Calculus exam consultation

Parametric functions

Let functions $\varphi(t)$ and $\psi(t)$ defined and continuous on the interval (α, β) and function $\varphi(t)$ strictly monotonic on the (α, β). Then the system of equations

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$$

defines unique continuous function

$$y(x) = \psi(\varphi^{-1}(x))$$

on the interval $(a; b)$, where $a = \lim_{t\to\alpha+} \varphi(t)$ and $b = \lim_{t\to\beta-} \psi(t)$

Assymptotes

Прямую $x = x_0$ называют вертикальной асимптотой кривой

$$x = x(t), \quad y = y(t),$$

если существует такое a (число, $+\infty$ или $-\infty$), что

$$\lim_{t\to a} x(t) = x_0, \quad \lim_{t\to a} y(t) = \infty,$$

или

$$\lim_{t\to a+0} x(t) = x_0, \quad \lim_{t\to a+0} y(t) = \infty,$$

или

$$\lim_{t\to a-0} x(t) = x_0, \quad \lim_{t\to a-0} y(t) = \infty.$$

Прямую $y = b$ называют горизонтальной асимптотой кривой $x = = x(t), y = y(t)$ при $x \to +\infty$, если существует такое a (число, $+\infty$ или $-\infty$), что

$$\lim_{t\to a} x(t) = +\infty, \quad \lim_{t\to a} y(t) = b,$$

или

$$\lim_{t\to a-0} x(t) = +\infty, \quad \lim_{t\to a-0} y(t) = b,$$

или

$$\lim_{t\to a+0} x(t) = +\infty, \quad \lim_{t\to a+0} y(t) = b.$$

Прямую $y = kx + b, k \neq 0$, называют наклонной асимптотой кривой $x = x(t), y = y(t)$ при $x \to +\infty$, если существует такое a (число, $+\infty$ или $-\infty$), что

$$\lim_{t\to a} x(t) = +\infty, \quad \lim_{t\to a} y(t) = \infty,$$

$$\lim_{t\to a} \frac{y(t)}{x(t)} = k,$$

$$\lim_{t\to a} (y(t) - kx(t)) = b,$$

Или рассматриваются пределы при $t \to a-$ или $t \to a+$
Аналогично даются определения асимптот при $x \to -\infty$.

Claim Calculating the derivatives

Suppose that the given system of equations defines unique continuous function $y(x)$

$$\begin{cases} x(t) = \varphi(t) \\ y(t) = \psi(t) \end{cases}$$

Then:

$$y'(x) = \frac{y'(t)}{x'(t)}$$

$$y''(x) = \frac{x'(t)y''(t) - y'(t)x''(t)}{(x'(t))^3}$$

Equivalent notation:

$$y'_x = \frac{y'_t}{x'_t}$$

$$y''_{xx} = \frac{x'_t y''_{tt} - y'_t x''_{tt}}{(x'_t)^3}$$

Exercise 1. (task 1.b from the homework)

$$\begin{cases} x = \varphi(t) = \frac{1}{4}(t-4)e^t \\ y = \psi(t) = \sqrt{t} \cdot e^t \end{cases}$$

1. Function $y(t)$ is defined, continuous and strictly monotonic on the interval $(0; +\infty)$ function $x(t)$ is also defined and continuous on the $(0; +\infty)$

2. Hence, given system of equations defines unique continuous function $x(y) = \varphi(\psi^{-1}(y))$ on the interval $(a; b)$, where $a = \lim\limits_{t \to 0+} \psi(t)$ and $b = \lim\limits_{t \to +\infty} \psi(t)$

3. $a = \lim\limits_{t \to 0+} \psi(t) = \lim\limits_{t \to 0+} \sqrt{t} \cdot e^t = 0$

$b = \lim\limits_{t \to +\infty} \psi(t) = \lim\limits_{t \to +\infty} \sqrt{t} \cdot e^t = +\infty$

Hence, $(a; b) = (0; +\infty)$

Answer: system defines continuous function $x(y) = \varphi(\psi^{-1}(y))$ on the interval $(0; +\infty)$

