

CMPT 419/983: Theoretical Foundations of Reinforcement Learning

Lecture 2

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Recap

- **Input:** K arms (possible actions), T rounds. $\mu_a := \mathbb{E}_{r \sim \nu_a}[r]$ is the (unknown) expected reward obtained by choosing action a .
- **Protocol:** In each round $t \in [T]$, the bandit algorithm chooses action $a_t \in [K]$ and observes reward $R_t \sim \nu_{a_t}$.
- **Objective:** Minimize $\text{Regret}(T) := \sum_{t=1}^T [\mu^* - \mathbb{E}[R_t]] = \sum_{a=1}^K \Delta_a \mathbb{E}[N_a(T)]$.
- **Assumption:** $\eta_t := R_t - \mu_{a_t}$ is 1 sub-Gaussian i.e. for all $\lambda \in \mathbb{R}$, $\mathbb{E}[\exp(\lambda \eta_t)] \leq \exp\left(\frac{\lambda^2}{2}\right)$.
- **Concentration for sub-Gaussian r.v.:** If X is centered and σ sub-Gaussian, then for any $\epsilon \geq 0$, $\Pr[X \geq \epsilon] \leq \exp\left(-\frac{\epsilon^2}{2\sigma^2}\right)$. For n i.i.d r.v's X_i s.t. $\mathbb{E}[X_i] = \mu$, if $\hat{\mu} := \frac{1}{n} \sum_{i=1}^n X_i$ and $X_i - \mu$ is σ sub-Gaussian, then $\Pr[|\hat{\mu} - \mu| \geq \epsilon] \leq \exp\left(-\frac{n\epsilon^2}{2\sigma^2}\right)$
- **Explore-then-Commit (ETC):** Under a sub-Gaussian assumption, ETC results in $O(\sqrt{KT})$ regret when exploring for $m = O\left(\frac{1}{\Delta^2}\right)$ rounds, while it can only result in $O(T^{2/3})$ regret when m is set independent of Δ .

ϵ -greedy Algorithm

Algorithm ϵ -greedy (EG)

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1: Input:  $\{\epsilon_t\}_{t=1}^T$ 
2: for  $t = 1 \rightarrow K$  do
3:   Select arm  $a_t = t$  and observe  $R_t$ 
4: end for
5: Calculate empirical mean reward for arm  $a \in [K]$  as  $\hat{\mu}_a(K) := \frac{\sum_{t=1}^K R_t \mathcal{I}\{a_t=a\}}{N_a(K)}$ 
6: for  $t = K + 1 \rightarrow T$  do
7:   Select arm  $\begin{cases} a_t = \arg \max_{a \in [K]} \hat{\mu}_a(t-1) \text{ w.p. } 1 - \epsilon_t \\ a_t \sim \mathcal{U}\{1, 2, \dots, K\} \text{ w.p. } \epsilon_t \end{cases}$ 
8:   Observe reward  $R_t$  and update for  $a \in [K]$ :
       
$$N_a(t) = N_a(t-1) + \mathcal{I}\{a_t = a\} \quad ; \quad \hat{\mu}_a(t) = \frac{N_a(t-1) \hat{\mu}_a(t-1) + R_t \mathcal{I}\{a_t = a\}}{N_a(t)}$$

9: end for
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- EG with $\epsilon_t = \epsilon$ can result in linear regret.
- For $K = 2$, EG with $\epsilon_t = O\left(\frac{1}{\Delta^2 t}\right)$ incurs $O\left(\frac{\log(T)}{\Delta^2}\right)$ regret.

Prove in Assignment 1!

Upper Confidence Bound (UCB) Algorithm

- Based on the principle of *optimism in the face of uncertainty*.

