CMPT 210: Probability and Computing

Lecture 19

Sharan Vaswani

March 21, 2024

Recap

Variance: Standard way to measure the deviation from the mean. For r.v. X, $\text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \sum_{x \in \text{Range}(X)} (x - \mu)^2 \Pr[X = x]$, where $\mu := \mathbb{E}[X]$.

Alternate Definition: $\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.

If $X \sim \text{Ber}(p)$, $\text{Var}[X] = p(1-p)$.

If $X \sim \text{Uniform}(\{v_1, v_2, \ldots v_n\})$, $\text{Var}[X] = \frac{[v_1^2 + v_2^2 + \ldots v_n^2]}{n} - \left(\frac{[v_1 + v_2 + \ldots v_n]}{n}\right)^2$.

Variance - Examples

Q: If $R \sim \text{Geo}(p)$, calculate $\text{Var}[R]$.

Variance - Examples

Q: If $R \sim \text{Geo}(p)$, calculate $\text{Var}[R]$.

$$\text{Var}[R] = \mathbb{E}[R^2] - (\mathbb{E}[R])^2 = \mathbb{E}[R^2] - \frac{1}{p^2}$$

Variance - Examples

Q: If $R \sim \text{Geo}(p)$, calculate $\text{Var}[R]$.

$$\text{Var}[R] = \mathbb{E}[R^2] - (\mathbb{E}[R])^2 = \mathbb{E}[R^2] - \frac{1}{p^2}$$

Recall that for a coin s.t. $\Pr[\text{heads}] = p$, R is the r.v. equal to the number of coin tosses we need to get the first heads. Let A be the event that we get a heads in the first toss. Using the law of total expectation,

$$\mathbb{E}[R^2] = \mathbb{E}[R^2|A] \Pr[A] + \mathbb{E}[R^2|A^c] \Pr[A^c]$$

$\mathbb{E}[R^2|A] = 1$ ($R^2 = 1$ if we get a heads in the first coin toss) and $\Pr[A] = p$. Hence,

$$\mathbb{E}[R^2] = (1)\,(p) + \mathbb{E}[R^2|A^c]\,(1-p) \quad ; \quad \mathbb{E}[R^2|A^c] = \sum_{k=1} k^2 \Pr[R = k|A^c]$$

Variance - Examples

Q: If $R \sim \text{Geo}(p)$, calculate $\text{Var}[R]$.

$$\text{Var}[R] = \mathbb{E}[R^2] - (\mathbb{E}[R])^2 = \mathbb{E}[R^2] - \frac{1}{p^2}$$

Recall that for a coin s.t. $\Pr[\text{heads}] = p$, R is the r.v. equal to the number of coin tosses we need to get the first heads. Let A be the event that we get a heads in the first toss. Using the law of total expectation,

$$\mathbb{E}[R^2] = \mathbb{E}[R^2|A] \Pr[A] + \mathbb{E}[R^2|A^c] \Pr[A^c]$$

$\mathbb{E}[R^2|A] = 1$ ($R^2 = 1$ if we get a heads in the first coin toss) and $\Pr[A] = p$. Hence,

$$\mathbb{E}[R^2] = (1)\,(p) + \mathbb{E}[R^2|A^c]\,(1-p) \quad ; \quad \mathbb{E}[R^2|A^c] = \sum_{k=1} k^2 \Pr[R = k|A^c]$$

Note that $\Pr[R = k|A^c] = \Pr[R = k| \text{ first toss is a tails}] = (1-p)^{k-2}\, p = \Pr[R = k-1]$

$$\implies \mathbb{E}[R^2|A^c] = \sum_{k=1} k^2 \Pr[R = k-1] = \sum_{t=0} (t+1)^2 \Pr[R = t] \qquad (t := k-1)$$

Variance - Examples

Continuing from the previous slide,

$$\begin{aligned}\mathbb{E}[R^2|A^c] &= \sum_{t=0}(t+1)^2 \Pr[R=t] = \sum_{t=0} t^2 \Pr[R=t] + 2\sum_{t=0} t \Pr[R=t] + \sum_{t=0} \Pr[R=t] \\ &= \sum_{t=1} t^2 \Pr[R=t] + 2\sum_{t=1} t \Pr[R=t] + \sum_{t=1} \Pr[R=t] = \mathbb{E}[R^2] + 2\mathbb{E}[R] + 1\end{aligned}$$

Variance - Examples

Continuing from the previous slide,

$$\begin{aligned}\mathbb{E}[R^2|A^c] &= \sum_{t=0}(t+1)^2 \Pr[R=t] = \sum_{t=0} t^2 \Pr[R=t] + 2\sum_{t=0} t \Pr[R=t] + \sum_{t=0} \Pr[R=t] \\ &= \sum_{t=1} t^2 \Pr[R=t] + 2\sum_{t=1} t \Pr[R=t] + \sum_{t=1} \Pr[R=t] = \mathbb{E}[R^2] + 2\mathbb{E}[R] + 1\end{aligned}$$

Putting everything together,

$$\mathbb{E}[R^2] = (1)\,(p) + (\mathbb{E}[R^2] + 2\mathbb{E}[R] + 1])\,(1-p) \implies p\,\mathbb{E}[R^2] = p + 2(1-p)\mathbb{E}[R] + (1-p)\mathbb{E}[1]$$

$$\implies p\,\mathbb{E}[R^2] = p + \frac{2(1-p)}{p} + (1-p) \qquad (\mathbb{E}[R] = \tfrac{1}{p},\ \mathbb{E}[1] = 1)$$

Variance - Examples

Continuing from the previous slide,

$$\mathbb{E}[R^2|A^c] = \sum_{t=0}(t+1)^2 \Pr[R=t] = \sum_{t=0} t^2 \Pr[R=t] + 2\sum_{t=0} t \Pr[R=t] + \sum_{t=0} \Pr[R=t]$$

$$= \sum_{t=1} t^2 \Pr[R=t] + 2\sum_{t=1} t \Pr[R=t] + \sum_{t=1} \Pr[R=t] = \mathbb{E}[R^2] + 2\mathbb{E}[R] + 1$$