Exercise Find intervals of monotonicity of the function $y(x)$ defined by the system of equations:

$$\begin{cases} x = \ln t \\ y = t^3 \end{cases}$$

1. $y'_t = \dfrac{1}{t}$
2. $x'_t = 3t^2$
3. $y'_x = \dfrac{y'_t}{x'_t} = \dfrac{1}{3t^3}$

Exercise Find derivatives and asymptotes of the curve defined by the system of equations:

$$\begin{cases} x = \frac{t^2}{1+t^3} \\ y = \frac{t^3}{1+t^3} \end{cases}$$

1. $y'_t = \left(1 - \dfrac{1}{1+t^3}\right)'_t = (-1)\cdot(-1)\cdot\left(1+t^3\right)^{-2}\cdot 3t^2 = \dfrac{3t^2}{(1+t^3)^2}$

$y''_{tt} = (y'_t)'_t = \left(\dfrac{3t^2}{(1+t^3)^2}\right)'_t = \dfrac{6t(1+t^3)^2 - 3t^2\cdot 2(1+t^3)\cdot 3t^2}{(1+t^3)^4} = \dfrac{6t-12t^4}{(1+t^3)^3}$

2. $x'_t = \left(\dfrac{t^2}{1+t^3}\right)'_t = \dfrac{2t(1+t^3) - t^2\cdot 3t^2}{(1+t^3)^2} = \dfrac{2t-t^4}{(1+t^3)^2} = \dfrac{t(2-t^3)}{(1+t^3)^2}$

$x''_{tt} = (x'_t)'_t = \left(\dfrac{t(2-t^3)}{(1+t^3)^2}\right)'_t = \dfrac{\left((2-t^3)+t(-3t^2)\right)(1+t^3)^2 - t(2-t^3)\cdot 2(1+t^3)\cdot 3t^2}{(1+t^3)^4} =$

$= \dfrac{\left(2-4t^3\right)(1+t^3) - t(2-t^3)\cdot 2\cdot 3t^2}{(1+t^3)^3} = \dfrac{\left(2-4t^3+2t^3-4t^6\right) - \left(12t^3-6t^6\right)}{(1+t^3)^3} =$

$= \dfrac{2-2t^3-4t^6-12t^3+6t^6}{(1+t^3)^3} = \dfrac{2-14t^3+2t^6}{(1+t^3)^3}$

3. $y'_x = \dfrac{y'_t}{x'_t} = \dfrac{3t}{2-t^3}$

$y''_{xx} = \dfrac{x'_t y''_{tt} - y'_t x''_{tt}}{(x'_t)^3} = \dfrac{-6(t^3+1)^3}{t(t^3-2)^3}$ (magic)

Asymptotes:

$\lim\limits_{t\to -1+} x(t) = +\infty$

$\lim\limits_{t\to -1+} y(t) = -\infty$

$\lim\limits_{t\to -1-} x(t) = -\infty$

$\lim\limits_{t\to -1-} y(t) = +\infty$

$\lim\limits_{t\to +\infty} x(t) = 0$

$\lim\limits_{t\to +\infty} y(t) = 1$

$\lim\limits_{t\to -\infty} x(t) = 0$

$\lim\limits_{t\to -\infty} y(t) = 1$

There are no horizontal nor vertical asymptotes

$\lim\limits_{t\to -1} \dfrac{y(t)}{x(t)} = \lim\limits_{t\to -1} t = -1 = k$

$\lim\limits_{t\to -1} y(t) - kx(t) = \lim\limits_{t\to -1} \dfrac{t^2(1+t)}{1+t^3} = \lim\limits_{t\to -1} \dfrac{t^2}{1-t+t^2} = \dfrac{1}{3} = b$

Oblique asymptote:

$y = -x + \dfrac{1}{3}$ as $t \to -1$

Taylor formula

Exercise Work out the limit using Taylor polynomial

$$\lim_{x\to 0} \frac{\sqrt[3]{1+x^2} - e^{\frac{x^2}{3}}}{\ln(1+3x^2) - 3x^2\cos(x)} =$$

