Семинар 18, 06.02.24 – Бельдиев

Смежные классы

$H \subset G \qquad gH = \{gh \mid$

$g_1H,\ g_2H$

$(g_1H)\circ(g_2H) = g_1 \cdot g_2 H$

1) $gH \circ eH = geH = gH$

2) $gH \circ g^{-1}H = eH = H$

$gH = Hg \iff H$ – нормальна

$\square$ Домножим на g^{-1}: $\quad g^{-1}gH = g^{-1}Hg$

$$H = g^{-1}Hg \quad \blacksquare$$

$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$

смежный класс: $\{z + kn \mid k \in \mathbb{Z}\}$

HSE

$\{r_1 + kn \mid k \in \mathbb{Z}\} \ni r_1$

$\{r_2 + kn \mid k \in \mathbb{Z}\} \ni r_2$

$r_1 + r_2 \in \{r_1 + r_2 + kn \mid k \in \mathbb{Z}\}$

Проверка изоморфизма:

$f : \mathbb{Z}/_n\mathbb{Z} \to \mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}$

$\{r + k\mathbb{Z}\} \mapsto r \bmod n$

Теорема о гомоморфизме

$f : G_1 \to G_2$

$\operatorname{Ker} f \in \{g \in G_1,\ f(g_1) = e_2\}$ (kernel)

1) $a, b \in \operatorname{Ker} \overset{?}{\Rightarrow} ab \in \operatorname{Ker} f$

$f(ab) = f(a) \cdot f(b) = e_2$

$\operatorname{Im} f = \{g_2 \in G_2 \mid \exists g_1 : f(g_1) = g_2\}$ (image)

2) $g \in \operatorname{Ker} f \Rightarrow xgx^{-1} \overset{!}{\in} \operatorname{Ker} f$; $f(xgx^{-1}) = f(x) f(g) f(x^{-1}) =$

$= f(x) e_2 f(x^{-1}) = f(x) f(x^{-1}) = e_2$

$\operatorname{Ker} f \trianglelefteq G \Rightarrow \exists$ факторгруппа $G/\operatorname{Ker} f$

$G/\operatorname{Ker} f \cong \operatorname{Im} f$

#58.30

(a) $\mathbb{Z}/_n\mathbb{Z} \cong \mathbb{Z}_n$

(?) $f : \mathbb{Z} \to \mathbb{Z}_n$

$r \mapsto r \bmod n$

$\left.\begin{array}{l} \operatorname{Ker} f = n\mathbb{Z} \\ \operatorname{Im} f = \mathbb{Z}_n \end{array}\right| \Rightarrow \mathbb{Z}/_n\mathbb{Z} \cong \mathbb{Z}_n$

#58.30в.

$4\mathbb{Z}/12\mathbb{Z} \cong \mathbb{Z}_3$

$f: 4\mathbb{Z} \to \mathbb{Z}_3$

$4k \mapsto 4k \bmod 3$

$\operatorname{Ker} f = \{4k \,\vdots\, 3\} \Leftrightarrow k \,\vdots\, 3$

$$\left.\begin{array}{l}\operatorname{Ker} f = 12\mathbb{Z} \\ \operatorname{Im} f = \mathbb{Z}_3\end{array}\right| \Rightarrow 4\mathbb{Z}/12\mathbb{Z} \cong \mathbb{Z}_3$$

#58.32.

(в) $\mathbb{C}^*/U \overset{?}{\cong} \mathbb{R}_+$

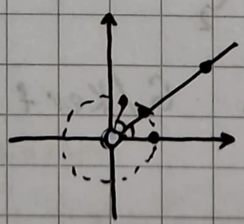
$f: \mathbb{C}^* \to \mathbb{R}_+$

$z \mapsto |z|$

$\operatorname{Ker} f = U$

$\operatorname{Im} f = \mathbb{R}_+$

(д) $\mathbb{C}^*/\mathbb{R}^* \overset{?}{\cong} U$



$f: \mathbb{C}^* \to U$

$z \mapsto \frac{|z|}{z} \mapsto \frac{z^2}{|z|^2}$

$\operatorname{Ker} f = \mathbb{R}_+ \Rightarrow \mathbb{C}^*/\mathbb{R}^*_+ \cong U$

$\mathbb{C}^* \to U \xrightarrow{z_1 \mapsto z_1^2} U \leftarrow g$

$z \mapsto \frac{z}{|z|} \longmapsto \frac{z^2}{|z|^2}$

$$\left.\begin{array}{l}\operatorname{Ker} g = \mathbb{R}^* \\ \operatorname{Im} g = U\end{array}\right| \Rightarrow \mathbb{C}^*/\mathbb{R}^* \cong U$$

$\left(\frac{z^2}{|z|^2} = 1\right) \Leftrightarrow z^2 = |z|^2$

(г) $U / U_n \cong U$

$U_n = \{ e^{\frac{2\pi k i}{n}} \mid k \in \mathbb{Z} \}$

$f : U \to U$

$z \mapsto z^n$

$\operatorname{Ker} f = U_n$

$\operatorname{Im} f = \{ z \in U \mid \exists a : a^n = z \}$

(а) $\mathbb{R} / \mathbb{Z} \cong U$
$(+) \quad (+) \quad (\times)$

$f : \mathbb{R} \to U$

$r \in \mathbb{R} \mapsto \cos(2\pi r) + \sin(2\pi r)$

$\operatorname{Ker} f = \{ z \in \mathbb{R} \mid \cos 2\pi r + i \sin 2\pi r = 1 \}$

$\Updownarrow$

$r = \frac{2\pi r}{2\pi} \in \mathbb{Z}$

#58.33

(а) $GL_n(\mathbb{R}) / SL_n(\mathbb{R}) \cong \mathbb{R}^*$

$GL_n(\mathbb{R}) \to \mathbb{R}^*$

$\det : X \mapsto \det(X)$

(г) $GL_n(\mathbb{C}) / B \cong U$

$z \longleftrightarrow \frac{z}{|z|}$

$\det : GL_n(\mathbb{C}) \to \mathbb{C}^* \longrightarrow U \longmapsto U$

$x \mapsto \det(x) \mapsto \frac{\det(x)}{|\det x|} \mapsto \frac{(\det x)^2}{|\det x|^2}$

HSE

#58.2:

$H \subset G$

$|G:H| = 2$

$gH \overset{?}{=} Hg$ – выполняется.

Сопряжёние:

$x \sim y: \quad y = gxg^{-1}$

$H \triangle G$ – норм. подгруппа

$h \in H \Rightarrow xhx^{-1} \in H$

#58.4a. $H \triangle S_3 = \{e, (12), (13), (23), (123), (132)\}$

(1) $(12) \in H$

$(13)(12)(13) = (1)(23) = (23)$

$(23)(12)(23) = (2)(13) = (13)$

(2) $(132)^2 = (123) \Rightarrow H = \{e, (123), (132)\}$

$A_n \subset S_n$

$\sigma \in A_n \overset{?}{\Rightarrow} \tau^{-1}\sigma\tau \in A_n$