Putting everything together,

$$\mathbb{E}[R^2] = (1)(p) + (\mathbb{E}[R^2] + 2\mathbb{E}[R] + 1])(1-p) \implies p\,\mathbb{E}[R^2] = p + 2(1-p)\mathbb{E}[R] + (1-p)\mathbb{E}[1]$$

$$\implies p\,\mathbb{E}[R^2] = p + \frac{2(1-p)}{p} + (1-p) \qquad (\mathbb{E}[R] = \tfrac{1}{p},\ \mathbb{E}[1] = 1)$$

$$\implies \mathbb{E}[R^2] = \frac{2(1-p)}{p^2} + \frac{1}{p} \implies \mathbb{E}[R^2] = \frac{2-p}{p^2}$$

$$\implies \mathrm{Var}[R] = \mathbb{E}[R^2] - (\mathbb{E}[R])^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

Variance - Examples

Continuing from the previous slide,

$$\mathbb{E}[R^2|A^c] = \sum_{t=0}(t+1)^2 \Pr[R=t] = \sum_{t=0} t^2 \Pr[R=t] + 2\sum_{t=0} t \Pr[R=t] + \sum_{t=0} \Pr[R=t]$$
$$= \sum_{t=1} t^2 \Pr[R=t] + 2\sum_{t=1} t \Pr[R=t] + \sum_{t=1} \Pr[R=t] = \mathbb{E}[R^2] + 2\mathbb{E}[R] + 1$$

Putting everything together,

$$\mathbb{E}[R^2] = (1)\,(p) + (\mathbb{E}[R^2] + 2\mathbb{E}[R] + 1])\,(1-p) \implies p\,\mathbb{E}[R^2] = p + 2(1-p)\mathbb{E}[R] + (1-p)\mathbb{E}[1]$$
$$\implies p\,\mathbb{E}[R^2] = p + \frac{2(1-p)}{p} + (1-p) \qquad (\mathbb{E}[R] = \tfrac{1}{p},\ \mathbb{E}[1] = 1)$$
$$\implies \mathbb{E}[R^2] = \frac{2(1-p)}{p^2} + \frac{1}{p} \implies \mathbb{E}[R^2] = \frac{2-p}{p^2}$$
$$\implies \mathrm{Var}[R] = \mathbb{E}[R^2] - (\mathbb{E}[R])^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

Standard Deviation

Standard Deviation: For r.v. X, the standard deviation in X is defined as:

$$\sigma_X := \sqrt{\mathrm{Var}[X]} = \sqrt{\mathbb{E}[X^2] - (\mathbb{E}[X])^2}$$

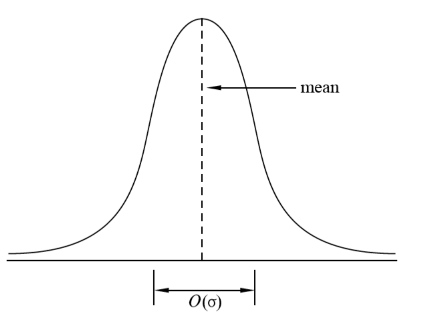
Standard deviation has the same units as expectation.

Standard Deviation

Standard Deviation: For r.v. X, the standard deviation in X is defined as:

$$\sigma_X := \sqrt{\mathrm{Var}[X]} = \sqrt{\mathbb{E}[X^2] - (\mathbb{E}[X])^2}$$

Standard deviation has the same units as expectation.



Standard deviation for a "bell"-shaped distribution indicates how wide the "main part" of the distribution is.

Properties of Variance

Q: For constants a, b and r.v. R, $\text{Var}[a\,R + b] = a^2\text{Var}[R]$.

Properties of Variance

Q: For constants a, b and r.v. R, $\mathrm{Var}[a\,R + b] = a^2\mathrm{Var}[R]$.

Proof:

$$
\begin{aligned}
\mathrm{Var}[aR+b] &= \mathbb{E}[(aR+b)^2] - (\mathbb{E}[aR+b])^2 = \mathbb{E}[a^2R^2 + 2abR + b^2] - (\mathbb{E}[aR] + \mathbb{E}[b])^2 \\
&= (a^2\mathbb{E}[R^2] + 2ab\mathbb{E}[R] + b^2) - (a\mathbb{E}[R] + b)^2 \\
&= (a^2\mathbb{E}[R^2] + 2ab\mathbb{E}[R] + b^2) - (a^2(\mathbb{E}[R])^2 + 2ab\mathbb{E}[R] + b^2) \\
&= a^2\left[\mathbb{E}[R^2] - (\mathbb{E}[R])^2\right] \\
\implies \mathrm{Var}[aR+b] &= a^2\mathrm{Var}[R]
\end{aligned}
$$

Properties of Variance

Q: For constants a, b and r.v. R, $\text{Var}[a\,R+b] = a^2\text{Var}[R]$.

Proof:

$$\begin{aligned}
\text{Var}[aR+b] &= \mathbb{E}[(aR+b)^2] - (\mathbb{E}[aR+b])^2 = \mathbb{E}[a^2R^2+2abR+b^2] - (\mathbb{E}[aR]+\mathbb{E}[b])^2 \\
&= (a^2\mathbb{E}[R^2]+2ab\mathbb{E}[R]+b^2) - (a\mathbb{E}[R]+b)^2 \\
&= (a^2\mathbb{E}[R^2]+2ab\mathbb{E}[R]+b^2) - (a^2(\mathbb{E}[R])^2+2ab\mathbb{E}[R]+b^2) \\
&= a^2\left[\mathbb{E}[R^2]-(\mathbb{E}[R])^2\right]
\end{aligned}$$

$$\implies \text{Var}[aR+b] = a^2\text{Var}[R]$$

Similarly, for the standard deviation,

$$\sigma_{aR+b} = \sqrt{\text{Var}[aR+b]} = \sqrt{a^2\text{Var}[R]} = |a|\,\sigma_R$$

Properties of Variance

Q: For constants a, b and r.v. R, $\text{Var}[a\,R + b] = a^2\text{Var}[R]$.

