Семинар 15, 23.01.24.

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \bar{o}(x^n)$$

$$\parallel$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \bar{o}(x^n)$$

$$f^{(k)}(0) = a_k \cdot k!$$

$$f(x) = 2 + 6x + 7x^2 - 9x^3 + \bar{o}(x^3)$$

$$f(0) = 2$$

$$f'(0) = 6$$

$$f''(0) = 14$$

$$f'''(0) = -54$$

$$\operatorname{arctg} x = a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n+1} x^{2n+1} + \bar{o}(x^{2n+1}) = g(x) \cdot x^{2n+1};\ g'(x)x^{2n+1} + g(x)(2n+1)x^{2n}$$

$$\frac{1}{1+x^2} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + (2n+1)a_{2n} x^{2n} + \bar{o}(x^{2n})$$

$$\parallel$$

$$1 - x^2 + x^4 - x^6 + x^8 + \ldots + \bar{o}(x^{2n})$$

$$a_1 = 1$$

$$2a_2 = 0$$

$$3a_3 = -1$$

$$2k\, a_{2k} = 0$$

$$(2k+1)a_{2k+1} = (-1)^k \Rightarrow a_{2k+1} = \frac{(-1)^k}{2k+1}$$

$$\operatorname{arctg} x = a_0 + x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \bar{o}(x^{2n+1})$$

$$a_0 = \operatorname{arctg} 0 = 0$$

$$f(x) = \operatorname{arctg} x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(2n+1)}(0)}{(2n+1)!}x^{2n+1} + \bar{o}(x^{2n+1})$$

$$f'(x) = g(x) = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 + \ldots + (-1)^n x^{2n} + \bar{o}(x^{2n})$$

$$\shortparallel$$

$$g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \ldots + \frac{g^{(2n)}(0)}{(2n)!}x^{2n} + \bar{o}(x^{2n})$$

$$g^{(2k+1)}(0) = 0$$

$$g^{(2k)}(0) = (-1)^k \cdot (2k)!$$

$$\shortparallel$$

$$f^{(2k+1)}(0) \qquad (f' = g)$$

Использование формулы Тейлора

① $A \cdot e^x - \frac{b}{1-x} = -\frac{1}{2}x^2 - \frac{5}{6}x^3 + \bar{o}(x^3)$

$$A\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right) - b\left(1 + x + x^2 + x^3\right) + \bar{o}(x^3) = (A-b) + (A-b)x + \left(\frac{A}{2} - b\right)x^2 + \left(\frac{A}{6} - b\right)x^3 + \bar{o}(x^3)$$

$$\begin{cases} A - b = 0 \\ \frac{A}{2} - b = -\frac{1}{2} \\ \frac{A}{6} - b = -\frac{5}{6} \end{cases} \qquad \begin{cases} A = b \\ \frac{A}{2} - A = -\frac{1}{2} \end{cases} \Rightarrow A = b = 1$$

$$\boxed{\text{Ответ: } A = 1;\ b = 1.}$$

② $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \qquad c \in (0; x)$

$$|\ln 1{,}3 - a| < 10^{-3}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n+1}\frac{x^n}{n} + \frac{f^{(n+1)}}{(n+1)!}x^{n+1}$$

$f(x) = \ln(1+x)$

$$x = 0{,}3: \quad \ln 1{,}3 = 0{,}3 - \frac{0{,}3^2}{2} + \frac{0{,}3^3}{3} + \ldots + (-1)^{n+1}\frac{0{,}3^n}{n} + \underbrace{\boxed{\frac{f^{(n+1)}}{(n+1)!}0{,}3^{n+1}}}_{r_n(x)}$$

$$f'(x) = \frac{1}{1+x} \qquad f''(x) = -\frac{1}{(1+x)^2} \qquad f'''(x) = \frac{2}{(1+x)^3} \qquad f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{(1+x)^n}$$

$$|f^{(n)}(c)| = (n-1)! \cdot \frac{1}{|(1+c)^n|} \leq (n-1)!$$

$c \in (0; 0,3)$

$$|f^{(n+1)}(c)| \leq n!$$

$$r_n(x) \leq \frac{n!}{(n+1)!} \cdot 0,3^{n+1} = \frac{0,3^{n+1}}{n+1}$$

$$n=3: \frac{0,3^4}{4} = \frac{0,0081}{4} \approx 0,002$$

$$n=4: \frac{0,3^5}{5} = \frac{0,0081}{5} \cdot 0,3 < 0,001$$

$$\ln 1,3 \approx 0,3 - \frac{0,3^2}{2} + \frac{0,3^3}{3} - \frac{0,3^4}{4} = \boxed{0,261975}$$

③ Найти:

$$\lim_{x\to 0} \frac{e^{\operatorname{arctg} x} - \frac{1}{1-x} + \frac{x^2}{2}}{\ln\left(\frac{1+x}{1-x}\right) - 2x} = \lim_{x\to 0} \frac{-\frac{7}{6}x^3 + \bar{o}(x^3)}{\frac{2}{3}x^3 + \bar{o}(x^3)} = \lim_{x\to 0} \frac{-\frac{7}{6} + \bar{o}(1)}{\frac{2}{3} + \bar{o}(1)} = -\frac{7}{6} \cdot \frac{3}{2} = \boxed{-\frac{7}{4}}$$

1) $$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \bar{o}(x^3) + x + \frac{x^2}{2} + \frac{x^3}{3} + \bar{o}(x^3) =$$

$$= 2x + \frac{2x^3}{3} + \bar{o}(x^3)$$

Знаменатель: $2x + \frac{2x^3}{3} + \bar{o}(x^3) - 2x = \frac{2x^3}{3} + \bar{o}(x^3)$

