CMPT 210: Probability and Computing

Lecture 24

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Recap

Chernoff Bound: Let $T_1, T_2, \ldots, T_n$ be mutually independent r.v's such that $0 \leq T_i \leq 1$ for all i. If $T := \sum_{i=1}^{n} T_i$, for all $c \geq 1$ and $\beta(c) := c \ln(c) - c + 1$,

$$\Pr[T \geq c\mathbb{E}[T]] \leq \exp(-\beta(c)\, \mathbb{E}[T])$$

Randomized Load Balancing

Fussbook is a new social networking site oriented toward unpleasant people. Like all major web services, Fussbook has a load balancing problem: it receives lots of forum posts that computer servers have to process. If any server is assigned more work than it can complete in a given interval, then it is overloaded and system performance suffers. That would be bad because Fussbook users are not a tolerant bunch.

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Fussbook receives 24000 forum posts in every 10-minute interval. Each post is assigned to one of several servers for processing, and each server works sequentially through its assigned tasks. It takes a server an average of $1/4$ second to process a post. No post takes more than 1 second.

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This implies that a server could be overloaded when it is assigned more than 600 units of work in a 10-minute interval. On average, for $24000 \times \frac{1}{4} = 6000$ units of work in a 10-minute interval, Fussbook requires at least 10 servers to ensure that no server is overloaded (with perfect load-balancing).

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Let m be the number of servers that Fussbook needs to use. Recall that a server may be overloaded if the load it is assigned exceeds 600 units. Let us first look at server 1 and define T to be the r.v. corresponding to the number of units of work assigned to the first server.

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Let T_i be the number of seconds server 1 spends on processing post i. $T_i = 0$ if the task is assigned to a different (not the first server). The maximum amount of time spent on post i is 1-second. Hence, $T_i \in [0, 1]$.

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Since there are $n := 24000$ posts in every 10-minute interval, the load (amount of units) assigned to the first server is equal to $T = \sum_{i=1}^{n} T_i$. Server 1 may be overloaded if $T \geq 600$, and hence we want to upper-bound the probability $\Pr[T \geq 600]$.

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Since the assignment of a post to a server is independent of the time required to process the post, the T_i r.v's are mutually independent. Hence, we can use the Chernoff bound.

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&= m\Pr[\text{server 1 is overloaded}] \leq m\exp\left(-\beta\left(\frac{m}{10}\right)\frac{6000}{m}\right) && \text{(All servers are equivalent)} \\
\implies &\Pr[\text{no server is overloaded}] \geq 1 - m\exp\left(-\beta\left(\frac{m}{10}\right)\frac{6000}{m}\right).
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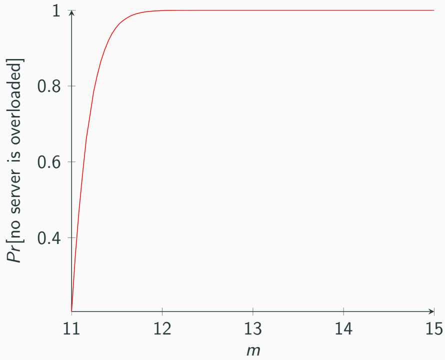
Plotting Pr[no server is overloaded] as a function of m.



Hence, as $m \geq 12$, the probability that no server gets overloaded tends to 1 and hence none of the Fussbook servers crash!

Questions?