Homework 23.

#1.

$\forall \varepsilon>0 \ \exists N_1 : \ \forall n > N_1 \ \ \forall x \in E \ \ |f_n(x) - f(x)| < \frac{\varepsilon}{2}$ $-$ $f_n(x) \rightrightarrows f(x)$

$\forall \varepsilon>0 \ \exists N_2 : \ \forall n > N_2 \ \ \forall x \in E \ \ |g_n(x) - g(x)| < \frac{\varepsilon}{2}$ $-$ $g_n(x) \rightrightarrows g(x)$

$\forall \varepsilon > 0 \quad \forall x \in E \quad |\alpha f_n(x) + \beta \cdot g_n(x) - (\alpha \cdot f(x) + \beta \cdot g(x))| =$

$= |\alpha(f_n(x) - f(x)) + \beta(g_n(x) - g(x))| \le |\alpha \cdot (f_n(x) - f(x))| + |\beta(g_n(x) - g(x))| <$

$< |\alpha| \cdot \frac{\varepsilon}{2} + |\beta| \cdot \frac{\varepsilon}{2} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ $-$ при $n > \max(N_1, N_2)$

$\Rightarrow \quad \alpha \cdot f_n(x) + \beta g_n(x) \rightrightarrows \alpha \cdot f(x) + \beta g(x)$ ч.т.д.

#2.

$f_n(x) \rightrightarrows f(x)$: $\forall \varepsilon > 0 \ \exists N_1 \ \forall n > N_1 \ \forall x \in E \quad |f_n(x) - f(x)| < \frac{\varepsilon}{M}$, где

M такое, что $|g(x)| \leq M$ ($g(x)$ – ограничена)

$\forall \varepsilon > 0 \ \forall x \in E \quad |f_n(x) \cdot g(x) - f(x) g(x)| = |(f_n(x) - f(x)) \cdot g(x)| \leq$

$\leq |f_n(x) - f(x)| \cdot |g(x)| < \frac{\varepsilon}{M} \cdot M = \varepsilon \quad \Rightarrow f_n(x) \cdot g(x) \rightrightarrows f(x) \cdot g(x)$ ч.т.д.

#3.

(a) $f_n(x) = (x-1)\arctan(x^n)$, $E = (0; +\infty)$

$\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} (x-1)\operatorname{arctg} x^n = [\operatorname{arctg} x^n \xrightarrow[n\to\infty]{x \geq 1} \frac{\pi}{2}] = \frac{\pi(x-1)}{2}; [\operatorname{arctg} x^n \xrightarrow[n\to\infty]{x<0} 0] = 0.$

(b) $f_n(x) = \sqrt[n]{1 + x^n + (x^2/2)^n}$, $E = [0; +\infty)$

$\cancel{\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \sqrt[n]{1 + x^n + (x^2/2)^n} = \lim\limits_{n\to\infty} \frac{(2 + 2x^n + x^2)^{\frac{1}{n}}}{\sqrt[n]{2}} = [\sqrt[n]{2} \xrightarrow[n\to\infty]{} 1] =}$

$\cancel{= \lim\limits_{n\to\infty} (2 + 2x^n + x^2)^{\frac{1}{n}} = \lim\limits_{n\to\infty} (e^{\frac{1}{n}\ln(2 + 2x^n + x^2)})}$

При $x=0$: $f_n(0) = \sqrt[n]{1} = 1$

При $0 < x < \sqrt{2}$: $f_n(x) \to \sqrt[n]{1 + x^n + (x^2/2)^n} \to [(\frac{x^2}{2})^n \xrightarrow[n\to\infty]{} 0] \to \sqrt[n]{1 + x^n} \to \sqrt[n]{x^n} = x \ (n\to\infty)$

При $x = \sqrt{2}$: $f_n(\sqrt{2}) \to \sqrt[n]{2 + x^n} \to \sqrt[n]{x^n} \to x \quad (n\to\infty)$

При $x > \sqrt{2}$: $f_n(x) = \sqrt[n]{1 + x^n + (x^2/2)^n} = \lim\limits_{n\to\infty} (\sqrt[n]{x^n + (x^2/2)^n}) = \lim\limits_{n\to\infty} \sqrt[n]{x^n(1 + \frac{x^n}{2^n})} =$

$= \lim\limits_{n\to\infty} \sqrt[n]{x^n \cdot \frac{x^n}{2^n}} = \lim\limits_{n\to\infty} \sqrt[n]{\frac{x^{2n}}{2^n}} = \lim\limits_{n\to\infty} \frac{x^2}{2} = \frac{x^2}{2}$

Ответ: (a) $\begin{cases} 0, & 0 < x < 1 \\ \frac{\pi}{2}(x-1), & 1 \leq x \end{cases}$ (b) $\begin{cases} 1, & x = 0 \\ x, & 0 < x \leq \sqrt{2} \\ \frac{x^2}{2}, & x > \sqrt{2} \end{cases}$

#4

(a) $f_n(x) = \frac{4n\sqrt{nx}}{3+4n^2x}$, $E = [\delta, +\infty)$, $\delta > 0$

$E_1 = (0; a]$, $a > 0$

$E_2 = (0; +\infty)$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{4n\sqrt{nx}}{3+4n^2x} = \lim_{n\to\infty} \frac{4\cdot\sqrt{\frac{x}{n}}}{\frac{3}{n^2}+4x} = 0$$

$$|f_n(x) - f(x)| = \left|\frac{4n\sqrt{nx}}{3+4n^2x}\right| = \frac{4n\sqrt{nx}}{3+4n^2x} \leq \frac{4n\sqrt{nx}}{4n^2x} = \frac{\sqrt{nx}}{nx} = \frac{1}{\sqrt{nx}} \leq \frac{1}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow f_n(x) \rightrightarrows 0$.

(b) $f_n(x) = \operatorname{arctg}\frac{n}{x}$, E_1, E_2.

$$\lim_{n\to\infty} \operatorname{arctg}\frac{n}{x} = \frac{\pi}{2}$$

при E_1

$$|f_n(x) - f(x)| = \left|\operatorname{arctg}\frac{n}{x} - \frac{\pi}{2}\right| = \frac{\pi}{2} - \operatorname{arctg}\frac{\pi}{x} \leq \frac{\pi}{2} - \frac{\pi}{2} = 0 \Rightarrow f_n(x) \rightrightarrows 0$$

$$|f_n(x) - f(x)| = [\text{При } x_n = n] = \left|\operatorname{arctg} 1 - \frac{\pi}{2}\right| = \left|\frac{\pi}{4} - \frac{\pi}{2}\right| = \frac{\pi}{4} \neq 0 \text{ на } E_2.$$