2) $$e^{\operatorname{arctg} x} = 1 + \operatorname{arctg} x + \frac{\operatorname{arctg}^2 x}{2} + \frac{\operatorname{arctg}^3 x}{6} + \bar{o}(\operatorname{arctg}^3 x) =$$

$$= 1 + \left(x - \frac{x^3}{3} + \bar{o}(x^3)\right) + \frac{1}{2}\left(x - \frac{x^3}{3} + \bar{o}(x^3)\right)^2 + \frac{1}{6}\left(x - \frac{x^3}{3} + \bar{o}(x^3)\right)^3 + \bar{o}(x^3) =$$

$$= 1 + x - \frac{x^3}{3} + \bar{o}(x^3) + \frac{1}{2}\left(x^2 + \bar{o}(x^3)\right) + \frac{1}{6}x^3 + \bar{o}(x^3) =$$

$$= 1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \bar{o}(x^3)$$

3) $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \bar{o}(x^3)$$

Числитель: $1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \bar{o}(x^3) - 1 - x - x^2 - x^3 + \bar{o}(x^3) + \frac{x^2}{2} = -\frac{7}{6}x^3 + \bar{o}(x^3)$

④ $\lim\limits_{x\to 0}\left(\cos(xe^x)-\ln(1-x)-x\right)^{\operatorname{ctg} x^3}$

1) $\cos(xe^x) = 1-\frac{x^2e^{2x}}{2}+\frac{x^4e^{4x}}{4!}+\bar{o}(x^4) =$

$= 1-\frac{x^2}{2}\left(1+\frac{2x}{1}+\frac{(2x)^2}{2}+\bar{o}(x^2)\right)+\frac{x^4}{4!}\left(1+2x+\bar{o}(x)\right)+\bar{o}(x^4)=$

$= 1-\frac{x^2}{2}-\frac{2x^3}{2}-\frac{23}{24}x^4+\bar{o}(x^4) = 1-\frac{x^2}{2}-x^3-\frac{23}{24}x^4+\bar{o}(x^4)$

2) $\ln(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}+\bar{o}(x^4)$

3) $1-\frac{x^2}{2}-x^3-\frac{23}{24}x^4+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}-x+\bar{o}(x^4) = 1-\frac{2}{3}x^3-\frac{17}{24}x^4+\bar{o}(x^4)$

4) $\operatorname{ctg} x^3 = \frac{\cos(x^3)}{\sin(x^3)} = \frac{\cos(x^3)}{x^3+\bar{o}(x^3)} = \frac{1+\bar{o}(x^3)}{x^3+\bar{o}(x^3)}$

3) $= 1-\frac{2}{3}x^3+\bar{o}(x^3)$

$\left(1-\frac{2}{3}x^3+\bar{o}(x^3)\right)^{\frac{1+\bar{o}(x^3)}{x^3+\bar{o}(x^3)}} = \left(\underbrace{\left(1-\frac{2}{3}x^3+\bar{o}(x^3)\right)^{\frac{1}{-\frac{2}{3}x^3+\bar{o}(x^3)}}}_{\to e}\right)^{\left(-\frac{2}{3}x^3+\bar{o}(x^3)\right)\cdot\frac{1+\bar{o}(x^3)}{x^3+\bar{o}(x^3)}} =$

$= e^{\left(-\frac{2}{3}x^3+\bar{o}(x^3)\right)\frac{1+\bar{o}(x^3)}{x^3+\bar{o}(x^3)}} = e^{\frac{\left(-\frac{2}{3}+\bar{o}(1)\right)\left(1+\bar{o}(x^3)\right)}{1+\bar{o}(1)}} = \boxed{e^{-\frac{2}{3}}}$

⑥ $\lim\limits_{x\to+\infty} x^{7/4}\left(\sqrt[4]{x+1}+\sqrt[4]{x-1}-2\sqrt[4]{x}\right)$

Замена $x=\frac{1}{y}$

$\left(\frac{1}{y}\right)^{\frac{7}{4}}\left(\sqrt[4]{\frac{1}{y}+1}+\sqrt[4]{\frac{1}{y}-1}-2\sqrt[4]{\frac{1}{y}}\right) =$

$= y^{-\frac{7}{4}}\left(\sqrt[4]{\frac{1+y}{y}}+\sqrt[4]{\frac{1-y}{y}}-2\frac{1}{\sqrt[4]{y}}\right) =$

$= y^{-\frac{7}{4}}\left(\frac{\sqrt[4]{1+y}}{\sqrt[4]{y}}+\frac{\sqrt[4]{1-y}}{\sqrt[4]{y}}-\frac{2}{\sqrt[4]{y}}\right) =$

$= \frac{\sqrt[4]{1+y}+\sqrt[4]{1-y}-2}{y^2}$ ⊜

$\sqrt[4]{1+y} = (1+y)^{\frac{1}{4}} = C^0_{\frac{1}{4}}+C^1_{\frac{1}{4}}y+C^2_{\frac{1}{4}}+\bar{o}(y^2) = 1+y\cdot\frac{1}{4}+\left(-\frac{3}{32}\right)\cdot y^2+\bar{o}(y^2)$

$$= \frac{1+\frac{y}{4}-\frac{3y^2}{16}+1-\frac{y}{4}-\frac{3y^2}{16}-2+\bar{o}(y^2)}{y^2} = \frac{-\frac{3}{16}y^2+\bar{o}(y^2)}{y^2} =$$

$$= -\frac{3}{16}+\bar{o}(1) \longrightarrow \boxed{-\frac{3}{16}}$$