CMPT 419/983: Theoretical Foundations of Reinforcement Learning

Lecture 8

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October 27, 2023

Recap

- Approximate policy iteration (API) aims to find an optimal policy without access to $\mathcal{P}$, r.
- API alternates between policy evaluation and policy improvement: at iteration k,
  - **Policy Evaluation**: Compute the estimate $\hat{q}^{\pi_k}$ (for example, using TD, Monte-Carlo).
  - **Policy Improvement**: $\forall s$, $\pi_{k+1}(s) = \arg\max_a \hat{q}^{\pi_k}(s,a)$.
- If the policy evaluation error at iteration k is controlled s.t. $\hat{q}^{\pi_k} = q^{\pi_k} + \epsilon_k$, then, API has the following convergence, $\left\|v^{\pi_{K+1}} - v^*\right\|_\infty \leq \gamma^K \left\|v^{\pi_0} - v^*\right\|_\infty + \frac{2\max_{k\in\{0,\ldots,K-1\}}\|\epsilon_k\|_\infty}{(1-\gamma)^2}$
- We have access to $\Phi \in \mathbb{R}^{SA\times d}$ s.t. for every π, there exists a θ^* such that, $\max_{(s,a)} |q^\pi(s,a) - \langle\theta^*, \phi(s,a)\rangle| \leq \varepsilon_b$.
- In order to control the policy evaluation error,
  - Choose $\mathcal{C} \subset \mathcal{S}\times\mathcal{A}$, and for each $z := (s,a) \in \mathcal{C}$, rollout m trajectories (truncated to horizon H) and calculate $\hat{R}(z)$. We can ensure that $|\hat{R}(z) - q^\pi(z)| \leq \varepsilon_s$ w.p. $1-\delta$ for all $z \in \mathcal{C}$.
  - Estimate $\hat{\theta} := \arg\min_\theta \frac{1}{2}\sum_{z\in\mathcal{C}} \zeta(z) \left[\langle\theta, \phi(z)\rangle - \hat{R}(z)\right]^2$.

Policy Evaluation for Approximate Policy Iteration

Claim: Assuming $V := \sum_{z \in \mathcal{C}} \zeta(z)\, \phi(z)\phi(z)^T \in \mathbb{R}^{d \times d}$ is invertible, for any $z \in \mathcal{S} \times \mathcal{A}$,

$$|q^\pi(z) - \langle \hat{\theta}, \phi(z) \rangle| \leq \varepsilon_{\mathbf{b}} + \|\phi(z)\|_{V^{-1}} \left[\varepsilon_{\mathbf{s}} + \varepsilon_{\mathbf{b}}\right]$$

Proof: Since $\hat{\theta}$ is computed by minimizing $\frac{1}{2} \sum_{z \in \mathcal{C}} \zeta(z) \left[\langle \theta, \phi(z) \rangle - \hat{R}(z)\right]^2$ and V is invertible,

$$\hat{\theta} = V^{-1} \left[\sum_{z' \in \mathcal{C}} \zeta(z')\, \hat{R}(z')\, \phi(z')\right]$$

$$|q^\pi(z) - \langle \hat{\theta}, \phi(z) \rangle| = |q^\pi(z) - \langle \theta^*, \phi(z) \rangle + \langle \theta^*, \phi(z) \rangle - \langle \hat{\theta}, \phi(z) \rangle|$$

(Add/subtract $\langle \theta^*, \phi(z) \rangle$)

$$\leq |q^\pi(z) - \langle \theta^*, \phi(z) \rangle| + |\langle \theta^*, \phi(z) \rangle - \langle \hat{\theta}, \phi(z) \rangle|$$

(Triangle inequality)

$$\implies |q^\pi(z) - \langle \hat{\theta}, \phi(z) \rangle| \leq \varepsilon_{\mathbf{b}} + |\langle \theta^*, \phi(z) \rangle - \langle \hat{\theta}, \phi(z) \rangle|$$

We will now bound $|\langle \theta^*, \phi(z) \rangle - \langle \hat{\theta}, \phi(z) \rangle|$.

Policy Evaluation for Approximate Policy Iteration

For $z' \in \mathcal{C}$, define $\mathcal{E}(z') := \hat{R}(z') - \langle \theta^*, \phi(z') \rangle$. Hence,

$$\hat{\theta} = V^{-1} \left[\sum_{z' \in \mathcal{C}} \zeta(z') \left[\langle \theta^*, \phi(z') \rangle + \mathcal{E}(z') \right] \phi(z') \right]$$

$$= V^{-1} \left[\sum_{z' \in \mathcal{C}} \zeta(z') \, \phi(z') \, \phi(z')^T \right] \theta^* + V^{-1} \left[\sum_{z' \in \mathcal{C}} \zeta(z') \, \mathcal{E}(z') \, \phi(z') \right]$$

$$\implies \hat{\theta} - \theta^* = V^{-1} \left[\sum_{z' \in \mathcal{C}} \zeta(z') \, \mathcal{E}(z') \, \phi(z') \right]$$

