Семинар 8, 07.11.23

$\lim\limits_{x \to x_0} f(x) = a$, если:

$\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon) : \forall x$ т.ч. $0 < |x - x_0| < \delta \Rightarrow |f(x) - a| < \varepsilon$

#1.

а) $\lim\limits_{x \to 5} x^2 = 25$

$\forall \varepsilon > 0 \quad \exists \delta : |x-5| < \delta \Rightarrow |x^2 - 25| < \varepsilon$

$|x^2 - 25| = |(x-5)(x+5)| = |x-5| \cdot |x+5| < \delta \cdot 11 = 11\delta \leq \varepsilon$

$\Downarrow$

$|x+5| < |x-5| + 10 < \delta + 10 \underset{\delta \leq 1}{\leq} 11$

$\delta = \min\left(1, \frac{\varepsilon}{11}\right)$

б) $\lim\limits_{x \to 2} \frac{2x^2 - x + 1}{x+1} = \frac{7}{3}$

$\frac{2x^2 - x + 1}{x+1} - \frac{7}{3} = \frac{6x^2 - 10x - 4}{3(x+1)}$

$0 < |x-2| < \delta \Rightarrow \left| \frac{6x^2 - 10x - 4}{3(x+1)} \right| < \varepsilon$

$\left| \frac{6x^2 - 10x - 4}{3(x+1)} \right| = \left| \frac{(x-2)(6x+2)}{3(x+1)} \right| < \frac{|x-2| \cdot |6x+2|}{3|x+1|} \leq \frac{\delta \cdot 20}{3 \cdot 2} = \frac{10\delta}{3} < \varepsilon$

$\Downarrow$

$|6x+2| = |6(x-2) + 14| \leq 6|x-2| + 14 < 6\delta + 14$

$\delta \leq \frac{3\varepsilon}{10}$

$x < 2 + \delta \Rightarrow 6x + 2 < 6(2+\delta) + 2 = 6\delta + 14 \underset{\delta \leq 1}{\leq} 6 \cdot 1 + 14 = 20$

$3|x+1| = 3|(x-2) + 3| \leq 3(|x-2| + 3) < 3\delta + 9$

$|x+1| = |3 + (x-2)| \geq 3 - |x-2| > 3 - \delta \underset{\delta \leq 1}{\geq} 2$

$\forall x_1, x_2, \ldots, x_n, \ldots \ (x_i \neq x_0) \xrightarrow[n \to \infty]{} x_0 \Rightarrow f(x_1), f(x_2), \ldots, f(x_n) \xrightarrow[n \to \infty]{} a$

а) $x_1, x_2, \ldots, x_n, \ldots \xrightarrow[n \to \infty]{} 5 \quad (x_i \neq 5)$

$x_1^2, x_2^2, \ldots, x_n^2, \ldots \xrightarrow[n \to \infty]{} 25$

б) $x_1, x_2, \ldots, x_n, \ldots \xrightarrow[n \to \infty]{} 2 \quad (x_i \neq 2)$

$\frac{2x_n^2 - x_n + 1}{x_n + 1} \xrightarrow[n \to \infty]{} \frac{2 \cdot 2^2 - 2 + 1}{2 + 1} = \frac{7}{3}$

#2.

$f(x) = \sin\frac{\pi}{x} \quad (x \neq 0)$

а) $\lim\limits_{x\to 0} \sin\frac{\pi}{x} = a$

$\forall \varepsilon > 0 \ \exists \delta : 0 < |x| < \delta \Rightarrow |\sin\frac{\pi}{x} - a| < \varepsilon$

$\sin\frac{\pi}{x} = 1 \Leftrightarrow \frac{\pi}{x} = \frac{\pi}{2} + 2\pi n$

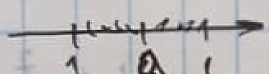
$\frac{x}{\pi} = \frac{1}{\frac{\pi}{2} + 2\pi n}$; $x = \frac{1}{\frac{1}{2} + 2n}$

