcome is equally likely, i.e. $\Pr[\{1\}] = \Pr[\{2\}] = \ldots = \Pr[\{6\}]$.

Since $\Pr[\mathcal{S}] = 1 \implies \sum_{\omega \in \mathcal{S}} \Pr[\omega] = 1 \implies [\Pr[\{1\}] + \Pr[\{2\}] + \ldots \Pr[\{6\}]] = 1$
$\implies \Pr[\{6\}] = \frac{1}{6}$.

Back to throwing dice

Q: Suppose we throw a standard dice. What is the probability that we get either a 3 or a 6?

$E = \{3\}$, $F = \{6\}$, $G = \{3, 6\}$. Since $E \cap F = \{\}$, E and F are mutually exclusive events, implying that $\Pr[G] = \Pr[E] + \Pr[F] = \Pr[\{3\}] + \Pr[\{6\}] = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

Hence, probability of getting either a 3 or a 6 is equal to $\frac{1}{3}$.

Q: Compute the probability of getting either 1, 2 or 3.

Q: Compute the probability of getting an even number.

Q: Compute the probability of getting either 1, 2, 3, 4, 5, 6

Probability Rules

Complement rule: $\Pr[E] = 1 - \Pr[E^c]$.

Proof: Recall that $E \cap E^c = \{\}$ and $E \cup E^c = \mathcal{S}$. Since E and E^c are disjoint,

$$\Pr[E \cup E^c] = \Pr[E] + \Pr[E^c] \implies \Pr[S] = \Pr[E] + \Pr[E^c] \implies \Pr[E^c] \quad = 1 - \Pr[E].$$

Inclusion-Exclusion rule: For any two events E, F, $\Pr[E \cup F] = \Pr[E] + \Pr[F] - \Pr[E \cap F]$.

Proof:

$$\Pr[E \cup F] = \sum_{\omega \in \{E \cup F\}} \Pr[\omega] = \sum_{\omega \in \{E-F\}} \Pr[\omega] + \sum_{\omega \in \{F-E\}} \Pr[\omega] + \sum_{\omega \in \{E \cap F\}} \Pr[\omega]$$

$$\text{(Since disjoint)}$$

$$= \left[\sum_{\omega \in \{E-F\}} \Pr[\omega] + \sum_{\omega \in \{E \cap F\}} \Pr[\omega] \right] + \left[\sum_{\omega \in \{F-E\}} \Pr[\omega] + \sum_{\omega \in \{E \cap F\}} \Pr[\omega] \right] - \sum_{\omega \in \{E \cap F\}} \Pr[\omega]$$

$$= \sum_{\omega \in E} \Pr[\omega] + \sum_{\omega \in F} \Pr[\omega] - \sum_{\omega \in \{E \cap F\}} \Pr[\omega] = \Pr[E] + \Pr[F] - \Pr[E \cap F]$$

Probability Rules

Union Bound: For any two events E, F, $\Pr[E \cup F] \le \Pr[E] + \Pr[F]$.

Proof: By the inclusion-exclusion rule, $\Pr[E \cup F] = \Pr[E] + \Pr[F] - \Pr[E \cap F]$. Since probabilities are non-negative, $\Pr[E \cap F] \ge 0$ and hence, $\Pr[E \cup F] \le \Pr[E] + \Pr[F]$.

Union Bound: For any events $E_1, E_2, E_3, \ldots E_n$,

$$\Pr[E_1 \cup E_2 \cup E_3 \ldots \cup E_n] \le \sum_{i=1}^{n} \Pr[E_i]$$

Monotonicity rule: For events A and B, if $A \subset B$, then $\Pr[A] < \Pr[B]$.

Proof:

$$\Pr[A] = \sum_{\omega \in A} \Pr[\omega] = \sum_{\omega \in B} \Pr[\omega] - \sum_{\omega \in \{B-A\}} \Pr[\omega] \implies \Pr[A] < \Pr[B]$$

(Since probabilities are non-negative.)

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Uniform Probability Spaces

Definition: A probability space is uniform if $Pr[\omega]$ is the same for every outcome $\omega \in \mathcal{S}$.

Since $\sum_{\omega \in \mathcal{S}} Pr[\omega] = 1 \implies Pr[\omega] = \frac{1}{|\mathcal{S}|}$ for all $\omega \in \mathcal{S}$.

Example: For a standard dice, $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$, $Pr[\{1\}] = Pr[\{2\}] = \ldots = Pr[\{6\}] = 1/6$.

$Pr[E] = \sum_{\omega \in E} Pr[\omega] = |E| \, Pr[\omega] = \frac{|E|}{|\mathcal{S}|}$.

Example: For a standard dice, if $E = \{3, 6\}$, then, $Pr[E] = \frac{|E|}{|\mathcal{S}|} = \frac{2}{6} = 1/3$.

Hence, for uniform probability spaces, computing the probability is equivalent to counting the outcomes we "care" about.