Hence, for an arbitrary $z \in \mathcal{S} \times \mathcal{A}$,

$$|\langle \theta^*, \phi(z) \rangle - \langle \hat{\theta}, \phi(z) \rangle| = \left| \left\langle V^{-1} \left[\sum_{z' \in \mathcal{C}} \zeta(z') \, \mathcal{E}(z') \, \phi(z') \right], \phi(z) \right\rangle \right|$$

$$= \left| \left\langle \sum_{z' \in \mathcal{C}} \zeta(z') \, \mathcal{E}(z') \, V^{-1} \phi(z'), \phi(z) \right\rangle \right| = \left| \sum_{z' \in \mathcal{C}} \zeta(z') \, \mathcal{E}(z') \, \langle \phi(z), V^{-1} \phi(z') \rangle \right|$$

Policy Evaluation for Approximate Policy Iteration

Recall that $|\langle\theta^*, \phi(z)\rangle - \langle\hat{\theta}, \phi(z)\rangle| = \left|\sum_{z'\in\mathcal{C}} \zeta(z')\,\mathcal{E}(z')\,\langle\phi(z), V^{-1}\phi(z')\rangle\right|$.

$$\begin{aligned}|\langle\theta^*, \phi(z)\rangle - \langle\hat{\theta}, \phi(z)\rangle| &\leq \sum_{z'\in\mathcal{C}} |\mathcal{E}(z')|\,\zeta(z')\,\left|\langle\phi(z), V^{-1}\phi(z')\rangle\right| \\ &\leq \left(\max_{z'\in\mathcal{C}}|\mathcal{E}(z')|\right) \sum_{z'\in\mathcal{C}} \zeta(z')\,\left|\langle\phi(z), V^{-1}\phi(z')\rangle\right|\end{aligned}$$

$$\sum_{z'\in\mathcal{C}} \zeta(z')\,\left|\langle\phi(z), V^{-1}\phi(z')\rangle\right| = \sqrt{\left(\mathbb{E}_{z'\sim\zeta}\,|\langle\phi(z), V^{-1}\phi(z')\rangle|\right)^2} \overset{\text{Jensen}}{\leq} \sqrt{\mathbb{E}_{z'}\,|\langle\phi(z), V^{-1}\phi(z')\rangle|^2}$$

$$= \sqrt{\mathbb{E}_{z'}\left[\phi(z)^T\, V^{-1}\,\phi(z')\phi(z')^T\, V^{-1}\,\phi(z)\right]} = \sqrt{\phi(z)^T\, V^{-1}\left[\sum_{z'}\zeta(z')\phi(z')\phi(z')^T\right] V^{-1}\,\phi(z)}$$

$$\implies \sum_{z'\in\mathcal{C}} \zeta(z')\,\left|\langle\phi(z), V^{-1}\phi(z')\rangle\right| = \sqrt{\phi(z)^T V^{-1}\phi(z)} = \|\phi(z)\|_{V^{-1}}$$

$$\implies |\langle\theta^*, \phi(z)\rangle - \langle\hat{\theta}, \phi(z)\rangle| \leq \max_{z'\in\mathcal{C}}|\mathcal{E}(z')|\;\|\phi(z)\|_{V^{-1}}$$

Policy Evaluation for Approximate Policy Iteration

Recall that $|\langle\theta^*, \phi(z)\rangle - \langle\hat{\theta}, \phi(z)\rangle| \leq \max_{z' \in \mathcal{C}} |\mathcal{E}(z')| \, \|\phi(z)\|_{V^{-1}}$. Bounding $\max_{z' \in \mathcal{C}} |\mathcal{E}(z')|$,

$$
\begin{aligned}
|\mathcal{E}(z')| &= |\hat{R}(z) - \langle\theta^*, \phi(z)\rangle| = |\hat{R}(z) - q^\pi(z) + q^\pi(z) - \langle\theta^*, \phi(z)\rangle| \\
&\hspace{20em} \text{(Add/subtract } q^\pi(z)\text{)} \\
&\leq |\hat{R}(z) - q^\pi(z)| + |q^\pi(z) - \langle\theta^*, \phi(z)\rangle| \qquad \text{(Triangle inequality)} \\
&\leq \varepsilon_{\mathbf{s}} + \varepsilon_{\mathbf{b}}
\end{aligned}
$$

$$
\implies |\langle\theta^*, \phi(z)\rangle - \langle\hat{\theta}, \phi(z)\rangle| \leq [\varepsilon_{\mathbf{s}} + \varepsilon_{\mathbf{b}}] \, \|\phi(z)\|_{V^{-1}}
$$

Putting everything together,

$$
|q^\pi(z) - \langle\hat{\theta}, \phi(z)\rangle| \leq \varepsilon_{\mathbf{b}} + [\varepsilon_{\mathbf{s}} + \varepsilon_{\mathbf{b}}] \, \|\phi(z)\|_{V^{-1}}
$$

Hence, in order to control the generalization error, we have to control $\|\phi(z)\|_{V^{-1}}$, while controlling the size of $\mathcal{C}$.

Policy Evaluation for Approximate Policy Iteration

Kiefer-Wolfowitz Theorem: There exists a $\mathcal{C} \subset \mathcal{S} \times \mathcal{A}$ and a distribution $\zeta \in \Delta_{|\mathcal{C}|}$ such that for $V := \sum_{z \in \mathcal{C}} \zeta(z)\, \phi(z)\phi(z)^T \in \mathbb{R}^{d \times d}$,

$$\sup_{z \in \mathcal{S} \times \mathcal{A}} \|\phi(z)\|_{V^{-1}} \leq \sqrt{d} \quad ; \quad |\mathcal{C}| \leq \frac{d\,(d+1)}{2}$$

- Intuitively, this means that we can find a *coreset* of feature vectors that captures most of the information in Φ. Finding such a coreset is referred to as *G-optimal design* in statistics.
- $\mathcal{C}$ and ζ can be approximately computed using a greedy algorithm that has access to Φ (Need to do this in Assignment 3!)