Proof:

$$\begin{aligned}
\text{Var}[aR+b] &= \mathbb{E}[(aR+b)^2] - (\mathbb{E}[aR+b])^2 = \mathbb{E}[a^2R^2 + 2abR + b^2] - (\mathbb{E}[aR] + \mathbb{E}[b])^2 \\
&= (a^2\mathbb{E}[R^2] + 2ab\mathbb{E}[R] + b^2) - (a\mathbb{E}[R] + b)^2 \\
&= (a^2\mathbb{E}[R^2] + 2ab\mathbb{E}[R] + b^2) - (a^2(\mathbb{E}[R])^2 + 2ab\mathbb{E}[R] + b^2) \\
&= a^2\left[\mathbb{E}[R^2] - (\mathbb{E}[R])^2\right]
\end{aligned}$$

$$\implies \text{Var}[aR+b] = a^2\text{Var}[R]$$

Similarly, for the standard deviation,

$$\sigma_{aR+b} = \sqrt{\text{Var}[aR+b]} = \sqrt{a^2\text{Var}[R]} = |a|\,\sigma_R$$

Note the difference from the property of expectation,

$$\mathbb{E}[aR+b] = a\mathbb{E}[R] + b$$

Properties of Variance

Recall that for r.v's R and S, $\mathbb{E}[R+S] = \mathbb{E}[R] + \mathbb{E}[S]$. In general, such a property is not true for the variance, i.e. variance of a sum is not necessarily equal to the sum of the variances.

If the r.v's R and S are *independent*, $\text{Var}[R+S] = \text{Var}[R] + \text{Var}[S]$.

Properties of Variance

Recall that for r.v's R and S, $\mathbb{E}[R+S] = \mathbb{E}[R] + \mathbb{E}[S]$. In general, such a property is not true for the variance, i.e. variance of a sum is not necessarily equal to the sum of the variances.

If the r.v's R and S are *independent*, $\mathrm{Var}[R+S] = \mathrm{Var}[R] + \mathrm{Var}[S]$.

Properties of Variance

Recall that for r.v's R and S, $\mathbb{E}[R+S] = \mathbb{E}[R] + \mathbb{E}[S]$. In general, such a property is not true for the variance, i.e. variance of a sum is not necessarily equal to the sum of the variances.

If the r.v's R and S are *independent*, $\mathrm{Var}[R+S] = \mathrm{Var}[R] + \mathrm{Var}[S]$.

Proof:

$$\begin{aligned}\mathrm{Var}[R+S] &= \mathbb{E}[(R+S)^2] - (\mathbb{E}[R+S])^2 = \mathbb{E}[R^2 + S^2 + 2RS] - (\mathbb{E}[R] + \mathbb{E}[S])^2 \\ &= \mathbb{E}[R^2 + S^2 + 2RS] - [(\mathbb{E}[R])^2 + (\mathbb{E}[S])^2 + 2\mathbb{E}[R]\,\mathbb{E}[S]]\end{aligned}$$

Properties of Variance

Recall that for r.v's R and S, $\mathbb{E}[R+S] = \mathbb{E}[R] + \mathbb{E}[S]$. In general, such a property is not true for the variance, i.e. variance of a sum is not necessarily equal to the sum of the variances.

If the r.v's R and S are *independent*, $\mathrm{Var}[R+S] = \mathrm{Var}[R] + \mathrm{Var}[S]$.

Proof:

$$\begin{aligned}
\mathrm{Var}[R+S] &= \mathbb{E}[(R+S)^2] - (\mathbb{E}[R+S])^2 = \mathbb{E}[R^2 + S^2 + 2RS] - (\mathbb{E}[R] + \mathbb{E}[S])^2 \\
&= \mathbb{E}[R^2 + S^2 + 2RS] - [(\mathbb{E}[R])^2 + (\mathbb{E}[S])^2 + 2\mathbb{E}[R]\,\mathbb{E}[S]] \\
&= [\mathbb{E}[R^2] - (\mathbb{E}[R])^2] + [\mathbb{E}[S^2] - (\mathbb{E}[S])^2] + 2(\mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]) \\
&= \mathrm{Var}[R] + \mathrm{Var}[S] + 2(\mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S])
\end{aligned}$$

Properties of Variance

Recall that for r.v's R and S, $\mathbb{E}[R+S] = \mathbb{E}[R] + \mathbb{E}[S]$. In general, such a property is not true for the variance, i.e. variance of a sum is not necessarily equal to the sum of the variances.

If the r.v's R and S are *independent*, $\text{Var}[R+S] = \text{Var}[R] + \text{Var}[S]$.

Proof:

$$\begin{aligned}
\text{Var}[R+S] &= \mathbb{E}[(R+S)^2] - (\mathbb{E}[R+S])^2 = \mathbb{E}[R^2 + S^2 + 2RS] - (\mathbb{E}[R] + \mathbb{E}[S])^2 \\
&= \mathbb{E}[R^2 + S^2 + 2RS] - [(\mathbb{E}[R])^2 + (\mathbb{E}[S])^2 + 2\mathbb{E}[R]\,\mathbb{E}[S]] \\
&= [\mathbb{E}[R^2] - (\mathbb{E}[R])^2] + [\mathbb{E}[S^2] - (\mathbb{E}[S])^2] + 2(\mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]) \\
&= \text{Var}[R] + \text{Var}[S] + 2(\mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S])
\end{aligned}$$

Recall that if r.v. are independent, $\mathbb{E}[RS] = \mathbb{E}[R]\,\mathbb{E}[S]$,

$$\implies \text{Var}[R+S] = \text{Var}[R] + \text{Var}[S]$$

Properties of Variance

Recall that for r.v's R and S, $\mathbb{E}[R+S] = \mathbb{E}[R] + \mathbb{E}[S]$. In general, such a property is not true for the variance, i.e. variance of a sum is not necessarily equal to the sum of the variances.

If the r.v's R and S are *independent*, $\mathrm{Var}[R+S] = \mathrm{Var}[R] + \mathrm{Var}[S]$.

