Homework - 11.

#5.

$a = (8, 4, 1) \qquad b = (2, -2, 1)$

$$\cos(\widehat{a, b}) = \frac{(a,b)}{|a|\cdot|b|} = \frac{8\cdot 2 - 2\cdot 4 + 1\cdot 1}{\sqrt{8^2+4^2+1^2}\cdot\sqrt{2^2+(-2)^2+1^2}} = \frac{9}{9\cdot 3} = \frac{1}{3}$$

$$\boxed{\widehat{a, b} = \arccos\left(\frac{1}{3}\right)}$$

#6.

$a = (-2, -2, -4) \qquad b = (5, 1, 6) \qquad c = (-3, 0, 2)$

Пусть $x = (\alpha, \beta, \gamma)$, тогда $(a, x) = 40$, $(b, x) = 0$, $(c, x) = 0$:

$$\begin{cases} -2\alpha - 2\beta - 4\gamma = 40 \\ 5\alpha + 1\beta + 6\gamma = 0 \\ -3\alpha + 0\beta + 2\gamma = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} -2 & -2 & -4 & 40 \\ 5 & 1 & 6 & 0 \\ -3 & 0 & 2 & 0 \end{array}\right) \underset{\text{I}-\text{III}}{\overset{\text{II}+\text{I}+\text{III}}{\rightsquigarrow}} \left(\begin{array}{ccc|c} 1 & -2 & -6 & 40 \\ 0 & -1 & 4 & 40 \\ -3 & 0 & 2 & 0 \end{array}\right) \overset{\text{III}-3\text{I}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & -2 & -6 & 40 \\ 0 & -1 & 4 & 40 \\ 0 & -6 & -16 & -120 \end{array}\right) \overset{\text{III}-6\text{II}}{\rightsquigarrow}$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & -2 & -6 & 40 \\ 0 & -1 & 4 & 40 \\ 0 & 0 & -40 & -120 \end{array}\right) \underset{\substack{\text{II}-4\text{III} \\ \text{I}+6\text{III}}}{\overset{\text{III}\cdot\left(-\frac{1}{40}\right)}{\rightsquigarrow}} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 58 \\ 0 & -1 & 0 & 28 \\ 0 & 0 & 1 & 3 \end{array}\right) \overset{\text{I}-2\text{II}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 28 \\ 0 & 0 & 1 & 3 \end{array}\right) \quad \begin{cases} \alpha = 2 \\ \beta = -28 \\ \gamma = 3 \end{cases}$$

$\boxed{\text{Ответ: } x = (2, -28, 3)}$.

#3.

$\frac{1}{x^3-1}$ над $\mathbb{R}$ $\qquad (x^3-1) = (x-1)(x^2+x+1)$

$$\frac{1}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \quad \Big| \cdot (x^3-1) = (x-1)(x^2+x+1)$$

$$1 = A(x^2+x+1) + (Bx+C)(x-1)$$

$$1 = Ax^2 + Ax + A + Bx^2 - Bx + Cx - C$$

$$1 = (A+B)x^2 + (A-B+C)x + (A-C)$$

$$\begin{cases} A+B=0 \\ A-B+C=0 \\ A-C=1 \end{cases} \quad \begin{cases} B=-A \\ A+A+A-1=0 \\ C=A-1 \end{cases} \quad \begin{cases} B=-\frac{1}{3} \\ A=\frac{1}{3} \\ C=-\frac{2}{3} \end{cases}$$

Ответ: $\frac{1}{x^3-1} = \frac{1/3}{x-1} + \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2+x+1}$

#4

$$(x^3-1) = (x-1)\left(x - \frac{-1-i\sqrt{3}}{2}\right)\left(x - \frac{-1+i\sqrt{3}}{2}\right)$$

$$\frac{1}{x^3-1} = \frac{\alpha}{x-1} + \frac{\beta}{x - \frac{-1-i\sqrt{3}}{2}} + \frac{\gamma}{x - \frac{-1+i\sqrt{3}}{2}} \quad \Big| \cdot (x^3-1)$$

$$1 = \alpha\left(x - \frac{-1-i\sqrt{3}}{2}\right)\left(x - \frac{-1+i\sqrt{3}}{2}\right) + \beta(x-1)\left(x - \frac{-1+i\sqrt{3}}{2}\right) + \gamma(x-1)\left(x - \frac{-1-i\sqrt{3}}{2}\right)$$

Подставляем $x = 1, \frac{-1-i\sqrt{3}}{2}, \frac{-1+i\sqrt{3}}{2}$.

1) $1 = \alpha\left(1 - \frac{-1-i\sqrt{3}}{2}\right)\left(1 - \frac{-1+i\sqrt{3}}{2}\right)$

$$1 = \alpha\left(1 - \frac{-1-i\sqrt{3}}{2} - \frac{-1+i\sqrt{3}}{2} + \frac{1+3}{4}\right)$$

$$1 = \alpha(2+1)$$

$$\alpha = \frac{1}{3}$$

2) $1 = \beta\left(\frac{-1-i\sqrt{3}}{2} - 1\right)\left(\frac{-1-i\sqrt{3}}{2} - \frac{-1+i\sqrt{3}}{2}\right)$

$$1 = \beta \cdot \frac{-3-i\sqrt{3}}{2} \cdot (-i\sqrt{3})$$

$$1 = \beta \cdot \frac{i3\sqrt{3} - 3}{2}$$

$$\beta = \frac{2}{i3\sqrt{3}-3} = \frac{-6-6\sqrt{3}i}{9\cdot 3+9} = \frac{-6(1+i\sqrt{3})}{+36} = \frac{1+i\sqrt{3}}{6}$$