Combining the Kiefer-Wolfowitz theorem with our previous result gives,

$$|q^\pi(z) - \hat{q}^\pi(z)| = |q^\pi(z) - \langle \hat{\theta}, \phi(z) \rangle| \leq \varepsilon_{\mathbf{b}} + \sqrt{d}\,[\varepsilon_{\mathbf{s}} + \varepsilon_{\mathbf{b}}] = \varepsilon_{\mathbf{b}}\left(1 + \sqrt{d}\right) + \varepsilon_{\mathbf{s}}\sqrt{d}$$

- Note that the $\sqrt{d}$ amplification in the error is tight.

- Algorithmically, we need to run Monte-Carlo estimation from $O(d^2)$ (s, a) pairs, and we can estimate $q^\pi(s, a)$ upto an $\varepsilon_{\mathbf{b}}\left(1 + \sqrt{d}\right) + \varepsilon_{\mathbf{s}}\sqrt{d}$ error for all (s, a) pairs.

Convergence of Approximate Policy Iteration

We have seen the following results:

$$\left\|v^{\pi_{k+1}} - v^*\right\|_\infty \leq \gamma^K \left\|v^{\pi_0} - v^*\right\|_\infty + \frac{2 \max_{k \in \{0,\ldots,K-1\}} \left\|\epsilon_k\right\|_\infty}{(1-\gamma)^2}$$

$$|q^\pi(s,a) - \hat{q}^\pi(s,a)| \leq \varepsilon_{\mathbf{b}} \left(1 + \sqrt{d}\right) + \varepsilon_{\mathbf{s}} \sqrt{d} \qquad \text{(for all } \pi \text{ and } (s,a) \text{ pairs)}$$

$$\implies \left\|v^{\pi_{k+1}} - v^*\right\|_\infty \leq \gamma^K \left\|v^{\pi_0} - v^*\right\|_\infty + \frac{2\varepsilon_{\mathbf{b}} \left(1 + \sqrt{d}\right) + 2\varepsilon_{\mathbf{s}} \sqrt{d}}{(1-\gamma)^2}$$

- If the q functions are exactly in the span of Φ, $\varepsilon_{\mathbf{b}} = 0$. For example, in the *tabular* setting where $d = S$ and the features are one hot vectors, the error depends on $\sqrt{S}\,\varepsilon_{\mathbf{s}}$.
- The algorithm for constructing $\mathcal{C}$ requires iterating through the states, and this can be inefficient. [YHAY+22] considers an online algorithm that does not require global access to the full Φ matrix, but has similar theoretical guarantees.
- Next, we will see an alternative algorithm – Politex that has slower convergence $[O(1/\sqrt{K})]$, but smaller error amplification $[O(1/(1-\gamma))]$.

Politex

Politex

- Like policy iteration, Politex alternates between evaluating the policy and updating it.
- Unlike policy iteration that uses a max over actions, Politex uses a softmax (multiplicative weights) to update the policy. This makes the resulting algorithm less aggressive.

Algorithm Politex

Input: MDP $M = (\mathcal{S}, \mathcal{A}, \rho)$, π_0, step-size η
for $k = 0 \to K - 1$ **do**
  Policy Evaluation: Compute the estimate $\hat{q}_k := \hat{q}^{\pi_k}$ (for example, using TD, Monte-Carlo) and define $\bar{q}_k = \sum_{i=0}^{k} \hat{q}_i$
  Policy Update: $\forall (s, a)$, $\pi_{k+1}(a|s) = \frac{\exp(\eta\, \bar{q}_k(s,a))}{\sum_{a'} \exp(\eta\, \bar{q}_k(s,a'))}$.
end for
Return the *mixture policy* $\bar{\pi}_K := \frac{\sum_{k=0}^{K-1} \pi_k}{K}$

- Politex returns the *mixture policy* $\bar{\pi}_K$ which corresponds to choosing a policy in $\{\pi_k\}_{k=0}^{K-1}$ uniformly at random.
- If $\bar{q}_k = \hat{q}_k$, Politex recovers policy iteration as $\eta \to \infty$ (Prove in Assignment 3!)

Convergence of Politex

Claim: If the policy evaluation error at iteration k is controlled s.t. $\hat{q}^k = q^{\pi_k} + \epsilon_k$, then Politex has the following convergence,

$$\left\| v^{\bar{\pi}_K} - v^* \right\|_\infty \leq \frac{\|\text{Regret}(K)\|_\infty}{(1-\gamma)\, K} + \frac{2 \max_{k \in \{0,\ldots,K-1\}} \|\epsilon_k\|_\infty}{(1-\gamma)},$$

where $\text{Regret}(K) = \sum_{k=0}^{K-1} \left[\mathcal{M}_{\pi^*} \hat{q}_k - \mathcal{M}_{\pi_k} \hat{q}_k\right] \in \mathbb{R}^S$ is the regret incurred by Politex on an online linear optimization problem for each state $s \in \mathcal{S}$.

