CMPT 210: Probability and Computing

Lecture 6

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Recap

- **Sample (outcome) space** $\mathcal{S}$: Nonempty (countable) set of possible outcomes. Example: When we threw one dice, the sample space is $\{1, 2, 3, 4, 5, 6\}$.
- **Outcome** $\omega \in \mathcal{S}$: Possible "thing" that can happen. Example: When we threw one dice, a possible outcome is $\omega = 1$.
- **Event** E: Any subset of the sample space. Example: When we threw one dice, a possible event is $E = \{6\}$ (first example) or $E = \{3, 6\}$ (second example).
- If $E_1, E_2, \ldots E_n$ are events in $\mathcal{S}$, $G = E_1 \cup E_2 \cup \ldots E_n$ consists of outcomes that lie in at least one E_i. Event G happens when *at least one* of the events E_i happen.
  Example: When rolling a single dice, if $E = \{3\}, F = \{6\}$, then, $G = E \cup F = \{3, 6\}$ and corresponds to the event that we get either a 3 or a 6.

Intersection of events

- Suppose E, F are two events in $\mathcal{S}$. Define the intersection $E \cap F$ to consist of outcomes that are in both E and F (this is just the definition of the intersection of two sets). Formally,

$$G = E \cap F = \{\omega | \omega \in E \text{ AND } \omega \in F\}$$

- Another way to interpret this is to say event G occurs if both events E and F occur.

Example: We threw two dice and cared about getting 6 in the first throw *and* 6 in the second throw. In this case, E is the event we get a 6 for the first dice.
$E = \{(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$, F is the event we get a 6 for the second dice.
$F = \{(1,6),(2,6),(3,6),(4,6),(5,6),(6,6)\}$, $G = E \cap F = \{(6,6)\}$. G happens when both E and F happen i.e. the first dice has a 6 and the second dice has 6.

- Can define intersection between more than two events in the same way we defined intersection between more than two sets. $G = E_1 \cap E_2 \cap \ldots E_n$. G happens when *all* of the events E_i happen.

Mutually exclusive and complement events

• **Mutually exclusive events**: If E and F are two events such that $E \cap F = \{\}$, then events E and F are mutually exclusive.

Example: We threw one dice and want to get both 3 *and* 6. This is not possible. Formally, $E = \{6\}$, $F = \{3\}$ and $E \cap F = \{\}$, hence, events E and F are mutually exclusive.

• **Complement of an event**: If E is an event, then its complement E^c is defined such that $E \cap E^c = \{\}$ and $E \cup E^c = \mathcal{S}$. Event E^c will occur if and only if event E does not occur.

Example: We threw one dice and want to get a 6 i.e. we define $E = \{6\}$. $E^c = \{1, 2, 3, 4, 5\}$.

• Two complement events are mutually exclusive, but two mutually exclusive events need not be the complements of each other. *Example*: $E = \{6\}$ and $F = \{3\}$ are mutually exclusive, but not complements.

• **Subset**: If $E \subset F$, then if E happens F will happen. *Example*: When we throw one dice, if $E = \{3\}$ and $F = \{1, 2, 3\}$ i.e. E is the event that we get 3 and F is the event that we can either $1, 2, 3$. Clearly, if E happens, F will happen.

Axioms of Probability

- **Probability function** on a sample space $\mathcal{S}$ is a total function $\Pr : \mathcal{S} \to [0, 1]$.

For any $\omega \in \mathcal{S}$, $0 \leq Pr[\omega] \leq 1 \quad ; \quad \sum_{\omega \in \mathcal{S}} \Pr[\omega] = 1$

- **Probability space**: The outcome space $\mathcal{S}$ together with the probability function.

Recall that we can define functions on sets. In this case, for an event E, $\Pr[E] = \sum_{\omega \in E} \Pr[\omega]$.

- **Union**: For mutually exclusive events $E_1, E_2, \ldots, E_n$ (sets $E_1, E_2, \ldots, E_n$ are disjoint), $\Pr[E_1 \cup E_2 \cup \ldots E_n] = \Pr[E_1] + \Pr[E_2] + \ldots + \Pr[E_n]$.

Proof:

$$\Pr[E_1 \cup E_2 \cup \ldots E_n] = \sum_{\omega \in \{E_1 \cup E_2 \cup \ldots E_n\}} \Pr[\omega]$$

Since E_i's are disjoint, any ω can only be in one of $E_1, E_2, \ldots E_n$

$$= \sum_{\omega \in E_1} \Pr[\omega] + \sum_{\omega \in E_2} \Pr[\omega] + \ldots + \sum_{\omega \in E_n} \Pr[\omega] = \Pr[E_1] + \Pr[E_2] + \ldots + \Pr[E_n].$$

Back to throwing dice

Q: Suppose we throw a standard dice. What is the probability that the number that comes up is 6?

