Семинар 17, 30.01.24 – Бельдиев

$H \subset G$ — **Смежные классы**

$g \in G \to \{gh \mid h \in H\} = gH$ $\boxed{\text{левый}}$

g_1, g_2 – в одном смежном классе $\Leftrightarrow g_1H = g_2H$

$\Leftrightarrow g_2^{-1}g_1H = H$

$\Leftrightarrow g_2^{-1}g_1 \in H$

$\Leftrightarrow g_1^{-1}g_2 \in H$

#56.37

(а) $n\mathbb{Z} \subset \mathbb{Z}$

$\{0, \pm n, \pm 2n, \ldots\} = \{kn \mid k \in \mathbb{Z}\}$

$\{1, \pm n+1, \pm 2n+1, \ldots\} = \{kn+1 \mid k \in \mathbb{Z}\}$

$\{2, \pm n+2, \pm 2n+2, \ldots\} = \{kn+2 \mid k \in \mathbb{Z}\}$

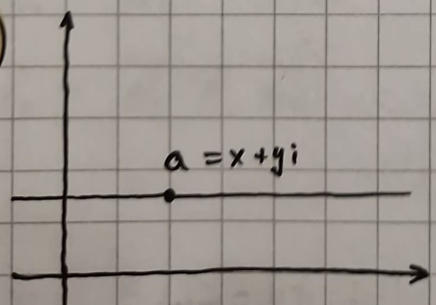
$\ldots$

$\{(n-1), \pm n+(n-1), \pm 2n+(n-1), \ldots\} = \{kn+n-1 \mid k \in \mathbb{Z}\}$

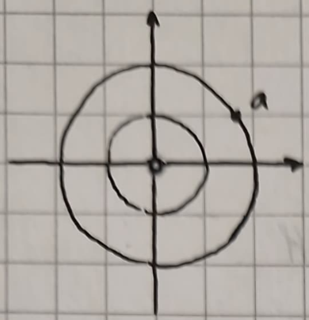
(в) $(\mathbb{Z}, +) \subset (\mathbb{R}, +)$

$a \in \mathbb{R} \to \{a+k \mid k \in \mathbb{Z}\}$

(г) $\mathbb{R} \subset \mathbb{C}$



(g) $\mathbb{C}^* / U \qquad U = \{ z \in \mathbb{C}^* \mid |z| = 1 \}$



#56.38.

$GL_n(\mathbb{C}) = \{ A \in Mat(n \times n, \mathbb{C}) \mid \det A \neq 0 \}$

$SL_n(\mathbb{C}) = \{ A \in Mat(n \times n, \mathbb{C}) \mid \det A = 1 \}$

$\underset{H}{\underset{\shortparallel}{SL_n(\mathbb{C})}} \subset \underset{G}{\underset{\shortparallel}{GL_n(\mathbb{C})}}$

(1) $a \in GL_n(\mathbb{C}) \to \{ ah \mid h \in SL_n(\mathbb{C}) \}$; $\det(ah) = \det(a) \cdot \det(h) = \det(a)$

(2) $a, b,\ \det(a) = \det(b)$

$a^{-1} b \overset{?}{\in} SL_n(\mathbb{C})$

$\det(a^{-1} b) = \det(a)^{-1} \det(b) = 1$

правый см. класс эл-та g: Hg

$Hg_1 = Hg_2 \Leftrightarrow H = Hg_2 g_1^{-1} \Leftrightarrow g_2 g_1^{-1} \in H$

$g_1^{-1} g_2 \in H \not\Rightarrow g_2^{-1} g_1 \in H$

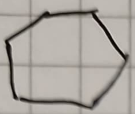
#56.44.

$H \subset G \to [G : H] =$ индекс H в G

$H_1 \subset H_2 \subset G,\ [G : H_2] = m,\ [H_2 : H_1] = n,\ [G : H_1] = mn$

если $|G| < \infty$: $[G:H_1] = \frac{|G|}{|H_1|} = \frac{|G|}{|H_2|} \cdot \frac{|H_2|}{|H_1|} = mn$

#56.45.