- The error amplification only depends on $1/1-\gamma$, and thus Politex has a better dependence on ϵ compared to approximate policy iteration.
- Compared to policy iteration that has an γ^K convergence, the convergence for Politex depends on $\frac{\text{Regret}(K)}{K}$. We will show that $\text{Regret}(K) = O(\sqrt{K})$, and hence, the Politex achieves the slower $O(1/\sqrt{K})$ convergence.
- The above claim does not depend on the specific update rule of Politex, and can be used to prove convergence for alternative algorithms that have sublinear regret.

Convergence of Politex

Proof: $v^{\pi^*} - v^{\pi_k} = (I - \gamma \mathbf{P}_{\pi^*})^{-1} [\mathcal{T}_{\pi^*} v^{\pi_k} - v^{\pi_k}]$ (Value difference lemma)

Summing up from $k = 0$ to $k = K - 1$ and dividing by K,

$$v^{\pi^*} - \frac{\sum_{k=0}^{K-1} v^{\pi_k}}{K} = \frac{1}{K} (I - \gamma \mathbf{P}_{\pi^*})^{-1} \sum_{k=0}^{K-1} [\mathcal{T}_{\pi^*} v^{\pi_k} - v^{\pi_k}]$$

$$\implies v^{\pi^*} - v^{\bar{\pi}_K} = (I - \gamma \mathbf{P}_{\pi^*})^{-1} \sum_{k=0}^{K-1} [\mathcal{T}_{\pi^*} v^{\pi_k} - v^{\pi_k}] = \frac{1}{K} (I - \gamma \mathbf{P}_{\pi^*})^{-1} \sum_{k=0}^{K-1} [\mathcal{T}_{\pi^*} v^{\pi_k} - \mathcal{T}_{\pi_k} v^{\pi_k}]$$

(Since $v^{\bar{\pi}_K} = \frac{\sum_{k=\mathbf{0}}^{K-\mathbf{1}} v^{\pi_k}}{K}$ (Prove in Assignment 3!) and $v^{\pi} = \mathcal{T}_{\pi} v^{\pi}$)

$$= \frac{1}{K} (I - \gamma \mathbf{P}_{\pi^*})^{-1} \sum_{k=0}^{K-1} [\mathcal{M}_{\pi^*} q^{\pi_k} - \mathcal{M}_{\pi_k} q^{\pi_k}]$$

(Since $\mathcal{T}_{\pi} v = \mathcal{M}_{\pi}[r + \gamma \mathbb{P} v] = \mathcal{M}_{\pi} q$)

$$= \frac{1}{K} (I - \gamma \mathbf{P}_{\pi^*})^{-1} \sum_{k=0}^{K-1} [\mathcal{M}_{\pi^*} \hat{q}_k - \mathcal{M}_{\pi_k} \hat{q}_k] + \frac{1}{K} (I - \gamma \mathbf{P}_{\pi^*})^{-1} \sum_{k=0}^{K-1} \left[(\mathcal{M}_{\pi_k} - \mathcal{M}_{\pi^*}) \underbrace{(\hat{q}_k - q^{\pi_k})}_{=\epsilon_k} \right]$$

Convergence of Politex

$v^{\pi^*} - v^{\bar{\pi}_K} = \frac{1}{K}(I - \gamma \mathbf{P}_{\pi^*})^{-1} \sum_{k=0}^{K-1} [\mathcal{M}_{\pi^*} \hat{q}_k - \mathcal{M}_{\pi_k} \hat{q}_k] + \frac{1}{K}(I - \gamma \mathbf{P}_{\pi^*})^{-1} \sum_{k=0}^{K-1} [(\mathcal{M}_{\pi_k} - \mathcal{M}_{\pi^*}) \epsilon_k]$

Using the definition of Regret(K) and taking norms,

$$\left\| v^{\pi^*} - v^{\bar{\pi}_K} \right\|_\infty = \left\| \frac{1}{K}(I - \gamma \mathbf{P}_{\pi^*})^{-1} \text{Regret}(K) + \frac{1}{K}(I - \gamma \mathbf{P}_{\pi^*})^{-1} \sum_{k=0}^{K-1} [(\mathcal{M}_{\pi_k} - \mathcal{M}_{\pi^*}) \epsilon_k] \right\|_\infty$$

$$\leq \frac{1}{K} \left\| (I - \gamma \mathbf{P}_{\pi^*})^{-1} \text{Regret}(K) \right\|_\infty + \frac{1}{K} \left\| (I - \gamma \mathbf{P}_{\pi^*})^{-1} \sum_{k=0}^{K-1} [(\mathcal{M}_{\pi_k} - \mathcal{M}_{\pi^*}) \epsilon_k] \right\|_\infty \qquad \text{(Triangle inequality)}$$

$$\leq \frac{\|\text{Regret}(K)\|_\infty}{K(1-\gamma)} + \frac{1}{K(1-\gamma)} \left\| \sum_{k=0}^{K-1} [(\mathcal{M}_{\pi_k} - \mathcal{M}_{\pi^*}) \epsilon_k] \right\|_\infty \qquad \text{(Neumann series)}$$

$$\leq \frac{\|\text{Regret}(K)\|_\infty}{K(1-\gamma)} + \frac{1}{K(1-\gamma)} \sum_{k=0}^{K-1} \left[\|\mathcal{M}_{\pi_k} \epsilon_k\|_\infty + \|\mathcal{M}_{\pi^*} \epsilon_k\|_\infty \right] \qquad \text{(Triangle inequality)}$$

$$\leq \frac{\|\text{Regret}(K)\|_\infty}{K(1-\gamma)} + \frac{2 \max_{k \in \{0, \dots, K-1\}} \|\epsilon_k\|_\infty}{(1-\gamma)} \quad \square \qquad (\mathcal{M}_\pi \text{ is non-expansive})$$

