Homework 24.

#1.

$$f(x)=\sum_{n=1}^{+\infty}\frac{\cos^2(nx)}{n(n+1)}$$

$$\frac{\cos^2(nx)}{n(n+1)} \le \frac{1}{n^2+n} < \frac{1}{n^2}$$ – сходится $\Rightarrow$ $f(x)$ непрерывна на $\mathbb{R}$

$$\int_0^{2\pi} f(x)dx = \int_0^{2\pi}\sum_{n=1}^{+\infty}\frac{\cos^2(nx)}{n(n+1)}dx = \sum_{n=1}^{+\infty}\int_0^{2\pi}\frac{\cos^2(nx)}{n(n+1)}dx = \sum_{n=1}^{+\infty}\frac{1}{n(n+1)}\cdot\frac{1}{n}\int_0^{2\pi}\cos^2(nx)d(nx) =$$

$$= \sum_{n=1}^{+\infty}\frac{2nx+\sin(2nx)}{4n^2(n+1)}\Bigg|_0^{2\pi} = \sum_{n=1}^{+\infty}\frac{4\pi n+0}{4n^2(n+1)} = \sum_{n=1}^{+\infty}\frac{\pi}{n(n+1)} = \pi\sum_{n=1}^{+\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) =$$

$$=\pi.$$

#3.

$$f_n(x) = nxe^{-nx^2} \quad \text{на } [0;1]$$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(nxe^{-nx^2}\right) = \lim_{n\to\infty}\frac{nx}{e^{nx^2}} \overset{x\in[0;1]}{=} 0 \Rightarrow \text{сходится}$$

$$\int_0^1\left(\lim_{n\to\infty} f_n(x)\right)dx = \int_0^1 0\,dx = 0$$

$$\nparallel$$

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx = \lim_{n\to\infty}\int_0^1 nxe^{-nx^2}dx = \lim_{n\to\infty}\left(\frac{1-e^{-n}}{2}\right) = \frac{1}{2}$$ ч.т.д.

#4.

$$f_n(x) = \frac{1}{n}\operatorname{arctg} x^n$$

$$\operatorname{arctg} x^n \le \frac{\pi}{2} \Rightarrow \frac{1}{n}\operatorname{arctg} x^n \le \frac{1}{n}\cdot\frac{\pi}{2};\ \lim_{n\to\infty}\frac{\pi}{2n} = 0 \Rightarrow \text{сход.}$$

$$\left(\lim_{n\to\infty} f_n(x)\right)' = \left(\lim_{n\to\infty}\frac{1}{n}\cdot\operatorname{arctg} x^n\right)' = (0)' = 0$$

$$\nparallel$$

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty}\left(\frac{1}{n}\cdot\operatorname{arctg} x^n\right)' = \lim_{n\to\infty}\frac{x^{n-1}}{1+x^{2n}} \overset{x=1}{=} \lim_{n\to\infty}\frac{1^{n-1}}{1+1^{2n}} = \frac{1}{2}$$ ч.т.д.

#5.

a) $\sum\limits_{n=1}^{+\infty} \frac{(x+2)^n \cos^2(nx)}{\sqrt{n^3+x^4}}$ $\quad x \in [-3; -1]$

$\left| \frac{(x+2)^n \cos^2(nx)}{\sqrt{n^3+x^4}} \right| \leq \left| \frac{1}{\sqrt{n^3+x^4}} \right| \leq \left| \frac{1}{n^{3/2}} \right|$ – сходится

b) $\sum\limits_{n=1}^{+\infty} \frac{(n+2)^3 (2x)^{2n}}{x^2+3n+4}$ $\quad x \in [-1/4; 1/4]$

$\left| \frac{(n+2)^3 (2x)^{2n}}{x^2+3n+4} \right| \leq \frac{(n+2)^3}{(x^2+3n+4) \cdot 2^{2n}} \leq \frac{(n+2)^3}{2^{2n}} \xrightarrow[n\to\infty]{} 0 \Rightarrow$ сходится