Homework 19.

#3.

a) $\int_0^{2\pi} \sin^4 x\, dx = \frac{1}{4}\int_0^{2\pi} (1-\cos 2x)^2 dx = \frac{1}{4}\int_0^{2\pi}(1 - 2\cos 2x + \cos^2 2x)\, dx =$

$= \frac{1}{4}\int_0^{2\pi}(1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x)\, dx = \frac{1}{4}\int_0^{2\pi}(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x)\, dx =$

$= \frac{1}{4}(\frac{3}{2}x - 2\cdot\frac{1}{2}\sin 2x + \frac{1}{4}\cdot\frac{1}{2}\cdot\sin 4x)\Big|_0^{2\pi} = \frac{1}{4}(\frac{3}{2}\cdot 2\pi - \sin 4\pi + \frac{1}{8}\sin 8\pi -$

$-(\frac{3}{2}\cdot 0 - \sin 0 + \frac{1}{8}\sin 0)) = \frac{1}{4}(3\pi + 0) = \frac{3}{4}\pi$

b) $\int_0^1 \frac{x^2}{1+x^6}dx = \frac{1}{3}\int_0^1 \frac{1}{1+x^6} d(x^3) = \frac{1}{3}\int_0^1 \frac{1}{t^2+1}dt = \frac{1}{3}(\text{arctg}\, t)\Big|_0^1 =$

$= \frac{1}{3}(\text{arctg}\, 1 - \text{arctg}\, 0) = \frac{1}{3}(\frac{\pi}{4} - 0) = \frac{\pi}{12}$

c) $\int_{\pi/4}^{\pi/3} \frac{x}{\sin^2 x}dx = -\int_{\pi/4}^{\pi/3} x\, d(\text{ctg}\, x) = (-x\cdot\text{ctg}\, x + \int \text{ctg}\, x\, dx)\Big|_{\pi/4}^{\pi/3} =$

$= (-x\,\text{ctg}\, x + \int \frac{1}{\sin x} d(\sin x))\Big|_{\pi/4}^{\pi/3} = (\ln|\sin x| - x\cdot\text{ctg}\, x)\Big|_{\pi/4}^{\pi/3} =$

$= (\ln|\sin\frac{\pi}{3}| - \frac{\pi}{3}\cdot\text{ctg}\frac{\pi}{3}) - (\ln|\sin\frac{\pi}{4}| - \frac{\pi}{4}\cdot\text{ctg}\frac{\pi}{4}) =$

$= \ln(\frac{\sqrt{3}}{2}) - \frac{\pi}{3}\cdot\frac{\sqrt{3}}{3} - \ln(\frac{\sqrt{2}}{2}) + \frac{\pi}{4} = \ln(\frac{\sqrt{3}}{\sqrt{2}}) + \pi(\frac{1}{4} - \frac{\sqrt{3}}{9})$

d) $\int_0^{\sqrt{3}} x\,\text{arctg}\, x\, dx = \frac{1}{2}\int_0^{\sqrt{3}} \text{arctg}\, x\, d(x^2) = \frac{1}{2}(x^2\cdot\text{arctg}\, x - \int x^2 d(\text{arctg}\, x))\Big|_0^{\sqrt{3}} =$

$= \frac{1}{2}(x^2\,\text{arctg}\, x - \int\frac{x^2}{1+x^2}dx)\Big|_0^{\sqrt{3}} = \frac{1}{2}(x^2\,\text{arctg}\, x - \int(1 - \frac{1}{1+x^2})dx)\Big|_0^{\sqrt{3}} =$

$= \frac{1}{2}(x^2\,\text{arctg}\, x - x + \text{arctg}\, x)\Big|_0^{\sqrt{3}} = \frac{1}{2}(3\,\text{arctg}\sqrt{3} - \sqrt{3} + \text{arctg}\sqrt{3}) =$

$= \frac{1}{2}(3\frac{\pi}{3} - \sqrt{3} + \frac{\pi}{3}) = \frac{1}{2}(\frac{4\pi}{3} - \sqrt{3}) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

e) $\int_{1/3}^{3} \frac{\operatorname{arctg} x}{x^2-x+1}dx = \int_{1/3}^{1} \frac{\operatorname{arctg} x}{x^2-x+1}dx + \int_{1}^{3} \frac{\operatorname{arctg} x}{x^2-x+1}dx =$

