Семинар 18, 13.02.24

Математические этюды

① $\int \frac{x^2-2x-5}{x^3-x^2+2x-2}\,dx$

$$\int \frac{1}{x^2-1}\,dx = \frac{1}{2}\int\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx$$

$$x^3-x^2+2x-2 = x^2(x-1)+2(x-1) = (x-1)(x^2+2)$$

$$\frac{x^2-2x-5}{(x^2+2)(x-1)} = \frac{bx+c}{x^2+2}+\frac{a}{x-1} = \frac{(bx+c)(x-1)+a(x^2+2)}{(x^2+2)(x-1)} = \frac{(a+b)x^2+(c-b)x+(2a-c)}{(x^2+2)(x+1)} \ominus$$

$$\begin{cases} a+b=1 \\ c-b=-2 \\ 2a-c=-5 \end{cases} \quad \begin{cases} b-2a=7 \\ a+b=1 \end{cases} \quad \begin{cases} a=-2 \\ b=3 \\ c=1 \end{cases}$$

$$\ominus \ \frac{3x+1}{x^2+2} + \frac{-2}{x-1}$$

$$\int \ldots dx = \underbrace{\int \frac{3x+1}{x^2+2}\,dx}_{\shortparallel} - \overbrace{\int \frac{2}{x-1}\,dx}^{2\ln|x-1|} = \frac{3}{2}\ln(x^2+2) + \frac{1}{\sqrt{2}}\operatorname{arctg}\left(\frac{x}{\sqrt{2}}\right) - 2\ln|x-1| + C$$

$$\underbrace{\int \frac{3x}{x^2+2}\,dx}_{\shortparallel} + \underbrace{\int \frac{1}{x^2+2}\,dx}_{\shortparallel}$$

$$\frac{3}{2}\int \frac{d(x^2)}{x^2+2} \qquad \int \frac{dx}{x^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\operatorname{arctg}\left(\frac{x}{\sqrt{2}}\right)$$

$$\shortparallel$$

$$[x^2=y]$$

$$\shortparallel$$

$$\frac{3}{2}\int \frac{dy}{y+2} = \frac{3}{2}\ln|y+2| = \frac{3}{2}\ln(x^2+2)$$

Answer: $\boxed{\frac{3}{2}\ln(x^2+2) + \frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}} - 2\ln|x-1| + C}$

$$\int \frac{P(x)}{Q(x)}dx\ ;\quad \frac{P(x)}{Q(x)} = \underset{\substack{\uparrow \\ \text{берётся}}}{S(x)} + \frac{R(x)}{Q(x)}\ ;\quad \deg R < \deg Q$$

$$Q(x) = (x-a_1)^{k_1}\cdot(x-a_2)^{k_2}\cdot\ldots\cdot(x-a_n)^{k_n}\cdot(x^2-p_1x+q_1)^{t_1}\cdot\ldots\cdot(x^2+p_mx+q_m)^{t_m}$$

Пример: $\frac{x^3}{x^2-1} = \frac{x^3-x+x}{x^2-1} = x + \frac{x}{x^2-1}$

$$\frac{R(x)}{Q(x)} = \frac{c_1}{x-a_1} + \frac{c_2}{(x-a_1)^2} + \ldots + \frac{c_{k_1}}{(x-a_1)^{k_1}} + \frac{d_1}{x-a_2} + \frac{d_2}{(x-a_2)^2} + \ldots + \frac{d_{k_2}}{(x-a_2)^{k_2}} +$$

$$+ \frac{e_1}{x-a_n} + \frac{e_2}{(x-a_n)^2} + \ldots + \frac{e_{k_n}}{(x-a_n)^{k_n}} + \ldots + \frac{g_1x+f_1}{x^2+p_1x+q_1} + \frac{g_2x+f_2}{(x^2+p_1x+q_1)^2} + \ldots + \frac{g_{t_1}x+f_{t_1}}{(x^2+p_1x+q_1)^{t_1}} + \ldots$$

$$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C$$

$$\int \frac{ex+f}{x^2+px+q}dx = \int \frac{ey+f}{y^2+a^2}dy = e\int\frac{y}{y^2+a^2}dy + f\int\frac{dy}{y^2+a^2} = e\int\frac{d(y^2)}{y^2+a^2} + f\int\frac{dy}{y^2+a^2}$$

