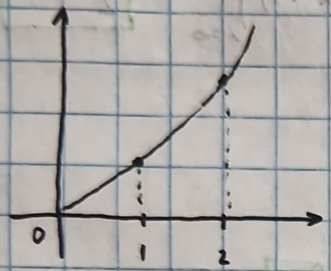
Семинар 19, 20.02.24

Вычисление опред. интеграла

① Вычислить интеграл как предел интегральной суммы.

(a) $\int_1^2 x^3 dx$



$1,\ 1+\frac{1}{n},\ 1+\frac{2}{n},\ 1+\frac{3}{n}, \dots, 1+\frac{n-1}{n},\ 2$

$[1; 1+\frac{1}{n}],\ [1+\frac{1}{n}; 1+\frac{2}{n}], \dots, [1+\frac{n-1}{n}; 2]$

$$\frac{1}{n}f(1)+\frac{1}{n}f\left(1+\frac{1}{n}\right)+\dots+\frac{1}{n}f\left(1+\frac{n-1}{n}\right)=$$

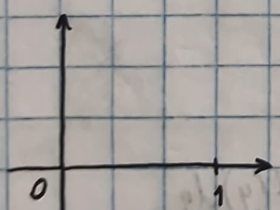
$$=\frac{1}{n}\left(1^3+\left(1+\frac{1}{n}\right)^3+\left(1+\frac{2}{n}\right)^3+\dots+\left(1+\frac{n-1}{n}\right)^3\right)=$$

$$=\frac{1}{n}\left(1^3+\left(1^3+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}\right)+\left(1^3+3\cdot\frac{2}{n}+3\cdot\frac{2^2}{n^2}+\frac{2^3}{n^3}\right)+\dots+\left(1^3+3\frac{n-1}{n}+3\frac{(n-1)^2}{n^2}+\frac{(n-1)^3}{n^3}\right)\right)=$$

$$=\frac{1}{n}\left(n+\frac{3}{n}\cdot\frac{n(n-1)}{2}+\frac{3}{n^2}\cdot\frac{(n-1)n(2n-1)}{6}+\frac{1}{n^3}\cdot\frac{n^2(n-1)^2}{4}\right)=$$

$$=1+\frac{3\cdot1\left(1-\frac{1}{n}\right)}{2}+\frac{\left(1-\frac{1}{n}\right)\cdot1\cdot\left(2-\frac{1}{n}\right)}{2}+\frac{\left(1-\frac{1}{n}\right)^2\cdot1^2}{4}\xrightarrow[n\to\infty]{}1+\frac{3}{2}+1+\frac{1}{4}=\boxed{\frac{15}{4}}$$

(b) $\int_0^1 e^x dx$



$\left(\frac{i-1}{n}; \frac{i}{n}\right),\ i=1,2,\dots,n$

$$\frac{1}{n}\left(e^0+e^{\frac{1}{n}}+e^{\frac{2}{n}}+\dots+e^{\frac{n-1}{n}}\right)=\frac{1}{n}\cdot\frac{e^{\frac{n}{n}}-1}{e^{\frac{1}{n}}-1}=\frac{e-1}{n(e^{\frac{1}{n}}-1)}$$

$$\lim_{n\to\infty}\left(n\cdot\left(e^{\frac{1}{n}}-1\right)\right)\underset{n=\frac{1}{x}}{=}\lim_{x\to0}\frac{e^x-1}{x}=\lim_{x\to0}\frac{1+x+\bar{o}(x)-1}{x}=\lim_{x\to0}(1+\bar{o}(1))=\boxed{1}$$

② Вычислить предел:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{k^2+n^2}$$

$$\sum_{k=1}^{n}\frac{k}{k^2+n^2}=\frac{1}{n}\sum_{k=1}^{n}\frac{k}{\frac{k^2}{n}+n}=\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\frac{k}{n}+\frac{n}{k}}\Rightarrow$$

$$\Rightarrow\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{k^2+n^2}=\int_0^1\frac{x\,dx}{x^2+1}=\frac{1}{2}\int_0^1\frac{d(x^2)}{x^2+1}=\frac{1}{2}\ln(x^2+1)\Big|_{x=0}^{x=1}=\boxed{\frac{1}{2}\ln 2}$$

③ (a) $\int_1^2 \ln x\,dx=(x\ln x-x)\Big|_{x=1}^{x=2}=\boxed{\ln 4-1}$

(b) $\int_e^{e^2}\frac{dx}{x\ln x}=\int_e^{e^2}\frac{d(\ln x)}{\ln x}\underset{\ln x=y}{=}\int_1^2\frac{dy}{y}=(\ln y)\Big|_{y=1}^{y=2}=\boxed{\ln 2}$