1. $\sqrt[3]{1+x^2} = 1 + \frac{1}{3}x^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2}x^4 + \overline{o}\left(x^4\right) = 1 + \frac{1}{3}x^2 - \frac{1}{9}x^4 + \overline{o}\left(x^4\right)$ as x $\to 0$

2. $e^{\frac{x^2}{3}} = 1 + \frac{x^2}{3} + \frac{1}{2}\left(\frac{x^2}{3}\right)^2 + \overline{o}\left(\left(\frac{x^2}{3}\right)^2\right) = 1 + \frac{x^2}{3} + \frac{x^4}{18} + \overline{o}\left(x^4\right)$ as x $\to 0$

3. $\ln\left(1+3x^2\right) = 3x^2 - \frac{(3x^2)^2}{2} + \overline{o}\left((3x^2)^2\right) = 3x^2 - \frac{9x^4}{2} + \overline{o}\left(x^4\right)$ as x $\to 0$

4. $3x^2\cos(x) = 3x^2\left(1 - \frac{x^2}{2} + \overline{o}\left(x^3\right)\right) = 3x^2 - \frac{3x^4}{2} + \overline{o}\left(x^5\right)$ as x $\to 0$

$$\lim_{x\to 0} \frac{\sqrt[3]{1+x^2} - e^{\frac{x^3}{4}}}{\ln(1+3x^2) - 3x^2\cos(x)} = \frac{\left(1 + \frac{1}{3}x^2 - \frac{1}{9}x^4 + \overline{o}\left(x^4\right)\right) - \left(1 + \frac{x^2}{3} + \frac{x^4}{18} + \overline{o}\left(x^4\right)\right)}{\left(3x^2 - \frac{9x^4}{2} + \overline{o}\left(x^4\right)\right) - \left(3x^2 - \frac{3x^4}{2} + \overline{o}\left(x^5\right)\right)} =$$

$$\lim_{x\to 0} \frac{\sqrt[3]{1+x^2} - e^{\frac{x^3}{4}}}{\ln(1+3x^2) - 3x^2\cos(x)} = \lim_{x\to 0} \frac{-\frac{1}{6}x^4 + \overline{o}\left(x^4\right)}{-3x^4 + \overline{o}\left(x^4\right)} = \frac{1}{18}$$

Answer: $\frac{1}{18}$

Exercise Work out the limit using Taylor polynomial

$$\lim_{x\to 0}\left(\sqrt[3]{1+2x+x^3}-\frac{2x}{2x+3}\right)^{\frac{1}{x^3}}=\lim_{x\to 0}e^{\left(\frac{1}{x^3}\cdot\ln\left(\sqrt[3]{1+2x+x^3}-\frac{2x}{2x+3}\right)\right)}$$

