Модуль 3.

Homework 15.

#1.

a) $x = \ln(1+e^{-t})$ $\qquad y = \ln(1+e^{t})$

$y(t) = \ln(1+e^{t})$

$y'(t) = \left(\ln(1+e^{t})\right)' = \frac{e^{t}}{1+e^{t}} = 1 - \frac{1}{1+e^{t}} > 0 \Rightarrow y(t) \uparrow$

Тогда $x(y)$ – непрерывная функция. ч.т.д.

b) $x = \frac{1}{4}(t-4)e^{t}$ $\qquad y = \sqrt{t} \cdot e^{t}$

$y(t) = \sqrt{t} \cdot e^{t} \uparrow$, т.к. при $t_1 < t_2$ $\quad y(t_1) < y(t_2)$

Тогда $x(y)$ – непрерывная функция. ч.т.д.

#2.

a) $x = a \cdot \cos t$ $\qquad y = b \cdot \sin t$ $\qquad t \in (0; \pi)$

$y'_x = y'(x)_t = y'(x(t)) = \frac{y'(t)}{x'(t)} = \frac{b\cos t}{-a\sin t} = \boxed{-\frac{b}{a}\operatorname{ctg} t}$

b) $x = (t-1)^2(t-2)$ $\qquad y = (t-1)^2(t-3)$ $\qquad t > \frac{5}{3}$

$y'_x = \frac{((t-1)^2(t-3))'}{((t-1)^2(t-2))'} = \frac{2(t-1)(t-3) + (t-1)^2}{2(t-1)(t-2) + (t-1)^2} = \frac{(t-1)(2t-6+t-1)}{(t-1)(2t-4+t-1)} =$

$= \boxed{\frac{3t-7}{3t-5}}$

#3.

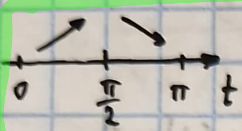
$x = \ln(\sin\frac{t}{2})$ $\qquad y = \ln(\sin t)$ $\qquad t \in (0; \pi)$

$y'(x)_t = \frac{(\ln(\sin t))'}{(\ln(\sin\frac{t}{2}))'} = \frac{\cos t}{\sin t} : \frac{\frac{1}{2}\cos\frac{t}{2}}{\sin\frac{t}{2}} = 2\operatorname{ctg} t \cdot \operatorname{tg}\frac{t}{2}$

$y'(x(t))=0$

$2\,tg\,t \cdot tg\frac{t}{2}=0$

$\left[\begin{array}{l} ctg\,t=0 \\ tg\frac{t}{2}=0 \end{array}\right. \qquad \left[\begin{array}{l} t=\pm\frac{\pi}{2}+2\pi k \\ \frac{t}{2}=0+\pi k \end{array}\right.$ С учётом $t\in(0;\pi)$, $t_1=\frac{\pi}{2}$ - экстремум



1) $x(\frac{\pi}{2})=\ln(\sin\frac{\pi}{4})=\ln\frac{\sqrt{2}}{2}$

2) $\lim\limits_{t\to 0}(\ln(\sin\frac{t}{2}))=\ln 0=-\infty$

3) $\lim\limits_{t\to \pi}(\ln(\sin\frac{t}{2}))=\ln 1=0$

#4.

$x=\frac{1+t}{t^3} \qquad y=\frac{3+t}{2t^2} \qquad M(2;2)$

$y'(t)=\frac{2t^2-(3+t)\cdot 4t}{4t^4}=\frac{-2t^2-12t}{4t^4}=\frac{t+6}{-2t^3}$

$x'(t)=\frac{t^3-(1+t)\cdot 3t^2}{t^6}=\frac{-2t^3-3t^2}{t^6}=\frac{2t+3}{-t^4}$

$y'(x(t))=\frac{y'(t)}{x'(t)}=\frac{t+6}{-2t^3}:\frac{2t+3}{-t^4}=\frac{(t+6)t^4}{2(2t+3)t^3}=\frac{t^2+6t}{4t+6}$

Найдём t, при котором x и y удовлетворяют т. M.

$\begin{cases} x=2 \\ y=2 \end{cases} \qquad \begin{cases} \frac{1+t}{t^3}=2 \\ \frac{3+t}{2t^2}=2 \end{cases} \qquad \begin{cases} 2t^3-t-1=0 \\ 4t^2-3-t=0 \end{cases} \qquad \begin{cases} 2t^3-t-1=0 \\ 4(t-1)(t+\frac{3}{4})=0 \end{cases}$

$t=1.$

$x'(1) = \frac{2\cdot 1+3}{-1} = -5 \neq 0 \Rightarrow$ касательная существует

$y'(x(1)) = \frac{1^2+6\cdot 1}{4\cdot 1+6} = \frac{7}{10}$

Касательная $y = kx + b$, где $k = y'(x(1)) = \frac{7}{10}$

Точка M $\in$ касательной, тогда

$2 = \frac{7}{10}\cdot 2 + b$; $b = \frac{6}{10}$ $\Rightarrow$ $\boxed{y = \frac{7}{10}x + \frac{6}{10} \text{ – касательная}}$

Нормаль: $y = -\frac{1}{k}x + l$; $y = -\frac{10}{7}x + l$

Точка M $\in$ нормали, тогда

$2 = -\frac{10}{7}\cdot 2 + l$; $l = \frac{34}{7}$ $\Rightarrow$ $\boxed{y = -\frac{10}{7}x + \frac{34}{7} \text{ – нормаль}}$

#5.

