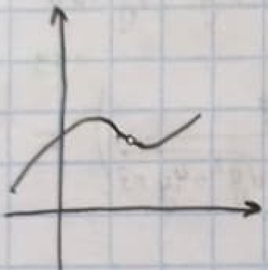
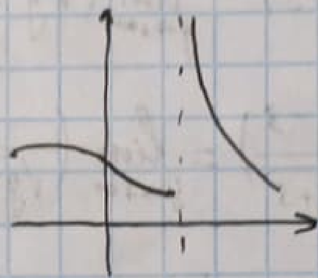
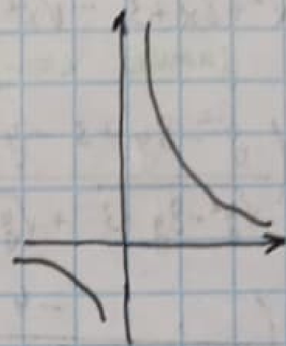
Семинар 10, 21.11.23

Асимптоты. O-символика

$f(x)$ $\quad x = a$

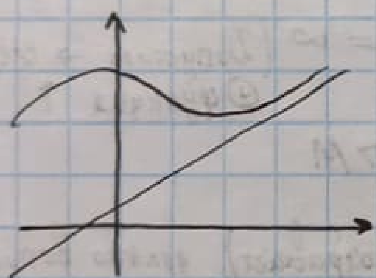


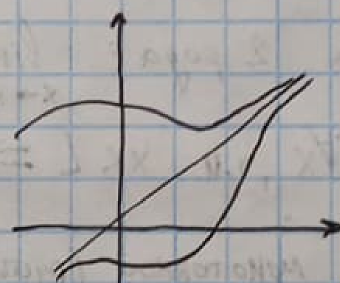




$$\lim_{x \to a^{\pm}} f(x) = \infty$$

$y = kx + b$





$$\lim_{x \to \pm\infty} (f(x) - kx - b) = 0 \Rightarrow \lim_{x \to \pm\infty} \left(\frac{f(x)}{x} - k - \frac{b}{x} \right) = 0$$

$$\lim_{x \to \pm\infty} \left(\frac{f(x)}{x} \right) = k$$

O-символика

$f(x) = o(g(x))$ при $x \to x_0$ $(\infty / +\infty / -\infty)$

$$\frac{f(x)}{g(x)} \xrightarrow[x \to x_0]{} 0$$

$|f(x)| \leq \varepsilon |g(x)|$ при $|x - x_0| < \delta$

$$\forall \varepsilon > 0 \quad \exists \delta: \quad \forall x: |x - x_0| < \delta \Rightarrow |f(x)| \leq \varepsilon |g(x)|$$

$f(x) = O(g(x)$

$\frac{|f(x)|}{|g(x)|} \leq C$ при $|x - x_0| < \delta$

$|f(x) \leq C \cdot |g(x)|$

#2.

(a) 1) $f(x) = O(x^3)$ при $x \to 0$

$f(x) \overset{?}{=} O(x^2)$ при $x \to 0$

1) $|f(x)| \leq C \cdot |x^3| \leq C \cdot |x^2|$

$|f(x)| \leq C \cdot |x^2|$

(b) $f(x) = O(x^3)$ при $x \to 0$

$f(x) \overset{?}{=} O(x^3)$ при $x \to +\infty$

$f(x) = x^4$

$\frac{x^4}{x^3} = x \xrightarrow[x \to 0]{} 0$ $\quad x \to +\infty$; $\frac{x^4}{x} = x \xrightarrow[x \to \infty]{} \infty$ (w)

(c) $f(x) = o(x^2)$ при $x \to 0$

$f(x) \overset{?}{=} o(x^3)$ при $x \to 0$

$\frac{f(x)}{x^2} \xrightarrow[x \to 0]{} 0$

$\frac{f(x)}{x^2 \cdot x} = \overbrace{\frac{f(x)}{x^2}}^{\to 0} \cdot \overbrace{\frac{1}{x}}^{} \xrightarrow[x \to 0]{??} 0$