Proof:

$$\begin{aligned}
\mathrm{Var}[R+S] &= \mathbb{E}[(R+S)^2] - (\mathbb{E}[R+S])^2 = \mathbb{E}[R^2 + S^2 + 2RS] - (\mathbb{E}[R] + \mathbb{E}[S])^2 \\
&= \mathbb{E}[R^2 + S^2 + 2RS] - [(\mathbb{E}[R])^2 + (\mathbb{E}[S])^2 + 2\mathbb{E}[R]\,\mathbb{E}[S]] \\
&= [\mathbb{E}[R^2] - (\mathbb{E}[R])^2] + [\mathbb{E}[S^2] - (\mathbb{E}[S])^2] + 2(\mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]) \\
&= \mathrm{Var}[R] + \mathrm{Var}[S] + 2(\mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S])
\end{aligned}$$

Recall that if r.v. are independent, $\mathbb{E}[RS] = \mathbb{E}[R]\,\mathbb{E}[S]$,

$$\implies \mathrm{Var}[R+S] = \mathrm{Var}[R] + \mathrm{Var}[S]$$

Properties of Variance

Pairwise Independence: Random variables $R_1, R_2, R_3, \ldots R_n$ are *pairwise* independent if for any pair R_i and R_j, for $x \in \text{Range}(R_i)$ and $y \in \text{Range}(R_j)$, events $\Pr[R_i = x]$ and $\Pr[R_j = y]$ are pairwise independent implying that $\Pr[(R_i = x) \cap (R_j = y)] = \Pr[R_i = x] \Pr[R_j = y]$.

Properties of Variance

Pairwise Independence: Random variables $R_1, R_2, R_3, \ldots R_n$ are *pairwise* independent if for any pair R_i and R_j, for $x \in \text{Range}(R_i)$ and $y \in \text{Range}(R_j)$, events $\Pr[R_i = x]$ and $\Pr[R_j = y]$ are pairwise independent implying that $\Pr[(R_i = x) \cap (R_j = y)] = \Pr[R_i = x] \Pr[R_j = y]$.

We can prove that for any pair of pairwise independent r.v's, R_i and R_j, $\mathbb{E}[R_i R_j] = \mathbb{E}[R_i] \, \mathbb{E}[R_j]$.

Properties of Variance

Pairwise Independence: Random variables $R_1, R_2, R_3, \ldots R_n$ are *pairwise* independent if for any pair R_i and R_j, for $x \in \text{Range}(R_i)$ and $y \in \text{Range}(R_j)$, events $\Pr[R_i = x]$ and $\Pr[R_j = y]$ are pairwise independent implying that $\Pr[(R_i = x) \cap (R_j = y)] = \Pr[R_i = x] \Pr[R_j = y]$.

We can prove that for any pair of pairwise independent r.v's, R_i and R_j, $\mathbb{E}[R_i R_j] = \mathbb{E}[R_i] \, \mathbb{E}[R_j]$.

For pairwise independent random variables $R_1, R_2, R_3, \ldots R_n$, $\text{Var}[\sum_{i=1}^n R_i] = \sum_{i=1}^n \text{Var}[R_i]$.

Properties of Variance

Pairwise Independence: Random variables $R_1, R_2, R_3, \dots R_n$ are *pairwise* independent if for any pair R_i and R_j, for $x \in \text{Range}(R_i)$ and $y \in \text{Range}(R_j)$, events $\Pr[R_i = x]$ and $\Pr[R_j = y]$ are pairwise independent implying that $\Pr[(R_i = x) \cap (R_j = y)] = \Pr[R_i = x] \Pr[R_j = y]$.

We can prove that for any pair of pairwise independent r.v's, R_i and R_j, $\mathbb{E}[R_i R_j] = \mathbb{E}[R_i]\,\mathbb{E}[R_j]$.

For pairwise independent random variables $R_1, R_2, R_3, \dots R_n$, $\text{Var}[\sum_{i=1}^n R_i] = \sum_{i=1}^n \text{Var}[R_i]$.

$$\begin{aligned}
\textit{Proof}: \text{Var}[R_1 + R_2 + \dots R_n] &= \mathbb{E}[(R_1 + R_2 + \dots R_n)^2] - (\mathbb{E}[R_1 + R_2 + \dots R_n])^2 \\
&= \sum_{i=1}^{n} [\mathbb{E}[R_i^2] - (\mathbb{E}[R_i])^2] + 2 \sum_{i,j \mid 1 \le i < j \le n} [\mathbb{E}[R_i R_j] - \mathbb{E}[R_i]\,\mathbb{E}[R_j]] \\
\implies \text{Var}[R_1 + R_2 + \dots R_n] &= \sum_{i=1}^{n} \text{Var}[R_i] \qquad \text{(Since the r.v's are pairwise independent)}
\end{aligned}$$

Properties of Variance

Pairwise Independence: Random variables $R_1, R_2, R_3, \ldots R_n$ are *pairwise* independent if for any pair R_i and R_j, for $x \in \text{Range}(R_i)$ and $y \in \text{Range}(R_j)$, events $\Pr[R_i = x]$ and $\Pr[R_j = y]$ are pairwise independent implying that $\Pr[(R_i = x) \cap (R_j = y)] = \Pr[R_i = x]\Pr[R_j = y]$.

We can prove that for any pair of pairwise independent r.v's, R_i and R_j, $\mathbb{E}[R_i R_j] = \mathbb{E}[R_i]\,\mathbb{E}[R_j]$.

For pairwise independent random variables $R_1, R_2, R_3, \ldots R_n$, $\text{Var}[\sum_{i=1}^{n} R_i] = \sum_{i=1}^{n} \text{Var}[R_i]$.

$$\begin{aligned} \textit{Proof}: \text{Var}[R_1 + R_2 + \ldots R_n] &= \mathbb{E}[(R_1 + R_2 + \ldots R_n)^2] - (\mathbb{E}[R_1 + R_2 + \ldots R_n])^2 \\ &= \sum_{i=1}^{n}[\mathbb{E}[R_i^2] - (\mathbb{E}[R_i])^2] + 2\sum_{i,j \mid 1 \le i < j \le n}[\mathbb{E}[R_i R_j] - \mathbb{E}[R_i]\,\mathbb{E}[R_j]] \\ \implies \text{Var}[R_1 + R_2 + \ldots R_n] &= \sum_{i=1}^{n} \text{Var}[R_i] \qquad \text{(Since the r.v's are pairwise independent)} \end{aligned}$$

Importantly, we do not require the r.v's to be mutually independent. Mutual independence $\implies$ pairwise independence, but pairwise independence $\not\Rightarrow$ mutual independence.

Variance - Examples

Q: If $R \sim \mathrm{Bin}(n, p)$, calculate $\mathrm{Var}[R]$.

Variance - Examples

Q: If $R \sim \text{Bin}(n, p)$, calculate $\text{Var}[R]$.

Define R_i to be the indicator random variable that we get a heads in toss i of the coin. Recall that R is the random variable equal to the number of heads in n tosses.