$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$. Since the dice is "standard", each outcome is equally likely, i.e. $\Pr[1] = \Pr[2] = \ldots = \Pr[6]$.

Since $\Pr[\mathcal{S}] = 1 \implies \sum_{\omega \in \mathcal{S}} \Pr[\omega] = 1 \implies \Pr[1] + \Pr[2] + \ldots \Pr[6] = 1$
$\implies \Pr[6] = \frac{1}{6}$.

Back to throwing dice

Q: Suppose we throw a standard dice. What is the probability that we get either a 3 or a 6?

$E = \{3\}$, $F = \{6\}$, $G = \{3, 6\}$. Since $E \cap F = \{\}$, E and F are mutually exclusive events, implying that $\Pr[G] = \Pr[E] + \Pr[F] = \Pr[\{3\}] + \Pr[\{6\}] = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

Hence, probability of getting either a 3 or a 6 is equal to $\frac{1}{3}$.

Q: Compute the probability of getting either 1, 2 or 3.

Q: Compute the probability of getting an even number.

Q: Compute the probability of getting either 1, 2, 3, 4, 5, 6

Probability Rules

- **Complement rule**: $\Pr[E] = 1 - \Pr[E^c]$.

Proof: Recall that $E \cap E^c = \{\}$ and $E \cup E^c = \mathcal{S}$. Since E and E^c are disjoint,

$$\Pr[E \cup E^c] = \Pr[E] + \Pr[E^c] \implies \Pr[S] = \Pr[E] + \Pr[E^c] \implies \Pr[E^c] = 1 - \Pr[E].$$

- **Inclusion-Exclusion rule**: For any two events E, F, $\Pr[E \cup F] = \Pr[E] + \Pr[F] - \Pr[E \cap F]$.

Proof:

$$\Pr[E \cup F] = \sum_{\omega \in \{E \cup F\}} \Pr[\omega] = \sum_{\omega \in \{E - F\}} \Pr[\omega] + \sum_{\omega \in \{F - E\}} \Pr[\omega] + \sum_{\omega \in \{E \cap F\}} \Pr[\omega]$$

(Since disjoint)

$$= \left[\sum_{\omega \in \{E - F\}} \Pr[\omega] + \sum_{\omega \in \{E \cap F\}} \Pr[\omega]\right] + \left[\sum_{\omega \in \{F - E\}} \Pr[\omega] + \sum_{\omega \in \{E \cap F\}} \Pr[\omega]\right] - \sum_{\omega \in \{E \cap F\}} \Pr[\omega]$$

$$= \sum_{\omega \in E} \Pr[\omega] + \sum_{\omega \in F} \Pr[\omega] - \sum_{\omega \in \{E \cap F\}} \Pr[\omega] = \Pr[E] + \Pr[F] - \Pr[E \cap F]$$

Probability Rules

- **Union Bound**: For any two events E, F, $\Pr[E \cup F] \leq \Pr[E] + \Pr[F]$.

Proof: By the inclusion-exclusion rule, $\Pr[E \cup F] = \Pr[E] + \Pr[F] - \Pr[E \cap F]$. Since probabilities are non-negative, $\Pr[E \cap F] \geq 0$ and hence, $\Pr[E \cup F] \leq \Pr[E] + \Pr[F]$.

- **Union Bound**: For any events $E_1, E_2, E_3, \ldots E_n$,

$$\Pr[E_1 \cup E_2 \cup E_3 \ldots \cup E_n] \leq \sum_{i=1}^{n} \Pr[E_i]$$

- **Monotonicity rule**: For events A and B, if $A \subset B$, then $\Pr[A] < \Pr[B]$.

Proof:

$$\Pr[A] = \sum_{\omega \in A} \Pr[\omega] = \sum_{\omega \in B} \Pr[\omega] - \sum_{\omega \in \{B - A\}} \Pr[\omega] \implies \Pr[A] < \Pr[B]$$

(Since probabilities are non-negative.)

Uniform Probability Spaces

- **Definition**: A probability space is uniform if $\Pr[\omega]$ is the same for every outcome $\omega \in \mathcal{S}$.

Since $\sum_{\omega \in \mathcal{S}} Pr[\omega] = 1 \implies \Pr[\omega] = \frac{1}{|\mathcal{S}|}$ for all $\omega \in \mathcal{S}$.