Algorithm Upper Confidence Bound

- 1: **Input:** δ
- 2: For each arm $a \in [K]$, initialize $U_a(0, \delta) := \infty$.
- 3: **for** $t = 1 \rightarrow T$ **do**
- 4: Select arm $a_t = \arg \max_{a \in [K]} U_a(t-1, \delta)$ (*Choose the lower-indexed arm in case of a tie*)
- 5: Observe reward R_t and update for $a \in [K]$:

$$N_a(t) = N_a(t-1) + \mathcal{I}\{a_t = a\} \quad ; \quad \hat{\mu}_a(t) = \frac{N_a(t-1) \hat{\mu}_a(t-1) + R_t \mathcal{I}\{a_t = a\}}{N_a(t)}$$

$$U_a(t, \delta) = \hat{\mu}_a(t) + \sqrt{\frac{2 \log(1/\delta)}{N_a(t)}}$$

- 6: **end for**
-

- Intuitively, UCB pulls a “promising” arm (with higher empirical mean $\hat{\mu}_a$) or one that has not been explored enough (with lower $N_a(t)$).

UCB – Regret Analysis

Claim: UCB with $\delta = \frac{1}{T^2}$ achieves the following problem-dependent bound on the regret,

$$\text{Regret}(\text{UCB}, T) \leq 2 \sum_{a=1}^K \Delta_a + \sum_{a \in [K] | \Delta_a > 0} \frac{16 \log(T)}{\Delta_a}$$

Proof: Without loss of generality, assume that arm 1 is the best arm. Using the regret decomposition, we know that $\text{Regret}(\text{UCB}, T) = \sum_a \Delta_a \mathbb{E}[N_a(T)]$. Define a threshold τ_a and $\hat{\mu}_{a, \tau_a}$ as the mean for arm a after pulling it for the first τ_a times. Define a “good” event G_a for each $a \neq 1$.

$$G_a = \left\{ \mu_1 < \min_{t \in [T]} U_1(t, \delta) \right\} \cap \left\{ \hat{\mu}_{a, \tau_a} + \sqrt{\frac{2 \log(1/\delta)}{\tau_a}} < \mu_1 \right\}$$

Consider two cases when bounding $\mathbb{E}[N_a(T)]$. Using the law of total expectation,

$$\begin{aligned} \mathbb{E}[N_a(T)] &= \mathbb{E}[N_a(T) | G_a] \Pr[G_a] + \mathbb{E}[N_a(T) | G_a^c] \Pr[G_a^c] \\ &\leq \underbrace{\mathbb{E}[N_a(T) | G_a]}_{\text{Term (i)}} + T \underbrace{\Pr[G_a^c]}_{\text{Term (ii)}} \quad (N_a(T) \leq T \text{ for all } a, \Pr[G_a] \leq 1) \end{aligned}$$

UCB – Regret Analysis

Recall that $G_a = \{\mu_1 < \min_{t \in [T]} U_1(t, \delta)\} \cap \left\{ \hat{\mu}_{a, \tau_a} + \sqrt{\frac{2 \log(1/\delta)}{\tau_a}} < \mu_1 \right\}$.

We will show that $\text{Term (i)} = \mathbb{E}[N_a(T) | G_a] \leq \tau_a$. To show this, we will prove (by contradiction) that $\Pr[N_a(T) > \tau | G_a] = 0$. Suppose, conditioned on the event G_a , $N_a(T) > \tau_a$, then there is a round t s.t. $N_a(t-1) = \tau_a$, $a_t = a$. Since $a_t = \arg \max_a U_a(t-1, \delta)$, it follows that $U_a(t-1, \delta) > U_1(t-1, \delta)$. However, we know that,

$$\begin{aligned} U_a(t-1, \delta) &= \hat{\mu}_a(t-1) + \sqrt{\frac{2 \log(1/\delta)}{N_a(t-1)}} = \hat{\mu}_a(t-1) + \sqrt{\frac{2 \log(1/\delta)}{\tau_a}} \\ &\hspace{15em} \text{(By assumption, } N_a(t-1) = \tau_a) \\ &= \hat{\mu}_{a, \tau_a} + \sqrt{\frac{2 \log(1/\delta)}{\tau_a}} \hspace{5em} \text{(Since arm } a \text{ has been pulled } \tau_a \text{ times)} \\ &\leq \mu_1 < U_1(t-1, \delta), \hspace{5em} \text{(Since we are conditioning on } G_a) \end{aligned}$$

which is a contradiction. Since, $\Pr[N_a(T) > \tau | G_a] = 0$, it implies that

$$\mathbb{E}[N_a(T) | G_a] = \sum_{n=0}^{\infty} \Pr[N_a(T) > n | G_a] = \sum_{n=0}^{\tau_a-1} \Pr[N_a(T) > n | G_a] \leq \tau_a.$$

