Семинар 26, 23.04.24

③ $f(x) = \ln(4+3x-x^2) = \ln((4-x)(1+x)) = \ln(4-x) + \ln(1+x)$; $x_0 = 2$

Замена: $x-2=y$; $x = y+2$

$$\ln(2-y) + \ln(y+3) = \ln 2 + \ln\left(1-\frac{y}{2}\right) + \ln 3 + \ln\left(1+\frac{y}{3}\right) =$$

$$= \ln 6 + \sum_{k=1}^{\infty} (-1)^{k+1} \cdot \frac{1}{k}\left(\frac{-y}{2}\right)^k + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \cdot \left(\frac{y}{3}\right)^k =$$

$$= \ln 6 + \sum_{k=1}^{\infty} \frac{1}{k}\left(-\frac{1}{2^k} + \frac{(-1)^{k+1}}{3^k}\right)(x-2)^k \qquad \ln(t+1): R = 1.$$

$R = 2.$

Предел функции многих переменных в точке

Последовательности:

$x_1, x_2, \dots, x_n, \dots \xrightarrow[n\to\infty]{} x$ $(\in \mathbb{R})$: $\forall \varepsilon > 0$ $\exists N$: $\forall n \geq N$: $|x_n - x| < \varepsilon$

в $\mathbb{R}^n$, основной случай $\mathbb{R}^2$

$(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n), \dots$ $\qquad (x_i, y_i) \in \mathbb{R}^2$

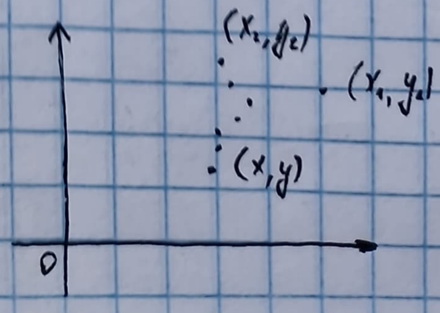
(1) $\lim\limits_{n\to\infty} (x_n, y_n) = (x, y) \iff \begin{cases} \lim x_n = x \\ \lim y_n = y \end{cases}$

(2) $\lim\limits_{n\to\infty} (x_n, y_n) = (x, y)$

$d((x_n, y_n), (x,y)) = \sqrt{(x-x_n)^2 + (y-y_n)^2}$

$\forall \varepsilon > 0$ $\exists N$: $\forall n \geq N$ $d((x_n, y_n), (x,y)) < \varepsilon$

$\iff \lim\limits_{n\to\infty} d((x_n, y_n), (x, y)) = 0$



$f(x,y)$: „ $f: \mathbb{R}^2 \to \mathbb{R}$ "

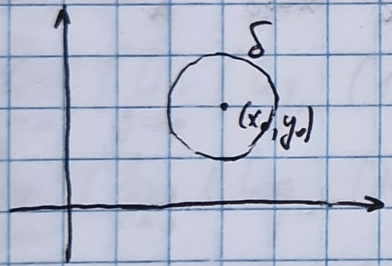
$f: U \to \mathbb{R}$, $U \subset \mathbb{R}^2$

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} f(x,y) = \lim_{(x,y)\to(x_0,y_0)} f(x,y) = a$$

(1) Гейне: $\forall (x_n, y_n) \xrightarrow[n\to\infty]{} (x_0, y_0)$, $(x_n, y_n) \neq (x_0, y_0)$ $\Rightarrow \lim\limits_{n\to\infty} f(x_n, y_n) = a$

(2) Коши: $\forall \varepsilon > 0 \quad \exists \delta > 0 : \quad \forall (x, y)$ т.ч. $0 < d((x_n, y_n), (x, y)) < \delta \Rightarrow$

$\Rightarrow \quad f(x, y) < \varepsilon$



① (a) $$\lim_{\substack{x \to 2 \\ y \to 1}} \frac{x^2 - 4y^2}{x^2 + 2x - 2xy - 4y} = \lim_{\substack{x \to 2 \\ y \to 1}} \frac{(x-2y)(x+2y)}{(x-2y)(x+2)} = \lim_{\substack{x \to 2 \\ y \to 1}} \frac{x+2y}{x+2} = \frac{4}{4} = 1$$

(b) $$\lim_{\substack{x \to 0 \\ y \to 2}} \frac{\sin xy}{x} \overset{?}{=} 2$$

$$\frac{\sin(x_n y_n)}{x_n} = \overbrace{\frac{\sin(x_n y_n)}{x_n y_n}}^{\to 1} \cdot \overset{\to 2}{y_n} \xrightarrow[n\to\infty]{} 1 \cdot 2 = 2$$