(c) $\int_{-2\sqrt{3}}^{2}\frac{dx}{(4+x^2)^2}=\left[\begin{aligned}&tg^2+1=\frac{1}{\cos^2 t}\\&x=2tg\,t\\&4+x^2=4+4tg^2t=\frac{4}{\cos^2 t}\\&dx=\frac{2}{\cos^2 t}dt\end{aligned}\right]=\int_{-\frac{\pi}{3}}^{\frac{\pi}{4}}\frac{2}{\cos^2 t}\cdot\frac{\cos^4 t}{16}\,dt=$

$=\frac{1}{8}\int_{-\frac{\pi}{3}}^{\frac{\pi}{4}}\cos^2 t\,dt$

#дальше всё понятно
(с) И. Бельдиев

(e) $\int_{-3}^{3}\sin x\cdot e^{-x^2}dx=\int_{-3}^{0}f(x)\,dx+\int_0^3 f(x)\,dx=\boxed{0}$

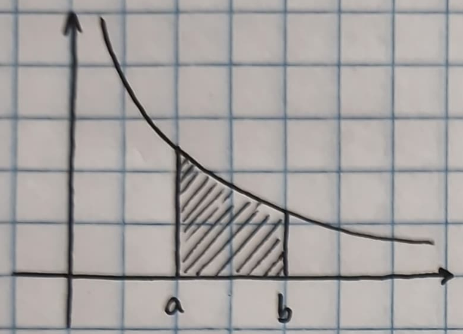
$\int_{-3}^{0}f(x)\,dx=[x=-y]=-\int_3^0 f(-y)\,dy=\int_0^3 f(-y)\,dy=-\int_0^3 f(y)\,dy$

(f) $\int_{0,1}^{10}\frac{\ln x}{1+x^2}dx=\int_{0,1}^{1}\frac{\ln x}{1+x^2}dx+\int_1^{10}\frac{\ln x}{1+x^2}dx=\boxed{0}$

$\int_{0,1}^{1}\frac{\ln x}{1+x^2}dx=\left[y=\frac{1}{x}\right]=-\int_{10}^{1}\frac{\ln(\frac{1}{y})}{1+\frac{1}{y^2}}\cdot\frac{dy}{y^2}=\int_1^{10}\frac{\ln(\frac{1}{y})}{1+y^2}dy=-\int_1^{10}\frac{\ln y}{1+y^2}dy$

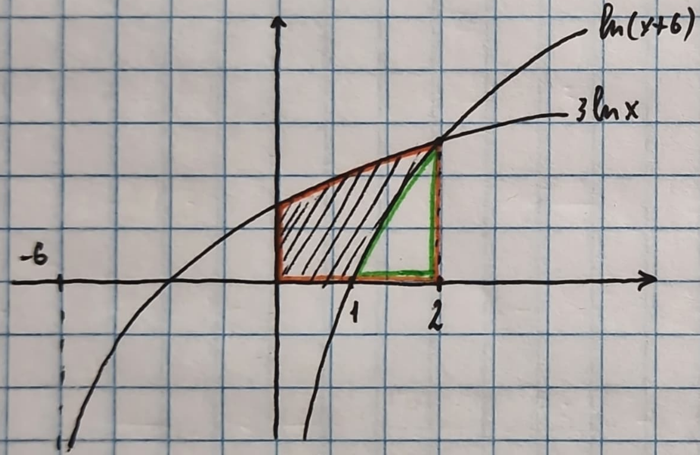
④ Найти площадь

(a) $y=\frac{1}{x}$, $y=0$, $x=a$, $x=b$, $0 \le a \le b$



$$S=\int_a^b \frac{dx}{x}$$

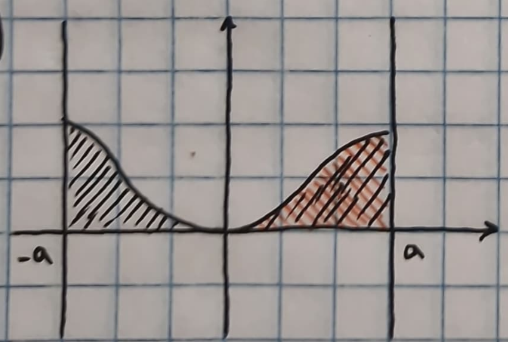
(d) $y=\ln(x+6)$, $y=3\ln x$, $x=0$, $y=0$



$$S = S - S$$

$$S=\int_0^2 \ln(x+6)\,dx - \int_1^2 3\ln x\,dx$$

(b)



$S = 2\cdot S$ (т.к. f – чётная)

$$S = 2\int_0^a x^3\cdot e^{-x^2}dx = \int_0^a x^2 e^{-x^2} d(x^2) =$$

$$= \int_0^{a^2} y e^{-y} dy$$