UCB – Regret Analysis

$$\text{Bounding Term (ii)} = \Pr[G_a^c] \leq \Pr[\mu_1 \geq \min_{t \in [T]} U_1(t, \delta)] + \Pr\left[\hat{\mu}_{a, \tau_a} + \sqrt{\frac{2 \log(1/\delta)}{\tau_a}} \geq \mu_1\right].$$

$$\left\{ \mu_1 \geq \min_{t \in [T]} U_1(t, \delta) \right\} = \left\{ \mu_1 \geq \min_{t \in [T]} \left\{ \hat{\mu}_1(t) + \sqrt{\frac{2 \log(1/\delta)}{N_1(t)}} \right\} \right\}$$

$$\subset \left\{ \mu_1 \geq \min_{s \in [T]} \left\{ \hat{\mu}_{1,s} + \sqrt{\frac{2 \log(1/\delta)}{s}} \right\} \right\}$$

$$= \bigcup_{s=1}^T \left\{ \mu_1 \geq \hat{\mu}_{1,s} + \sqrt{\frac{2 \log(1/\delta)}{s}} \right\}$$

$$\Rightarrow \Pr\left[\mu_1 \geq \min_{t \in [T]} U_1(t, \delta)\right] \leq \sum_{s=1}^T \Pr\left[\mu_1 \geq \hat{\mu}_{1,s} + \sqrt{\frac{2 \log(1/\delta)}{s}}\right] \quad (\text{Union Bound})$$

$$\leq \sum_{s=1}^T \delta = \delta T \quad (\text{Using concentration for sub-Gaussian r.v's})$$

UCB – Regret Analysis

Recall that Term (ii) = $\Pr[G_a^c] \leq \delta T + \Pr\left[\hat{\mu}_{a,\tau_a} + \sqrt{\frac{2\log(1/\delta)}{\tau_a}} \geq \mu_1\right]$. Assume that τ_a is chosen such that $\Delta_a - \sqrt{\frac{2\log(1/\delta)}{\tau_a}} \geq \frac{\Delta_a}{2}$.

$$\begin{aligned}\Pr\left[\hat{\mu}_{a,\tau_a} + \sqrt{\frac{2\log(1/\delta)}{\tau_a}} \geq \mu_1\right] &= \Pr\left[\hat{\mu}_{a,\tau_a} - \mu_a + \sqrt{\frac{2\log(1/\delta)}{\tau_a}} \geq \Delta_a\right] \leq \Pr\left[\hat{\mu}_{a,\tau_a} - \mu_a \geq \frac{\Delta_a}{2}\right] \\ &\leq \exp\left(-\frac{\tau_a \Delta_a^2}{8}\right)\end{aligned}$$

(Using concentration for sub-Gaussian r.v's)

Putting everything together,

$$\begin{aligned}\implies \Pr[G_a^c] &\leq \delta T + \exp\left(-\frac{\tau_a \Delta_a^2}{8}\right) \\ \implies \mathbb{E}[N_a(T)] &\leq \tau_a + T \left[\delta T + \exp\left(-\frac{\tau_a \Delta_a^2}{8}\right) \right]\end{aligned}$$

UCB – Regret Analysis

Recall that $\mathbb{E}[N_a(T)] \leq \tau_a + T \left[\delta T + \exp\left(-\frac{\tau_a \Delta_a^2}{8}\right) \right]$.