Q: Suppose we have a loaded (not "standard") dice such that the probability of getting an even number is twice that of getting an odd number (all even numbers are equally likely, and so are the odd numbers). What is the probability of getting an even number?

Let p be the probability of getting an odd number. Probability of getting an even number $= 2p$.

$\sum_{\omega \in \mathcal{S}} \Pr[\omega] = 1 \implies 3p + 3(2p) = 1 \implies p = \frac{1}{9}$. Hence, probability of getting an odd number $= \frac{1}{9}$. Probability of getting an even number $= \frac{2}{9}$.

Q: What is the probability that the number that comes up is 6?

Q: What is the probability that we get either a 3 or a 6?

Q: What is the probability that we get a prime number

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Questions?

Probability Examples

Q: Suppose we select a card at random from a standard deck of 52 cards. What is the probability of getting:

- A spade
- A spade facecard
- A black card
- The queen of hearts
- An ace

Probability Examples

Q: A class consists of 6 men and 4 women. An exam is given and the students are ranked according to their performance. Assuming that no two students obtain the same scores and all rankings are considered equally likely, what is the probability that women receive the top 4 scores?

In general, let the number of men be m and let the number of women be w.

Number of possible rankings = Number of permutations = $(m + w)!$.

The event of interest is that where the women achieve the top scores. In a possible ranking, let's fix the top w slots for women. The w women can be arranged in $w!$ ways. And the m men can be arranged in $m!$ ways. Hence, total number of rankings where women receive the top scores = $m!\ w!$.

Since all rankings are equally likely, probability that women receive the top w scores = $\frac{m!w!}{(m+w)!}$. In this case, since $m = 6$ and $w = 4$, probability that women receive the top 4 scores = $\frac{6!\ 4!}{10!}$.

Probability Examples

Q: A class consists of m men and w women. An exam is given and the students are ranked according to their performance. Assuming that no two students obtain the same scores and all rankings are considered equally likely, what is probability that women receive the top t ($t \leq w$) scores?

Number of ways to select the t women that have top scores $= \binom{w}{t}$. The top t women can be arranged in $t!$ ways. The number of remaining students is equal to $m + w - t$. These can be arranged in $(m + w - t)!$ ways. Hence, total number of rankings where women receive the top t scores $= \binom{w}{t} \ (m + w - t)! \ t!$.

As before, the total number of rankings $= (m + w)!$. Since all rankings are equally likely, the probability that women receive the top t scores $= \frac{\binom{w}{t}(m+w-t)! \, t!}{(m+w)!} = \frac{w! \, (m+w-t)!}{(w-t)!(m+w)!}$

Probability Examples

Q: A committee of size 5 is to be selected from a group of 6 CS and 9 Math students (no double majors allowed). If the selection is made randomly, what is the probability that the committee consists of 3 CS and 2 Math students?

Number of possible ways of selecting the committee = $|\mathcal{S}| = \binom{15}{5}$.

The event of interest (E) requires choosing 3 CS and 2 Math students. Number of ways we can select the CS students = $\binom{6}{3}$. Similarly, number of ways we can select the Math students = $\binom{9}{2}$.

Hence, $|E| = \binom{6}{3}\binom{9}{2} \implies \Pr[E] = \frac{|E|}{|\mathcal{S}|} = \frac{\binom{6}{3}\binom{9}{2}}{\binom{15}{5}}$.

Q: From a set of n items a random sample of size k is to be selected. What is the probability a given item (α) will be among the k selected items?

Number of ways of choosing the sample $= \binom{n}{k}$.

If we want a particular item in the sample, number of ways of choosing the other items $= \binom{n-1}{k-1}$.

Hence, probability that a given item will be among the k selected $= \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}$.

Probability Examples

Q: From a set of n items a random sample of size k is to be selected. Given two items of interest: α and β, what is the probability that (i) both α and β will be among the k selected (ii) at least one of α or β will be among the k selected (iii) neither α nor β will be among the k selected?

(i) If we want both α and β to be in the sample, number of ways of choosing the other items $= \binom{n-2}{k-2}$. Hence, probability that both α and β will be in the sample $= \frac{\binom{n-2}{k-2}}{\binom{n}{k}} = \frac{k(k-1)}{n(n-1)}$.

(ii) Let A be the event that item α is in the selection. $\Pr[A] = \frac{k}{n}$. Similarly B be the event that item β is in the selection. $\Pr[B] = \frac{k}{n}$. We want to compute $\Pr[A \cup B]$. By the union-rule, $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$. Hence, probability that either α or β will be among the k selected items $= \frac{2k}{n} - \frac{k(k-1)}{n(n-1)}$.

(iii) If we want neither α nor β to be in the sample, number of ways of choosing the items $= \binom{n-2}{k}$. Hence, probability that neither α nor β will be in the sample $= \frac{\binom{n-2}{k}}{\binom{n}{k}} = \frac{(n-k)(n-k-1)}{n(n-1)}$.

Q: Let us consider random permutations of the letters (i) ABBA (ii) ABBA'. What is the probability that the third letter is B?

Questions?