Рассмотрим $\int_{1/3}^{1} \frac{\operatorname{arctg} x}{x^2-x+1}dx$: $[x=\frac{1}{y}]$, тогда

$$\int_{3}^{1} \frac{\operatorname{arctg}(\frac{1}{y})}{\frac{1}{y^2}-\frac{1}{y}+1} d\left(\frac{1}{y}\right) = -\int_{3}^{1} \frac{\operatorname{arctg}(\frac{1}{y})}{\frac{1}{y^2}-\frac{1}{y}+1} \frac{dy}{y^2} = -\int_{3}^{1} \frac{\operatorname{arctg}(\frac{1}{y})}{y^2-y+1} d(y) = \int_{1}^{3} \frac{\frac{\pi}{2}-\operatorname{arctg} y}{y^2-y+1}$$

$$= \frac{\pi}{2}\int_{1}^{3} \frac{dy}{y^2-y+1} = \frac{\pi}{2}\int_{1}^{3} \frac{d(y-\frac{1}{2})}{(y-\frac{1}{2})^2+\frac{3}{4}} = \frac{2}{\sqrt{3}}\frac{\pi}{2} \operatorname{arctg}\left(\left(y-\frac{1}{2}\right)\cdot\frac{2}{\sqrt{3}}\right)\Big|_{1}^{3} =$$

$$= \frac{\pi}{\sqrt{3}}\left(\operatorname{arctg}\left(\frac{5}{\sqrt{3}}\right) - \operatorname{arctg}\left(\frac{1}{\sqrt{3}}\right)\right) = \frac{\pi}{\sqrt{3}}\left(\operatorname{arctg}\frac{5}{\sqrt{3}} - \frac{\pi}{6}\right)$$

f) $\int_{0}^{2\pi} \frac{dx}{4+\cos^2 x} = \int_{0}^{2\pi} \frac{dx}{4\sin^2 x + 5\cos^2 x} = \int_{0}^{2\pi} \frac{\frac{1}{\cos^2 x}dx}{5+4\operatorname{tg}^2 x} = [\operatorname{tg} x = t]$

$$= \int_{0}^{2\pi} \frac{dt}{5+4t^2} = \left[s=\frac{2}{\sqrt{5}}t\right] = \int_{0}^{2\pi} \frac{\frac{\sqrt{5}}{2}ds}{5+5s^2} = \frac{\sqrt{5}}{2\cdot 5}\int_{0}^{2\pi} \frac{ds}{1+s^2} =$$

$$= \frac{1}{2\sqrt{5}}(\operatorname{arctg} s)\Big|_{0}^{2\pi} = \frac{1}{2\sqrt{5}}\left(\operatorname{arctg}\left(\frac{2}{\sqrt{5}}\operatorname{tg} x\right)\right)\Big|_{0}^{2\pi} = \frac{1}{2\sqrt{5}}\cdot 2\pi - 0 = \frac{\pi}{\sqrt{5}}$$

#2a.

$$\lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{n}\sqrt{k(n-k)} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sqrt{\frac{k}{n}\left(1-\frac{k}{n}\right)} = \int_{0}^{1}\sqrt{x(1-x)}\,dx = \int_{0}^{1}\sqrt{x-x^2}\,dx =$$

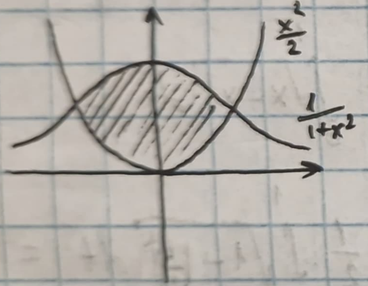
$$= \int_{0}^{1}\sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^2}\,dx = \left[t=x-\frac{1}{2}\right] = \int_{0}^{1}\sqrt{\frac{1}{4}-t^2}\,dt = \frac{1}{2}\int_{0}^{1}\sqrt{1-4t^2}\,dt =$$