Convergence of Politex

Our aim now is to control $\|\text{Regret}(K)\|_\infty$ where $\text{Regret}(K) = \sum_{k=0}^{K-1} [\mathcal{M}_{\pi^*} \hat{q}_k - \mathcal{M}_{\pi_k} \hat{q}_k]$. By definition, for an arbitrary vector $u \in \mathbb{R}^{S \times A}$, $(\mathcal{M}_\pi u)(s) = \sum_a \pi(a|s)\, u(s,a)$. Hence,

$$\|\text{Regret}(K))\|_\infty = \max_s \left| \sum_{k=0}^{K-1} \left[\sum_a \pi^*(a|s)\, \hat{q}_k(s,a) - \sum_a \pi_k(a|s)\, \hat{q}_k(s,a) \right] \right|$$

Define $R_K(\pi^*, s) := \sum_{k=0}^{K-1} \langle \pi^*(\cdot|s), \hat{q}_k(s,\cdot) \rangle - \langle \pi_k(\cdot|s), \hat{q}_k(s,\cdot) \rangle$

$$\implies \|\text{Regret}(K))\|_\infty = \max_s |R_K(\pi^*, s)|$$

To bound $R_K(\pi^*, s)$, we will cast Politex as an online linear optimization for each state $s \in \mathcal{S}$:

- In each iteration $k \in [K]$, Politex chooses a distribution $\pi_k(\cdot|s) \in \Delta_A$ for each state s.
- The "environment" chooses and reveals the vector $\hat{q}_k(s,\cdot) \in \mathbb{R}^A$ and Politex receives a reward $\langle \pi_k(\cdot|s), \hat{q}_k(s,\cdot) \rangle$.
- The aim is to do as well as the optimal policy π^* that receives a reward $\langle \pi^*(\cdot|s), \hat{q}_k(s,\cdot) \rangle$

Digression – Online Optimization

Online Optimization

1: **Input**: w_0, Algorithm $\mathcal{A}$, Convex set $\mathcal{W}$
2: **for** $k = 0, \ldots, K-1$ **do**
3: Algorithm $\mathcal{A}$ chooses point (decision) $w_k \in \mathcal{W}$
4: Environment chooses and reveals the (potentially adversarial) function $f_k : \mathcal{W} \to \mathbb{R}$
5: Algorithm receives a reward $f_k(w_k)$
6: **end for**

Application: Prediction from Expert Advice – Given n experts,
$\mathcal{W} = \Delta_n = \{w_i | w_i \geq 0 \; ; \; \sum_{i=1}^n w_i = 1\}$ and $f_k(w_k) = \langle c_k, w_k \rangle$ where c_k is the reward vector.

Application: Imitation Learning – Given access to an expert that knows what action $a \in [A]$ to take in each state $s \in \mathcal{S}$, learn a policy $\pi : \mathcal{S} \to \mathcal{A}$ that imitates the expert, i.e. we want that $\pi(a|s) \approx \pi_{\text{expert}}(a|s)$. Here, $w = \pi$ and $\mathcal{W} = \Delta_A \times \Delta_A \ldots \Delta_A$ (simplex for each state) and f_k is a measure of the (negative) discrepancy between π_k and π_{expert}.

Q: What is w, $\mathcal{W}$, f_k for Politex (for state s)?

Digression – Online Optimization

Recall that the sequence of functions $\{f_k\}_{k=0}^{K-1}$ is potentially adversarial and can depend on w_k.

Objective: Do well against the *best fixed decision in hindsight*, i.e. if we knew the entire sequence of functions beforehand, we would choose $w^* := \arg\max_{w \in \mathcal{W}} \sum_{k=0}^{K-1} f_k(w)$.

Regret: $R_K(w^*) := \sum_{k=0}^{K-1} [f_k(w^*) - f_k(w_k)]$

We want to design algorithms that achieve a *sublinear regret* (that grows as $o(T)$). A sublinear regret implies that the performance of our sequence of decisions is approaching that of w^*.

Q: What is "best" decision we want to compare against in Politex (for state s)?

Hence, bounding $R_K(\pi^*, s)$ for Politex is equivalent to bounding the regret for a sequence of linear functions of the form: $f_k(w) = \langle g_k, w \rangle$.

Digression – Online Optimization

The simplest algorithm that results in sublinear regret is *Online Gradient Ascent*.

Online Gradient Ascent: At iteration k, the algorithm chooses the point w_k. After the function f_k is revealed, the algorithm receives a reward $f_k(w_k)$ and uses it to compute

$$w_{k+1} = \Pi_{\mathcal{W}}[w_k + \eta_k \nabla f_k(w_k)]$$

where $\Pi_{\mathcal{W}}[x] = \arg\min_{y \in \mathcal{W}} \frac{1}{2} \|y - x\|_2^2$ is the Euclidean projection onto $\mathcal{W}$.