Variance - Examples

Q: If $R \sim \text{Bin}(n, p)$, calculate $\text{Var}[R]$.

Define R_i to be the indicator random variable that we get a heads in toss i of the coin. Recall that R is the random variable equal to the number of heads in n tosses.

Hence,

$$R = R_1 + R_2 + \ldots + R_n \implies \text{Var}[R] = \text{Var}[R_1 + R_2 + \ldots + R_n]$$

Since $R_1, R_2, \ldots, R_n$ are mutually independent indicator random variables,

$$\text{Var}[R] = \text{Var}[R_1] + \text{Var}[R_2] + \ldots + \text{Var}[R_n]$$

Since the variance of an indicator (Bernoulli) r.v. is $p(1-p)$,

$$\text{Var}[R] = n\,p\,(1-p).$$

Questions?

Matching Birthdays

Q: In a class of n students, what is the probability that two students share the same birthday? Assume that (i) each student is equally likely to be born on any day of the year, (ii) no leap years and (iii) student birthdays are independent of each other.

Matching Birthdays

Q: In a class of n students, what is the probability that two students share the same birthday? Assume that (i) each student is equally likely to be born on any day of the year, (ii) no leap years and (iii) student birthdays are independent of each other.

For $d := 365$ (since no leap years),

$$\Pr[\text{two students share the same birthday}] = 1 - \frac{d \times (d-1) \times (d-2) \times \ldots (d-(n-1))}{d^n}$$

Matching Birthdays

Q: In a class of n students, what is the probability that two students share the same birthday? Assume that (i) each student is equally likely to be born on any day of the year, (ii) no leap years and (iii) student birthdays are independent of each other.

For $d := 365$ (since no leap years),

$$\Pr[\text{two students share the same birthday}] = 1 - \frac{d \times (d-1) \times (d-2) \times \ldots (d-(n-1))}{d^n}$$

Q: On average, how many pairs of students have matching birthdays?

Matching Birthdays

Q: In a class of n students, what is the probability that two students share the same birthday? Assume that (i) each student is equally likely to be born on any day of the year, (ii) no leap years and (iii) student birthdays are independent of each other.

For $d := 365$ (since no leap years),

$$\Pr[\text{two students share the same birthday}] = 1 - \frac{d \times (d-1) \times (d-2) \times \ldots (d-(n-1))}{d^n}$$

Q: On average, how many pairs of students have matching birthdays?

Define M to be the number of pairs of students with matching birthdays. For a fixed ordering of the students, let $X_{i,j}$ be the indicator r.v. corresponding to the event $E_{i,j}$ that the birthdays of students i and j match. Hence,

$$M = \sum_{i,j|1 \leq i < j \leq n} X_{i,j} \implies \mathbb{E}[M] = \mathbb{E}[\sum_{i,j|1 \leq i < j \leq n} X_{i,j}] = \sum_{i,j|1 \leq i < j \leq n} \mathbb{E}[X_{i,j}] = \sum_{i,j|1 \leq i < j \leq n} \Pr[E_{i,j}]$$

(Linearity of expectation)

Matching Birthdays

For a pair of students i, j, let B_i be the r.v. equal to the day of student i's birthday. Range(B_i) = {1,2,. . . , d}. For all $k \in [d]$, $\Pr[B_i = k] = 1/d$ (each student is equally likely to be born on any day of the year).

Matching Birthdays

For a pair of students i, j, let B_i be the r.v. equal to the day of student i's birthday. Range(B_i) = {1,2,. . . , d}. For all $k \in [d]$, $\Pr[B_i = k] = 1/d$ (each student is equally likely to be born on any day of the year).

$$E_{i,j} = (B_i = 1 \cap B_j = 1) \cup (B_i = 2 \cap B_j = 2) \cup \ldots$$

$$\implies \Pr[E_{i,j}] = \sum_{k=1}^{d} \Pr[B_i = k \cap B_j = k] = \sum_{k=1}^{d} \Pr[B_i = k] \Pr[B_j = k] = \sum_{k=1}^{d} \frac{1}{d^2} = \frac{1}{d}$$

(student birthdays are independent of each other)

$$\implies \mathbb{E}[M] = \sum_{i,j|1 \le i < j \le n} \Pr[E_{i,j}] = \frac{1}{d} \sum_{i,j|1 \le i < j \le n} (1) = \frac{1}{d}[(n-1) + (n-2) + \ldots + 1] = \frac{n(n-1)}{2d}$$

Matching Birthdays

For a pair of students i, j, let B_i be the r.v. equal to the day of student i's birthday. Range(B_i) = {1,2,. . . , d}. For all $k \in [d]$, $\Pr[B_i = k] = 1/d$ (each student is equally likely to be born on any day of the year).

$$E_{i,j} = (B_i = 1 \cap B_j = 1) \cup (B_i = 2 \cap B_j = 2) \cup \dots$$

$$\implies \Pr[E_{i,j}] = \sum_{k=1}^{d} \Pr[B_i = k \cap B_j = k] = \sum_{k=1}^{d} \Pr[B_i = k] \Pr[B_j = k] = \sum_{k=1}^{d} \frac{1}{d^2} = \frac{1}{d}$$

(student birthdays are independent of each other)

$$\implies \mathbb{E}[M] = \sum_{i,j|1 \le i < j \le n} \Pr[E_{i,j}] = \frac{1}{d} \sum_{i,j|1 \le i < j \le n} (1) = \frac{1}{d}[(n-1) + (n-2) + \dots + 1] = \frac{n(n-1)}{2d}$$

Hence, in our class of 75 students, on average, there are $\frac{(21)(41)}{365} = 7.60$ students with matching birthdays.

Matching Birthdays

Q: Are the $X_{i,j}$ r.v's mutually independent?

Matching Birthdays

Q: Are the $X_{i,j}$ r.v's mutually independent?

No, because if $X_{i,j} = 1$ and $X_{j,k} = 1$, then,
$\Pr[X_{i,k} = 1 | X_{j,k} = 1 \cap X_{i,j} = 1] = 1 \neq \frac{1}{d} = \Pr[X_{i,k} = 1]$.

Matching Birthdays

Q: Are the $X_{i,j}$ r.v's mutually independent?