$$= [\arcsin(2t)=s] = \frac{1}{2}\int_{0}^{1}\frac{\cos^2 s}{2}ds = \frac{1}{4}\int_{0}^{1}\frac{\cos 2s+1}{2}ds = \frac{1}{8}\int_{0}^{1}(\cos 2s+1)\,ds =$$

$$= \frac{1}{8}\int_{0}^{1}\cos 2s\,ds + \frac{1}{8}\int_{0}^{1}ds = \frac{1}{8}\left(\left(\frac{\sin 2s}{2}\right)+s\right)\Big|_{0}^{1} = \frac{1}{8}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{8}$$

#4

b) $y = \frac{x^2}{2}$ $\qquad y = \frac{1}{1+x^2}$

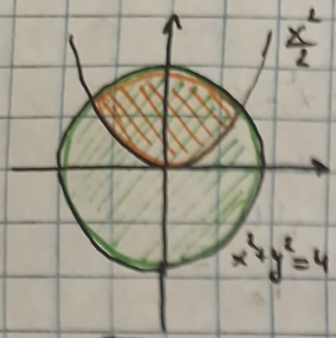


$$S = \int_{-1}^{1}\left(\frac{1}{1+x^2} - \frac{x^2}{2}\right)dx = \left(\operatorname{arctg} x - \frac{x^3}{6}\right)\Big|_{-1}^{1} = \operatorname{arctg} 1 - \frac{1}{6} - \operatorname{arctg}(-1) - \frac{1}{6} =$$

$$= \frac{\pi}{4} - \frac{1}{6} + \frac{\pi}{4} - \frac{1}{6} = \frac{\pi}{2} - \frac{1}{3}$$

c) $x^2+y^2=4$ $\quad y=\frac{x^2}{2}$ $\quad y \geqslant \frac{x^2}{2}$

$y=\sqrt{4-x^2}$



$$S=\int_{-\sqrt{2\sqrt5-2}}^{\sqrt{2\sqrt5-2}}\left(\sqrt{4-x^2}-\frac{x^2}{2}\right)=\left(\frac{x}{2}\sqrt{4-x^2}+2\arcsin\frac{x}{2}-\frac{x^3}{6}\right)\Bigg|_{x=-\sqrt{2\sqrt5-2}}^{x=\sqrt{2\sqrt5-2}}=$$

$$=\sqrt{\frac{\sqrt5}{2}-\frac12}\sqrt{4-2\sqrt5+2}+2\arcsin\left(\sqrt{\frac{\sqrt5}{2}-\frac12}\right)-\frac{(2\sqrt5-2)\sqrt{2\sqrt5-2}}{6}-$$

$$+\sqrt{\frac{\sqrt5}{2}-\frac12}\sqrt{4-2\sqrt5+2}+2\arcsin\left(\sqrt{\frac{\sqrt5}{2}-\frac12}\right)-\frac{(2\sqrt5-2)\sqrt{2\sqrt5-2}}{6}=$$

$$\cancel{2\sqrt{\left(\frac{\sqrt5}{2}-\frac12\right)(6-2\sqrt5)}=\sqrt{(2\sqrt5-2)(6-2\sqrt5)}=\sqrt{12\sqrt5-12-20+4\sqrt5}}=$$

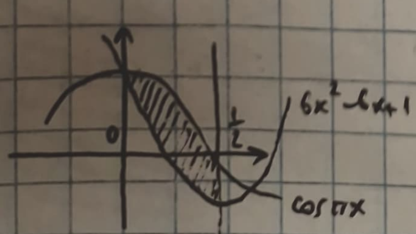
$$=\cancel{\sqrt{16\sqrt5-32}=4\sqrt{(\sqrt5-2)}}$$

$$=\cancel{4\sqrt{\sqrt5-2}+4\arcsin\left(\sqrt{\frac{\sqrt5}{2}-\frac12}\right)-\frac{(2\sqrt5-2)\sqrt{2\sqrt5-2}}{3}}=$$

$$=2\cdot\frac{1}{\sqrt2}\sqrt{\sqrt5-1}\cdot(\sqrt5-1)+4\arcsin\left(\sqrt{\frac{\sqrt5}{2}-\frac12}\right)-\frac{2(\sqrt5-1)\sqrt2\sqrt{\sqrt5-1}}{3}=$$