The Online Gradient Ascent update at iteration k can also be written as:

$$w_{k+1} = \underset{w \in \mathcal{W}}{\arg\max} \left[\langle \nabla f_k(w_k), w \rangle - \frac{1}{2\eta_k} \|w - w_k\|_2^2 \right]$$

In other words, gradient ascent moves in the direction of the gradient $\nabla f_k(w_k)$, while remaining "close" (in the Euclidean norm) to the previous iterate w_k.

Instead of using the Euclidean norm, we could measure the distance to w_k differently.

Digression – Online Optimization

- Online Mirror Ascent generalizes gradient ascent by choosing a strictly convex, differentiable function $\psi : \mathbb{R}^d \to \mathbb{R}$ to induce a distance metric. ψ is referred to as the *mirror map*.
- ψ induces the *Bregman divergence* $D_\psi(\cdot, \cdot)$, a distance metric between points x, y,

$$D_\psi(y, x) := \psi(y) - \psi(x) - \langle \psi(x), y - x \rangle .$$

Geometrically, $D_\psi(y, x)$ is the distance between the function $\psi(y)$ and the line $\psi(x) + \langle \nabla \psi(x), y - x \rangle$ which is tangent to the function at x.

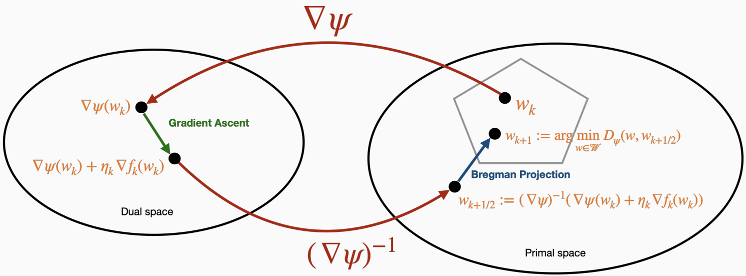
Using this distance measure results in the mirror ascent update:

$$w_{k+1} = \operatorname*{arg\,max}_{w \in \mathcal{W}} \left[\langle \nabla f_k(w_k), w \rangle - \frac{1}{\eta_k} D_\psi(w, w_k) \right]$$

- Setting $\psi(x) = \frac{1}{2} \|x\|^2$ results in $D_\psi(y, x) = \frac{1}{2} \|y - x\|^2$ and recovers gradient ascent.

The mirror ascent update can be equivalently written as:

$$w_{k+1/2} = (\nabla\psi)^{-1}\left(\nabla\psi(w_k) + \eta_k \nabla f_k(w_k)\right) \; ; \quad w_{k+1} = \arg\min_{w \in \mathcal{W}} D_\psi(w, w_{k+1/2})$$



Prove in Assignment 3!

Digression – Online Optimization

In order to analyze mirror ascent, we will make some assumptions on f_k and ψ.

• We will assume that $\{f_k\}_{k=0}^{K-1}$ are linear i.e. for some vector g_k, $f_k(w) = \langle g_k, w \rangle$. We will also assume that $\{f_k\}_{k=0}^{K-1}$ are G-Lipschitz continuous.

Lipschitz continuous functions: f is *Lipschitz continuous* iff f can not change arbitrarily fast meaning that its gradient is bounded. Formally, for any $w \in \mathcal{W}$,

$$\|\nabla f(w)\|_\infty \leq G$$

where G is the Lipschitz constant.

• We will assume that ψ is ν strongly-convex.

Strongly-convex functions: If f is differentiable, it is ν-strongly convex iff its domain $\mathcal{D}$ is a convex set and for all $x, y \in \mathcal{D}$ and $\nu > 0$,

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\nu}{2} \|y - x\|_1^2$$

i.e. for all y, the function is lower-bounded by the quadratic defined in the RHS.

Digression – Online Optimization

Claim: For G-Lipschitz linear functions $\{f_k\}_{k=0}^{K-1}$ such that $f_k(w) = \langle g_k, w \rangle$, online mirror ascent with a ν strongly-convex mirror map ψ, $\eta_k = \eta = \sqrt{\frac{2\nu}{K}} \frac{D}{G}$ where $D^2 := \max_{u \in \mathcal{W}} D_\psi(u, w_0)$ has the following regret for all $u \in \mathcal{W}$,

$$R_K(u) \leq \frac{\sqrt{2}\, DG}{\sqrt{\nu}} \sqrt{K}\,,$$

Proof: Recall the mirror ascent update: $\nabla \phi(w_{k+1/2}) = \nabla \phi(w_k) + \eta_k \nabla f_k(w_k)$.
Setting $\eta_k = \eta$ and using the definition of regret

$$R_K(u) = \textstyle\sum_{k=0}^{K-1} [\langle g_k, u \rangle - \langle g_k, w_k \rangle] = \sum_{k=0}^{K-1} \frac{1}{\eta} \left\langle \nabla \psi(w_{k+1/2}) - \nabla \psi(w_k), u - w_k \right\rangle.$$