Example: For a standard dice, $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$, $\Pr[1] = \Pr[2] = \ldots = \Pr[6] = 1/6$.

$\Pr[E] = \sum_{\omega \in E} Pr[\omega] = |E|\, Pr[\omega] = \frac{|E|}{|\mathcal{S}|}$.

Example: For a standard dice, if $E = \{3, 6\}$, then, $\Pr[E] = \frac{|E|}{|\mathcal{S}|} = \frac{2}{6} = 1/3$.

Hence, for uniform probability spaces, computing the probability is equivalent to counting the outcomes we "care" about.

Back to throwing dice

Q: Suppose we have a loaded (not "standard") dice such that the probability of getting an even number is twice that of getting an odd number (all even numbers are equally likely, and so are the odd numbers). What is the probability of getting a 6?

Let p be the probability of getting an odd number. Probability of getting an even number $= 2p$.

$\sum_{\omega \in \mathcal{S}} \Pr[\omega] = 1 \implies 3p + 3(2p) = 1 \implies p = \frac{1}{9}$. Hence, probability of getting an odd number $= \frac{1}{9}$. Probability of getting a 6 $=$ Probability of getting an even number $= \frac{2}{9}$.

Q: What is the probability that we get either a 3 or a 6?

Q: What is the probability that we get a prime number

Probability Examples

Q: Suppose we select a card at random from a standard deck of 52 cards. What is the probability of getting:

- A spade
- A spade facecard
- A black card
- The queen of hearts
- An ace

Probability Examples

Q: A class consists of 6 men and 4 women. An exam is given and the students are ranked according to their performance. Assuming that no two students obtain the same scores and all rankings are considered equally likely, what is the probability that women receive the top 4 scores?

In general, let the number of men be m and let the number of women be w.

Number of possible rankings = Number of permutations = $(m+w)!$.

The event of interest is where the women achieve the top scores. In a possible ranking, let's fix the top w slots for women. The w women can be arranged in $w!$ ways. And the m men can be arranged in $m!$ ways. Hence, total number of rankings where women receive the top scores = $m!\,w!$.

Since all rankings are equally likely, probability that women receive the top w scores = $\frac{m!w!}{(m+w)!}$.
In this case, since $m = 6$ and $w = 4$, probability that women receive the top 4 scores = $\frac{6!\,4!}{10!}$.

Probability Examples

Q: A class consists of m men and w women. An exam is given and the students are ranked according to their performance. Assuming that no two students obtain the same scores and all rankings are considered equally likely, what is probability that women receive the top t ($t \leq w$) scores?

Number of ways to select the t women that have top scores $= \binom{w}{t}$. The top t women can be arranged in $t!$ ways. The number of remaining students is equal to $m + w - t$. These can be arranged in $(m + w - t)!$ ways. Hence, total number of rankings where women receive the top t scores $= \binom{w}{t} \quad (m + w - t)! \quad t!$.

As before, the total number of rankings $= (m + w)!$. Since all rankings are equally likely, the probability that women receive the top t scores $= \frac{\binom{w}{t}\,(m+w-t)!\,t!}{(m+w)!} = \frac{w!\,(m+w-t)!}{(w-t)!(m+w)!}$

Probability Examples

Q: A committee of size 5 is to be selected from a group of 6 CS and 9 Math students (no double majors allowed). If the selection is made randomly (all selections are equally likely), what is the probability that the committee consists of 3 CS and 2 Math students?

Number of possible ways of selecting the committee $= |\mathcal{S}| = \binom{15}{5}$.

The event of interest (E) requires choosing 3 CS and 2 Math students. Number of ways we can select the CS students $= \binom{6}{3}$. Similarly, number of ways we can select the Math students $= \binom{9}{2}$.

Hence, $|E| = \binom{6}{3}\binom{9}{2} \implies \Pr[E] = \frac{|E|}{|\mathcal{S}|} = \frac{\binom{6}{3}\binom{9}{2}}{\binom{15}{5}}$.

Probability Examples

Q: From a set of n items a random sample of size k is to be selected (all selections are equally likely). What is the probability a given item (α) will be among the k selected items?

Number of ways of choosing the sample $= \binom{n}{k}$.

If we want a particular item in the sample, number of ways of choosing the other items $= \binom{n-1}{k-1}$.

Hence, probability that a given item will be among the k selected $= \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}$.

Probability Examples

Q: From a set of n items a random sample of size k is to be selected (all selections are equally likely). Given two items of interest: α and β, what is the probability that (i) both α and β will be among the k selected (ii) at least one of α or β will be among the k selected (iii) neither α nor β will be among the k selected?