1. $\sqrt[3]{1+2x+x^3}=1+\frac{1}{3}\left(2x+x^3\right)+\frac{\left(\frac{1}{3}\left(\frac{1}{3}-1\right)\right)}{2}\left(2x+x^3\right)^2+\frac{\left(\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\right)}{6}\left(2x+x^3\right)^3$
$+\overline{o}\left(\left(2x+x^3\right)^3\right)=1+\frac{x^3+2x}{3}-\frac{1}{9}x^2\left(2+x^2\right)^2+\frac{5}{81}x^3\left(2+x^2\right)^3+\overline{o}\left(x^3\right)=1+\frac{x^3+2x}{3}-\frac{1}{9}x^2\cdot 4+$
$+\frac{5}{81}x^3\cdot 8+\overline{o}\left(x^3\right)=1+\frac{67x^3-36x^2+54x}{81}+\overline{o}\left(x^3\right)$ as x $\longrightarrow 0$
2. $\frac{2x}{2x+3}=\frac{2x}{3}\left(1+\frac{2x}{3}\right)^{-1}=\frac{2x}{3}\left(1-\frac{2x}{3}+\left(\frac{2x}{3}\right)^2+\overline{o}\left(\left(\frac{2x}{3}\right)^2\right)\right)=\frac{18x-12x^2+8x^3}{27}+\overline{o}\left(x^3\right)$ as x $\longrightarrow 0$
3. $\sqrt[3]{1+2x+x^3}-\frac{2x}{2x+3}=1+\frac{67x^3-36x^2+54x}{81}+\overline{o}\left(x^3\right)-\frac{18x-12x^2+8x^3}{27}-\overline{o}\left(x^3\right)=$
$=1+\frac{67x^3-36x^2+54x}{81}+\overline{o}\left(x^3\right)-\frac{54x-36x^2+24x^3}{81}-\overline{o}\left(x^3\right)=1+\frac{43x^3}{81}+\overline{o}\left(x^3\right)$
4. $\ln\left(\sqrt[3]{1+2x+x^3}-\frac{2x}{2x+3}\right)=\ln\left(1+\frac{43x^3}{81}+\overline{o}\left(x^3\right)\right)=\frac{43x^3}{81}+\overline{o}\left(x^3\right)+\overline{o}\left(\frac{43x^3}{81}+\overline{o}\left(x^3\right)\right)=$
$=\frac{43x^3}{81}+\overline{o}\left(x^3\right)$ as x $\longrightarrow 0$
5. $\lim_{x\to 0}\frac{1}{x^3}\ln\left(\sqrt[3]{1+2x+x^3}-\frac{2x}{2x+3}\right)=\lim_{x\to 0}\frac{1}{x^3}\left(\frac{43x^3}{81}+\overline{o}\left(x^3\right)\right)=\frac{43}{81}$
6. $\lim_{x\to 0}e^{\left(\frac{1}{x^3}\cdot\ln\left(\sqrt[3]{1+2x+x^3}-\frac{2x}{2x+3}\right)\right)}=e^{\frac{43}{81}}$

Answer: $e^{\frac{43}{81}}$

Indefinite integrals

Exercise Work out indefinite integral

$$\int\frac{x^3dx}{x^8-2}=\frac{1}{4}\int\frac{d(x^4)}{x^8-2}=\frac{1}{4}\left(\frac{1}{2\sqrt{2}}\ln\left|\frac{x^4-\sqrt{2}}{x^4+\sqrt{2}}\right|\right)+C=\frac{\sqrt{2}}{16}\ln\left|\frac{x^4-\sqrt{2}}{x^4+\sqrt{2}}\right|+C,C\in\mathbb{R}$$

Answer: $\frac{\sqrt{2}}{16}\ln\left|\frac{x^4-\sqrt{2}}{x^4+\sqrt{2}}\right|+C,C\in\mathbb{R}$

Exercise Work out indefinite integral

$$\int\frac{xdx}{x^4-2x^2-1}=\frac{1}{2}\int\frac{d(x^2-1)}{(x^2-1)^2-2}=\frac{1}{2}\left(\frac{1}{2\sqrt{2}}\ln\left|\frac{x^2-1-\sqrt{2}}{x^2-1+\sqrt{2}}\right|\right)+C=\frac{1}{4\sqrt{2}}\ln\left|\frac{x^2-1-\sqrt{2}}{x^2-1+\sqrt{2}}\right|+C,C\in\mathbb{R}$$

Answer: $\frac{1}{4\sqrt{2}}\ln\left|\frac{x^2-1-\sqrt{2}}{x^2-1+\sqrt{2}}\right|+C,C\in\mathbb{R}$

Exercise Work out 2 indefinite integrals

$$I_1 = \int e^{ax} \cos bx \, dx$$
$$I_2 = \int e^{ax} \sin bx \, dx$$

1. By applying the integration by parts rule:

$$I_1 = \frac{1}{a}\int \cos bx \, d(e^{ax}) = \frac{1}{a}\left(e^{ax}\cos bx - \int e^{ax}\, d(\cos bx)\right) = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\, dx =$$
$$= \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}I_2$$
$$I_2 = \frac{1}{a}\int \sin bx \, d(e^{ax}) = \frac{1}{a}\left(e^{ax}\sin bx - \int e^{ax}\, d(\sin bx)\right) = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\, dx =$$
$$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}I_1$$

