Homework 15.

① Найти A и B, чтобы при $x \to 0$

$$(A + B\cos x)\cdot\sin x = x + \bar{o}(x^4)$$

$$\left(A + B\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \bar{o}(x^4)\right)\right)\left(x - \frac{x^3}{6} + \bar{o}(x^4)\right) =$$

$$= \left(A + B - \frac{B}{2}x^2 + \frac{B}{24}x^4\right)\left(x - \frac{x^3}{6}\right) + \bar{o}(x^4) =$$

$$= Ax + Bx - \frac{B}{2}x^3 + \underbrace{\frac{B}{24}x^5}_{\bar{o}(x^4)} - \frac{A}{6}x^3 - \frac{B}{6}x^3 + \underbrace{\frac{B}{12}x^5}_{\bar{o}(x^4)} - \underbrace{\frac{B}{144}x^7}_{\bar{o}(x^4)} + \bar{o}(x^4) =$$

$$= (A+B)x + \left(-\frac{B}{2} - \frac{A}{6} - \frac{B}{6}\right)x^3 + \bar{o}(x^4)$$

$$\begin{cases} A + B = 1 \\ -\frac{B}{2} - \frac{A}{6} - \frac{B}{6} = 0 \quad |\cdot(-6) \end{cases} \quad \begin{cases} A = 1 - B \\ 4B + A = 0 \end{cases} \quad \begin{cases} A = 1 - B \\ 3B = -1 \end{cases} \quad \begin{cases} A = 4/3 \\ B = -1/3 \end{cases}$$

$$\Rightarrow x + \bar{o}(x^4) = \left(\frac{4}{3} - \frac{1}{3}\cos x\right)\sin x + \bar{o}(x^4)$$

$$\boxed{\text{Ответ: } A = 4/3,\ B = -1/3.}$$

② Вычислить с точностью 10^{-3} $\cos 72°$ $\left(72° = \frac{2\pi}{5}\right)$

Формула Тейлора с ост. членом в форме Лагранжа:

$$\cos x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + \frac{f^{(2n+1)}(x)}{(2n+1)!}x^{2n+1}$$

$$f^{(2n+1)}(x) = (-1)^{\frac{n(n+1)}{2}} \cdot \sin x$$

$$|f^{(2n+1)}(x)| = \left|(-1)^{\frac{n(n+1)}{2}}\sin x\right| = |\sin x| \leq 1$$

$$|r_n(x)| \Rightarrow |r_n(x)| = \left|\frac{f^{(2n+1)}(x)}{(2n+1)!}x^{2n+1}\right| \leq \left|\frac{x^{2n+1}}{(2n+1)!}\right|$$

$$r_n\left(\frac{2\pi}{5}\right) \le \left|\frac{\left(\frac{2\pi}{5}\right)^{2n+1}}{(2n+1)!}\right| = \frac{(2\pi)^{2n+1}}{(2n+1)!\cdot 5^{2n+1}}$$

$$n=3: \quad \frac{(2\pi)^7}{5^7\cdot 7!} = \frac{2\cdot 4\cdot 2^4\pi^7}{5^7\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} = \frac{2^4\pi^7}{5^8\cdot 3\cdot 6\cdot 7} < \frac{2^4\pi^6}{5^8\cdot 6\cdot 7} = \frac{8\cdot\pi^6}{5^8\cdot 21} = \left(\frac{2}{10}\right)^8\cdot\frac{2^3\pi^6}{21} =$$

$$= \left(\frac{1}{10}\right)^8\cdot\frac{2^{11}\pi^6}{21} < \frac{2^{11}\cdot 10^3}{21\cdot 10^8} = \frac{2048}{21}\cdot 10^{-5} < 0{,}001$$

$$\cos 72^\circ = \cos\left(\frac{2\pi}{5}\right) = \left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}\right] = 1-\left(\frac{2\pi}{5}\right)^2\cdot\frac{1}{2}+\frac{1}{24}\left(\frac{2\pi}{5}\right)^4-\frac{1}{720}\left(\frac{2\pi}{5}\right)^6 \approx$$

$$\approx 0{,}3088 \approx 0{,}309$$

$\boxed{\text{Ответ: } 0{,}309}$.