$$\mathbb{E}[N_a(T)] \leq \frac{8 \log(1/\delta)}{\Delta_a^2} + T [\delta T + \delta] \quad (\text{Setting } \tau_a = \frac{8 \log(1/\delta)}{\Delta_a^2})$$

$$\leq \frac{8 \log(1/\delta)}{\Delta_a^2} + 2\delta T^2$$

$$= \frac{16 \log(T)}{\Delta_a^2} + 2 \quad (\text{Setting } \delta = 1/T^2)$$

$$\implies \text{Regret}(\text{UCB}, T) \leq \sum_a \Delta_a \mathbb{E}[N_a(T)] = 2 \sum_{a=1}^K \Delta_a + \sum_{a=2}^K \frac{16 \log(T)}{\Delta_a} \quad \square$$

UCB – Regret Analysis

Claim: For $\Delta \leq 1$, UCB with $\delta = \frac{1}{T^2}$ achieves the following worst-case regret,

$$\text{Regret}(\text{UCB}, T) \leq 2K + 8\sqrt{K T \log(T)}$$

Proof: Define $C > 0$ to be a constant to be tuned later. From the regret decomposition result,

$$\begin{aligned} \text{Regret}(\text{UCB}, T) &= \sum_{a=1}^K \Delta_a \mathbb{E}[N_a(T)] = \sum_{a|\Delta_a < C} \Delta_a \mathbb{E}[N_a(T)] + \sum_{a|\Delta_a \geq C} \Delta_a \mathbb{E}[N_a(T)] \\ &\leq CT + \sum_{a|\Delta_a \geq C} \Delta_a \mathbb{E}[N_a(T)] && \text{(Since } \sum_{a=1}^K N_a(T) = T \text{)} \\ &\leq CT + \sum_{a|\Delta_a \geq C} \left[\frac{16 \log(T)}{\Delta_a} + 2\Delta_a \right] && \text{(From the previous slide)} \\ &\leq CT + \left[\frac{16K \log(T)}{C} + \sum_{a|\Delta_a \geq C} 2\Delta_a \right] && \text{(Setting } C = \sqrt{\frac{16K \log(T)}{T}} \text{)} \end{aligned}$$

$$\implies \text{Regret}(\text{UCB}, T) \leq 8\sqrt{K T \log(T)} + 2K\Delta_a \leq 2K + 8\sqrt{K T \log(T)}$$

UCB vs ETC

- Similar to best-tuned ETC, UCB results in an $\tilde{O}(\sqrt{KT})$ problem-independent regret.
- Unlike best-tuned ETC, UCB does not need to know the gaps Δ to set algorithm parameters, but does require knowledge of the horizon T .

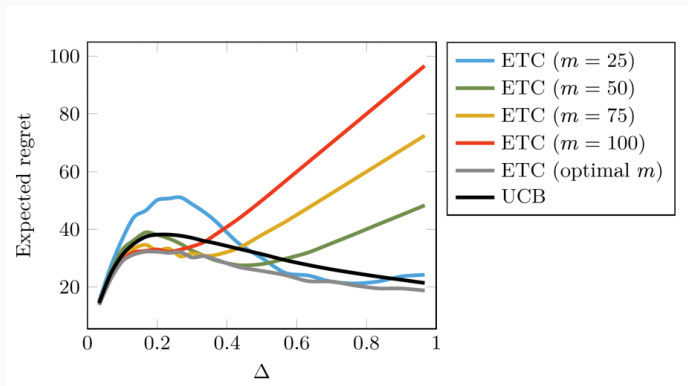


Figure 1: For $K = 2$, $T = 1000$, Gaussian rewards, comparing UCB and ETC(m) as a function of the gap Δ .

Improvements to UCB

- **Problem:** UCB requires knowledge of T and hence, the number of rounds needs to be fixed.
- *Sol:* Define UCB as $\hat{\mu}_a(t) + \sqrt{\frac{2 \log(f(t))}{N_a(t)}}$ where $f(t) := 1 + t \log^2(t)$. No dependence on T , but results in the same $O(\sqrt{KT \log(T)})$ worst-case regret. (see [LS20, Chapter 8])
- **Lower-Bound:** For a fixed T and for every bandit algorithm, there exists a stochastic bandit problem with rewards in $[0, 1]$ such that $\text{Regret}(T) = \Omega(\sqrt{KT})$. (see [LS20, Chapter 15]).
- **Problem:** UCB is sub-optimal by a $\sqrt{\log(T)}$ factor compared to the lower-bound. Is it possible to develop an algorithm that does not incur this log factor?
- *Sol:* [Lat18, MG17] propose modifications of UCB that achieve $O(\sqrt{KT})$ regret.