(i) If we want both α and β to be in the sample, number of ways of choosing the other items $= \binom{n-2}{k-2}$. Hence, probability that both α and β will be in the sample $= \frac{\binom{n-2}{k-2}}{\binom{n}{k}} = \frac{k(k-1)}{n(n-1)}$.

(ii) Let A be the event that item α is in the selection. $\Pr[A] = \frac{k}{n}$. Similarly B be the event that item β is in the selection. $\Pr[B] = \frac{k}{n}$. We want to compute $\Pr[A \cup B]$. By the inclusion-exclusion rule, $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$. Hence, probability that either α or β will be among the k selected items $= \frac{2k}{n} - \frac{k(k-1)}{n(n-1)}$.

(iii) If we want neither α nor β to be in the sample, number of ways of choosing the items $= \binom{n-2}{k}$. Hence, probability that neither α nor β will be in the sample $= \frac{\binom{n-2}{k}}{\binom{n}{k}} = \frac{(n-k)(n-k-1)}{n(n-1)}$.

Probability - Examples

Q: Let us consider random permutations (all permutations are equally likely) of the letters (i) ABBA (ii) ABBA'. What is the probability that the third letter is B?

Questions?

Birthday Paradox

Q: There are 75 students in a class. What is the probability that two students have their birthdays in the same week?

Q: In this class, what is the probability that two students share the same birthday? Assume that (i) each student is equally likely to be born on any day of the year, (ii) no leap years and (iii) student birthdays are independent of each other.

Let n be the number of students, and let d be the number of days in the year. Let's order the students according to their ID. A birthday sequence is (11 Feb, 23 April, 31 August, ...). First let's count the number of possible birthday sequences.

The first student's birthday can be one of d days. Similarly, the second student's birthday can be one of d days, and so on. By the product rule, the total number of birthday sequences = $d \times d \times \ldots = d^n$.

Birthday Paradox

The event of interest is that two students share the same birthday. Let us compute the probability of the event that NO two students share the same birthday, and then use the complement rule.

The first birthday can be chosen in d ways, the second in $d-1$ ways, and so on. By the generalized product rule, the number of birthday sequences such that no birthday is shared $=$ $d \times (d-1) \times (d-2) \times \dots (d-(n-1))$.

Hence, the probability that no two students share the same birthday
$= \frac{\text{the number of birthday sequences such that no birthday is shared}}{\text{total number of birthday sequences}} = \frac{d \times (d-1) \times (d-2) \times \dots (d-(n-1))}{d^n}$

$$= \left(1 - \frac{0}{d}\right) \times \left(1 - \frac{1}{d}\right) \dots \left(1 - \frac{n-1}{d}\right) \leq \exp(-0/d) \times \exp(-1/d) \dots \exp(-(n-1)/d)$$

$$(\text{for } x > 0,\ 1 - x \leq \exp(-x))$$

$$= \exp\left(\frac{-0}{d} + \frac{-1}{d} + \dots \frac{-(n-1)}{d}\right) = \exp\left(-\frac{n(n-1))}{2d}\right)$$

Birthday Paradox

Probability that two students share a birthday $\geq 1 - \exp\left(-\frac{n(n-1))}{2d}\right)$. Let's plot for $d = 365$.

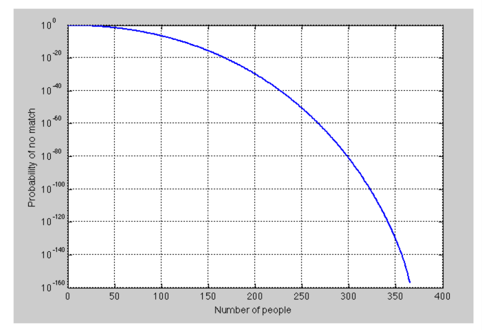

Figure 1: Plotting $\exp\left(-\frac{n(n-\mathbf{1}))}{\mathbf{2}d}\right)$ for $d = 365$

In our class, there is $> 99\%$ that two students have the same birthday!

Birthday Principle

If there are n pigeons and d pigeonholes, then the probability that two pigeons occupy the same hole is $\geq 1 - \exp\left(-\frac{n(n-1))}{2d}\right)$

For $n = \lceil\sqrt{2d}\rceil$, probability that two pigeons occupy the same hole is about $1 - \frac{1}{e} \approx 0.632$.

Example: If we are randomly throwing $\lceil\sqrt{2d}\rceil$ balls into d bins, then the probability that two balls land in the same bin is around 0.632.

Later in the course, we will see applications of this principle to load balancing.

Questions?