Homework #3.

#1.

(a) $\lim\limits_{n\to\infty} \dfrac{3n^2+2n-1}{3-n-4n^2} = -\dfrac{3}{4}$

$$\frac{3n^2+2n-1}{3-n-4n^2} = \frac{\overset{\to 3}{3} + \overset{\to 0}{\frac{2}{n}} - \overset{\to 0}{\frac{1}{n^2}}}{\underset{\to 0}{\frac{3}{n^2}} - \underset{\to 0}{\frac{1}{n}} - \underset{\to -4}{4}} \xrightarrow[n\to\infty]{} \frac{3}{-4}$$

(b) $\lim\limits_{n\to\infty} n - \dfrac{3}{\frac{3}{n} - \frac{3}{n^2} + \frac{1}{n^3}} = -1$

$$n - \frac{3}{\underset{\to 0}{\frac{3}{n}} - \underset{\to -3}{\frac{3}{n^2}} + \frac{1}{n^3}} = n - \frac{3n^3}{3n^2-3n+1} = \frac{3n^3-3n^2+n-3n^3}{3n^2-3n+1} = \frac{n-3n^2}{3n^2-3n+1} =$$

$$= \frac{\frac{1}{n}-3}{\underset{\to 3}{3} - \underset{\to 0}{\frac{3}{n}} + \underset{\to 0}{\frac{1}{n^2}}} \xrightarrow[n\to\infty]{} \frac{-3}{3} = -1$$

(c) $\lim\limits_{n\to\infty} \frac{2n-\sqrt{4n^2-1}}{\sqrt{n^2+3}-n} = \frac{1}{6}$

$$\frac{2n-\sqrt{4n^2-1}}{\sqrt{n^2+3}-n} = \frac{(4n^2-4n^2+1)(\sqrt{n^2+3}+n)}{(n^2+3-n^2)(2n+\sqrt{4n^2-1})} = \frac{\sqrt{n^2+3}+n}{6n+3\sqrt{4n^2-1}} = \frac{\overset{\to 1}{\sqrt{1+\frac{3}{n^2}}} + \overset{\to 0}{\frac{1}{n^3}}}{\underset{\to 0}{\frac{6}{n^3}} + \underset{\to 6}{3\sqrt{4-\frac{1}{n^2}}}} \to$$

$$\xrightarrow[n\to\infty]{} \frac{1}{6}$$

(d) $\lim\limits_{n\to\infty} \sqrt[3]{n+1} - \sqrt[3]{n-1} = 0$

$$\sqrt[3]{n+1} - \sqrt[3]{n-1} = \frac{(\sqrt[3]{n+1}-\sqrt[3]{n-1})(\sqrt[3]{(n+1)^2}+\sqrt[3]{n^2-1}+\sqrt[3]{(n-1)^2})}{\sqrt[3]{n+1}+\sqrt[3]{n^2-1}+\sqrt[3]{(n-1)^2}} =$$

$$= \frac{n+1-n+1}{\sqrt[3]{(n+1)^2}+\sqrt[3]{n^2-1}+\sqrt[3]{(n-1)^2}} = \frac{\overset{\to 2}{2}}{\underset{\to\infty}{\sqrt[3]{(n+1)^2}+\sqrt[3]{n^2-1}+\sqrt[3]{(n-1)^2}}} \xrightarrow[n\to\infty]{} \frac{2}{\infty} = 0$$

(e) $\lim\limits_{n\to\infty} \frac{10^n+n!}{2^n+(n+1)!} = 0$

$$\frac{10^n+n!}{2^n+(n+1)!} = \frac{\overset{\to 0}{\frac{10^n}{n!}} + \overset{\to 1}{1}}{\underset{\to 0}{\frac{2^n}{n!}} + \underset{\to\infty}{n}} \longrightarrow \frac{1}{\infty} = 0$$

(f) $\lim\limits_{n\to\infty} \frac{\sqrt[n]{8}-1}{\sqrt[n]{16}-1} = \frac{3}{4}$

$$\sqrt[n]{2} = t; \quad \frac{t^3-1}{t^4-1} = \frac{(t-1)(t^2+t+1)}{(t^2-1)(t^2+1)} = \frac{(t-1)(t^2+t+1)}{(t-1)(t+1)(t^2+1)} = \frac{t^2+t+1}{(t+1)(t^2+1)} =$$

$$= \frac{\overset{\to 1}{\sqrt[n]{4}} + \overset{\to 1}{\sqrt[n]{2}} + \overset{\to 1}{1}}{\underset{\to 2}{(\sqrt[n]{2}+1)}\underset{\to 2}{(\sqrt[n]{4}+1)}} \xrightarrow[n\to\infty]{} \frac{3}{4}$$

#2.

1) $\exists C\ \exists n_0\ \forall n > n_0\ (y_n \leq C)$

2) $\lim\limits_{n\to\infty} x_n = -\infty \iff \forall M\ \exists N\ \forall n > N\ \ x_n < M$

Док-ть: $\lim\limits_{n\to\infty} (x_n + y_n) = -\infty$

$\forall K\ \exists N_1\ \forall n > N_1\ \ (x_n + y_n) < K$

$y_n \leq C;\ \ x_n < K - y_n \leq K - C;\ \ x_n < K - C \Rightarrow M = K - C \Rightarrow$

$\Rightarrow$ Если $K = M + C$, то $\exists N_1\ \forall n > N_1\ \ (x_n + y_n) < K \iff$

$\iff \lim\limits_{n\to\infty} (x_n + y_n) = -\infty$ Q.E.D.

#3.

1) $\lim\limits_{n\to\infty} x_n = a$, где $a = \pm\infty \iff \forall M\ \exists N\ \forall n > N\ |x_n| > M$

2) $\exists C\ \exists n_0\ \forall n > n_0\ (y_n \geq C > 0)$

Док-ть: $\lim\limits_{n\to\infty} (x_n \cdot y_n) = a \iff \forall K\ \exists N_1\ \forall n > N_1\ |x_n \cdot y_n| > K$

$|x_n| \cdot |y_n| > K$

$|x_n| > \frac{K}{|y_n|} \geq \frac{K}{C}\ \ \ (y_n \geq C > 0)$

По условию 1, $|x_n| > M \Rightarrow M = \frac{K}{C};\ \ K = M \cdot C$

Т.е. если $K = M \cdot C$, то $\exists N_1\ \forall n > N_1\ \ |x_n \cdot y_n| > K \Rightarrow \lim\limits_{n\to\infty} (x_n \cdot y_n) = a$,

где $a = \pm\infty$ Q.E.D.