Семинар 23, 12.03.24 – Бельдиев

$f(x) \in \mathbb{C}[x]$

$f(x) = a(x-x_1)(x-x_2)\cdot\ldots\cdot(x-x_n)$

$f(x) \in \mathbb{R}[x]$

$f(x) = a(x-x_1)\cdot\ldots\cdot(x-x_n)\cdot(x^2+p_1x+q_1)\cdot\ldots\cdot(x^2-p_mx-q_m)$

F – поле; $f \in F[x]$

$f = f_1 \cdot f_2 \cdot \ldots \cdot f_n$, где f_i неприводимый, $\deg f_i \geq 1$

Пример: $F_2 = \mathbb{Z}_2$; 1) $x^2+1 = x^2-1 = (x-1)(x+1) = (x+1)^2$

2) $\underset{f(x)}{\underset{\parallel}{x^2+x+1}} = \cancel{(x-x_1)(x-x_2)}$

$f(0) = 1 \quad f(1)(\equiv 3) \equiv 1$

3) $x^3+2 = x^3 = x\cdot x\cdot x$

4) $x^3+1 = (x+1)(x^2+x+1)$

5) x^3+x^2+1 – неприводимый

6) x^3+x+1 – неприводимый

7) $x^3+x^2+x+1 = x^2(x+1)+(x+1) = (x+1)(x^2+1) = (x+1)^3$

8) $x^4+x^2+1 = x^4+(x+1)^2 = (x^2+x+1)^2$

#68.5. $\mathbb{F}_5[x]$

b) $x^3+2x^2+4x+1 = f(x)$

$f(2) = 8+8+8+1 = 25 = 0$

$$
\begin{array}{r|l}
x^3+2x^2+4x+1 & x-2 \\
\underline{x^3-2x^2} & \overline{x^2-x+2} \\
4x^2+4x & \\
\underline{-x^2+2x} & \\
2x+1 & \\
\underline{2x-4} & \\
0 &
\end{array}
$$

$f(x) = (x-2)(x^2-x+2)$

$\boxed{\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n}$ – остатки mod n

$\mathbb{F}[x]/(f(x))$ – остатки при делении на $f(x)$

$f(x) \neq 0$

Пример: $\mathbb{R}[x]/(x^2+1)$

$x \cdot x = x^2 = -1 \qquad (x^2 \equiv -1 \mod (x^2+1))$

$(x+1)(x^2-2) = (x+1)(-1-2) = -3x-3$

$\boxed{\mathbb{F}[x]/(f(x)) \text{ – поле} \iff f(x) \text{ неприводим}}$

(1) $f(x)$ приводим, $f = g \cdot h$

$\mathbb{F}[x]/(f)$: g^{-1} – ?

$g(x) \cdot s(x) = 1$ в $\mathbb{F}[x]/(f)$

$g(x) \cdot s(x) - 1 \,\vdots\, f$ в $\mathbb{F}[x]$

$g(x) \cdot s(x) - 1 = f(x) \cdot q(x)$

$\underset{\vdots g}{g(x) \cdot s(x)} - \underset{\vdots g}{f(x) \cdot q(x)} = \underset{\vdots g}{1}$

(2) $f(x)$ неприводим

$\mathbb{F}[x]/(f)$ — поле

$g(x) \in \mathbb{F}[x]/(f(x))$; $g(x) \not\vdots f(x)$

$(g(x), f(x)) = 1$

лин. представление НОД: $\exists a(x), b(x)$: $g(x)a(x) + f(x)b(x) = 1$ в $\mathbb{F}[x]$

в $\mathbb{F}[x]/(f)$: $g(x)a(x) = 1$

$\mathbb{F}_7[x]/(x^2+2)$

$f(x) = x^2 + 2$ корней в $\mathbb{F}_7$ нет $\Rightarrow \mathbb{F}_7[x]/(x^2+2)$ — поле

$$\frac{x^3+4}{5x^2+4x+1} = \frac{x^3+4}{5\cdot(-2)+4x+1} = \frac{x^3+4}{4x-2} = \frac{-2x+4}{4x-2} = (-2x+4)(-4x-2) =$$

$$= x^2 - 16x + 4x - 8 =$$
$$= -2 - 12x - 8 =$$
$$= \boxed{2x+4} \text{ — Answer}$$

$$\frac{1}{5x^2+4x+1} = \frac{1}{4x-2} = g(x)$$

$g(x)(4x-2) - 1 \vdots (x^2+2) \Leftrightarrow$

$\Leftrightarrow g(x)(4x-2) + a(x)(x^2+2) = 1$

$(x^2+2, 4x-2) = (4, 4x-2)$

$$\begin{array}{r|l} x^2+2 & 4x-2 \\ \cline{2-2} -\;8x^2-4x & 2x+1 \\ \hline 4x+2 & \\ -\;4x-2 & \\ \hline 4 & \end{array}$$

$1 = -(4x-2)(4x+2)$ — в $\mathbb{F}_7[x]/(x^2+2)$

$\frac{1}{4x-2} = -4x-2$

$(x^2+2) = (4x-2)(2x+1) + 4$; $4 = (x^2+2)\cdot 1 - (4x-2)(2x+1)$

$\frac{1}{4} = 2$ (в $\mathbb{F}_7$) $\qquad 1 = (x^2+2)\cdot 2 - (4x-2)\cdot(4x+2)$

$\mathbb{F}_2[x]/(x^3+x^2+1)$ "$x^3 = x^2 - 1 = x^2 + 1$"

$$\frac{x^2+1}{x^2+x+1} + x^2(x^2+x+1) - \frac{x^3+x}{x^2} \ominus$$

(1) $x^2(x^2+x+1) = x^4 + x^3 + x^2 = x^4 + 1 = x \cdot x^3 + 1 =$

$= x(x^2+1) + 1 = x^3 + x + 1 = x^2 + x + 2 = x^2 + x$

(2) $\frac{x^2+1}{x^2+x+1} = \frac{x^2+1}{x^3+x} = \frac{x^2+1}{x(x^2+1)} = \frac{1}{x} = g(x)$

$= \frac{x^3+x^2}{x} = x^2 + x$

$x\,g(x) + a(x)(x^3+x+1) = 1$

(x, x^3+x+1)

$$\begin{array}{l|l} x^3 + x^2 + 1 & x \\ \hline -x^3 & x^2+x \\ \quad x^2 & \\ \quad -x^2 & \\ \hline \qquad 1 & \end{array}$$

$x^3 + x^2 + 1 = x(x^2+1) + 1$

$0 = x(x^2+1) + 1$ in $\mathbb{F}_2[x]/(x^3+x^2+1)$

$x(x^2+1) = 1$

(3) $\frac{x^3+x}{x^2} = x + \frac{1}{x} = x^2 + 2x = x^2$

$\ominus\ x^2 + x + x^2 + x + x^2 = 3x^2 + 2x = x^2$