③ $$\lim_{x\to 0}\frac{\ln(e^{2x}+\sin x)-3\arcsin x+5\frac{x^2}{2}}{\sqrt[3]{8+x^3}-2} \,\ominus$$

1) $$\ln(e^{2x}+\sin x) = \ln\left(1+2x+\frac{4x^2}{2}+\frac{8x^3}{6}+\bar{o}(x^3)+x-\frac{x^3}{6}+\bar{o}(x^3)\right) =$$

$$= \ln\left(1+3x+2x^2+\frac{7}{6}x^3+\bar{o}(x^3)\right) = \left[3x+2x^2+\frac{7}{6}x^3+\bar{o}(x^3)=y\right] =$$

$$= \ln(1+y) = y-\frac{y^2}{2}+\frac{y^3}{3}+\bar{o}(y^3) =$$

$$= 3x+2x^2+\frac{7}{6}x^3+\bar{o}(x^3)-\frac{1}{2}\left(3x+2x^2+\frac{7}{6}x^3+\bar{o}(x^3)\right)^2 +$$

$$+\frac{1}{3}\left(3x+2x^2+\frac{7}{6}x^3+\bar{o}(x^2)\right)^3+\bar{o}(x^3) =$$

$$= 3x+2x^2+\frac{7}{6}x^3-\frac{1}{2}(9x^2+12x^3)+\frac{1}{3}\cdot 27x^3+\bar{o}(x^3) =$$

$$= 3x-\frac{5}{2}x^2+\frac{25}{6}x^3+\bar{o}(x^3)$$

2) $$\arcsin x = x+\frac{x^3}{6}+\bar{o}(x^3)$$

3) $$\sqrt[3]{8+x^3} = 2\sqrt[3]{1+\left(\frac{x}{2}\right)^3} = 2\left(1+\left(\frac{x}{2}\right)^3\right)^{\frac{1}{3}} = 2\left(1+C^1_{\frac{1}{3}}\left(\frac{x}{2}\right)^3+C^2_{\frac{1}{3}}\left(\frac{x}{2}\right)^6+C^3_{\frac{1}{3}}\left(\frac{x}{2}\right)^9+\bar{o}(x^3)\right) =$$

$$= 2\left(1+C^1_{\frac{1}{3}}\left(\frac{x}{2}\right)^3\right)+\bar{o}(x^3) = 2+\frac{2}{3}\cdot\frac{x^3}{8}+\bar{o}(x^3) = 2+\frac{x^3}{12}+\bar{o}(x^3)$$

$$\ominus \lim_{x\to 0}\frac{3x-\frac{5}{2}x^2+\frac{25}{6}x^3-3x-\frac{x^3}{2}+\frac{5}{2}x^2+\bar{o}(x^3)}{2+\frac{x^3}{12}+\bar{o}(x^3)-2} = \lim_{x\to 0}\frac{\frac{22}{6}x^3+\bar{o}(x^3)}{\frac{x^3}{12}+\bar{o}(x^3)} = \frac{22}{6}:\frac{1}{12} = \boxed{44}$$

④ $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{1+3x+x^2} + \sin(\ln(1-x)) - e^{-\frac{7x^2}{6}}}{x - \operatorname{arctg} x} \; (=)$

1) $\sqrt[3]{1+3x+x^2} = (1+(3x+x^2))^{\frac{1}{3}} = 1 + C_{\frac{1}{3}}^{1}(3x+x^2) + C_{\frac{1}{3}}^{2}(3x+x^2)^2 +$

$+ C_{\frac{1}{3}}^{3}(3x+x^2)^3 + \bar{o}(x^3) = 1 + \frac{1}{3}(3x+x^2) - \frac{1}{9}(3x+x^2)^2 +$

$+ (+\frac{5}{81})(3x+x^2)^3 + \bar{o}(x^3) =$

$= 1 + x + \frac{x^2}{3} + (-1)x^2 + \frac{-2}{3}x^3 + \frac{5}{3}x^3 + \bar{o}(x^3) =$

$= 1 + x + (-\frac{2}{3})x^2 + 1 \cdot x^3 + \bar{o}(x^3)$

2) $\sin(\ln(1-x)) = \ln(1-x) - \frac{(\ln(1-x))^3}{6} + \bar{o}(x^3) =$

$= -x - \frac{x^2}{2} - \frac{x^3}{3} + \bar{o}(x^3) - \frac{1}{6}(-x - \frac{x^2}{2} - \frac{x^3}{3} + \bar{o}(x^3))^3 + \bar{o}(x^3) =$