2. Hence:

$$\begin{cases} I_1 = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}I_2 \\ I_2 = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}I_1 \end{cases}$$
$$\begin{cases} I_1 = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}I_2 \\ I_2 = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}I_1 \end{cases}$$
$$I_1 = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\left(\frac{1}{a}e^{ax}\sin bx - \frac{b}{a}I_1\right)$$
$$I_1 = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a^2}e^{ax}\sin bx - \frac{b^2}{a^2}I_1$$
$$I_1 + \frac{b^2}{a^2}I_1 = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a^2}e^{ax}\sin bx$$
$$I_1 = \frac{ae^{ax}\cos bx + be^{ax}\sin bx}{a^2+b^2}$$
$$I_1 = \frac{e^{ax}\left(a\cos bx + b\sin bx\right)}{a^2+b^2}$$
$$I_2 = \frac{e^{ax}\left(a\sin bx - b\cos bx\right)}{a^2+b^2}$$

Since I_1 and I_2 are indefinite integrals, we should add constants:

$$I_1 = \frac{e^{ax}\left(a\cos bx + b\sin bx\right)}{a^2+b^2} + C_1, C_1 \in \mathbb{R}$$
$$I_2 = \frac{e^{ax}\left(a\sin bx - b\cos bx\right)}{a^2+b^2} + C_2, C_2 \in \mathbb{R}$$

(such strange situatuin happend because we used sets of functions like they are just functions)

Answer:

$$I_1 = \frac{e^{ax}\left(a\cos bx + b\sin bx\right)}{a^2+b^2} + C_1, C_1 \in \mathbb{R}$$
$$I_2 = \frac{e^{ax}\left(a\sin bx - b\cos bx\right)}{a^2+b^2} + C_2, C_2 \in \mathbb{R}$$

Definite integrals

Exercise Work out definite integral

$$\int_e^{e^2} \frac{dx}{x\ln x} = \int_e^{e^2} \frac{d(\ln(x))}{\ln x} = \ln|\ln(x)|\,|_e^{e^2} = \ln\left|\ln\left(e^2\right)\right| - \ln|\ln(e)| = \ln 2$$

Answer: $\ln 2$

Convergence of the integrals

Exercise Determine whether the given integral converges or diverges

$$\int_0^1 \frac{\cos^2(x)}{\sqrt[3]{x}}dx$$

1. $\forall x \in (0;1] : 0 < \cos^2(x) < 1 \implies \forall x \in (0;1] : 0 < \dfrac{\cos^2(x)}{\sqrt[3]{x}} < 1$

2. $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}}$ converges $\implies \displaystyle\int_0^1 \frac{\cos^2(x)}{\sqrt[3]{x}}dx$ converges by the comparision test

Answer: converges

Exercise Determine whether the given integral converges or diverges

$$I = \int_1^{+\infty} x^x e^{-x^n} dx, n \in \mathbb{N}$$

1. $n = 1$

$$I = \int_1^{+\infty} \left(\frac{x}{e}\right)^x dx$$

$\forall x \geq e : \left(\dfrac{x}{e}\right)^x \geq 1 \implies I$ diverges by the comparision test

2. $n \geq 2$

$$x^x e^{-x^n} = e^{x\ln x}e^{-x^n} = \frac{1}{e^{x^n - x\ln x}}$$

$\forall x \geq e : x \geq 1 + \ln x \implies \forall x \geq e : x^{n-1} \geq 1 + \ln x \implies \forall x \geq e : x^n - x\ln x \geq x$

$\implies \forall x \geq e : e^{x^n - x\ln x} \geq e^x \implies \forall x \geq e : \dfrac{1}{e^{x^n - x\ln x}} \geq \dfrac{1}{e^x}$

