

CMPT 210: Probability and Computing

Lecture 3

Sharan Vaswani

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Recap - Counting

Product Rule: For sets A_1, A_2, \dots, A_m , $|A_1 \times A_2 \times \dots \times A_m| = \prod_{i=1}^m |A_i|$ (E.g: Selecting one course each from every subject.)

Sum rule: If A_1, A_2, \dots, A_m are disjoint sets, then, $|A_1 \cup A_2 \cup \dots \cup A_m| = \sum_{i=1}^m |A_i|$ (E.g Number of rainy, snowy or hot days in the year).

Generalized product rule: If S is the set of length k sequences such that the first entry can be selected in n_1 ways, after the first entry is chosen, the second one can be chosen in n_2 ways, and so on, then $|S| = n_1 \times n_2 \times \dots \times n_k$. (E.g Number of ways n people can be arranged in a line = $n!$)

Division rule: $f : A \rightarrow B$ is a k -to-1 function, then, $|A| = k|B|$. (E.g. For arranging people around a round table, $f : \text{seatings} \rightarrow \text{arrangements}$ is an n -to-1 function).

Counting subsets (Combinations)

Q: How many size- k subsets of a size- n set are there?

Example: How many ways can we select 5 books from 100?

Let us form a permutation of the n elements, and pick the first k elements to form the subset. Every size k subset can be generated this way. There are $n!$ total such permutations.

The order of the first k elements in the permutation does not matter in forming the subset, and neither does the order of the remaining $n - k$ elements.

The first k elements can be ordered in $k!$ ways and the remaining $n - k$ elements can be ordered in $(n - k)!$ ways. Using the product rule, $k! \times (n - k)!$ permutations map to the same size k subset.

Hence, the function $f : \text{permutations} \rightarrow \text{size } k \text{ subsets}$ is a $k! \times (n - k)!$ -to-1 function. By the division rule, $|\text{permutations}| = k! \times (n - k)! |\text{size } k \text{ subsets}|$. Hence, the total number of size k subsets $= \frac{n!}{k! \times (n - k)!}$.

$$n \text{ choose } k = \binom{n}{k} := \frac{n!}{k! \times (n - k)!}.$$

Counting subsets (Combinations)

Q: Prove that $\binom{n}{k} = \binom{n}{n-k}$ - both algebraically (using the formula for $\binom{n}{k}$) and combinatorially (without using the formula)

Q: Which is bigger? $\binom{8}{4}$ vs $\binom{8}{5}$?

Counting subsets – Example

Q: How many m -bit binary sequences contain exactly k ones?

Consider set $A = \{1, \dots, m\}$ and selecting S , a subset of size k . For example, say $m = 10, k = 3$ and $S = \{3, 7, 10\}$. S records the positions of the 1's, and can be mapped to the sequence 0010001001. Similarly, every m -bit sequence with exactly k ones can be mapped to a subset S of size k . Hence, there is a bijection:

$f : m\text{-bit sequence with exactly } k \text{ ones} \rightarrow \text{subsets of size } k \text{ from size } m\text{-set, and}$
 $|m\text{-bit sequence with exactly } k \text{ ones}| = |\text{subsets of size } k| = \binom{m}{k}.$

Q: Suppose we want to buy 10 donuts. There are 5 donut varieties – chocolate, lemon-filled, sugar, glazed, plain. What is the number of ways to select the 10 donuts?

Recall that the number of ways of selecting 10 donuts with 5 varieties = number of 14-bit sequences with exactly 4 ones = $\binom{14}{4} = 1001$.

Q: What is the number of ways of choosing n things with k varieties?

Counting subsets – Example

Q: What is the number of n -bit binary sequences with at least k ones?

Q: What is the number of n -bit binary sequences with less than k ones?

Q: What is the total number of n -bit binary sequences?

Binomial Theorem

For all $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$,

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

Example: If $a = b = 1$, then $\sum_{k=0}^n \binom{n}{k} = 2^n$ (result from previous slide).

If $n = 2$, then $(a + b)^2 = \binom{2}{0}a^2 + \binom{2}{1}ab + \binom{2}{2}b^2 = a^2 + 2ab + b^2$.

Q: What is the coefficient of the terms with ab^3 and a^2b^3 in $(a + b)^4$?

Q: For $a, b > 0$, what is the coefficient of $a^{2n-7}b^7$ and $a^{2n-8}b^8$ in $(a + b)^{2n} + (a - b)^{2n}$?

Q: A fair die (with numbers $\{1, 2, 3, 4, 5, 6\}$) is rolled 6 times in succession.

- How many rolls will have no 6?
- How many rolls will have each number once?
- How many rolls will have 6 come up exactly once?
- How many rolls will have 6 come up exactly k times (for $k \leq 6$)?

Q: How many 5 digit numbers are there which contain at least one zero? Note that a number is different from a string, i.e. 01234 is not a 5-digit number and is hence not allowed.

Q: How many non-negative integer solutions ($x_1, x_2, x_3 \geq 0$) are there to the following equation:

$$x_1 + x_2 + x_3 = 40$$

Questions?