Семинар 23, 02.04.24

Сходимость функциональных послед.

$f_n(x)$, $n \in N$

$\lim\limits_{n\to\infty} f_n(x) = f(x)$

① (a) $\lim\limits_{n\to\infty} \frac{nx^2}{x+3n+2} = \lim\limits_{n\to\infty} \frac{x^2}{\frac{x}{n}+3+\frac{2}{n}} = \frac{x^2}{3}$, $x \in [0;+\infty)$

(b) $\lim\limits_{n\to\infty} \sqrt[n]{1+x^n}$, $x \in [0;2]$

$x^n + 1 \sim x^n$

$\underset{\substack{\downarrow \\ x}}{\sqrt[n]{x^n}} \le \sqrt[n]{1+x^n} \le \underset{\substack{\downarrow \\ x}}{\sqrt[n]{2x^n}}$ при $x > 1$

при $x = 1$: $\to 1$

при $x < 1$: $\underset{\substack{\downarrow \\ 1}}{1} \le \sqrt[n]{x^n+1} \le \underset{\substack{\downarrow \\ 1}}{1+x^n}$

$$\lim\limits_{n\to\infty} \sqrt[n]{1+x^n} = \begin{cases} 1, & x \in [0;1] \\ x, & x \in (1;2] \end{cases}$$

(c) $n(x^{\frac{1}{n}} - 1)$, $x \in [1;3]$ "$o\left(\frac{\ln x}{n}\right)$"

$x^{\frac{1}{n}} = e^{\frac{1}{n}\ln x} = 1 + \frac{\ln x}{n} + o\left(\frac{1}{n}\right)$

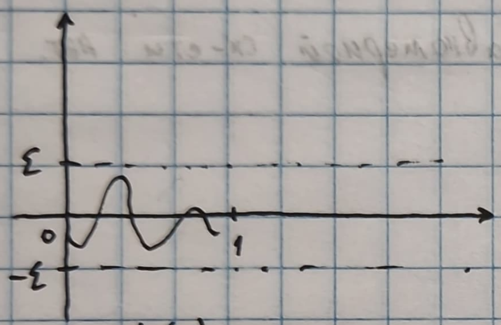
$f_n(x) = n\left(1 + \frac{\ln x}{n} - 1 + \bar{o}\left(\frac{1}{n}\right)\right) = \ln(x + \bar{o}(1)) \xrightarrow[n\to\infty]{} \ln x = f(x)$

Поточечная сходимость:

$$\lim_{n\to\infty} f_n(x) = f(x) : \quad \forall x\in E \ \forall \varepsilon>0 \ \exists N : \forall n>N : |f_n(x)-f(x)|<\varepsilon$$

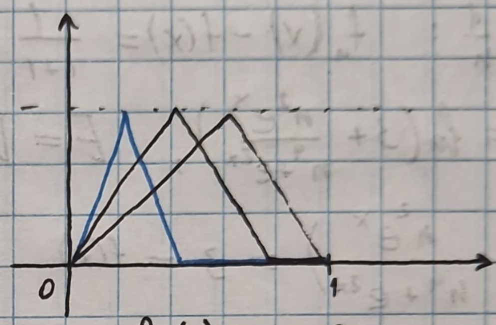
Равномерная сходимость:

$$f_n(x) \underset{n\to\infty}{\rightrightarrows} f(x) : \quad \forall \varepsilon>0 \ \exists N : \forall n>N \ \forall x\in E \ |f_n(x)-f(x)|<\varepsilon$$



$f_n(x) \rightrightarrows 0$

$x\in[0;1]=E$



$f_n(x) \xrightarrow[n\to\infty]{} 0$

② (a) $f_n(x) = \dfrac{\operatorname{arctg}(nx)}{\sqrt{n+x}}, \quad x\in E=[0;+\infty)$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\operatorname{arctg}(nx)}{\sqrt{n+x}} = 0$$

$$|f_n(x)-f(x)| = \frac{\operatorname{arctg}(nx)}{\sqrt{n+x}} \le \frac{\pi}{2}\cdot\frac{1}{\sqrt{n}} = \frac{\pi}{2\sqrt{n}} \xrightarrow[n\to\infty]{} 0$$

(b) $f_n(x) = n\cdot\sin\frac{1}{nx}, \quad E=[1;+\infty)$

$$\lim_{n\to\infty}\left(n\cdot\sin\frac{1}{nx}\right) = \lim_{n\to\infty}\left(\frac{n\cdot\sin\frac{1}{nx}}{\frac{1}{nx}\cdot nx}\right) = \lim_{n\to\infty}\left(\frac{\sin\frac{1}{nx}}{\frac{1}{nx}}\cdot\frac{1}{x}\right) = \frac{1}{x}$$

$$|f_n(x)-f(x)| = \left|n\sin\frac{1}{nx}-\frac{1}{x}\right| = \left|\frac{1}{x}-\frac{\sin c}{2nx^2}-\frac{1}{x}\right| = \frac{|\sin c|}{2nx^2} \le \frac{1}{2n} \to 0$$

$$\sin\frac{1}{nx} = \frac{1}{nx} - \frac{\sin c}{2}\cdot\frac{1}{n^2x^2}$$

③ (a) $f_n(x) = \frac{nx}{1+n^2x^2}$, $E=[0;1]$

$$\lim_{n\to\infty} \frac{nx}{1+n^2x^2} = \lim_{n\to\infty} \frac{\frac{x}{n}}{\frac{1}{n^2}+x^2} = 0, \quad x\in[0;1]$$

$$|f_n(x) - f(x)| = \frac{nx}{1+n^2x^2}$$

при $x_n = \frac{1}{n}$: $f_n(x) - f(x) = \frac{1}{1+1} = \frac{1}{2} \Rightarrow$ равномерной сх-сти нет.

(b) $f_n(x) = \ln\left(3 + \frac{n^2e^x}{n^4+e^{2x}}\right)$, $E=[0; +\infty)$

$$\lim_{n\to\infty} \ln\left(3+\frac{n^2e^x}{n^4+e^{2x}}\right) = 3 = f(x)$$

$$f_n(x) - f(x) = \ln\left(3+\frac{n^2e^x}{n^4+e^{2x}}\right) - \ln 3$$

$x_n = 2\ln n$: $f_n(2\ln n) = \ln\left(3+\frac{n^2n^2}{n^4+n^4}\right) = \ln\left(3+\frac{1}{2}\right) = \ln\frac{7}{2}$

$|f_n(x) - f(x)| = \left|\ln\frac{7}{2} - \ln 3\right| = \ln\frac{7}{6}$. $\Rightarrow$ расх.