Семинар 17, 06.02.24

$a > 0$ $(a < 0 : a \to -a)$

① $\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{dx}{a\sqrt{1-\left(\frac{x}{a}\right)^2}} = \int \frac{d\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \arcsin \frac{x}{a} + C$

Вычислить:

$$\int \frac{dx}{\sqrt{2-x-4x^2}} = \int \frac{dx}{\sqrt{\left(\frac{\sqrt{33}}{\sqrt{16}}\right)^2 - \left(2x+\frac{1}{4}\right)^2}} = \frac{1}{2}\int \frac{d\left(2x+\frac{1}{4}\right)}{\sqrt{\left(\frac{\sqrt{33}}{4}\right)^2 - \left(2x+\frac{1}{4}\right)^2}} = \frac{1}{2}\arcsin\left(\frac{2x+\frac{1}{4}}{\frac{\sqrt{33}}{4}}\right) + C$$

$$2 - x - 4x^2 = -\left(\underbrace{(2x)^2 + 2\cdot(2x)\cdot\frac{1}{4} + \left(\frac{1}{4}\right)^2}\, - \left(\frac{1}{4}\right)^2 - 2\right) = \boxed{-\left(2x+\frac{1}{4}\right)^2 + \frac{33}{16}}$$

② $\int \sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} - \int x\, d\left(\sqrt{a^2-x^2}\right) = x\sqrt{a^2-x^2} - \int x \cdot \frac{-2x}{2\sqrt{a^2-x^2}}\,dx =$

$$= x\sqrt{a^2-x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = x\sqrt{a^2-x^2} + \int \frac{x^2-a^2+a^2}{\sqrt{a^2-x^2}}\,dx =$$

$$= x\sqrt{a^2-x^2} + \int -\sqrt{a^2-x^2}\,dx + a^2\int \frac{dx}{\sqrt{a^2-x^2}} =$$

$$= x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\,dx + a^2 \cdot \arcsin\left(\frac{x}{a}\right)$$

$$2\int \sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} + a^2 \cdot \arcsin \frac{x}{a}$$

$$\int \sqrt{a^2-x^2}\,dx = \boxed{\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\cdot \arcsin \frac{x}{a} + C}$$

Вычислить:

$$\int \sqrt{3-2x-x^2}\,dx = \int \sqrt{4-(x+1)^2}\,dx = \boxed{\frac{x+1}{2}\sqrt{4-(x+1)^2} + 2\arcsin\frac{x+1}{2} + C}$$

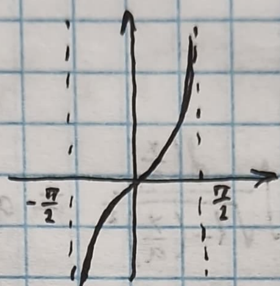
3) $$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{d(a\,tg\,t)}{\sqrt{a^2 tg^2 t + a^2}} = \int \frac{a}{\cos^2 t}\cdot\frac{\cos t}{a}\,dt = \int \frac{dt}{\cos t} \ominus$$

Замена:

$x = x(t)$

$x(t)^2 + 1 = y(t)^2$

$\sin^2 t +$



$tg^2 t + 1 = \frac{1}{\cos^2 t}$

$x = a\cdot tg\,t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$

$$\sqrt{x^2+a^2} = \sqrt{a^2 tg^2 t + a^2} = a\sqrt{\frac{1}{\cos^2 t}} = \frac{a}{|\cos t|} = \frac{a}{\cos t}$$

$$\ominus \int \frac{\cos t\,dt}{\cos^2 t} = \int \frac{d(\sin t)}{\cos^2 t} = \int \frac{d(\sin t)}{1-\sin^2 t} \underset{y=\sin t}{=} \int \frac{dy}{1-y^2} = \frac{1}{2}\int\left(\frac{1}{1+y} + \frac{1}{1-y}\right)dy =$$

$$= \frac{1}{2}\left(\ln|1+y| - \ln|1-y|\right) + C = \frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| + C =$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin t}{1-\sin t}\right| + C = \frac{1}{2}\ln\left(\frac{(1+\sin t)^2}{1-\sin^2 t}\right) + C =$$

$$= \frac{1}{2}\ln\left(\frac{1+\sin t}{\cos t}\right)^2 + C = \ln\left(\frac{1+\sin t}{\cos t}\right) + C =$$

$$= \ln\left(\frac{1}{\cos t} + tg\,t\right) + C = \ln\left(\sqrt{tg^2 t + 1} + tg\,t\right) + C =$$

$$= \ln\left(\sqrt{\frac{x^2}{a^2}+1} + \frac{x}{a}\right) + C = \ln\left(\frac{\sqrt{x^2+a^2}+x}{a}\right) + C =$$

$$= \ln\left(x + \sqrt{x^2+a^2}\right) - \ln a + C = \boxed{\ln\left(x+\sqrt{x^2+a^2}\right) + C}$$

Гиперболический косинус: $\operatorname{ch} x = \cosh x = \frac{e^x + e^{-x}}{2}$

Гиперболический синус: $\operatorname{sh} x = \sinh x = \frac{e^x - e^{-x}}{2}$

$(\operatorname{ch} x)' = \operatorname{sh} x \qquad (\operatorname{sh} x)' = \operatorname{ch} x$

$\operatorname{ch}^2 x - \operatorname{sh}^2 x = 1 \qquad \operatorname{ch}^2 x - 1 = \operatorname{sh}^2 x$

④ $\sqrt{x^2 - a^2} \ominus$

Замена: $x = a \cdot \operatorname{ch} t \quad (x > a)$

$\ominus\ a|\operatorname{sh} t| = a \cdot \operatorname{sh} t$ (т.к. $\operatorname{sh} t > 0$ при $x > 0$)

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{d(a \operatorname{ch} t)}{a \cdot \operatorname{sh} t} = \int \frac{a \cdot \operatorname{sh} t \, dt}{a \cdot \operatorname{sh} t} = \int dt = t + C \ominus$$

$x = a \cdot \operatorname{ch} t = \frac{a}{2} \cdot (e^t + e^{-t}) = \frac{a}{2}\left(y + \frac{1}{y}\right)$

$[e^t = y]$

$2xy = ay^2 + a$

$ay^2 - 2xy + a = 0$

$D = 4x^2 - 2a^2 \cdot 2 = 4x^2 - 4a^2$

$$y = \frac{2x \pm \sqrt{4x^2 - 4a^2}}{2a} = \frac{x \pm \sqrt{x^2 - a^2}}{a} = e^t; \quad e^t = \frac{x + \sqrt{x^2 - a^2}}{a}$$

$t = \ln(x + \sqrt{x^2 - a^2}) - \ln a$

$\ominus\ \boxed{\ln(x + \sqrt{x^2 - a^2}) + C}$