Семинар 9, 14.11.23

① $\lim\limits_{x\to-\infty} f(x) = +\infty$

$\forall M \in \mathbb{R} \quad \exists L = L(M) : \forall x$ т.ч. $x < L \Rightarrow f(x) > M$

② $x \to x_0$

$f(x) \leq g(x) \leq h(x)$

$\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} h(x) = a \Rightarrow \lim\limits_{x\to x_0} g(x) = a$

$x_1, x_2, \dots, x_n, \dots \to x_0 \quad , \quad x_i \neq x_0$

$g(x_1), g(x_2), \dots, g(x_n), \dots \xrightarrow{?} a$

$$\begin{array}{ccccc} f(x_i) & \leq & g(x_i) & \leq & h(x_i) \\ \downarrow{\scriptstyle i\to\infty} & & \downarrow{\scriptstyle i\to\infty} & & \downarrow{\scriptstyle i\to\infty} \\ a & & a & & a \end{array}$$

$x \to \infty / +\infty / -\infty$

$f(x) \leq g(x) \leq h(x)$

$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} h(x) = a \Rightarrow$

$\Rightarrow \lim\limits_{x\to+\infty} g(x) = a$

$x_1, x_2, \dots, x_n, \dots \to +\infty$

$g(x_1), g(x_2), \dots, g(x_n), \dots \xrightarrow{?} a$

$$\begin{array}{ccccc} f(x_i) & \leq & g(x_i) & \leq & h(x_i) \\ \downarrow{\scriptstyle i\to+\infty} & & \downarrow{\scriptstyle i\to+\infty} & & \downarrow{\scriptstyle i\to+\infty} \\ a & & a & & a \end{array}$$

③ a) $\lim\limits_{x\to+\infty} \left(1 + \frac{1}{x}\right)^x = e$

$n \leq x < n+1 \quad ; \quad n = [x]$

$$\begin{array}{ccccccccccc} \left(1+\frac{1}{[x]+1}\right)^{[x]} & = & \left(1+\frac{1}{n+1}\right)^n & \leq & \left(1+\frac{1}{x}\right)^x & \leq & \left(1+\frac{1}{x}\right)^{n+1} & \leq & \left(1+\frac{1}{n}\right)^{n+1} & = & \left(1+\frac{1}{[x]}\right)^{[x]+1} \\ \downarrow{\scriptstyle x\to+\infty} & & & & \downarrow{\scriptstyle x\to+\infty} & & & & & & \downarrow{\scriptstyle x\to+\infty} \\ e & & & & e & & & & & & e \end{array}$$

b) $\lim\limits_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e$

$x = -y$

$\lim\limits_{y\to+\infty}\left(1+\frac{1}{-y}\right)^{-y}$

$\left(1-\frac{1}{y}\right)^{-y} = \left(\frac{y-1}{y}\right)^{-y} = \left(\frac{y}{y-1}\right)^y = \left(1+\frac{1}{y-1}\right)^y = \overset{\to e}{\left(1+\frac{1}{y-1}\right)^{y-1}} \cdot \overset{\to 1}{\left(1+\frac{1}{y-1}\right)} \to e$

c) $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$

$x = \frac{1}{y} \to \infty$

$\left(1+\frac{1}{y}\right)^y \xrightarrow[y\to\infty]{} e$

d) $\lim\limits_{x\to+\infty}\left(1+\frac{a}{x}\right)^x = e^a$

$\left(1+\frac{1}{\left(\frac{x}{a}\right)}\right)^x = \left(\left(1+\frac{1}{\left(\frac{x}{a}\right)}\right)^{\frac{x}{a}}\right)^a = \left\{y = \frac{x}{a} \to \infty\right\} = \left(\left(1+\frac{1}{y}\right)^y\right)^a \longrightarrow e^a$

4 a) $P(x) = a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0$

$\lim\limits_{x\to x_0} P(x) \overset{?}{=} P(x_0)$

$x_1, x_2, \ldots, x_n, \ldots \longrightarrow x_0$

$P(x_i) = a_m \cdot x_i^m + a_{m-1}\cdot x_i^{m-1} + \ldots + a_1 \cdot x_i + a_0 \xrightarrow[i\to\infty]{} a_m x_0^m + \ldots + a_1 x_0 + a_0 = P(x_0)$

$x_i \to x_0 \Rightarrow x_i^k \to x_0^k \; ; \; a_k x_i^k \to a_k \cdot x_0^k$

б) $\lim\limits_{x\to x_0}\frac{P(x)}{Q(x)} = \frac{\lim\limits_{x\to x_0} P(x)}{\lim\limits_{x\to x_0} Q(x)} = \frac{P(x_0)}{Q(x_0)}$

в) $f(x) = \sin x$

$\lim\limits_{x\to x_0}\sin x \overset{?}{=} \sin x_0$

$|\sin x| \leq |x|$

$$\sin x - \sin x_0 = \left|2 \sin \frac{x-x_0}{2} \cdot \cos \frac{x-x_0}{2}\right| = 2 \underbrace{\left|\sin \frac{x-x_0}{2}\right|}_{\substack{\downarrow x \to x_0 \\ 0}} \cdot \underbrace{\left|\cos \frac{x-x_0}{2}\right|}_{\leq 1} \xrightarrow[x \to x_0]{} 0$$

$$\cos x = -\sin\left(x - \frac{\pi}{2}\right); \quad \operatorname{tg} x = \frac{\sin x}{\cos x}$$