$(x_n, y_n) \to (0, 2)$

(c) $$\lim_{\substack{x \to 0 \\ y \to 1}} (1 + x)^{\frac{1}{x + x^2 y}} = \lim_{\substack{x \to 0 \\ y \to 1}} \left((1+x)^{\frac{1}{x}}\right)^{\frac{1}{1+xy}} \overset{?}{=} e$$

$x_n \to 0$, $y_n \to 0$ $$(1 + x_n)^{\frac{1}{x_n + x_n^2 y_n}} = \Big(\underbrace{(1 + x_n)^{\frac{1}{x_n}}}_{\to e}\Big)^{\underbrace{\frac{1}{1 + x_n y_n}}_{\to 1}} \xrightarrow[n\to\infty]{} e$$

② $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ a, & (x,y) = (0,0) \end{cases}$

(a) f непрерывна по x в точке $(0,0)$: $g_0(x) = f(x,0)$

f непрерывна по x в точке (x_0, y_0), если непр. $g_{y_0} = f(x, y_0)$

$g_0(x) = \dfrac{x^2}{x^2} = 1 \quad (x \neq 0) \longrightarrow \boxed{a = 1}$

(b) f непрерывна по y в т. $(0,0)$: $h_0(y) = f(0,y) = -\dfrac{y^2}{y^2} = -1$

$\longrightarrow \boxed{a = -1}$

(c) $y = \alpha\sqrt{x}$: $f(x, \alpha\sqrt{x}) = \dfrac{x^2 - \alpha^2 x}{x^2 + \alpha^2 x} = \dfrac{x - \alpha^2}{x + \alpha^2} \xrightarrow[x\to 0]{} \dfrac{-\alpha^2}{\alpha^2} = -1.$

$\longrightarrow \boxed{a = -1}$

(d) нельзя выбрать такое a

③ (a) $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} \dfrac{x^2 y}{x^2+y^2}\right) = \lim\limits_{x\to 0} 0 = 0$

(b) $\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0} \dfrac{x^2 y}{x^2+y^2}\right) = \lim\limits_{y\to 0} 0 = 0$

(c) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x^2 y}{x^2+y^2} \overset{?}{=} 0$

$\left|\dfrac{x^2 y}{x^2+y^2} - 0\right| = \dfrac{|x^2 y|}{x^2+y^2} \leq \underbrace{\dfrac{x^2}{x^2+y^2}}_{\leq 1} \cdot |y| \leq 1 \cdot |y| = |y| < \varepsilon$

берём (x,y) т.ч. $|y| < \varepsilon$

$\Rightarrow \lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x^2 y}{x^2+y^2} = 0.$

(b) $f(x,y) = x + y\sin\frac{1}{y}$

$$\lim_{x\to 0}\left(\lim_{y\to 0}\left(x + y\cdot\sin\frac{1}{y}\right)\right) = \lim_{x\to 0} x = 0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\left(x + y\cdot\sin\frac{1}{y}\right)\right) = \lim_{y\to 0}\left(y\cdot\sin\frac{1}{y}\right) = 0$$

$$\lim_{\substack{x\to 0\\ y\to 0}}\left(x + y\cdot\sin\frac{1}{y}\right) = 0$$

$$\left|x + y\cdot\sin\frac{1}{y}\right| \le |x| + |y| \xrightarrow[(x,y)\to(0,0)]{} 0$$

$$\begin{matrix} x_n \to 0 \\ y_n \to 0 \end{matrix} \Rightarrow |x_n| + |y_n| \xrightarrow[n\to\infty]{} 0$$

(c) $f(x,y) = \log_{1+x}(1+x+y)$

$$\lim_{x\to 0}\left(\lim_{y\to 0}\left(\log_{1+x}(1+x+y)\right)\right) = \lim_{x\to 0}\left(\log_{1+x}(1+x)\right) = \lim_{x\to 0} 1 = 1$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\left(\log_{1+x}(1+x+y)\right)\right) = \lim_{y\to 0}\infty = \infty$$

$x \to 0+$; $(x>0)$, $y>0$

$$x=\delta; \quad \log_{1+x}(1+y+x) \ge \log_{1+x}(1+y) = \frac{\ln(1+y)}{\ln(1+x)} \longrightarrow +\infty$$

$$\lim_{\substack{x\to 0\\ y\to 0}}\left(\log_{1+x}(1+x+y)\right) = \lim_{\substack{x\to 0\\ y\to 0}}\frac{\ln(1+x+y)}{\ln(1+x)}$$

$y=0$: $f(x,0) = 1$

$$y = x+x^2 \Rightarrow f(x, x+x^2) = \frac{\ln(1+2x+x^2)}{\ln(1+x)} = \frac{2\ln(1+x)}{\ln(1+x)} = 2$$