Stochastic Linear Bandits

Stochastic Linear Bandits

- MAB treat each arm (e.g. drug choice) independently. But the arms (and their rewards) can be dependent. E.g., drugs with similar chemical composition can have similar side-effects.
- Stochastic Linear Bandits can model linear dependence between different arms. For this, we require *feature vectors* $X_a \in \mathbb{R}^d$ for each arm $a \in [K]$.
- **Reward Model:** For an unknown vector $\theta^* \in \mathbb{R}^d$, the mean reward for arm a is given as: $\mu_a = \langle X_a, \theta^* \rangle$. Hence, arms with similar feature vectors will have similar mean rewards.
- Similar to the MAB setting, on pulling arm a_t at round t , we observe the reward $R_t = \mu_{a_t} + \eta_t = \langle X_{a_t}, \theta^* \rangle + \eta_t$. We will assume that η_t is conditionally 1 sub-Gaussian, i.e. if $\mathcal{H}_{t-1} := \{X_1, R_1, \dots, X_t\}$ is the *history* of interactions until round t , then for all $\lambda \in \mathbb{R}$, $\mathbb{E}[\exp(\lambda \eta_t) | \mathcal{H}_{t-1}] \leq \exp(\lambda^2/2)$.
- $\text{Regret}(T) := \sum_{t=1}^T [\max_{a \in [K]} \langle X_a, \theta^* \rangle - \mathbb{E}[R_t]] = T \max_{a \in [K]} \langle X_a, \theta^* \rangle - \sum_{t=1}^T \mathbb{E}[R_t]$.
- In the special case, when all the arms are independent, i.e. $d = K$ and $\forall a \in [K]$, $X_a = e_a$ where $\forall i \in [d], i \neq a, e_a[i] = 0$ and $e_a[a] = 1$. Hence, $\mu_a = \theta_a^*$ and the linear bandit setup strictly generalizes MAB.

Stochastic Linear Bandits – Estimating $\hat{\mu}_a(t)$

At round t , we have collected the following data: $\{X_s, R_s\}_{s=1}^t$. **Q:** How do we estimate $\hat{\mu}_a(t)$?

By solving regularized ridge regression, i.e. for a regularization parameter $\lambda \geq 0$,

$$\hat{\theta}_t := \arg \min_{\theta} \left\{ \frac{1}{2} \sum_{s=1}^t [\langle X_s, \theta \rangle - R_s]^2 + \frac{\lambda}{2} \|\theta\|^2 \right\}$$

Setting the derivative to zero to solve the above minimization problem,

$$\begin{aligned} \sum_{s=1}^t \left[X_s \left[\langle X_s, \hat{\theta}_t \rangle - R_s \right] \right] + \lambda \hat{\theta}_t &= 0 \\ \Rightarrow \underbrace{\left[\sum_{s=1}^t X_s X_s^T + \lambda I_d \right]}_{:= V_t \in \mathbb{R}^{d \times d}} \hat{\theta}_t &= \underbrace{\sum_{s=1}^t X_s R_s}_{:= b_t \in \mathbb{R}^{d \times 1}} \Rightarrow V_t \hat{\theta}_t = b_t \Rightarrow \hat{\theta}_t = V_t^{-1} b_t \end{aligned}$$

Hence, the empirical mean for each arm after t rounds: $\hat{\mu}_a = \langle X_a, \hat{\theta}_t \rangle = X_a^T V_t^{-1} b_t$