$x = \frac{1}{t(t+1)}$ $\qquad$ $y = \frac{(t+1)^2}{t}$

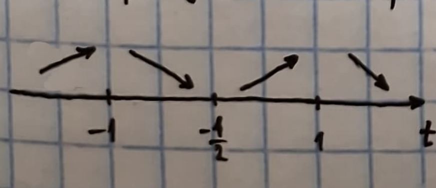
$$x'(t) = \frac{-(t+1+t)}{t^2(t+1)^2} = \frac{-2t-1}{t^2(t+1)^2}$$

$$y'(t) = \frac{2(t+1)t - (t+1)^2}{t^2} = \frac{(t+1)(2t-t-1)}{t^2} = \frac{t^2-1}{t^2}$$

$$y'(x(t)) = \frac{y'(t)}{x'(t)} = \frac{t^2-1}{t^2}\cdot\frac{t^2(t+1)^2}{-(2t+1)} = \frac{t^4-t^2+2t^3-2t+t^2-1}{-(2t+1)} =$$

$$= \frac{t^4+2t^3-2t-1}{-2t-1}$$

Экстремумы: ± 1, в $0, -\frac{1}{2}$ – прерывается



$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} \frac{1}{t(t+1)} = 0 \qquad \lim_{t\to+\infty} \frac{(t+1)^2}{t} = \lim_{t\to+\infty}\left(t+2+\frac{1}{t}\right) = +\infty \Rightarrow x=0 \text{ – вертик. асимпт.}$$

$$x(1) = \frac{1}{1(1+1)} = \frac{1}{2} \qquad y(1) = \frac{(1+1)^2}{1} = 4$$

$$x\left(-\frac{1}{2}\right) = \frac{1}{-\frac{1}{2}\left(-\frac{1}{2}+1\right)} = -4 \qquad y\left(-\frac{1}{2}\right) = \frac{\left(-\frac{1}{2}+1\right)^2}{-\frac{1}{2}} = -\frac{1}{2}$$

$$\lim_{t\to-1} \frac{1}{t(t+1)} = -\infty \qquad \lim_{t\to-1} \frac{(+t+1)^2}{+t} = 0 \Rightarrow x = 0 \text{ – гориз. асимптота}$$

$$\lim_{t\to-\infty} \frac{1}{t(t+1)} = 0 \qquad \lim_{t\to-\infty} \frac{(t+1)^2}{t} = \lim_{t\to-\infty}\left(t+2+\frac{1}{t}\right) = -\infty \Rightarrow x=0 \text{ – вертик. асимптота}$$

$$\lim_{t\to0^+} \frac{1}{t(t+1)} = +\infty \qquad \lim_{t\to0^+} \frac{(t+1)^2}{t} = +\infty$$

$$\lim_{t\to0^-} \frac{1}{t(t+1)} = -\infty \qquad \lim_{t\to0^-} \frac{(t+1)^2}{t} = -\infty$$

$$\cancel{y''(x(t))} = \lim_{t\to0} \frac{y(t)}{x(t)} = \lim_{t\to0}\left(\frac{(t+1)^2}{t}\cdot\frac{t(t+1)}{1}\right) = \lim_{t\to0}(t+1)^3 = 1$$

$$\lim_{t\to0}\left(y(t) - 1\cdot x(t)\right) = \lim_{t\to0}\left(\frac{(t+1)^2}{t} - \frac{1}{t(t+1)}\right) = \lim_{t\to0}\left(\frac{(t+1)^3-1}{t(t+1)}\right) =$$

$$= \lim_{t\to0}\left(\frac{t^3+3t^2+3t}{t(t+1)}\right) = \lim_{t\to0}\left(\frac{t^2+3t+3}{t+1}\right) = 3$$

$y = x + 3$ – наклонная асимптота.

$$y''(x(t)) = \left(\frac{y'(t)}{x'(t)}\right)' = \left(\frac{t^4+2t^3-2t-1}{-2t-1}\right)' =$$

$$= \frac{(4t^3+6t^2-2)(-2t-1)-(t^4+2t^3-2t-1)(-2)}{(-2t-1)^2} =$$

$$= \frac{-8t^4-12t^3+4t-4t^3-6t^2+2+2t^4+4t^3-4t-2}{(-2t-1)^2} =$$

$$= \frac{-6t^4-12t^3-6t^2}{(2t+1)^2} = \frac{-6t^2(1t^2+2t+1)}{(2t+1)^2} = \frac{-6t^2(t+1)^2}{(2t+1)^2}$$

Нули: -1; 0; $-\frac{1}{2}$.