Семинар 11, 28.11.23.

Подсчёт пределов функций. Производная

① a) $\lim\limits_{x\to\infty}\left(\frac{x}{2x+1}\right)^x = \begin{cases} 0, & x\to+\infty \\ +\infty, & x\to-\infty \end{cases}$

$\frac{x}{2x+1} \xrightarrow[x\to\infty]{} \frac{1}{2}$

$\frac{1}{3} < \frac{x}{2x+1} < \frac{2}{3}$ $\qquad\qquad$ $\frac{1}{3} < \frac{x}{2x+1} < \frac{2}{3}$

$\left(\frac{1}{3}\right)^x < \left(\frac{x}{2x+1}\right)^x < \left(\frac{2}{3}\right)^x$ $\qquad$ $\left(\frac{1}{3}\right)^x > \left(\frac{x}{2x+1}\right)^x > \left(\frac{2}{3}\right)^x$

$\downarrow x\to+\infty$ $\quad (x>0) \quad$ $\downarrow x\to+\infty$ $\qquad$ $\downarrow x\to-\infty$ $\quad (x<0) \quad$ $\downarrow x\to-\infty$

$0 \qquad\qquad 0 \qquad\qquad +\infty \qquad\qquad +\infty$

b) $\lim\limits_{x\to0}(1+3x^4)^{\frac{1}{\sin^2 x}} = 1$

$$\Big(\underbrace{(1+3x^4)^{\frac{1}{3x^4}}}_{\xrightarrow[x\to0]{} e}\Big)^{\overbrace{\frac{3x^4}{\sin^2 x}}^{\to 0}} \xrightarrow[x\to0]{} e^0 = 1$$

$$\frac{3x^4}{\sin^2 x} = 3x^2\cdot\left(\frac{x}{\sin x}\right)^2 \xrightarrow[x\to0]{} 0$$

$$\lim_{x\to x_0}\left(f(x)^{g(x)}\right)=\left(\lim_{x\to x_0} f(x)\right)^{\lim_{x\to x_0} g(x)}$$

(c) $\displaystyle\lim_{x\to 0}\left(\frac{\ln(1+3x+x^2)+\ln(1-3x+x^2)}{x^2}\right)$

...(c_0) $\displaystyle\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\left(\ln(1+x)^{\frac{1}{x}}\right)=\lim_{x\to 0}(\ln e)=1$

$$\frac{\ln(1+3x+x^2)+\ln(1-3x+x^2)}{x^2}=\frac{\ln(1-7x^2+x^4)}{x^2}=\ln(1-7x^2+x^4)^{\frac{1}{x^2}}=$$

$$=\ln\Bigg(\underbrace{(1-7x^2+x^4)^{\frac{1}{x^4-7x^2}}}_{e}\Bigg)^{\frac{x^4-7x^2}{x^2}}=\ln e^{\overbrace{\frac{x^4-7x^2}{x^2}}^{-7}}\to\ln e^{-7}=-7$$

(d) $\displaystyle\lim_{x\to 0}\frac{\ln(\cos 5x)}{\ln(\cos 4x)}$

$$\frac{\ln(\cos 5x)}{\ln(\cos 4x)}=\frac{\ln(1-2\sin^2\frac{5x}{2})}{\ln(1-2\sin^2 2x)}=\overbrace{\frac{\ln(1-2\sin^2\frac{5x}{2})}{-2\sin^2\frac{5x}{2}}}^{\to 1}\cdot\overbrace{\frac{-2\sin^2 2x}{\ln(1-2\sin^2 2x)}}^{\to 1}\cdot\overbrace{\frac{-2\sin^2\frac{5x}{2}}{-2\sin^2 2x}}^{\to\frac{25}{16}}$$

$\to\frac{25}{16}$

(e) $\displaystyle\lim_{x\to 0}\frac{e^{x^2}-\cos x}{\sin^2 x}=1+\frac{1}{2}=\frac{3}{2}$

$$\frac{e^{x^2}-1+2\sin^2\frac{x}{2}}{\sin^2 x}=\frac{e^{x^2}-1}{\sin^2 x}+\overbrace{\frac{2\sin^2\frac{x}{2}}{\sin^2 x}}^{\xrightarrow{x\to 0}\ 2\cdot\left(\frac{1}{2}\right)^2=\frac{1}{2}}$$

$$\frac{e^{x^2}-1}{\sin^2 x}=\overbrace{\frac{e^{x^2}-1}{x^2}}^{\to 1}\cdot\overbrace{\frac{x^2}{\sin^2 x}}^{\to 1}$$

$x^2=t$, $\quad t=\ln s$ ($t\to 0$, $s\to 1$), тогда $\displaystyle\frac{e^{\ln s}-1}{\ln s}=\frac{s-1}{\ln s}=\frac{\overbrace{s-1}^{\to 0}}{\underbrace{\ln(1+(s-1))}_{\to 0}}\to 1$