Algorithm Linear Upper Confidence Bound

- 1: **Input:** $\{\beta_t\}_{t=2}^{T+1}$, $V_0 = \lambda I_d \in \mathbb{R}^{d \times d}$, $b = 0 \in \mathbb{R}^d$
- 2: For each arm $a \in [K]$, initialize $U_a(1, \delta) := \infty$.
- 3: **for** $t = 1 \rightarrow T$ **do**
- 4: Select arm $a_t = \arg \max_{a \in [K]} U_a(t, \delta)$ (*Choose the lower-indexed arm in case of a tie*)
- 5: Observe reward R_t and update:

$$V_t = V_{t-1} + X_t X_t^T \quad ; \quad b_t = b_{t-1} + R_t X_t \quad ; \quad \hat{\theta}_t = V_t^{-1} b_t$$
$$U_a(t+1) = \langle X_a, \hat{\theta}_t \rangle + \sqrt{\beta_{t+1}} \|X_a\|_{V_t^{-1}} \quad \quad \text{(where } \|x\|_A := \sqrt{x^T A x}\text{)}$$

6: **end for**

In the special case, when all the arms are independent, Linear UCB with $\beta_t = \beta = 2 \log(1/\delta)$ is equivalent to UCB, and hence, Linear UCB strictly generalizes UCB.

Prove this in Assignment 1!

Linear UCB – Regret Analysis

Claim: $U_a(t+1) := \langle X_a, \hat{\theta}_t \rangle + \sqrt{\beta_{t+1}} \|X_a\|_{V_t^{-1}} = \max_{\theta \in \mathcal{C}_{t+1}} \langle \theta, X_a \rangle$ where $\mathcal{C}_{t+1} = \left\{ \theta \mid \left\| \theta - \hat{\theta}_t \right\|_{V_t}^2 \leq \beta_{t+1} \right\}$.

\mathcal{C}_{t+1} is an ellipsoid centered at $\hat{\theta}_t$ with the principle axes being the eigenvectors of V_t and the corresponding lengths being the reciprocal of the eigenvalues. As t increases, the eigenvalues of matrix V_t increases and the volume of the ellipsoid decreases.

Prove this in Assignment 1! For the subsequent proof, we will use this equivalence.

Claim: Assuming (i) $\|\theta^*\| \leq 1$, (ii) $\|X_a\| \leq 1$ for all a and (iii) $R_t \in [0, 1]$, UCB with $\sqrt{\beta_t} = \sqrt{d \log \left(\frac{\lambda d + t}{\lambda d} \right) + 2 \log(1/\delta) + \sqrt{\lambda}}$ achieves the following worst-case bound on the regret,

$$\text{Regret}(\text{LinUCB}, T) \leq O\left(d\sqrt{T} \log(T)\right)$$

Linear UCB – Regret Analysis

Proof: Define a “good” event $G := \{\forall t \in [T] | \theta^* \in \mathcal{C}_t := \left\{ \theta \mid \left\| \theta - \hat{\theta}_{t-1} \right\|_{V_{t-1}}^2 \leq \beta_t \right\},$ and denote the instantaneous expected regret at round t as $r_t = \max_a \langle X_a, \theta^* \rangle - \langle X_t, \theta^* \rangle$. Using the law of total expectation,

$$\begin{aligned} \text{Regret}(\text{LinUCB}, T) &= \mathbb{E}[\text{Regret}(\text{LinUCB}, T) | G] \Pr[G] + \mathbb{E}[\text{Regret}(T) | G^c] \Pr[G^c] \\ &\leq \mathbb{E}[\text{Regret}(\text{LinUCB}, T) | G] + T \Pr[G^c] \\ &\quad (\text{Regret}(\text{LinUCB}, T) \leq T \text{ and } \Pr[G] \leq 1) \\ &= \sum_{t=1}^T \mathbb{E}[r_t | G] + T \Pr[G^c] \leq \sqrt{T \sum_{t=1}^T [\mathbb{E}[r_t | G]]^2} + T \Pr[G^c] \\ &\quad (\text{Cauchy Schwarz inequality: } \langle x, y \rangle \leq \|x\| \|y\| \text{ with } x, y \in \mathbb{R}^T \text{ and } x[t] = 1, y[t] = r_t) \end{aligned}$$