No, because if $X_{i,j} = 1$ and $X_{j,k} = 1$, then,
$\Pr[X_{i,k} = 1 | X_{j,k} = 1 \cap X_{i,j} = 1] = 1 \neq \frac{1}{d} = \Pr[X_{i,k} = 1]$.

Q: Are the $X_{i,j}$ pairwise independent?

Matching Birthdays

Q: Are the $X_{i,j}$ r.v's mutually independent?

No, because if $X_{i,j} = 1$ and $X_{j,k} = 1$, then,
$\Pr[X_{i,k} = 1 | X_{j,k} = 1 \cap X_{i,j} = 1] = 1 \neq \frac{1}{d} = \Pr[X_{i,k} = 1]$.

Q: Are the $X_{i,j}$ pairwise independent?

Yes, because for all i, j and i', j' (where $i \neq i'$), $\Pr[X_{i,j} = 1 | X_{i',j'} = 1] = \Pr[X_{i,j} = 1]$ because if students i' and j' have matching birthdays, it does not tell us anything about whether i and j have matching birthdays.

Matching Birthdays

Q: If M is the random variable equal to the number of pairs of students with matching birthdays, calculate $\mathrm{Var}[M]$.

Matching Birthdays

Q: If M is the random variable equal to the number of pairs of students with matching birthdays, calculate $\text{Var}[M]$.

$$\text{Var}[M] = \text{Var}[\sum_{i,j|1 \leq i < j \leq n} X_{i,j}]$$

Since $X_{i,j}$ are pairwise independent, the variance of the sum is equal to the sum of the variance.

$$\implies \text{Var}[M] = \sum_{i,j|1 \leq i < j \leq n} \text{Var}[X_{i,j}] = \sum_{i,j|1 \leq i < j \leq n} \frac{1}{d}\left(1 - \frac{1}{d}\right) = \frac{1}{d}\left(1 - \frac{1}{d}\right)\frac{n(n-1)}{2}$$

(Since $X_{i,j}$ is an indicator (Bernoulli) r.v. and $\Pr[X_{i,j} = 1] = \frac{1}{d}$)

Matching Birthdays

Q: If M is the random variable equal to the number of pairs of students with matching birthdays, calculate $\mathrm{Var}[M]$.

$$\mathrm{Var}[M] = \mathrm{Var}[\sum_{i,j|1 \le i < j \le n} X_{i,j}]$$

Since $X_{i,j}$ are pairwise independent, the variance of the sum is equal to the sum of the variance.

$$\implies \mathrm{Var}[M] = \sum_{i,j|1 \le i < j \le n} \mathrm{Var}[X_{i,j}] = \sum_{i,j|1 \le i < j \le n} \frac{1}{d}\left(1 - \frac{1}{d}\right) = \frac{1}{d}\left(1 - \frac{1}{d}\right)\frac{n(n-1)}{2}$$

(Since $X_{i,j}$ is an indicator (Bernoulli) r.v. and $\Pr[X_{i,j} = 1] = \frac{1}{d}$)

Hence, in our class of 75 students, the standard deviation for the matching birthdays is equal to $\sqrt{\frac{(37)(75)}{365}\frac{364}{365}} \approx 2.75$.

Questions?

Covariance

For two random variables R and S, the covariance between R and S is defined as:

$$\mathrm{Cov}[R, S] := \mathbb{E}[(R - \mathbb{E}[R])\,(S - \mathbb{E}[S])] = \mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]$$

Covariance

For two random variables R and S, the covariance between R and S is defined as:

$$\mathrm{Cov}[R,S] := \mathbb{E}[(R-\mathbb{E}[R])\,(S-\mathbb{E}[S])] = \mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]$$

$$\begin{aligned}
\mathrm{Cov}[R,S] &= \mathbb{E}[(R-\mathbb{E}[R])\,(S-\mathbb{E}[S])] \\
&= \mathbb{E}\left[RS - R\,\mathbb{E}[S] - S\,\mathbb{E}[R] + \mathbb{E}[R]\,\mathbb{E}[S]\right] \\
&= \mathbb{E}[RS] - \mathbb{E}[R\,\mathbb{E}[S]] - \mathbb{E}[S\,\mathbb{E}[R]] + \mathbb{E}[R]\,\mathbb{E}[S] \\
\implies \mathrm{Cov}[R,S] &= \mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S] - \mathbb{E}[S]\,\mathbb{E}[R] + \mathbb{E}[R]\,\mathbb{E}[S] = \mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]
\end{aligned}$$

Covariance

For two random variables R and S, the covariance between R and S is defined as:

$$\mathrm{Cov}[R,S] := \mathbb{E}[(R-\mathbb{E}[R])\,(S-\mathbb{E}[S])] = \mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]$$

$$\begin{aligned}
\mathrm{Cov}[R,S] &= \mathbb{E}[(R-\mathbb{E}[R])\,(S-\mathbb{E}[S])] \\
&= \mathbb{E}\left[RS - R\,\mathbb{E}[S] - S\,\mathbb{E}[R] + \mathbb{E}[R]\,\mathbb{E}[S]\right] \\
&= \mathbb{E}[RS] - \mathbb{E}[R\,\mathbb{E}[S]] - \mathbb{E}[S\,\mathbb{E}[R]] + \mathbb{E}[R]\,\mathbb{E}[S] \\
\implies \mathrm{Cov}[R,S] &= \mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S] - \mathbb{E}[S]\,\mathbb{E}[R] + \mathbb{E}[R]\,\mathbb{E}[S] = \mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]
\end{aligned}$$

Covariance generalizes the notion of variance to multiple random variables.

$$\mathrm{Cov}[R,R] = \mathbb{E}[R\,R] - \mathbb{E}[R]\,\mathbb{E}[R] = \mathrm{Var}[R]$$

Covariance

For two random variables R and S, the covariance between R and S is defined as:

$$\mathrm{Cov}[R,S] := \mathbb{E}[(R-\mathbb{E}[R])\,(S-\mathbb{E}[S])] = \mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]$$

$$\begin{aligned}
\mathrm{Cov}[R,S] &= \mathbb{E}[(R-\mathbb{E}[R])\,(S-\mathbb{E}[S])] \\
&= \mathbb{E}\left[RS - R\,\mathbb{E}[S] - S\,\mathbb{E}[R] + \mathbb{E}[R]\,\mathbb{E}[S]\right] \\
&= \mathbb{E}[RS] - \mathbb{E}[R\,\mathbb{E}[S]] - \mathbb{E}[S\,\mathbb{E}[R]] + \mathbb{E}[R]\,\mathbb{E}[S] \\
\implies \mathrm{Cov}[R,S] &= \mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S] - \mathbb{E}[S]\,\mathbb{E}[R] + \mathbb{E}[R]\,\mathbb{E}[S] = \mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]
\end{aligned}$$

Covariance generalizes the notion of variance to multiple random variables.

$$\mathrm{Cov}[R,R] = \mathbb{E}[R\,R] - \mathbb{E}[R]\,\mathbb{E}[R] = \mathrm{Var}[R]$$

If R and S are independent r.v's, $\mathbb{E}[RS] = \mathbb{E}[R]\,\mathbb{E}[S]$ and $\mathrm{Cov}[R,S] = 0$.