3) $1 = \gamma\left(\frac{-1+i\sqrt{3}}{2} - 1\right)\left(\frac{-1+i\sqrt{3}}{2} - \frac{-1-i\sqrt{3}}{2}\right)$

$$1 = \gamma \cdot \frac{-3+i\sqrt{3}}{2} \cdot i\sqrt{3}$$

$$1 = \gamma \cdot \frac{-i3\sqrt{3}-3}{2}$$

$$\gamma = \frac{2}{-i3\sqrt{3}-3} = \frac{-6+6\sqrt{3}i}{9+9\cdot 3} = \frac{-1+\sqrt{3}i}{6}$$

Ответ: $\boxed{\frac{1}{x^3-1} = \frac{1/3}{x-1} - \frac{\frac{1+i\sqrt{3}}{6}}{x-\frac{-1-i\sqrt{3}}{2}} + \frac{\frac{-1+i\sqrt{3}}{6}}{x-\frac{-1+i\sqrt{3}}{2}}}$

#1.

$x^{2n} + x^n + 1$

$(x^n)^2 + x^n + 1 = 0$

$D = 1 - 4 = -3; \quad x^n = \frac{-1 \pm i\sqrt{3}}{2}$

$x_1^n = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = e^{i\frac{2\pi}{3}} \Rightarrow x_1 = e^{i\left(\frac{\frac{2\pi}{3}+2\pi k}{n}\right)},$

$x_2^n = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = e^{i\frac{4\pi}{3}} \Rightarrow x_2 = e^{i\left(\frac{\frac{4\pi}{3}+2\pi k}{n}\right)},$ $k=\overline{0,n-1}$

Тогда $x^n - e^{i\frac{2\pi}{3}} = \prod_{k=0}^{n-1}\left(x - e^{i\frac{\frac{2\pi}{3}+2\pi k}{n}}\right)$

$x^n - e^{i\frac{4\pi}{3}} = \prod_{k=0}^{n-1}\left(x - e^{i\frac{\frac{4\pi}{3}+2\pi k}{n}}\right)$

Ответ: $\boxed{\prod_{k=0}^{n-1}\left(x - e^{i\left(\frac{\frac{4\pi}{3}+2\pi k}{n}\right)}\right)\left(x - e^{i\left(\frac{\frac{2\pi}{3}+2\pi k}{n}\right)}\right).}$

$\boxed{\#2.}$

Из #2 над $\mathbb{C}$ $x^{2n}+x^n+1=\prod\limits_{k=0}^{n-1}\left(x-e^{i\left(\frac{2\pi}{3n}+\frac{2\pi k}{n}\right)}\right)\left(x-e^{i\left(-\frac{2\pi}{3n}+\frac{2\pi k}{n}\right)}\right)=$

$=\prod\limits_{k=0}^{n-1}\left(x-e^{i\left(\frac{2\pi}{3n}+\frac{2\pi k}{n}\right)}\right)\left(x-e^{i\left(-\frac{2\pi}{3n}+\frac{2\pi(n-k)}{n}\right)}\right)=$

$=\prod\limits_{k=0}^{n-1}\left(x-e^{i\left(\frac{2\pi}{3n}+\frac{2\pi k}{n}\right)}\right)\left(x-e^{i\left(-\frac{2\pi}{3n}-\frac{2\pi k}{n}\right)}\right)$

Тогда произведение $\left(x-e^{i\left(\frac{2\pi}{3n}+\frac{2\pi k}{n}\right)}\right)\left(x-e^{i\left(-\frac{2\pi}{3n}-\frac{2\pi k}{n}\right)}\right)=$

$=x^2-\left(e^{i\left(\frac{2\pi}{3n}+\frac{2\pi k}{n}\right)}+e^{i\left(-\frac{2\pi}{3n}-\frac{2\pi k}{n}\right)}\right)x+e^{i\left(\frac{2\pi}{3n}+\frac{2\pi k}{n}\right)-i\left(\frac{2\pi}{3n}+\frac{2\pi k}{n}\right)}=$

$$= x^2 - \left(\cos\left(\frac{2\pi}{3n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi}{3n} + \frac{2\pi k}{n}\right) + \cos\left(-\frac{2\pi}{3n} - \frac{2\pi k}{n}\right) + i\sin\left(-\frac{2\pi}{3n} + \frac{2\pi k}{n}\right)\right)x +$$

$$+ e^0 = x^2 - \left(2\cos\left(\frac{2\pi}{3n} + \frac{2\pi k}{n}\right) + i\left(\sin\left(\frac{2\pi}{3n} + \frac{2\pi k}{n}\right) - \sin\left(\frac{2\pi}{3n} + \frac{2\pi k}{n}\right)\right)\right)x + 1 =$$

$$= x^2 - 2\cos\left(\frac{2\pi}{3n} + \frac{2\pi k}{n}\right)x + 1 \quad \in \mathbb{R}$$

Выбираем числа для пары с аргументами $\frac{2\pi}{3n} + \frac{2\pi k}{n}$ и $-\frac{2\pi k}{3n} + \frac{2\pi(n-k)}{n}$, т.к. эти числа будут сопряжёнными.

Тогда получится n множителей вида $x^2 - 2\cdot Re\left(e^{i\left(\frac{2\pi}{3n} + \frac{2\pi k}{n}\right)}\right)x + 1$, которые $\in \mathbb{R}$.

$$\boxed{\text{Ответ: } \prod_{k=0}^{n-1}\left(x^2 - 2Re\left(e^{\frac{2\pi}{3n} + \frac{2\pi k}{n}}\right)x + 1\right)}$$