$$x^2+px+q = \left(x+\frac{p}{2}\right)^2 + q - \frac{p^2}{4} = \left[y = x+\frac{p}{2}\right] = y^2+a^2$$

$$q - \frac{p^2}{4} = a^2$$

① $\int \frac{dx}{(x+2)(4x^2+8x+7)}$ ⊜

$$\frac{1}{(x+2)(4x^2+8x+7)} = \frac{bx+c}{4x^2+8x+7} + \frac{a}{x+2} = \frac{(bx+c)(x+2)+a(4x^2+8x+7)}{(x+2)(4x^2+8x+7)} =$$

$$= \frac{(4a+b)x^2+(8a+2b+c)x+(2c+7a)}{(x+2)(4x^2+8x+7)} = \frac{-\frac{4}{7}x}{4x^2+8x+7} + \frac{\frac{1}{7}}{x+2}$$

$$\begin{cases} 4a+b=0 \\ 8a+2b+c=0 \\ 2c+7a=1 \end{cases} \qquad \begin{cases} a=\frac{1}{7} \\ b=-\frac{4}{7} \\ c=0 \end{cases}$$

$$-\frac{4}{7}\int\frac{x\,dx}{4x^2+8x+7} = -\frac{4}{7}\int\frac{x\,dx}{4(x+1)^2+3} = [2x+2=y] = -\frac{4}{7}\int\frac{\frac{y-2}{2}\cdot\frac{1}{2}dy}{y^2+(\sqrt{3})^2} =$$

$$= -\frac{1}{7}\int\frac{(y-2)\,dy}{y^2+(\sqrt{3})^2} = -\frac{1}{7}\int\frac{y\,dy}{y^2+(\sqrt{3})^2} + \frac{2}{7}\int\frac{dy}{y^2+(\sqrt{3})^2} =$$

$$= -\frac{1}{7}\int\frac{\frac{1}{2}d(y^2)}{y^2+(\sqrt{3})^2} + \frac{2}{7}\int\frac{dy}{y^2+(\sqrt{3})^2} = -\frac{1}{14}\ln|y^2+3| + \frac{2}{7}\cdot\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{y}{\sqrt{3}} + C$$

$$\ominus \boxed{-\frac{1}{14}\ln(4x^2+8x+7) + \frac{2\sqrt{3}}{21}\operatorname{arctg}\left(\frac{2x+2}{\sqrt{3}}\right) + \frac{1}{7}\ln|x+2| + C} \text{ - Answer}$$

$$\int\frac{(bx+c)}{(x^2+1)^n}dx = \underbrace{b\int\frac{x\,dx}{(x^2+1)^n}} + c\int\frac{dx}{(x^2+1)^n}$$

$$\frac{b}{2}\int\frac{d(x^2)}{(x^2+1)^n} = \frac{b}{2}\int\frac{dy}{(y+1)^n} = \frac{b}{2}\cdot\frac{1}{-n+1}\cdot\frac{1}{(y+1)^{n-1}} + C$$

$$\int\frac{dx}{(x^2+1)^n} = \frac{x}{(x^2+1)^n} - \int x\,d\left(\frac{1}{(x^2+1)^n}\right) = \frac{x}{(x^2+1)^n} + \int\frac{2nx^2dx}{(x^2+1)^{n+1}} =$$

$$= \frac{x}{(x^2+1)^n} + 2n\int\frac{x^2+1-1}{(x^2+1)^{n+1}}dx = \frac{x}{(x^2+1)^n} + 2n\int\frac{dx}{(x^2+1)^n} - 2n\int\frac{dx}{(x^2+1)^{n+1}}$$

$$\boxed{2n\int\frac{dx}{(x^2+1)^{n+1}} = \frac{x}{(x^2+1)^n} + 2n\int\frac{dx}{(x^2+1)^n} - \int\frac{dx}{(x^2+1)^n}}$$