$\sin\frac{\pi}{x} = -1 \Leftrightarrow \frac{\pi}{x} = -\frac{\pi}{2} + 2\pi n$

$\frac{x}{\pi} = \frac{1}{-\frac{\pi}{2} + 2\pi n}$; $x = \frac{1}{2n - \frac{1}{2}}$

$\forall \delta \ \exists N : \forall n > N \ \frac{1}{2n \pm \frac{1}{2}} < \delta$

$\begin{cases} |1 - a| < \varepsilon \\ |-1 - a| < \varepsilon \end{cases}$



б) $x_n = \frac{1}{\frac{1}{2} + 2n} \xrightarrow[n\to\infty]{} 0 \Rightarrow f(x) = 1 \xrightarrow[n\to\infty]{} 1$

$\widetilde{x}_n = \frac{1}{2n - \frac{1}{2}} \xrightarrow[n\to 0]{} 0 \Rightarrow f(\widetilde{x}_n) = -1 \xrightarrow[n\to\infty]{} -1$

$a_1, a_2, a_3 \ldots, \ f(a_i) = 1$

$b_1, b_2, b_3 \ldots, \ f(b_i) = -1$

$a_1, b_1, a_2, b_2, a_3, b_3 \ldots$

#3.

а) $\lim\limits_{x\to 2} \frac{x^2 + 4x - 5}{x^2 - 1} = \frac{2^2 + 4\cdot 2 - 5}{2^2 - 1} = \frac{7}{3}$

б) $\lim\limits_{x\to 2} \frac{x^3 - 8}{x^3 - 3x - 2} = \lim\limits_{x\to 2} \frac{(x-2)(x^2 + 2x + 4)}{(x-2)(x^2 + 2x + 1)} = \lim\limits_{x\to 2} \frac{x^2 + 2x + 4}{x^2 + 2x + 1} = \frac{12}{9} = \frac{4}{3}$

c) $\lim\limits_{x\to 0} \frac{2\sqrt{x^2+x+1}-2-x}{x^2} = \lim\limits_{x\to 0} \frac{4x^2+4x+4-4-4x-x^2}{x^2(2\sqrt{x^2+x+1}+2+x)} =$

$= \lim\limits_{x\to 0} \frac{3x^2}{x^2(2\sqrt{x^2+x+1}+2+x)} = \lim\limits_{x\to 0} \frac{3}{2\sqrt{x^2+x+1}+2+x} = \frac{3}{2\cdot 1+2+0} = \frac{3}{4}$

e) $\lim\limits_{x\to 0} \frac{\sin 3x}{x} = \lim\limits_{x\to 0} \frac{3\overbrace{\sin 3x}^{\to 1}}{3x} = 3$

f) $\lim\limits_{x\to 0} \frac{\sin x}{\sin 6x - \sin 7x} = \frac{1}{-1} = -1$

$\frac{\sin 6x - \sin 7x}{\sin x} = \frac{\sin 6x}{\sin x} - \frac{\sin 7x}{\sin x} = \left[\frac{\sin 6x}{6x} \cdot \frac{6x}{\sin x} \xrightarrow[x\to 0]{} 6\cdot 1\cdot 1 \right] =$

$= 6 - 7 = -1$

(g) $\lim\limits_{x\to 1} \frac{\sin 7\pi x}{\sin 2\pi x} = \lim\limits_{y\to 0} \frac{\sin 7\pi(y+1)}{\sin 2\pi(y+1)} = \lim\limits_{y\to 0} \frac{\sin(7\pi y+7\pi)}{\sin(2\pi y+2\pi)} = \lim\limits_{y\to 0} \frac{-\sin 7\pi y}{\sin 2\pi y} =$

$y = x-1 \to 0$

$x = y+1$

$= -\lim\limits_{y\to 0} \frac{\sin 7\pi y}{7\pi y} \cdot \frac{2\pi y}{\sin 2\pi y} \cdot \frac{7}{2} = -1\cdot 1\cdot \frac{7}{2} = -\frac{7}{2}$