$$=\frac{5\sqrt2}{3}\sqrt{\sqrt5-1}(\sqrt5-1)+4\arcsin\left(\sqrt{\frac{\sqrt5}{2}-\frac12}\right)$$

a) $y=6x^2-6x+1$ $\quad y=\cos\pi x$ $\quad 0\leqslant x\leqslant\frac12$

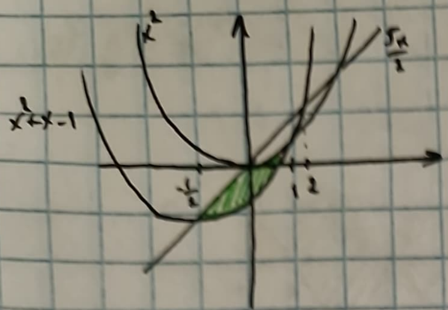


$$S=\int_0^{\frac12}(\cos\pi x-6x^2+6x-1)dx=$$

$$=\left(\frac{\sin\pi x}{\pi}-2x^3+3x^2-x\right)\Bigg|_0^{\frac12}=$$

$$=\frac{\sin\frac{\pi}{2}}{\pi}-2\cdot\frac18+3\cdot\frac14-\frac12=\frac1\pi-\frac14+\frac34-\frac12=\frac1\pi$$

d) $y=x^2$ $\quad y=x^2+x-1$ $\quad y=\frac{5x}{2}$ $\quad y\le x^2$

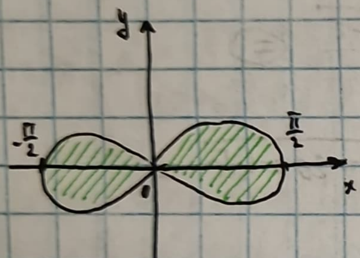


$$S=\int_{-1/2}^{0}\left(\frac{5x}{2}-x^2-x+1\right)dx+\int_{0}^{1}\left(x^2-x^2-x+1\right)dx=$$

$$=\left(-\frac{x^3}{3}+\frac{3}{4}x^2+x\right)\Big|_{-\frac{1}{2}}^{0}+\left(-\frac{x^2}{2}+x\right)\Big|_{0}^{1}=$$

$$=\frac{-1}{8\cdot 3}-\frac{3}{4\cdot 4}+\frac{1}{2}-\frac{1}{2}+1=\frac{37}{48}.$$

e) $y^2=\sin^2 x\cos x$ $\qquad -\pi/2\le x\le \pi/2$



Функция симметрична относительно осей Ox и Oy

$\Rightarrow$ $S=0.$

#2.

b) $$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{3n}\right)=\lim_{n\to\infty}\sum_{k=n+1}^{3n}\frac{1}{k}=\lim_{n\to\infty}\int_{n+1}^{3n}\frac{1}{x}dx=$$

$$=\lim_{n\to\infty}\left(\ln|x|\right)\Big|_{x=n+1}^{x=3n}=\lim_{n\to\infty}\left(\ln|3n|-\ln|n+1|\right)=$$

$$=\lim_{n\to\infty}\left(\ln\left(\frac{3n}{n+1}\right)\right)=\lim_{n\to\infty}\left(\ln\left(\frac{3}{1+\frac{1}{n}}\right)\right)=\ln\frac{3}{1}=\ln 3.$$

a) $$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\sqrt{k(n-k)}=\lim$$

#1.

a) $\int_{1}^{2}\frac{1}{x^2}dx=\frac{1}{n}\sum_{k=1}^{n-1}f\left(\frac{1}{k^2}\right)$, $\quad \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{1}{k^2}\right)=\frac{1}{2}$

b) $\int_{1}^{e}\ln x\,dx=\frac{e-1}{n}\sum_{k=1}^{n}f\left(1+\frac{(e-1)k}{n}\right)$ $\quad \lim_{n\to\infty}\frac{e-1}{n}\cdot\ln\left(1+\frac{(e-1)k}{n}\right)=1$