Using the three point Bregman property: for any 3 points x, y, z,
$\langle \nabla \psi(z) - \nabla \psi(y), z - x \rangle = D_\psi(x, z) + D_\psi(z, y) - D_\psi(x, y)$,

$$\left\langle \nabla \psi(w_{k+1/2}) - \nabla \psi(w_k), u - w_k \right\rangle = D_\psi(u, w_k) + D_\psi(w_k, w_{k+1/2}) - D_\psi(u, w_{k+1/2})$$

$$\implies R_K(u) = \sum_{k=0}^{K-1} \frac{1}{\eta} \left[D_\psi(u, w_k) + D_\psi(w_k, w_{k+1/2}) - D_\psi(u, w_{k+1/2}) \right]$$

Digression – Online Optimization

$R_K(u) = \sum_{k=0}^{K-1} \frac{1}{\eta} \left[D_\psi(u, w_k) + D_\psi(w_k, w_{k+1/2}) - D_\psi(u, w_{k+1/2})\right]$, $w_{k+1} = \arg\min_{w \in \mathcal{W}} D_\psi(w, w_{k+1/2})$.

Recall the optimality condition: if $x^* = \arg\min_{x \in \mathcal{X}} f(x)$, then $\forall x \in \mathcal{X}$, $\langle f(x^*), x^* - x\rangle \leq 0$.
Using the above condition for $f = D_\psi(w, w_{k+1/2})$ and $x^* = w_{k+1}$, we infer that for any $w \in \mathcal{W}$,

$$\left\langle \nabla\psi(w_{k+1}) - \nabla\psi(w_{k+1/2}), w_{k+1} - w \right\rangle \leq 0$$

$$\implies D_\psi(w, w_{k+1}) + D_\psi(w_{k+1}, w_{k+1/2}) - D_\psi(w, w_{k+1/2}) \leq 0 \quad \text{(3 point Bregman property)}$$

$$\implies -D_\psi(u, w_{k+1/2}) \leq -D_\psi(u, w_{k+1}) - D_\psi(w_{k+1}, w_{k+1/2}) \quad \text{(Setting } w = u\text{)}$$

Putting everything together,

$$R_K(u) \leq \sum_{k=0}^{K-1} \frac{1}{\eta} \left[D_\psi(u, w_k) - D_\psi(u, w_{k+1})\right] + \left[D_\psi(w_k, w_{k+1/2}) - D_\psi(w_{k+1}, w_{k+1/2})\right]$$

$$\leq \frac{1}{\eta} D_\psi(u, w_0) + \frac{1}{\eta} \sum_{k=0}^{K-1} \left[D_\psi(w_k, w_{k+1/2}) - D_\psi(w_{k+1}, w_{k+1/2})\right]$$

Digression – Online Optimization

Recall that $R_K(u) \leq \frac{1}{\eta} D_\psi(u, w_0) + \frac{1}{\eta} \sum_{k=0}^{K-1} \left[D_\psi(w_k, w_{k+1/2}) - D_\psi(w_{k+1}, w_{k+1/2})\right]$. By def. of D_ψ,

$$
\begin{aligned}
D_\psi(w_k, w_{k+1/2}) - D_\psi(w_{k+1}, w_{k+1/2}) &= \psi(w_k) - \psi(w_{k+1}) - \langle \nabla\psi(w_{k+1/2}), w_k - w_{k+1}\rangle \\
&\leq \langle \nabla\psi(w_k) - \nabla\psi(w_{k+1/2}), w_k - w_{k+1}\rangle - \frac{\nu}{2} \left\|w_k - w_{k+1}\right\|_1^2 \\
&\qquad \text{(Using strong-convexity of } \psi \text{ with } y = w_{k+1} \text{ and } x = w_k) \\
&= -\eta\langle g_k, w_k - w_{k+1}\rangle - \frac{\nu}{2} \left\|w_k - w_{k+1}\right\|_1^2 \qquad \text{(Using the mirror ascent update)} \\
&\leq \eta G \left\|w_k - w_{k+1}\right\|_1 - \frac{\nu}{2} \left\|w_k - w_{k+1}\right\|_1^2 \\
&\qquad \text{(Holder's inequality: } \langle x, y\rangle \leq \|x\|_\infty \|y\|_1 \text{ and since } f_k \text{ is } G\text{-Lipschitz)} \\
&\leq \frac{\eta^2 G^2}{2\nu} \qquad \text{(For all } z,\ a\,z - bz^2 \leq \tfrac{a^2}{4b})
\end{aligned}
$$

$$
\implies R_K(u) \leq \frac{1}{\eta} D_\psi(u, w_0) + \frac{\eta G^2 K}{2\nu} \leq \frac{D^2}{\eta} + \frac{\eta G^2 K}{2\nu} \qquad \text{(Since } D_\psi(u, w_1) \leq D^2)
$$

$$
R_K(u) \leq \frac{\sqrt{2} D G}{\sqrt{\nu}} \sqrt{K} \quad \square \qquad \text{(Setting } \eta = \sqrt{\tfrac{2\nu}{K} \tfrac{D}{G}})
$$

Convergence of Politex

- We have proved that: For G-Lipschitz linear functions $\{f_k\}_{k=0}^{K-1}$ such that $f_k(w) = \langle g_k, w \rangle$, online mirror ascent with a ν strongly-convex mirror map ψ, $\eta_k = \eta = \sqrt{\frac{2\nu}{K}} \frac{D}{G}$ where $D^2 := \max_{u \in \mathcal{W}} D_\psi(u, w_1)$ has the following regret for all $u \in \mathcal{W}$, $R_K(u) \leq \frac{\sqrt{2}DG}{\sqrt{\nu}} \sqrt{K}$.
- For Politex (for $s \in \mathcal{S}$), $w = \pi_s := \pi(\cdot|s)$, $\mathcal{W} = \Delta_A$, $g_k = \hat{q}_k(s, \cdot)$ and $u = \pi_s^* := \pi^*(\cdot|s)$.