Linear UCB – Regret Analysis

Recall that $\text{Regret}(\text{LinUCB}, T) \leq \sqrt{T \sum_{t=1}^T [\mathbb{E}[r_t|G]]^2} + T \Pr[G^c]$. Let us first bound $\mathbb{E}[r_t|G]$. If event G happens, then $\theta^* \in \mathcal{C}_t$. Hence, for all $a \in [K]$,

$$\langle \theta^*, X_a \rangle \leq \max_{\theta \in \mathcal{C}_t} \langle \theta, X_a \rangle = U_a(t) \leq U_{a_t}(t)$$

(Using the equivalence on Slide 15 and the algorithm)

$$\implies \max_a \langle \theta^*, X_a \rangle \leq U_{a_t}(t) = \max_{\theta \in \mathcal{C}_t} \langle \theta, X_t \rangle = \langle \tilde{\theta}_t, X_t \rangle \quad (\tilde{\theta}_t := \arg \max_{\theta \in \mathcal{C}_t} \langle \theta, X_t \rangle)$$

$$\implies \mathbb{E}[r_t|G] = \mathbb{E}[\max_a \langle X_a, \theta^* \rangle - \langle X_t, \theta^* \rangle | G] \leq \mathbb{E}[\langle \tilde{\theta}_t - \theta^*, X_t \rangle | G]$$

$$\leq \mathbb{E} \left[\left\| \tilde{\theta}_t - \theta^* \right\|_{V_{t-1}^{-1}} \left\| X_t \right\|_{V_{t-1}^{-1}} | G \right]$$

(Cauchy Schwarz inequality with $x, y \in \mathbb{R}^d$ and $x = V_{t-1}^{1/2}(\tilde{\theta}_t - \theta^*)$, $y = V_{t-1}^{-1/2} X_t$)

$$\leq \mathbb{E} \left[\left[\left\| \tilde{\theta}_t - \hat{\theta}_{t-1} \right\|_{V_{t-1}^{-1}} + \left\| \theta^* - \hat{\theta}_{t-1} \right\|_{V_{t-1}^{-1}} \right] \left\| X_t \right\|_{V_{t-1}^{-1}} | G \right] \quad (\Delta \text{ inequality})$$

$$\implies \mathbb{E}[r_t|G] \leq 2\sqrt{\beta_t} \mathbb{E} \left[\left\| X_t \right\|_{V_{t-1}^{-1}} | G \right] \quad (\text{Since } \theta^*, \tilde{\theta}_t \in \mathcal{C}_t)$$

Linear UCB – Regret Analysis

Putting everything together,

$$\begin{aligned}\text{Regret}(\text{LinUCB}, T) &\leq \sqrt{T \sum_{t=1}^T [\mathbb{E}[r_t | G]]^2} + T \Pr[G^c] \leq 2 \sqrt{T \sum_{t=1}^T \beta_t \mathbb{E}[\|X_t\|_{V_{t-1}^{-1}}^2]} + T \Pr[G^c] \\ &\leq 2 \sqrt{T \beta_T \mathbb{E}\left[\sum_{t=1}^T \|X_t\|_{V_{t-1}^{-1}}^2 \mid G\right]} + T \Pr[G^c] \\ &\hspace{20em} (\text{Since } \beta_t \leq \beta_T \text{ for all } t \in [T])\end{aligned}$$

We will prove the following results: (i) $\sum_{t=1}^T \|X_t\|_{V_{t-1}^{-1}}^2 \leq 2d \log\left(\frac{\lambda d + T}{\lambda d}\right)$ deterministically and (ii) $\sqrt{\beta_t} = \sqrt{d \log\left(\frac{\lambda d + t}{\lambda d}\right) + 2 \log(T) + \sqrt{\lambda}}$, $\Pr[G^c] \leq \frac{1}{T}$.