$= -x - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^3}{6} + \bar{o}(x^3) = -x - \frac{x^2}{2} - \frac{x^3}{6} + \bar{o}(x^3)$

3) $e^{-\frac{7x^2}{6}} = 1 - \frac{7x^2}{6} + \frac{1}{2}\left(\frac{7x^2}{6}\right)^2 - \frac{1}{6}\cdot\left(\frac{7x^2}{6}\right)^3 + \bar{o}(x^3)$

$= 1 - \frac{7}{6}x^2 + \bar{o}(x^3)$

4) $\operatorname{arctg} x = x - \frac{x^3}{3} + \bar{o}(x^3)$

$(=) \lim\limits_{x\to 0} \dfrac{1 + x - \frac{2}{3}x^2 + 1\cdot x^3 + \bar{o}(x^3) - x - \frac{x^2}{2} - \frac{x^3}{6} - 1 + \frac{7}{6}x^2}{x - x + \frac{x^3}{3} + \bar{o}(x^3)} =$

$= \lim\limits_{x\to 0} \dfrac{+\frac{5}{6}x^3 + \bar{o}(x^3)}{\frac{x^3}{3} + \bar{o}(x^3)} = \lim\limits_{x\to 0} \dfrac{+\frac{5}{6} + \bar{o}(1)}{\frac{1}{3} + \bar{o}(1)} = \frac{5}{6} \cdot \frac{3}{1} = \frac{5}{2} = \boxed{2,5}$

⑤ $\lim\limits_{x\to 0}\left(\dfrac{2e^{x-x^2}-2}{2x-x^2}\right)^{\frac{\sin x}{x^3}} \ominus$

1) $\dfrac{\sin x}{x^3} = \dfrac{1}{x^3}\left(x - \dfrac{x^3}{6} + \bar{o}(x^3)\right) = \dfrac{1}{x^2} - \dfrac{1}{6} + \bar{o}(1)$

2) $e^{x-x^2} = 1 + (x-x^2) + \dfrac{(x-x^2)^2}{2} + \dfrac{(x-x^2)^3}{6} + \bar{o}(x^3) =$

$= 1 + x - x^2 + \dfrac{x^2}{2} - x^3 + \dfrac{x^3}{6} + \bar{o}(x^3) = 1 + x - \dfrac{x^2}{2} - \dfrac{5}{6}x^3 + \bar{o}(x^3)$

3) $\dfrac{2e^{x-x^2}-2}{2x-x^2} = \dfrac{2+2x-x^2-\frac{5}{3}x^3+\bar{o}(x^3)-2}{2x-x^2} = \dfrac{2x-x^2}{2x-x^2} + \dfrac{\bar{o}(x^3)-\frac{5}{3}x^3}{2x-x^2} =$

$= 1 + \dfrac{\bar{o}(x^3)-\frac{5}{3}x^3}{2x-x^2}$

$\ominus \lim\limits_{x\to 0}\left(1 + \dfrac{\bar{o}(x^3)-\frac{5}{3}x^3}{2x-x^2}\right)^{\frac{\sin x}{x^3}} = \lim\limits_{x\to 0}\left(\underbrace{\left(1 + \dfrac{\bar{o}(x^3)-\frac{5}{3}x^3}{2x-x^2}\right)^{\frac{1}{\frac{\bar{o}(x^3)-\frac{5}{3}x^3}{2x-x^2}}}}_{\to e}\right)^{\frac{\bar{o}(x^3)-\frac{5}{3}x^3}{2x-x^2}\cdot\left(\frac{1}{x^2}-\frac{1}{6}+\bar{o}(1)\right)} =$

$= \lim\limits_{x\to 0} e^{\left(\frac{-\frac{5}{3}x+\frac{5}{18}x^3+\bar{o}(x^3)}{2x-x^2}\right)} = \lim\limits_{x\to 0} e^{\left(\frac{-\frac{5}{3}+\frac{5}{18}x^2+\bar{o}(x^2)}{2-x}\right)} = \boxed{e^{-\frac{5}{6}}}$

