Homework 14.

1. Представить формулой Маклорена с $O(x^n)$

а) $$f(x) = \frac{x^2+3e^x}{e^{2x}} = x^2\cdot e^{-2x} + 3e^{-x} =$$

$$= x^2\left(\sum_{k=0}^{n}\frac{(-2x)^k}{k!} + \bar{o}(x^n)\right) + 3\left(\sum_{k=0}^{n}\frac{(-x)^k}{k!} + \bar{o}(x^n)\right) =$$

$$= \sum_{k=0}^{n}\frac{(-2)^k}{k!}x^{k+2} + \bar{o}(x^{n+2}) + 3\sum_{k=0}^{n}\frac{(-1)^k}{k!}x^k + 3\bar{o}(x^n) =$$

$$= 3 - 3x + 3\sum_{k=2}^{n}\frac{(-1)^k}{k!}x^k + \sum_{k=2}^{n}\frac{(-2)^{k-2}}{(k-2)!}x^k + \bar{o}(x^n) =$$

$$= \boxed{3 - 3x + \sum_{k=2}^{n}\left(\frac{3\cdot(-1)^k}{k!} + \frac{(-2)^{k-2}}{(k-2)!}\right)x^k + \bar{o}(x^n)} \qquad x\to 0$$

б) $$f(x) = x\sqrt[3]{4-4x+x^2} = x\cdot(x-2)^{\frac{2}{3}} = x\cdot 2^{\frac{2}{3}}\left(\frac{x}{2}-1\right)^{\frac{2}{3}} =$$

$$= -x\cdot\sqrt[3]{4}\left(1-\frac{x}{2}\right)^{\frac{2}{3}} = -x\sqrt[3]{4}\sum_{k=0}^{n}C_{\frac{2}{3}}^{k}\cdot\left(-\frac{x}{2}\right)^k = -\sqrt[3]{4}\sum_{k=0}^{n}C_{\frac{2}{3}}^{k}\left(-\frac{x}{2}\right)^k\cdot x =$$

$$= \boxed{\sqrt[3]{4}\sum_{k=0}^{n-1}C_{\frac{2}{3}}^{k}\frac{(-x)^{k+1}}{2^k} + \bar{o}(x^n)} \qquad x\to 0$$

2. Представить ф. Маклорена с $\bar{o}(x^{2n})$.

$$f(x) = \sin(x)\cdot\cos(2x) = \frac{1}{2}\left(\sin(-x) + \sin(3x)\right) =$$

$$= \frac{1}{2}\left(\sum_{k=0}^{n-1}\frac{(-1)^k}{(2k+1)!}(-x)^{2k+1} + \bar{o}(x^{2n+1}) + \sum_{k=0}^{n-1}\frac{(-1)^k}{(2k+1)!}(3x)^{2k+1} + \bar{o}(x^{2n+1})\right) =$$

$$= \boxed{\frac{1}{2}\sum_{k=0}^{n-1}\frac{(-1)^{k+1} + (-1)^k\cdot 3^{2k+1}}{(2k+1)!}x^{2k+1} + \bar{o}(x^{2n})} \qquad x\to 0$$

③ Представить ф. Тейлора в окр. т. $x_0=-1$ с $\bar{o}((x+1)^{2n+1})$

$$f(x)=(x+1)^2\cdot 2^{x^2+2x} =$$

$$\left[\,x+1=y\to 0;\ x=y-1;\ x^2+2x=y^2-2y+1+2y-2=y^2-1\,\right]$$

$$= f(y)=y^2\cdot 2^{y^2-1}=\frac{1}{2}y^2\cdot 2^{y^2} =$$

$$\left[\,2^t=\sum_{k=0}^{n}\frac{(\ln(2))^k}{k!}t^k+\bar{o}(t^n)\,\right]$$

$$= \frac{1}{2}y^2\left(\sum_{k=0}^{n}\frac{(\ln(2))^k}{k!}y^{2k}+\bar{o}(y^{2n})\right)=\frac{1}{2}\sum_{k=0}^{n}\frac{(\ln(2))^k}{k!}y^{2k+2}+\bar{o}(y^{2n+2})=$$

$$=\frac{1}{2}\sum_{k=0}^{n}\frac{(\ln(2))^k}{k!}y^{2k+2}+\bar{o}(y^{2n+1})=\frac{1}{2}\sum_{k=0}^{n}\frac{(\ln(2))^k}{k!}(x+1)^{2k+2}+\bar{o}((x+1)^{2n+1})=$$

$$=\boxed{\frac{1}{2}\sum_{k=0}^{n}\frac{(\ln(2))^k}{k!}\sum_{i=0}^{2k+2}C_{2k+2}^{i}x^k+\bar{o}((x+1)^{2n+1})}\qquad x\to x_0=-1$$

⑤ Представить ф. Маклорена с $\bar{o}(x^4)$

$$f(x)=\sin(\operatorname{arctg}x)=\frac{x}{\sqrt{x^2+1}}=x(1+x^2)^{-\frac{1}{2}}=$$

$$=x\left(\sum_{k=0}^{n}C_{-\frac{1}{2}}^{k}x^{2k}+\bar{o}(x^n)\right)=\sum_{k=0}^{n}C_{-\frac{1}{2}}^{k}x^{2k+1}+\bar{o}(x^n)$$