Given these results,

$$\text{Regret}(\text{LinUCB}, T) \leq 2 \sqrt{2d T \beta_T \log\left(\frac{\lambda d + T}{\lambda d}\right)} + 1 = O\left(d\sqrt{T} \log(T)\right) \quad \square$$

Linear UCB – Regret Analysis

Claim: If $\|X_a\| \leq 1$ for all a , $\sum_{t=1}^T \|X_t\|_{V_{t-1}^{-1}}^2 \leq 2d \log\left(\frac{\lambda d + T}{\lambda d}\right)$.

Proof:

$$\begin{aligned} V_t &= V_{t-1} + X_t X_t^\top = V_{t-1}^{1/2} \left[I_d + V_{t-1}^{-1/2} X_t X_t^\top V_{t-1}^{-1/2} \right] V_{t-1}^{1/2} \\ \implies \det[V_t] &= \det[V_{t-1}^{1/2}] \det \left[I_d + V_{t-1}^{-1/2} X_t X_t^\top V_{t-1}^{-1/2} \right] \det[V_{t-1}^{1/2}] \\ &\hspace{20em} (\det[XY] = \det[X] \det[Y]) \\ &= \det[V_{t-1}] \det \left[I_d + V_{t-1}^{-1/2} X_t [V_{t-1}^{-1/2} X_t]^\top \right] (\det[X^{1/2}] = \sqrt{\det[X]}) \\ &= \det[V_{t-1}] \left(1 + \left\| V_{t-1}^{-1/2} X_t \right\|^2 \right) = \det[V_{t-1}] \left(1 + \|X_t\|_{V_{t-1}^{-1}}^2 \right) \\ &\hspace{2em} (\text{Matrix Determinant Lemma: } \det[I_d + x x^\top] = 1 + x^\top x = 1 + \|x\|^2) \\ \implies \ln \left(1 + \|X_t\|_{V_{t-1}^{-1}}^2 \right) &= \ln \left(\frac{\det[V_t]}{\det[V_{t-1}]} \right) \end{aligned}$$

Linear UCB – Regret Analysis




Recall that $\ln \left(1 + \|X_t\|_{V_{t-1}^{-1}}^2 \right) = \ln \left(\frac{\det[V_t]}{\det[V_{t-1}]} \right)$.

Hence, $\sum_{t=1}^T \ln \left(1 + \|X_t\|_{V_{t-1}^{-1}}^2 \right) = \ln \left(\frac{\det[V_T]}{\det[V_0]} \right)$. For any $x \geq 0$, $x \leq 2 \ln(1 + x)$. Hence, $\sum_{t=1}^T \|X_t\|_{V_{t-1}^{-1}}^2 \leq 2 \sum_{t=1}^T \ln(1 + \|X_t\|_{V_{t-1}^{-1}}^2)$, implying,

$$\sum_{t=1}^T \|X_t\|_{V_{t-1}^{-1}}^2 \leq 2 \sum_{t=1}^T \ln(1 + \|X_t\|_{V_{t-1}^{-1}}^2) = 2 \ln \left(\frac{\det[V_T]}{\det[V_0]} \right)$$

$$\begin{aligned} \det[V_T] &\leq \left(\frac{\text{Tr}[V_T]}{d} \right)^d \quad (\det[A] = \prod \lambda_i = \left((\prod \lambda_i)^{1/d} \right)^d \leq \left(\frac{\sum \lambda_i}{d} \right)^d = \left(\frac{\text{Tr}[A]}{d} \right)^d) \\ &= \left(\frac{\text{Tr}[V_0] + \sum_{t=1}^T X_t X_t^\top}{d} \right)^d \leq \left(\frac{\text{Tr}[V_0] + T}{d} \right)^d = \left(\frac{d\lambda + T}{d} \right)^d \\ &\quad \text{(Since } \|X_t\| \leq 1) \end{aligned}$$

$$\Rightarrow \sum_{t=1}^T \|X_t\|_{V_{t-1}^{-1}}^2 \leq 2 \ln \left(\left(\frac{(d\lambda + T)/d}{(\det[V_0])^{1/d}} \right)^d \right) = 2d \log \left(\frac{\lambda d + T}{\lambda d} \right) \quad \square$$

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