Семинар 25, 16.04.24

Ряды Тейлора

$f(x)$, $(x_0-\varepsilon; x_0+\varepsilon)$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x_0) + f'(x_0)(x-x_0) + \ldots$$ – ряд Тейлора

$$f(x) \overset{?}{=} \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \quad (\text{при } |x|<1)$$

$$f(x) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + r_n(x; x_0) \to 0$$

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1) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad (x \in \mathbb{R})$

2) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad (x \in \mathbb{R})$

3) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \quad (x \in \mathbb{R})$

$$f(x) = e^x \; ; \; r_n(x; 0) = \frac{f^{(n+1)}(c)}{(n+1)!} \cdot x^{n+1} = \frac{e^c \cdot x^{n+1}}{(n+1)!} \xrightarrow[n\to\infty]{} 0 \quad (e^c \le e^x)$$

$c \in [0; x]$

$$(1+x)^\alpha = \sum_{k=0}^{\infty} C_k^\alpha x^k \quad (|x|<1) \text{ при } \alpha \notin \mathbb{N}, \alpha \neq 0.$$

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} \quad (x \in (-1; 1])$$

$x = 1$: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ – сходится

$a_k = \frac{(-1)^{k+1}}{k}$

$x = -1$: $-1 - \frac{1}{2} - \frac{1}{3} - \ldots$ – расходится

$$\frac{1}{R} = \lim \sqrt[k]{\frac{1}{k}} = 1, \quad R = 1$$

① a) $\operatorname{arctg} x = \sum_{k=0}^{\infty} a_k x^k = a_0 + x - \frac{x^3}{3} + \frac{x^5}{5} - \dots = \sum_{k=0}^{\infty} (-1) \frac{x^{2k+1}}{2k+1} + a_0$

Продифф.: $\frac{1}{1+x^2} = \sum_{k=0}^{\infty} k \cdot a_k \cdot x^{k-1}$

$= 1 - x^2 + x^4 - x^6 + \dots \Rightarrow a_k = \begin{cases} 0, & k \vdots 2 \\ \frac{(-1)^{k+1}}{k}, & k \not\vdots 2 \end{cases}$ $\quad R = 1.$

Надо найти a_0.

$a_0 = \operatorname{arctg} 0 = 0 \Rightarrow \operatorname{arctg} x = \sum_{k=0}^{\infty} (-1) \frac{x^{2k+1}}{2k+1}$

b) $\arccos x$

$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}} = -(1-x^2)^{-\frac{1}{2}} = -\sum_{k=0}^{\infty} C_{-\frac{1}{2}}^{k} (-x^2)^k =$

$= \sum_{k=0}^{\infty} (-1)^{k+1} C_{-\frac{1}{2}}^{k} x^{2k}$

$\arccos x = a_0 + \sum_{k=0}^{\infty} (-1)^{k+1} C_{-\frac{1}{2}}^{k} \frac{x^{2k+1}}{2k+1}$ $\quad R = 1$

$a_0 = \arccos 0 = \frac{\pi}{2}$

$\arccos x = \frac{\pi}{2} + \sum_{k=0}^{\infty} (-1)^{k+1} C_{-\frac{1}{2}}^{k} \frac{x^{2k+1}}{2k+1}$

② $f(x) = \frac{3x+8}{(2x-3)(x^2+4)} = \frac{a}{2x-3} + \frac{bx+c}{x^2+4} = \frac{ax^2 + 4a + 2bx^2 + 2cx - 3bx - 3c}{(2x-3)(x^2+4)} =$

$= \frac{(a+2b)x^2 + (2c-3b)x + 4a - 3c}{(2x-3)(x^2+4)} = \frac{2}{2x-3} - \frac{x}{x^2+4}$ ⊜

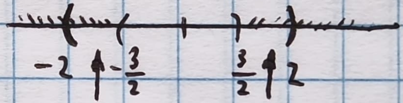
$\begin{cases} a+2b=0 \\ 2c-3b=3 \\ 4a-3c=8 \end{cases}$ $\begin{cases} a=-2b \\ 2c-3b=3 \\ -8b-3c=8 \end{cases}$ $\begin{cases} a=-2b \\ 2c-3b=3 \\ -25b=25 \end{cases}$ $\begin{cases} a=2 \\ b=-1 \\ c=0 \end{cases}$

$\frac{2}{2x-3} = -\frac{2}{3} \cdot \frac{1}{1-\frac{2}{3}x} = -\frac{2}{3} \sum_{k=0}^{\infty} \frac{2^k x^k}{3^k} = -\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^{k+1} x^k$

$\frac{x}{x^2+4} = \frac{x}{4} \cdot \frac{1}{1+\frac{x^2}{4}} = \frac{x}{4} \cdot \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{4^k} = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^{2k+1}}{4^{k+1}}$

$$= -\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^{k+1} x^k - \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{4^{k+1}}$$

$\frac{2}{2x-3} \longrightarrow R_1 = \frac{3}{2}$



$\Rightarrow R = \frac{3}{2}$

$-\frac{x}{x^2+4} \longrightarrow R_2 = 2$

сход. + расход = расход.

3) $\ln(4+3x-x^2) = \ln((4-x)(x+1)) = \ln(4-x) + \ln(1+x) =$

$= \ln 4 + \ln\left(1-\frac{x}{4}\right) + \ln(1+x)$

4) $\sin^4 x = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1}{4} - \cos 2x + \frac{\cos^2 2x}{4}$