Тогда при $\bar{o}(x^4)$: $f(x)=x-\frac{x^3}{2}+\frac{3}{8}x^5+\bar{o}(x^4)=\boxed{x-\frac{x^3}{2}+\bar{o}(x^4)}$ $x\to 0$

④ Представить ф. Маклорена с $\bar{o}(x^3)$

$$f(x)=\sqrt{1+2x-x^2}-\sqrt[3]{1-3x+x^3}=(1+(2x-x^2))^{\frac{1}{2}}-(1+(-3x+x^3))^{\frac{1}{3}}=$$

$$=\sum_{k=0}^{3}C_{\frac{1}{2}}^{k}(2x-x^2)^k+\bar{o}((2x-x^2))-\sum_{k=0}^{3}C_{\frac{1}{3}}^{k}(x^3-3x)^k+\bar{o}((x^3-3x))=$$

$$=1+\frac{1}{2}(2x-x^2)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2}(2x-x^2)^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(2x-x^2)^3+\bar{o}((2x-x^2))-$$

$$-1-\frac{1}{3}(x^3-3x)-\frac{\frac{1}{3}(\frac{1}{3}-1)}{2}(x^3-3x)^2-\frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!}(x^3-3x)^3+\bar{o}((x^3-3x))=$$

$$= x - \frac{x^2}{2} - \frac{1}{8}\left(4x^2 - 4x^3 + \cancel{x^4}\right) + \frac{1}{16}\left(8x^3 - \cancel{12x^4} + \cancel{6x^5} - \cancel{x^6}\right) + \bar{o}\left((2x-x^2)\right) -$$

$$- \frac{x^3}{3} + x + \frac{1}{9}\left(\cancel{x^6} - \cancel{6x^4} + 9x^2\right) - \frac{5}{81}\left(\cancel{x^9} - \cancel{9x^7} + \cancel{27x^5} - 27x^3\right) + \bar{o}\left((x^3-3x)\right) =$$

$$= x - \frac{x^2}{2} - \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^3}{2} + \bar{o}(x^3) - \frac{x^3}{3} + x + x^2 + \frac{5}{3}x^3 + \bar{o}(x^3) =$$

$$= \boxed{2x + \frac{7}{3}x^3 + \bar{o}(x^3)} \qquad x \to 0$$

6 Представить ф. Маклорена с $\bar{o}(x^5)$

$$f(x) = (1+x)^{\sin x} = \sum_{k=0}^{5} C_{\sin x}^{k} x^k = 1 + C_{\sin x}^{1}x + C_{\sin x}^{2}x^2 + C_{\sin x}^{3}x^3 +$$

$$+ C_{\sin x}^{4}x^4 + C_{\sin x}^{5}x^5 + \bar{o}(x^5) = 1 + \sin x \cdot x + \frac{\sin x \cdot (\sin x - 1)}{2}x^2 +$$

$$+ \frac{\sin x(\sin x-1)(\sin x-2)}{6}x^3 + \frac{\sin x(\sin x-1)(\sin x-2)(\sin x-3)}{24}x^4 +$$

$$+ \frac{\sin x(\sin x-1)(\sin x-2)(\sin x-3)(\sin x-4)}{120}x^5 + \bar{o}(x^5) =$$

$$= 1 + x\left(x - \frac{x^3}{6}\right) + \frac{x^2}{2}\left(x - \frac{x^3}{6}\right)\left(x - \frac{x^3}{6} - 1\right) + \frac{x^3}{6}\left(x-\frac{x^3}{6}\right)\left(x-\frac{x^3}{6}-2\right)\left(x-\frac{x^3}{6}-1\right) +$$

$$+ \frac{x^4}{24}\left(x-\frac{x^3}{6}\right)\left(x-\frac{x^3}{6}-1\right)\left(x-\frac{x^3}{6}-2\right)\left(x-\frac{x^3}{6}-3\right) + \left(x-\frac{x^3}{6}\right)\left(x-\frac{x^3}{6}-1\right)\left(x-\frac{x^3}{6}-2\right)\left(x-\frac{x^3}{6}-3\right) \times$$

$$\times \left(x - \frac{x^3}{6} - 4\right)\frac{x^5}{120} + \bar{o}(x^5) =$$

$$= 1 + \left(x^2 - \frac{x^4}{6}\right) + \left(\frac{x^4}{2} - \frac{x^3}{2} - \frac{x^5}{12}\right) + \left(\frac{x^5}{6} + \frac{x^4}{3} - \frac{6x^5}{24} + \bar{o}(x^5) - \frac{x^5}{2}\right) =$$

$$= \boxed{1 + x^2 - \frac{x^3}{2} + \frac{2}{3}x^4 - \frac{2}{3}x^5 + \bar{o}(x^5)} \qquad x \to 0$$