$D_n \supset R_n$

$|D_n| = 2n$ $\qquad |D_n / R_n| = [D_n : R_n] = 2$

$|R_n| = n$

Произведение

$G_1, G_2 \longrightarrow G_1 \times G_2 = \{(g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2\}$

$(g_1, g_2)(a_1, a_2) = (g_1 a_1, g_2 a_2)$

$|G_1 \times G_2| = |G_1| \times |G_2|$

$\mathrm{ord}((g_1, g_2)) = \text{НОК}(\mathrm{ord}(g_1), \mathrm{ord}(g_2))$

$(g_1, g_2)^n = (g_1^n, g_2^n) = (e_1, e_2)$

$g_1^n = e_1 \Rightarrow n \vdots \mathrm{ord}(g_1)$

$g_2^n = e_2 \Rightarrow n \vdots \mathrm{ord}(g_2)$

$\Rightarrow n \vdots [\mathrm{ord}(g_1), \mathrm{ord}(g_2)]$

$\mathbb{Z}_3 \times \mathbb{Z}_5 = \mathbb{Z}_{15}$

$\mathrm{ord}((1,1)) = [3,5] = 3 \cdot 5 = 15$

$\mathbb{Z}_m \times \mathbb{Z}_n = \mathbb{Z}_{mn} \iff \text{НОД}(m,n) = 1 \iff \text{НОК}(m,n) = mn$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \ncong \mathbb{Z}_4$

$(m,n) = 1 \Rightarrow \mathrm{ord}((1,1)) = mn$

$(m,n) \neq 1$; $\mathrm{ord}((a,b)) = \text{НОК}(\mathrm{ord}(a), \mathrm{ord}(b))$ $\qquad \forall a \in \mathbb{Z}_m : a^m = e$

$\mathrm{ord}((1,1)) = \text{НОК}(m,n) = \frac{mn}{\text{НОД}(m,n)} < mn$ $\qquad \forall b \in \mathbb{Z}_n : b^n = e$

$$(a,b)^{[m,n] < mn} = \left(a^{[m,n]}, b^{[m,n]}\right) = (1, 1)$$

#60.5

(а) $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$

(б) $\mathbb{Z}_{60} \cong \mathbb{Z}_{20} \times \mathbb{Z}_3 \cong \mathbb{Z}_{12} \times \mathbb{Z}_5 \cong \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_5$

$\ncong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{10}$

#58.28

(а) $f: \mathbb{Z}_6 \to \mathbb{Z}_6$

$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$

$f(0) = 0$

$f(1) = a$

$f(2) = f(1+1) = f(1) + f(1) = 2a$

$f(3) = 3a$

$f(4) = 4a$

$f(5) = 5a$

$f(k+l) \overset{?}{=} f(k) + f(l), \quad 0 \leq k, l \leq 5$

$f(k+l) = (k+l)a = ka + la$

$f: \mathbb{Z}_m \to \mathbb{Z}_n$

$f(0) = 0$

$f(1) = a$

$f(k) = ka$

$f(m) = f(0) = ma = 0$ (в $\mathbb{Z}_n$)

$ma \vdots n$

(г) $f: \mathbb{Z}_{12} \to \mathbb{Z}_{15}$

$\mathbb{Z}_{12} = \{0, 1, 2, \dots, 11\}$ $\quad$ $\mathbb{Z}_{15} = \{0, 1, 2, \dots, 14\}$

$f(0) = 0$

$f(1) = a$

$f(2) = 2a$

$f(3) = 3a$

$\dots$ $f(11) = 11a$

$f(12) = 12a = f(0) = 0$

$12a = 0$ (в $\mathbb{Z}_{15}$) $\Leftrightarrow 12a \vdots 15$; $4a \vdots 5$; $a \vdots 5$

Проверка: $f(k+l) \overset{?}{=} f(k) + f(l)$, $0 \leq k, l \leq 11$

$f(k+l) \overset{?}{=} (k+l)a$, $\quad f(k) = ak$, $\quad f(l) = al$

Знаем: $f(s) = as$, если $0 \leq s \leq 11$

$s_1, s_2 : s_1 - s_2 \vdots 12$

$f(s_1) - f(s_2) = as_1 - as_2 \vdots 15$

$\underbrace{a}_{\vdots 5}\underbrace{(s_1 - s_2)}_{\vdots 12} \vdots 15$