⑥ $\lim\limits_{x\to+\infty}\left(\frac{\sqrt{x^2-x}}{x}+\frac{1}{4}\sin\frac{2}{x}\right)^{x^2+\sin 3x} \ominus$

Замена: $x=\frac{1}{y}$

$\ominus \lim\limits_{y\to 0}\left(\frac{\sqrt{\frac{1}{y^2}-\frac{1}{y}}}{\frac{1}{y}}+\frac{1}{4}\sin(2y)\right)^{\frac{1}{y^2}+\sin\frac{3}{y}} \ominus$

1) $\frac{\sqrt{\frac{1}{y^2}-\frac{1}{y}}}{1/y}=y\sqrt{\frac{1}{y^2}-\frac{1}{y}}=\sqrt{1-y}=(1-y)^{\frac{1}{2}}=$

$=1+C^1_{\frac{1}{2}}\cdot(-y)+C^2_{\frac{1}{2}}\cdot(-y)^2+C^3_{\frac{1}{2}}\cdot(-y)^3+\bar{o}(y^3)=$

$=1-\frac{1}{2}y-\frac{1}{8}y^2-\frac{1}{16}y^3+\bar{o}(y^3)$

2) $\sin(2y)=2y-\frac{(2y)^3}{3!}+\bar{o}(y^3)=2y-\frac{4}{3}y^3+\bar{o}(y^3)$

3) $\sin\left(\frac{3}{y}\right)=\frac{3}{y}-\frac{(3/y)^3}{3!}+\bar{o}(y^3)=\frac{3}{y}-\frac{9}{2y^3}+\bar{o}(y^3)$

4) $\frac{\sqrt{\frac{1}{y^2}-1/y}}{1/y}+\frac{1}{4}\sin 2y=1-\frac{1}{2}y-\frac{1}{8}y^2-\frac{1}{16}y^3+\left(2y-\frac{4}{3}y^3\right)\cdot\frac{1}{4}+\bar{o}(y^3)=$

$=1-\frac{1}{8}y^2-\frac{19}{48}y^3+\bar{o}(y^3)$

$\ominus \lim\limits_{y\to 0}\left(1-\frac{1}{8}y^2-\frac{19}{48}y^3+\bar{o}(y^3)\right)^{\frac{1}{y^2}+\sin\frac{3}{y}}$

$=\lim\limits_{y\to 0}\left(\underbrace{\left(1-\frac{1}{8}y^2-\frac{19}{48}y^3+\bar{o}(y^3)\right)^{\frac{1}{-\frac{1}{8}y^2-\frac{19}{48}y^3+\bar{o}(y^3)}}}_{\to e}\right)^{\left(-\frac{1}{8}y^2-\frac{19}{48}y^3+\bar{o}(y^3)\right)\left(\frac{1}{y^2}+\sin\frac{3}{y}\right)}=$

$=\lim\limits_{y\to 0}e^{\left(-\frac{1}{8}-\frac{19}{48}y+\bar{o}(y)-\frac{1}{8}\sin\left(\frac{3}{y}\right)\cdot y^2-\frac{19}{48}y^3\cdot\sin\frac{3}{y}\right)}=\boxed{e^{-\frac{1}{8}}}$

⑦ $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\ln(ctg\,x)+2x-\frac{\pi}{2}}{(1-tg\,x)^2} =$

Замена: $x = t+\frac{\pi}{4}$; $t = x-\frac{\pi}{4} \to 0$

$= \lim\limits_{t\to 0} \dfrac{\ln(ctg(t+\frac{\pi}{4}))+2(t+\frac{\pi}{4})-\frac{\pi}{2}}{(1-tg(t+\frac{\pi}{4}))^2} = \lim\limits_{t\to 0} \dfrac{\ln\left(\frac{ctg\,t-1}{ctg\,t+1}\right)+2t}{\left(1-\frac{tg+1}{1-tg\,t}\right)^2} =$

1) $\dfrac{tg\,t+1}{1-tg\,t} = \dfrac{\frac{\sin t}{\cos t}+1}{1-\frac{\sin t}{\cos t}} = \dfrac{\sin t+\cos t}{\cos t-\sin t}$