Claim 1: With the *negative entropy mirror map*: $\psi(\pi_s) = \sum_{a \in \mathcal{A}} \pi(a|s) \log(\pi(a|s))$, the corresponding Bregman divergence $D_\psi(\pi_s, \tilde{\pi}_s)$ is equal to the KL divergence: $\mathrm{KL}(\pi_s || \tilde{\pi}_s) = \sum_{a \in \mathcal{A}} \pi(a|s) \log\left(\pi(a|s) / \tilde{\pi}(a|s)\right)$.

Claim 2: Online mirror ascent with $w = \pi_s$, *negative entropy mirror map*, $\eta_k = \eta$ on linear losses $f_k(\pi_s) = \langle \pi(\cdot|s), \hat{q}_k(s, \cdot) \rangle$ is equivalent to the update for Politex (for state $s \in \mathcal{S}$).

Prove in Assignment 3!

Hence, Politex (for state $s \in \mathcal{S}$) has the following regret: $R_K(\pi_s^*) \leq \frac{\sqrt{2}DG}{\sqrt{\nu}} \sqrt{K}$. We now need to characterize the constants D, G, ν.

Convergence of Politex

Recall that Politex (for state $s \in \mathcal{S}$) has the following regret: $R_K(\pi_s^*) \leq \frac{\sqrt{2}DG}{\sqrt{\nu}} \sqrt{K}$.

- Recall that $D^2 = \max D_\psi(u, w_0) = \text{KL}(\pi^*(\cdot|s)||\pi_0(\cdot|s))$. For all $a \in \mathcal{A}$, choose $\pi_0(a|s) = \frac{1}{A}$ i.e. for each state, π_0 is a uniform distribution over actions. With this choice,

$$\text{KL}(\pi^*(\cdot|s)||\pi_0(\cdot|s)) = \sum_a \pi^*(a|s) \log\left(A\,\pi^*(a|s)\right) \leq \log\left(A \max_a \pi^*(a|s)\right) \sum_a \pi^*(a|s) \leq \log\left(A\right)$$

- Recall that $\|\nabla f(x)\|_\infty \leq G$. If the $\hat{q}_k(s,a)$ functions are constrained to lie in the $[0, 1/1-\gamma]$ interval, then $G = \frac{1}{1-\gamma}$.

- Recall that ν is the strong-convexity of ψ, i.e. the following inequality holds: $\psi(y) \geq \psi(x) + \langle \nabla\psi(x), y - x\rangle + \frac{\nu}{2}\|y - x\|_1^2$.

$$\psi(y) - \psi(x) - \langle \nabla\psi(x), y - x\rangle = D_\psi(y, x) = \text{KL}(y||x) \geq \frac{1}{2}\|y - x\|_1^2 \quad \text{(Pinsker's inequality)}$$

Hence, $\nu = 1$.

Convergence of Politex

Putting everything together, we can prove the following claim:

Claim: If $\hat{q}(s,a) \in [0, 1/1-\gamma]$ for all (s,a), Politex with $\pi_0(a|s) = \frac{1}{A}$ for all (s,a) and $\eta_k = \eta = \sqrt{\frac{2\,\log(A)}{K}}\,(1-\gamma)$ has the following regret,

$$R_K(\pi^*, s) \leq \frac{\sqrt{2\,\log(A)}}{1-\gamma}\sqrt{K} \implies \|\text{Regret}(K)\|_\infty = \frac{\sqrt{2\,\log(A)}}{1-\gamma}\sqrt{K}$$

Combining the above bound with the general result for Politex,

$$\left\|v^{\bar{\pi}_K} - v^*\right\|_\infty \leq \frac{\sqrt{2\,\log(A)}}{(1-\gamma)^2\sqrt{K}} + \frac{2\max_{k\in\{0,\dots,K-1\}}\|\boldsymbol{\epsilon}_k\|_\infty}{(1-\gamma)}$$

Controlling the policy evaluation error using G experimental design and Monte-Carlo estimation ensures that $\max_{k\in\{0,\dots,K-1\}}\|\boldsymbol{\epsilon}_k\|_\infty \leq \varepsilon_{\mathbf{b}}\left(1+\sqrt{d}\right) + \varepsilon_{\mathbf{s}}\sqrt{d}$.

$$\implies \left\|v^{\bar{\pi}_K} - v^*\right\|_\infty \leq \frac{\sqrt{2\,\log(A)}}{(1-\gamma)^2\sqrt{K}} + \frac{2\varepsilon_{\mathbf{b}}\left(1+\sqrt{d}\right) + 2\varepsilon_{\mathbf{s}}\sqrt{d}}{(1-\gamma)}$$

Dong Yin, Botao Hao, Yasin Abbasi-Yadkori, Nevena Lazić, and Csaba Szepesvári, *Efficient local planning with linear function approximation*, International Conference on Algorithmic Learning Theory, PMLR, 2022, pp. 1165–1192.