Covariance

For two random variables R and S, the covariance between R and S is defined as:

$$\mathrm{Cov}[R, S] := \mathbb{E}[(R - \mathbb{E}[R])\,(S - \mathbb{E}[S])] = \mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]$$

$$\begin{aligned}
\mathrm{Cov}[R, S] &= \mathbb{E}[(R - \mathbb{E}[R])\,(S - \mathbb{E}[S])] \\
&= \mathbb{E}\left[RS - R\,\mathbb{E}[S] - S\,\mathbb{E}[R] + \mathbb{E}[R]\,\mathbb{E}[S]\right] \\
&= \mathbb{E}[RS] - \mathbb{E}[R\,\mathbb{E}[S]] - \mathbb{E}[S\,\mathbb{E}[R]] + \mathbb{E}[R]\,\mathbb{E}[S] \\
\implies \mathrm{Cov}[R, S] &= \mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S] - \mathbb{E}[S]\,\mathbb{E}[R] + \mathbb{E}[R]\,\mathbb{E}[S] = \mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]
\end{aligned}$$

Covariance generalizes the notion of variance to multiple random variables.

$$\mathrm{Cov}[R, R] = \mathbb{E}[R\,R] - \mathbb{E}[R]\,\mathbb{E}[R] = \mathrm{Var}[R]$$

If R and S are independent r.v's, $\mathbb{E}[RS] = \mathbb{E}[R]\,\mathbb{E}[S]$ and $\mathrm{Cov}[R, S] = 0$.

The covariance between two r.v's is symmetric i.e. $\mathrm{Cov}[R, S] = \mathrm{Cov}[S, R]$.

Covariance

For two arbitrary (not necessarily independent) r.v's, R and S,

$$\mathrm{Var}[R+S] = \mathrm{Var}[R] + \mathrm{Var}[S] + 2\,\mathrm{Cov}[R,S]$$

Covariance

For two arbitrary (not necessarily independent) r.v's, R and S,

$$\mathrm{Var}[R+S] = \mathrm{Var}[R] + \mathrm{Var}[S] + 2\,\mathrm{Cov}[R,S]$$

Recall from Slide 6, where we showed that,

$$\mathrm{Var}[R+S] = \mathrm{Var}[R] + \mathrm{Var}[S] + 2(\mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]) = \mathrm{Var}[R] + \mathrm{Var}[S] + 2\,\mathrm{Cov}[R,S].$$

If R and S are independent, $\mathrm{Cov}[R,S] = 0$ and we recover the formula for the sum of independent variables.

Covariance

For two arbitrary (not necessarily independent) r.v's, R and S,

$$\mathrm{Var}[R+S] = \mathrm{Var}[R] + \mathrm{Var}[S] + 2\,\mathrm{Cov}[R,S]$$

Recall from Slide 6, where we showed that,

$$\mathrm{Var}[R+S] = \mathrm{Var}[R] + \mathrm{Var}[S] + 2(\mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]) = \mathrm{Var}[R] + \mathrm{Var}[S] + 2\,\mathrm{Cov}[R,S].$$

If R and S are independent, $\mathrm{Cov}[R,S] = 0$ and we recover the formula for the sum of independent variables.

For $R = S$, $\mathrm{Var}[R+R] = \mathrm{Var}[R] + \mathrm{Var}[R] + 2\mathrm{Cov}[R,R] = \mathrm{Var}[R] + \mathrm{Var}[R] + 2\mathrm{Var}[R] = 4\mathrm{Var}[R]$ which is consistent with our previous formula that $\mathrm{Var}[2R] = 2^2\mathrm{Var}[R]$.

Covariance

For two arbitrary (not necessarily independent) r.v's, R and S,

$$\text{Var}[R+S] = \text{Var}[R] + \text{Var}[S] + 2\,\text{Cov}[R,S]$$

Recall from Slide 6, where we showed that,

$$\text{Var}[R+S] = \text{Var}[R] + \text{Var}[S] + 2(\mathbb{E}[RS] - \mathbb{E}[R]\,\mathbb{E}[S]) = \text{Var}[R] + \text{Var}[S] + 2\,\text{Cov}[R,S].$$

If R and S are independent, $\text{Cov}[R,S] = 0$ and we recover the formula for the sum of independent variables.

For $R = S$, $\text{Var}[R+R] = \text{Var}[R] + \text{Var}[R] + 2\text{Cov}[R,R] = \text{Var}[R] + \text{Var}[R] + 2\text{Var}[R] = 4\text{Var}[R]$ which is consistent with our previous formula that $\text{Var}[2R] = 2^2\text{Var}[R]$.

Generalization to multiple random variables $R_1, R_2, \ldots R_n$ (Recall from Slide 7):

$$\text{Var}\left[\sum_{i=1}^{n} R_i\right] = \sum_{i=1}^{n} \text{Var}[R_i] + 2\sum_{1 \le i < j \le n} \text{Cov}[R_i, R_j]$$

Covariance - Example

Q: If X and Y are indicator r.v's for events A and B respectively, calculate the covariance between X and Y

Covariance - Example

Q: If X and Y are indicator r.v's for events A and B respectively, calculate the covariance between X and Y

We know that $\text{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$. Note that $X = \mathcal{I}_A$ and $Y = \mathcal{I}_B$. We can conclude that $XY = \mathcal{I}_{A\cap B}$ since $XY = 1$ iff both events A and B happen.