② $$\int\cos^5x\,dx = \int\cos^4x\cdot d(\sin x) = \int(1-\sin^2x)^2d(\sin x) = [y=\sin x] =$$

$$= \int(1-y^2)^2dy = \int dy - 2\int y^2dy + \int y^4dy = y - \frac{2y^3}{3} + \frac{y^5}{5} + C =$$

$$= \sin x - \frac{2}{3}\sin^3x + \frac{1}{5}\sin^5x + C$$

$$\int\sin^6x\,dx = \frac{1}{8}\int(1-\cos 2x)^3dx = \frac{1}{8}\int(1-3\cos 2x+3\cos^2 2x-\cos^3 2x)\,dx =$$

$$= \frac{1}{8}\left(x - 3\cdot\frac{1}{2}\sin 2x + 3\int\frac{1}{2}\cdot(1+\cos 4x)\,dx - \int\cos^2 2x\,d(\sin 2x)\right) =$$

$$= \frac{1}{8}\left(x - \frac{3}{2}\sin 2x + \frac{3x}{2} + \frac{1}{4}\cdot\frac{3}{2}\sin 4x - \int (1-\sin^2 2x)\, d(\sin\ )\right) =$$

$$= \frac{1}{8}\left(x - \frac{3}{2}\sin 2x + \frac{3x}{2} + \frac{3}{8}\sin 4x - \sin\right.$$

$$=$$

③ $\int e^{\sqrt{x}} dx = \int 2y e^{y} dy = 2\int y\, d(e^{y}) = 2e^{y} y - 2\int e^{y} dy = 2ye^{y} - 2e^{y} + C =$

$$\begin{bmatrix} y = \sqrt{x} \\ y^2 = x \\ dx = 2y\,dy \end{bmatrix}$$

$$\boxed{= 2e^{\sqrt{x}}\cdot\sqrt{x} - 2e^{\sqrt{x}} + C}$$

④ $\int \frac{\ln x\, dx}{x\sqrt{1+\ln x}} = \int \frac{\ln x\, d(\ln x)}{\sqrt{x + \ln x}} \underset{\ln x = y}{=} \int \frac{y\,dy}{\sqrt{1+y}} \underset{\sqrt{1+y}=t}{=} \int \frac{(t^2-1)\, dt \cdot 2t}{t} =$

$$= \int 2(t^2-1)\, dt = \frac{2}{3}t^3 - 2t + C = \boxed{\frac{2}{3}\left(\sqrt{1+\ln(x)}\right)^3 - 2\sqrt{1+\ln(x)} + C}$$

⑥ $\int \frac{x^2}{(1+x^2)^2}dx = -\frac{1}{2}\int x\, d\left(\frac{1}{1+x^2}\right) = -\frac{1}{2}x\cdot\frac{1}{1+x^2} + \frac{1}{2}\int\frac{1}{1+x^2}dx =$

$$\boxed{= \frac{-x}{2+2x^2} + \frac{1}{2}\operatorname{arctg} x + C}$$

⑤ $\int \sin(\ln x) dx = \begin{bmatrix} \ln x = y \\ x = e^{y} \\ dx = e^{y} dy \end{bmatrix} = \int e^{y}\cdot \sin y\, dy = \int \sin y\, d(e^{y}) =$

$$= e^{y}\sin y - \int e^{y} d(\sin y) = e^{y}\sin y - \int \cos y \cdot e^{y} dy =$$

$$= e^{y}\sin y - \int \cos y\, d(e^{y}) = e^{y}\sin y - e^{y}\cos y + \int e^{y} d(\cos y) =$$

$$= e^{y}\sin y - e^{y}\cos y - \int \sin y \cdot e^{y} dy$$

$$\int \sin y \cdot e^{y}\cdot dy = \frac{1}{2}e^{y}(\sin y - \cos y) + C = \frac{1}{2}e^{\ln x}(\sin(\ln x) - \cos(\ln x)) + C =$$

$$= \boxed{\frac{x}{2}(\sin(\ln x) - \cos(\ln x)) + C}$$

Нечестний спосіб:

$$\int e^y \sin y\, dy = \mathrm{Im}\int e^y(\cos y + i\sin y)\, dy = \mathrm{Im}\int e^y \cdot e^{iy}\, dy =$$

$$= \mathrm{Im}\int e^{y(1+i)}\, dy = \mathrm{Im}\left(\frac{1}{1+i} e^{y(1+i)}\right) + C =$$

$$= \mathrm{Im}\left(\frac{1}{1+i} e^{y+iy}\right) + C = \mathrm{Im}\left(\frac{1}{1+i} e^y(\cos y + i\sin y)\right) + C =$$

$$= \mathrm{Im}\left(\frac{1}{2}(1-i)e^y(\cos y + i\sin y)\right) + C = \mathrm{Im}\left((\dots) + i\left(\frac{e^y}{2}(\sin y - \cos y)\right)\right) + C =$$

$$= \boxed{\frac{e^y}{2}(\sin y - \cos y) + C}$$