3. $\displaystyle\int_1^{+\infty} \frac{1}{e^x}dx$ converges $\implies \displaystyle\int_e^{+\infty} \frac{1}{e^x}dx$ converges

$\implies \displaystyle\int_e^{+\infty} \frac{1}{e^{x^n - x\ln x}}dx$ converges by the comparision test $\implies I$ converges

Answer: converges (absolutely) if and only if $n \geq 2$

Exercise Determine whether the given integral converges or diverges

$$\int_0^{+\infty} x^2\cos(e^x)dx$$

$$\int_0^{+\infty} x^2\cos(e^x)dx = \left| e^x = t, x = \ln(t), dx = \frac{dt}{t} \right| = \int_1^{+\infty} \frac{\ln^2(t)}{t}\cos(t)dt$$

2. Let's consider the function $f(t) = \dfrac{\ln^2(t)}{t}$:

$\lim\limits_{t\to+\infty} f(t) = 0$

Monotonicity of the $f(t)$:

$$\left(\frac{\ln^2(t)}{t}\right)' = \frac{2\ln(t) - \ln^2(t)}{t^2}$$

$\forall t \in (e^2; +\infty) f'(t) < 0$

3. Let's consider the function $g(t) = \cos(t)$

$G(t) = \sin(t) + C$ is bounded on the $(0; +\infty)$

4. Functions $f(x)$ and $g(x)$ satisfy the preconditions of the Dirichlet test on the interval $(e^2; +\infty)$ and thus integral $\displaystyle\int_{e^2}^{+\infty} \frac{\ln^2(t)}{t}\cos(t)dt$ converges by the Dirichlet test

5. Absolute convergence:

$$I_1 = \int_1^{+\infty} \frac{\ln^2(t)}{t}|\cos(t)|dt$$

$$\forall t \in \mathbb{R} : |\cos(t)| \geq \cos^2(t) \implies \forall t > 1 : \frac{\ln^2(t)}{t}|\cos(t)| \geq \frac{\ln^2(t)}{t}\cos^2(t)$$

$$I_2 = \int_1^{+\infty} \frac{\ln^2(t)}{t}\cos^2(t)dt = \frac{1}{2}\int_1^{+\infty} \frac{\ln^2(t)}{t}dt + \frac{1}{2}\int_1^{+\infty} \frac{\ln^2(t)}{t}\cos(2t)dt$$

$\displaystyle\int_1^{+\infty} \frac{\ln^2(t)}{t}\cos(2t)dt$ converges (proof the same as one for the convergence of the I)

$$\int_1^{+\infty} \frac{\ln^2(t)}{t}dt = \int_1^{+\infty} \ln^2(t)d(\ln(t)) = \frac{\ln^3(t)}{3}\Big|_1^{+\infty} = +\infty \implies \text{ integral diverges } \implies$$

$\implies$ I_2 diverges $\implies$ I_1 diverges by the comparision test $\implies$ I converges conditionally

Answer: converges conditionally

Exercise Determine whether the given integral converges or diverges

$$I = \int_0^1 \frac{\ln(1+x)}{e^{\sin x^2}-1}dx$$

1. $\lim\limits_{x\to 0} \frac{\left(\frac{\ln(1+x)}{e^{\sin x^2}-1}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to 0} \frac{x\ln(1+x)}{\frac{e^{\sin x^2}-1}{\sin x^2}} \cdot \frac{1}{\sin x^2} = \lim\limits_{x\to 0} \frac{x^2}{\frac{e^{\sin x^2}-1}{\sin x^2}} \cdot \frac{1}{\sin x^2} =$

$= \lim\limits_{x\to 0} \frac{x^2}{\sin x^2} = 1 \implies \frac{\ln(1+x)}{e^{\sin x^2}-1} \sim \frac{1}{x}$ as $x \to 0$

2. $\int_0^1 \frac{1}{x}dx$ diverges $\implies I$ diverges by the limit comparision test

Answer: diverges