2) $\dfrac{ctg\,t-1}{ctg\,t+1} = \dfrac{\frac{\cos t}{\sin t}-1}{\frac{\cos t}{\sin t}+1} = \dfrac{\cos t-\sin t}{\cos t+\sin t}$

3) $\sin t+\cos t = t-\frac{t^3}{3!}+1-\frac{t^2}{2}+\bar{o}(t^3) = 1+t-\frac{t^2}{2}-\frac{t^3}{6}+\bar{o}(t^3)$

4) $\cos t-\sin t = 1-\frac{t^2}{2}-t+\frac{t^3}{3!}+\bar{o}(t^3) = 1-t-\frac{t^2}{2}+\frac{t^3}{6}+\bar{o}(t^3)$

Числитель: $\ln\left(\frac{\cos t-\sin t}{\cos t+\sin t}\right)+2t = \ln(\cos t-\sin t)-\ln(\cos t+\sin t)+2t =$

$= \ln\left(1-t-\frac{t^2}{2}+\frac{t^3}{6}+\bar{o}(t^3)\right)-\ln\left(1+t-\frac{t^2}{2}-\frac{t^3}{6}+\bar{o}(t^3)\right)+2t =$

$=\left(\left(-t-\frac{t^2}{2}+\frac{t^3}{6}+\bar{o}(t^3)\right)-\frac{1}{2}\left(-t-\frac{t^2}{2}+\frac{t^3}{6}+\bar{o}(t^3)\right)^2\right)-$

$-\left(\left(t-\frac{t^2}{2}-\frac{t^3}{6}+\bar{o}(t^3)\right)-\frac{1}{2}\left(t-\frac{t^2}{2}-\frac{t^3}{6}+\bar{o}(t^3)\right)^2\right)+2t=$

$=\left(-t-\frac{t^2}{2}+\frac{t^3}{6}+\bar{o}(t^3)-\frac{t^2}{2}-\frac{t^3}{2}\right)-\left(t-\frac{t^2}{2}-\frac{t^3}{6}+\bar{o}(t^3)+\frac{t^3}{2}-\frac{t^2}{2}\right)+2t=$

$=-\frac{2}{3}t^3+\bar{o}(t^3)$

Знаменатель: $\left(1-e^{\ln\left(\frac{tg+1}{1-tg\,t}\right)}\right)^2 = \left(1-e^{\ln\left(\frac{\sin t+\cos t}{\cos t-\sin t}\right)}\right)^2 =$

$=\left(1-e^{\ln(\sin t+\cos t)-\ln(\cos t-\sin t)}\right)^2 = \left(1-e^{\left(t-\frac{t^2}{2}-\frac{t^3}{6}+\bar{o}(t^3)+\frac{t^3}{2}-\frac{t^2}{2}\right)-\left(-t-\frac{t^2}{2}+\frac{t^3}{6}+\bar{o}(t^3)-\frac{t^2}{2}-\frac{t^3}{2}\right)}\right)^2=$

$=\left(1-e^{2t+\frac{2}{3}t^3+\bar{o}(t^3)}\right)^2 = \left(1-\left(1+\left(2t+\frac{2}{3}t^3+\bar{o}(t^3)\right)+\frac{1}{2}\left(2t+\frac{2}{3}t^3+\bar{o}(t^3)\right)^2+\right.\right.$

$\left.\left.+\left(2t+\frac{2}{3}t^3+\bar{o}(t^3)\right)^3\cdot\frac{1}{3!}\right)\right)^2 = \left(2t+\frac{2}{3}t^3+\bar{o}(t^3)+2t^2+\frac{4}{3}t^3\right)^2 = \left(2t+2t^2+2t^3+\bar{o}(t^3)\right)^2 =$

$$= 4(t+t^2+t^3)^2 + \bar{o}(t^6) = 4(t^2+t^3+t^3) + \bar{o}(t^6)$$

$$= \lim_{t\to 0} \frac{-\frac{2}{3}t^3 + \bar{o}(t^3)}{4(t^2+t^3+t^3)+\bar{o}(t^6)} = \lim_{t\to 0} \frac{-\frac{2}{3}t + \bar{o}(t)}{4(1+t+t)+\bar{o}(t^4)} = \boxed{\frac{0}{1} = 0}$$