Homework - 8.

#1.

a) $\lim\limits_{n\to\infty}\left(\frac{n+2}{n+1}\right)^{2n} = \lim\limits_{n\to\infty}\left(\left(1+\frac{1}{n+1}\right)^{n+1}\right)^{\frac{2n}{n+1}} \xrightarrow[n\to\infty]{} e^{\frac{2n}{n+1}} = e^{\frac{2}{1+\frac{1}{n}}} \xrightarrow[n\to\infty]{} e^{2}$

b) $\lim\limits_{n\to\infty}\left(1-\frac{4}{n}\right)^{3n-2} = \lim\limits_{n\to\infty}\left(\left(1-\frac{1}{\frac{n}{4}}\right)^{\frac{n}{4}}\right)^{\frac{4(3n-2)}{n}} \xrightarrow[n\to\infty]{} e^{-\frac{4(3n-2)}{n}} = e^{\frac{-12+\frac{8}{n}}{1}} \xrightarrow[n\to\infty]{} e^{-12}$

c) $\lim\limits_{n\to\infty}\left(\frac{n^2+3}{n^2+2}\right)^{4n^2+1} = \lim\limits_{n\to\infty}\left(\left(1+\frac{1}{n^2+2}\right)^{n^2+2}\right)^{\frac{4n^2+1}{n^2+2}} \xrightarrow[n\to\infty]{} e^{\frac{4n^2+1}{n^2+2}} = e^{\frac{4+\frac{1}{n^2}}{1+\frac{2}{n^2}}} \xrightarrow[n\to\infty]{} e^{4}$

d) $\lim\limits_{n\to\infty}\left(\frac{n+2}{3n-1}\right)^{n} = \lim\limits_{n\to\infty}\left(\frac{1+\frac{2}{n}}{3-\frac{1}{n}}\right)^{n} \xrightarrow[n\to\infty]{} \left(\frac{1}{3}\right)^{n} \xrightarrow[n\to\infty]{} 0$

#2.

a) $\lim\limits_{x\to 2}(2x^2-3x+1)=3$

По Коши: $\forall \varepsilon>0 \quad \exists \delta=\delta(\varepsilon) \quad \forall x \in \mathring{U}_{\delta(\varepsilon)} \quad |2x^2-3x+1-3|<\varepsilon$

$|2x^2-3x-2| = |(x-2)(2x+1)| \le |x-2|\cdot|2x+1| < \varepsilon$

$0<|x-2|<\delta$

$|2x+1| = |2(x-2)+5| \le 2|x-2|+5 \le 2\delta+5 \underset{\delta\le 1}{\le} 2+5=7$

$|x-2|\cdot|2x+1| \le \delta\cdot 7 < \varepsilon \Rightarrow \delta \le \frac{\varepsilon}{7}$

$\delta(\varepsilon) = \min(1; \frac{\varepsilon}{7})$

По Гейне: $\forall \begin{matrix} x_n \to 2 \\ x_n \ne 2 \end{matrix} \quad f(x_n) = 2\cdot x_n^2 - 3x_n + 1 \xrightarrow[n\to\infty]{} 2\cdot 2^2 - 3\cdot 2 + 1 = 3$

b) $\lim\limits_{x\to 1}\frac{x^2-3x}{x+1} = -1$

По Коши: $\forall \varepsilon>0 \quad \exists \delta=\delta(\varepsilon) \quad \forall x \in \mathring{U}_{\delta(\varepsilon)} \quad |\frac{x^2-3x}{x+1}+1|<\varepsilon$

$\left|\frac{x^2-3x}{x+1}+1\right| = \left|\frac{x^2-2x+1}{x+1}\right| = \left|\frac{(x-1)^2}{x+1}\right| < \varepsilon$

$0<|x-1|<\delta$

$|x+1| = |(x-1)+2| \le |x-1|+2 \le \delta+2 \underset{\delta\le 1}{\le} 3$

$\left|\frac{(x-1)^2}{x+1}\right| \le \frac{\delta^2}{3} < \varepsilon \Rightarrow \delta \le \sqrt{3\varepsilon}$

$\delta(\varepsilon) = \min(1; \sqrt{3\varepsilon})$

По Гейне: $\forall \begin{matrix} x_n \to 1 \\ x_n \ne -1 \end{matrix} \quad f(x_n) = \frac{x_n^2-3x_n}{x_n+1} \xrightarrow[n\to\infty]{} \frac{1^2-3\cdot 1}{1+1} = \frac{-2}{2} = -1$

#3.

a) $\lim\limits_{x\to 1}\frac{x^2+4x-5}{x^2-1} = \lim\limits_{x\to 1}\frac{(x-1)(x+5)}{(x-1)(x+1)} = \lim\limits_{x\to 1}\frac{x+5}{x+1} = \frac{6}{2} = 3$

b) $$\lim_{x\to 3}\frac{x^3-5x^2+3x+9}{x^3-8x^2+21x-18}=\frac{(x-3)(x^2-2x-3)}{(x-3)(x^2-5x+6)}=\lim_{x\to 3}\frac{(x-3)(x+1)}{(x-3)(x-2)}=\lim_{x\to 3}\frac{x+1}{x-2}=4$$

c) $$\lim_{x\to 5}\frac{\sqrt{6-x}-1}{3-\sqrt{4+x}}=\lim_{x\to 5}\frac{(6-x-1)(\sqrt{4+x}+3)}{(\sqrt{6-x}+1)(9-4-x)}=\lim_{x\to 5}\frac{(5-x)(\sqrt{4+x}+3)}{(5-x)(\sqrt{6-x}+1)}=\lim_{x\to 5}\frac{\sqrt{4+x}+3}{\sqrt{6-x}+1}=3$$

d) $$\lim_{x\to 3}\frac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3}=\lim_{x\to 3}\frac{(x^2-2x+6-x^2-2x+6)}{(x-1)(x-3)(\sqrt{x^2-2x+6}+\sqrt{x^2+2x-6})}=$$

$$\lim_{x\to 3}\frac{-4(x-3)}{(x-3)(x-1)\sqrt{x^2-2x+6}+\sqrt{x^2+2x-6})}=\lim_{x\to 3}\frac{-4}{(x-1)(\sqrt{x^2-2x+6}+\sqrt{x^2+2x-6})}=\frac{-4}{2(3+3)}=-\frac{1}{3}$$

e) $$\lim_{x\to 0}\frac{tg 4x}{\sin x}=\lim_{x\to 0}\left(\frac{\sin 4x}{\cos 4x\cdot\sin x}\right)=\lim_{x\to 0}\left(\frac{\sin 4x}{4x}\cdot\frac{4x}{x}\cdot\frac{x}{\sin x}\cdot\frac{1}{\cos 4x}\right)=1\cdot 4\cdot 1\cdot 1=4$$

f) $$\lim_{x\to 0}\frac{\cos 3x^3-1}{\sin^6 2x}=\lim_{x\to 0}\left(\frac{(2x)^6\cdot(\cos^2 3x^3-1)}{\sin^6 2x\cdot(2x)^6\cdot(\cos 3x^3+1)}\right)=\lim_{x\to 0}\left(\frac{(2x)^6\cdot(-\sin^2 3x^3)\cdot(3x^3)}{\sin^6(2x)\,(2x)^6\cdot(3x^3)^2(\cos 3x^3+1)}\right)=$$

$$=1\cdot(-1)\cdot\frac{1}{2}\cdot\frac{9}{64}=-\frac{9}{128}$$