Модуль 3

Лекция 15, 12.01.23

Формула Тейлора в ф. Пеано

$\lim\limits_{x \to x_0} f(x) = A \qquad f(x) = \underbrace{A}_{}^{\in \mathbb{R}} + \bar{o}(1), \quad x \to x_0$

$\exists f'(x) \qquad f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\in T_1(x)} + \bar{o}((x - x_0)), \quad x \to x_0$

$\vdots$

$\exists f^{n}(x) \qquad f(x) = T_n(x) + \bar{o}((x - x_0)^n)$

Теорема: Если $f(x)$ n раз дифф. в т. X_0, то

$$f(x) = T_n(x) + \bar{o}((x - x_0)^n), \text{ где}$$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \text{ — мн-н Тейлора}$$

Свойства мн-на Тейлора

1) $T_n^{(k)}(x_0) = f^{(k)}(x_0) \qquad \forall\ 0 \le k \le n$

Док-во: $T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(k+1)}(x_0)}{(k+1)!}(x - x_0)^{k+1} + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$

$T_n^{(k)}(x_0) = 0 + \ldots + 0 + \underbrace{\qquad}_{k+1 \text{ слаг.}} + 0 + \ldots + 0$

$[((x - x_0)^k)' = k(x - x_0)^{k-1}]$

$\text{„}k+1 \text{ слаг.“} = \left(\frac{f^{(k)}(x_0)}{k!}(x - x_0)^k\right)^{(k)} = \frac{f^{(k)}(x_0)}{k!} \cdot k! \cdot 1 = f^{(k)}(x_0)$

Примеры:

1) $f(x) = \sin x, \quad x_0 = 0$

$f^{(k)}(0) = ?$

$f^{(2k)}(0) = 0$

$f^{(2k+1)}(0) = (-1)^k$

$$\sin x = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} \cdot x^{2k+1} + \bar{o}(x^{2n+1}), \quad x \to 0$$

2) $f(x) = \cos x, \quad x_0 = 0$

$f^{(2k+1)}(0) = 0$

$f^{(2k)}(0) = (-1)^k$

$$\cos x = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k} + \bar{o}(x^{2n}), \quad x \to 0$$

3) $f(x) = e^x, \quad x_0 = 0$

$f^{(k)}(0) = 1$

$$f(x) = \sum_{k=0}^{n} \frac{x^k}{k!} + \bar{o}(x^n), \quad x \to 0$$

Док-во: $R_n(x) = f(x) - T_n(x)$ – остаточный член

$R_n(x) = \bar{o}((x-x_0)^n)$?

$R_n(x) = (x-x_0)^n \cdot \bar{o}(1)$

$\frac{R_n(x)}{(x-x_0)^n} = \bar{o}(1)$

$f'(x_0) - T_n'(x_0) = 0$

$x \to x_0$

лопит.

$$\lim_{x \to x_0} \frac{R_n(x)}{(x-x_0)^n} = \lim_{x \to x_0} \frac{R_n'(x)}{n(x-x_0)^{n-1}} = \ldots = \lim_{x \to x_0} \frac{f^{(n-1)}(x) - T_n^{(n-1)}(x)}{n!(x-x_0)} =$$

$$= \lim_{x \to x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0) - f^{(n)}(x_0)(x-x_0)}{n!(x-x_0)} =$$

$$= \frac{1}{n!}\left(\lim_{x \to x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0)}{x - x_0} - \cancel{\lim_{x \to x_0}} f^{(n)}(x_0)\right) = 0$$

ч.т.д.

Пример:

$$\lim_{x \to 0} \underbrace{\frac{x - \sin x}{e^x - 1 - x - \frac{x^2}{2}}}_{\shortparallel}$$

$$\frac{x - (x + \bar{o}(x))}{(1 + x + \bar{o}(x)) - 1 - x - \frac{x^2}{2}} = \frac{\bar{o}(x)}{-\frac{x^2}{2} + \bar{o}(x)} \Big|^{:x} = \frac{\bar{o}(1)}{-\frac{x}{2} + \bar{o}(1)}$$

$$\frac{x - (x - \frac{x^3}{6} + \bar{o}(x_3))}{(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \bar{o}(x^3)) - 1 - x - \frac{x^2}{2}} = \frac{\frac{x^3}{6} + \bar{o}(x^3)}{\frac{x^3}{6} + \bar{o}(x^3)} = 1$$

Теорема (Формула Тейлора с ост. членом в форме Лагранжа):

Если $f(x)$ дифф. $(n+1)$ раз на $(a;b)$ и $a < x, x_0 < b$, то

$\exists \xi \in (x_0; x)$:

$$\underset{\underset{f(x) - T_n(x)}{\shortparallel}}{R_n(x)} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

Пример:

$x_0 = 0$

$$|\sin x - T_n^{\sin}(x)| = |R_n(x)| \leq \left|\frac{1}{(n+1)!} \cdot x^{n+1}\right| \xrightarrow[\forall x]{n \to \infty} 0$$

Док-во:

$$\gamma(t) = f(x) - T_n(\underset{\substack{\uparrow \\ \text{вместо } x_0}}{t}; x) - (x-t)^{n+1} \frac{R_n(x)}{(x - x_0)^{n+1}}$$

$$\gamma(x_0) = f(x) - T_n(x) - R_n(x) = 0$$

$$\gamma(x) = f(x) - T_n(x_0; x) - 0 = f(x) - f(x) = 0$$

$\gamma(x_0) = \gamma(x) = 0$ $\quad$ $\gamma(t)$ дифф. на $(a; b)$ $\Rightarrow \exists \xi \in (x_0; x):$

$$T_n(t; x) = \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k \qquad \gamma'(\xi) = 0$$