Covariance - Example

Q: If X and Y are indicator r.v's for events A and B respectively, calculate the covariance between X and Y

We know that $\text{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$. Note that $X = \mathcal{I}_A$ and $Y = \mathcal{I}_B$. We can conclude that $XY = \mathcal{I}_{A\cap B}$ since $XY = 1$ iff both events A and B happen.

$$\implies \mathbb{E}[X] = \Pr[A]\;;\, \mathbb{E}[Y] = \Pr[B];\, \mathbb{E}[XY] = \Pr[A \cap B]$$

$$\implies \text{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] = \Pr[A \cap B] - \Pr[A]\Pr[B]$$

If $\text{Cov}[X, Y] > 0 \implies \Pr[A \cap B] > \Pr[A]\Pr[B]$. Hence,

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} > \frac{\Pr[A]\Pr[B]}{\Pr[B]} = \Pr[A]$$

If $Cov[X, Y] > 0$, it implies that $\Pr[A|B] > \Pr[A]$ and hence, the probability that event A happens increases if B is going to happen/has happened. Similarly, if $Cov[X, Y] < 0$, $\Pr[A|B] < \Pr[A]$. In this case, if B happens, then the probability of event A decreases.

Correlation

The correlation between two r.v's R_1 and R_2 is defined as:

$$\mathrm{Corr}[R_1, R_2] = \frac{\mathrm{Cov}[R_1, R_2]}{\sqrt{\mathrm{Var}[R_1]\,\mathrm{Var}[R_2]}}$$

$\mathrm{Corr}[R_1, R_2] \in [-1, 1]$ and indicates the strength of the relationship between R_1 and R_2.

Correlation

The correlation between two r.v's R_1 and R_2 is defined as:

$$\mathrm{Corr}[R_1, R_2] = \frac{\mathrm{Cov}[R_1, R_2]}{\sqrt{\mathrm{Var}[R_1]\,\mathrm{Var}[R_2]}}$$

$\mathrm{Corr}[R_1, R_2] \in [-1, 1]$ and indicates the strength of the relationship between R_1 and R_2.

If $\mathrm{Corr}[R_1, R_2] > 0$, then R_1 and R_2 are said to be positively correlated, else if $\mathrm{Corr}[R_1, R_2] < 0$, the r.v's are negatively correlated.

Correlation

The correlation between two r.v's R_1 and R_2 is defined as:

$$\mathrm{Corr}[R_1, R_2] = \frac{\mathrm{Cov}[R_1, R_2]}{\sqrt{\mathrm{Var}[R_1]\,\mathrm{Var}[R_2]}}$$

$\mathrm{Corr}[R_1, R_2] \in [-1, 1]$ and indicates the strength of the relationship between R_1 and R_2.

If $\mathrm{Corr}[R_1, R_2] > 0$, then R_1 and R_2 are said to be positively correlated, else if $\mathrm{Corr}[R_1, R_2] < 0$, the r.v's are negatively correlated.

If $R_1 = R_2 = R$, then, $\mathrm{Corr}[R, R] = \frac{\mathrm{Cov}[R,R]}{\sqrt{\mathrm{Var}[R]\,\mathrm{Var}[R]}} = \frac{\mathrm{Var}[R]}{\mathrm{Var}[R]} = 1$.

Correlation

The correlation between two r.v's R_1 and R_2 is defined as:

$$\text{Corr}[R_1, R_2] = \frac{\text{Cov}[R_1, R_2]}{\sqrt{\text{Var}[R_1]\,\text{Var}[R_2]}}$$

$\text{Corr}[R_1, R_2] \in [-1, 1]$ and indicates the strength of the relationship between R_1 and R_2.

If $\text{Corr}[R_1, R_2] > 0$, then R_1 and R_2 are said to be positively correlated, else if $\text{Corr}[R_1, R_2] < 0$, the r.v's are negatively correlated.

If $R_1 = R_2 = R$, then, $\text{Corr}[R, R] = \frac{\text{Cov}[R,R]}{\sqrt{\text{Var}[R]\,\text{Var}[R]}} = \frac{\text{Var}[R]}{\text{Var}[R]} = 1$.

If R_1 and R_2 are independent, $\text{Cov}[R_1, R_2] = 0$ and $\text{Corr}[R_1, R_2] = 0$.

Correlation

The correlation between two r.v's R_1 and R_2 is defined as:

$$\mathrm{Corr}[R_1, R_2] = \frac{\mathrm{Cov}[R_1, R_2]}{\sqrt{\mathrm{Var}[R_1]\,\mathrm{Var}[R_2]}}$$

$\mathrm{Corr}[R_1, R_2] \in [-1, 1]$ and indicates the strength of the relationship between R_1 and R_2.

If $\mathrm{Corr}[R_1, R_2] > 0$, then R_1 and R_2 are said to be positively correlated, else if $\mathrm{Corr}[R_1, R_2] < 0$, the r.v's are negatively correlated.

If $R_1 = R_2 = R$, then, $\mathrm{Corr}[R, R] = \frac{\mathrm{Cov}[R,R]}{\sqrt{\mathrm{Var}[R]\,\mathrm{Var}[R]}} = \frac{\mathrm{Var}[R]}{\mathrm{Var}[R]} = 1$.

If R_1 and R_2 are independent, $\mathrm{Cov}[R_1, R_2] = 0$ and $\mathrm{Corr}[R_1, R_2] = 0$.

If $R_1 = -R_2 = R$, then,

$$\begin{aligned}\mathrm{Corr}[R, -R] &= \frac{\mathrm{Cov}[R, -R]}{\sqrt{\mathrm{Var}[R]\,\mathrm{Var}[-R]}} = \frac{\mathrm{Cov}[R, -R]}{\sqrt{\mathrm{Var}[R]\,(-1)^2\mathrm{Var}[R]}} = \frac{\mathrm{Cov}[R, -R]}{\mathrm{Var}[R]} \\ &= \frac{\mathbb{E}[-R^2] - \mathbb{E}[R]\,\mathbb{E}[-R]}{\mathrm{Var}[R]} = \frac{-\mathbb{E}[R^2] + \mathbb{E}[R]\,\mathbb{E}[R]}{\mathrm{Var}[R]} = \frac{-\mathrm{Var}[R]}{\mathrm{Var}[R]} = -1\end{aligned}$$

Questions?