② $f(x) = x^{\alpha} \cdot \sin\frac{1}{x}$ $\quad (x \neq 0)$

$f(0) = 0$

(1) непрерывна:

$\alpha = 0$ – не непрерывна

$\alpha < 0$: $x^{\alpha} \xrightarrow[x \to 0+]{} \infty$

$\alpha > 0$: $\underbrace{\left|x^{\alpha} \cdot \sin\frac{1}{x}\right|}_{\downarrow \atop 0} \leq |x|^{\alpha} \xrightarrow[x \to 0]{} 0$

(2) дифференцируема

$$f'(x) = \lim_{h \to 0} \frac{f(h+x) - f(x)}{h}$$

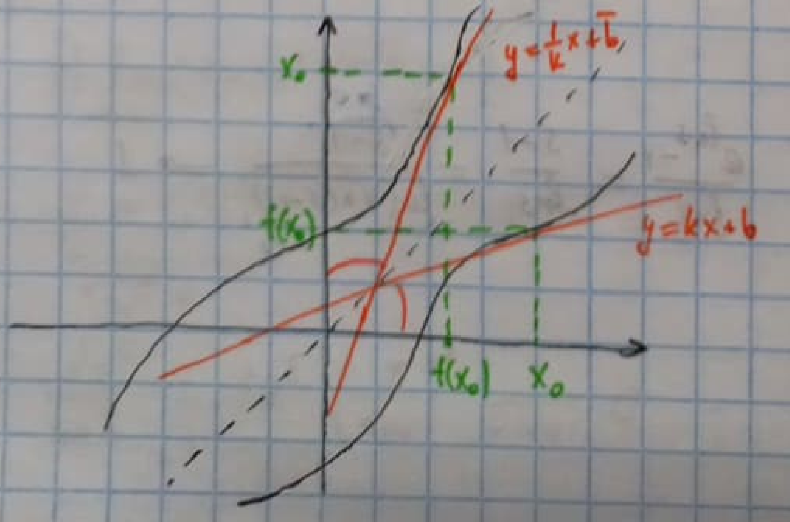
$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{\alpha} \cdot \sin\frac{1}{h}}{h} = \lim_{h \to 0} \left(h^{\alpha-1} \cdot \sin\frac{1}{h}\right)$$

If $\alpha > 1$, then $f'(0) = 0$

$\alpha \leq 1$: $f'(0)$ не сущ.

$a \in (0; 1)$: f непрер. в $x = 0$, но $f'(0)$ не сущ.

Производная обратной

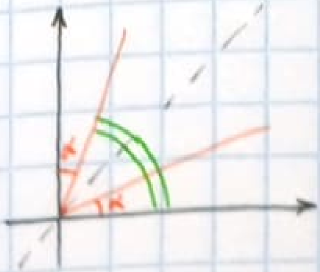


$y = f(x)$

$x = (g)$

$f(g(x)) = x$

$g(f(x)) = x$



$$tg\left(\frac{\pi}{2}-\alpha\right) = \frac{1}{tg\,\alpha}$$

(3) $g'(f(x_0)) = \dfrac{1}{f'(x_0)}$

$f(x) = \arcsin x \qquad g(x) = \sin x \qquad g'(x) = \cos x$

$g'(f(x_0)) = \cos(f(x_0)) = \cos(\arcsin x_0)$

$f'(x_0) = \dfrac{1}{\cos(\arcsin x_0)} = \dfrac{1}{\sqrt{1-x^2}}$

$\sin(\arcsin x) = x$

$$\boxed{\begin{array}{ll} (\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}} & (\mathrm{arctg}\, x)' = \dfrac{1}{1+x^2} \\[2ex] (\arccos x)' = \dfrac{1}{-\sqrt{1-x^2}} & (\mathrm{arcctg}\, x)' = \dfrac{-1}{1+x^2} \end{array}}$$

(4) (d) $f(x) = 3^{\mathrm{arctg}(2x+\pi)}$ $\qquad (f(g(x)))' = f'(g(x)) \cdot g'(x)$

$f'(x) = \left(3^{\mathrm{arctg}(2x+\pi)}\right)' = \qquad (a^x)' = \dfrac{a^x}{\ln a}$

$= \dfrac{3^{\mathrm{arctg}(2x+\pi)}}{\ln 3} \cdot (\mathrm{arctg}(2x+\pi))' =$

$= \dfrac{3^{\mathrm{arctg}(2x+\pi)}}{\ln 3} \cdot 2 \cdot \dfrac{1}{1+(2x+\pi)^2}$

(e) $f(x) = \sin(\cos^2 x) \cdot \cos(\sin^2 x)$ $\qquad (gh)' = g'h + gh'$

$(\sin(\cos^2 x))' = \cos(\cos^2 x)\,(\cos^2 x)' = \cos(\cos^2 x) \cdot 2\cos x \cdot (-\sin x)$

(f) $x^{e^x} = e^{\ln x \cdot e^x}$