$f(x) = x^{\frac{5}{2}}$ $\quad (2 < \frac{5}{2} < 3)$

$\frac{x^{\frac{5}{2}}}{x^2} = x^{\frac{1}{2}} = \sqrt{x} \xrightarrow[x \to 0]{} 0$ $\quad \frac{x^{\frac{5}{2}}}{x^3} = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}} \xrightarrow[x \to 0]{} \infty$

(d) $f(x) = O(x^3)$ при $x \to 0$

$f(x) \overset{?}{=} o(x^2)$ при $x \to 0$

$|f(x)| \leqslant C|x^3| \overset{?}{\Rightarrow} \frac{f(x)}{x^2} \xrightarrow[x\to 0]{} 0$

$\left|\frac{f(x)}{x^2} \cdot \frac{x}{x}\right| \leqslant C \cdot |x| \xrightarrow[x\to 0]{} 0$

$f(x) = O(x^n) \Rightarrow f(x) = o(x^{n-1})$ $x \to 0$

#4.

$f(x) = 1 + 2x + o(x)$ при $x \to 0$

$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} (1 + 2x + o(x)) = \lim\limits_{x\to 0} 1 + \lim\limits_{x\to 0} 2x + \underbrace{\lim\limits_{x\to 0} o(x)}_{\overset{?}{=}\, 0} =$

$= 1 + 0 + 0 = 1$

если $f(x) = o(x)$, т.е. $\frac{f(x)}{x} \xrightarrow[x\to 0]{} 0$, то $f(x) \xrightarrow[x\to 0]{} 0$

#5.

$f(x) = 1 + 3x + O(x^2)$ при $x \to 0$

$\lim\limits_{x\to 0} \frac{f(x) - 1}{x} = \lim\limits_{x\to 0} \frac{3x + O(x^2)}{x} = \lim\limits_{x\to 0} \Big(\underbrace{3}_{\to 3} + \underbrace{\frac{O(x^2)}{x}}_{\to 0}\Big) = 3$

$|f(x)| \leqslant C|x^2|$

$\frac{f(x)}{|x|} \leqslant C|x| \xrightarrow[x\to 0]{} 0$

Док-во: $\frac{O(g(x))}{x} = O\left(\frac{g(x)}{x}\right)$

$f = O(g) \overset{x}{\Rightarrow} \frac{f}{x} = O\left(\frac{g}{x}\right)$

$|f| \leqslant C \cdot |g| \overset{x}{\Rightarrow} \left|\frac{f}{x}\right| \leqslant C \cdot \left|\frac{g}{x}\right|$

$\frac{O(g(x))}{x} = O\left(\frac{g(x)}{x}\right)$

$x \cdot O(g(x)) = O(x \cdot g(x))$

$h(x) \cdot O(g(x)) = O(h(x) \cdot g(x))$

#6.

$$f(y) = y + 2y^2 + o(y^2) \quad \text{при } y \to 0$$

$$f(3x + x^2) = a_0 + a_1 x + a_2 x^2 + o(x^2)$$

$$f(3x + x^2) = (3x + x^2) + 2(3x + x^2)^2 + o((3x + x^2)^2) =$$

$$= \underbrace{3x + x^2 + 18x^2}_{} + \underbrace{12x^3 + 2x^4 + o((3x + x^2)^2)}_{\overset{?}{=}\, o(x^2)}$$

$$:x^2$$

$$12x + 2x^2 + \frac{o((3x + x^2)^2)}{x^2} \longrightarrow 0$$

$$\underset{0}{\downarrow} \;\; x \to 0 \;\; \underset{0}{\downarrow}$$

$$\parallel$$

$$o\left(\left(\frac{3x + x^2}{x}\right)^2\right) = o((3 + x)^2) \xrightarrow[x \to 0]{?} 0$$

$$\Uparrow ?!$$

$$\frac{f(x)}{(3+x)^2} \xrightarrow[x \to 0]{} 0 \Rightarrow f(x) \xrightarrow[x \to 0]{} 0$$