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Формула Тейлора

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\frac{f'''(x_0)}{3!}(x-x_0)^3+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\bar{o}((x-x_0)^n)$$

(при $x\to x_0$)

При $x_0=0$ это – формула Маклорена.

1) $\boxed{e^x}=\sum\limits_{k=0}^{n}\frac{x^k}{k!}+\bar{o}(x^n)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\bar{o}(x^n)$ (при $x\to x_0$)

2) $\log_e(1+x)=\boxed{\ln(1+x)}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+(-1)^{n+1}\cdot\frac{x^n}{n}+\bar{o}(x^n)$

3) $\boxed{(1+x)^\alpha}=1+C_\alpha^1\cdot x+C_\alpha^2\cdot x^2+\ldots+C_\alpha^n\cdot x^n+\bar{o}(x^n)$, где

$$C_\alpha^k=\frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-k+1)}{k!}$$

$\alpha=-1$: $\boxed{\frac{1}{1+x}}=1-x+x^2-x^3+\ldots+(-1)^n x^n+\bar{o}(x^n)$

$$C_{-1}^k=\frac{(-1)\cdot(-2)\cdot(-3)\cdot\ldots\cdot(-k)}{k!}=(-1)^k$$

$$\boxed{\frac{1}{1-x}}=1+x+x^2+x^3+\ldots+x^n+\bar{o}(x^n)$$

4) $\boxed{\mathrm{arctg}\,x}=x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots+(-1)^n\cdot\frac{x^{2n+1}}{2n+1}+\bar{o}(x^{2n+1})$

① Представить ф. Маклорена с $\bar{o}(x^n)$.

a) $f(x)=(x+5)\cdot e^{2x}$

$$e^{2x}=1+2x+\frac{2^2x^2}{2!}+\frac{2^3x^3}{3!}+\ldots+\frac{2^n\cdot x^n}{n!}+\underbrace{\bar{o}(2^n\cdot x^n)}_{=\bar{o}(x^n)}$$

$$(x+5)e^{2x}=(x+5)\left(1+2x+\frac{2^2x^2}{2!}+\frac{2^3x^3}{3!}+\ldots+\frac{2^nx^n}{n!}+\bar{o}(x^n)\right)=$$

$$=x+2x^2+\frac{2^2x^3}{2!}+\ldots+\frac{2^nx^{n+1}}{n!}+5+10x+5\cdot\frac{2^2x^2}{2!}+\ldots+5\cdot\frac{2^nx^n}{n!}+\bar{o}(x^n)\ \ominus$$

$$[(x+5)\bar{o}(x^n) = x\cdot\bar{o}(x^n) + 5\cdot\bar{o}(x^n) = \bar{o}(x^{n+1}) + \bar{o}(x^n) = \bar{o}(x^n)+\bar{o}(x^n) = \bar{o}(x^n)]$$

$$\Leftrightarrow 5 + (1+5\cdot 2)x + \left(2 + 5\cdot\frac{2^2}{2}\right)x^2 + \left(\frac{2^3}{2!} + 5\cdot\frac{2^3}{3!}\right)x^3 + \ldots + \bar{o}(x^n) \Leftrightarrow$$

$$\left[\text{coef при } x^k = \frac{2^{k-1}}{(k-1)!} + \frac{5\cdot 2^k}{k!}\right]$$

$$\Leftrightarrow \boxed{\sum_{k=1}^{n}\left(\frac{2^{k-1}}{(k-1)!} + \frac{5\cdot 2^k}{k!}\right)x^k + 5 + \bar{o}(x^n)}$$

б) $$f(x) = \ln\frac{3+x}{2-x} = \ln(3+x) - \ln(2-x) = \ln\left(3\left(1+\frac{x}{3}\right)\right) - \ln\left(2\left(1-\frac{x}{2}\right)\right) =$$

$$= \ln 3 + \ln\left(1+\frac{x}{3}\right) - \ln(2) - \ln\left(1-\frac{x}{2}\right) =$$

$$= \ln\frac{3}{2} + \ln\left(1+\frac{x}{3}\right) - \ln\left(1-\frac{x}{2}\right) =$$

$$= \ln\frac{3}{2} + \sum_{k=1}^{n}(-1)^{k+1}\cdot\frac{x^k}{3^k\cdot k} + \bar{o}(x^n) - \sum_{k=1}^{n}(-1)^{k+1}\frac{(-x)^k}{2^k\cdot k} - \bar{o}(x^n) =$$

$$= \ln\frac{3}{2} + \sum_{k=1}^{n}\left(\frac{(-1)^{k+1}}{3^k\cdot k} - (-1)^{k+1}\cdot(-1)^k\cdot\frac{1}{2^k\cdot k}\right)x^k + \bar{o}(x^n) =$$

$$= \boxed{\ln\frac{3}{2} + \sum_{k=1}^{n}\left(\frac{(-1)^{k+1}}{3^k\cdot k} + \frac{1}{2^k\cdot k}\right)x^k + \bar{o}(x^n)}$$

(2) Представить ф. Маклорена с $\bar{o}(x^{2n+1})$

$$f(x) = \sin^2 x\cdot\cos^2 x = (\sin x\cdot\cos x)^2 = \left(\frac{1}{2}\sin 2x\right)^2 = \frac{\sin^2 2x}{4} =$$

$$= \frac{1}{4}\cdot\frac{1-\cos 4x}{2} = \frac{1}{8} - \frac{1}{8}\cos 4x =$$

$$= \frac{1}{8} - \frac{1}{8}\sum_{k=0}^{n}(-1)^k\cdot\frac{4^{2k}\cdot x^{2k}}{(2k)!} + \bar{o}(x^{2n}) =$$

$$= \frac{1}{8} - \frac{1}{8}\sum_{k=0}^{n}(-1)^k\cdot\frac{4^{2k}\cdot x^{2k}}{(2k)!} + \bar{o}(x^{2n+1}) =$$

$$= \boxed{\frac{1}{8}\sum_{k=1}^{n}(-1)^k\cdot\frac{4^{2k}\cdot x^{2k}}{(2k)!} + \bar{o}(x^{2n+1})}$$

③ Представить формулой Маклорена с $\bar{o}(x^3)$.

$$f(x) = e^{x\cdot\cos x}$$

$$e^{x\cdot\cos x} = 1 + x\cos x + \frac{x^2\cdot\cos^2 x}{2} + \frac{x^3\cdot\cos^3 x}{6} + \underbrace{\bar{o}(x^3\cdot\cos^3 x)}_{\bar{o}(x^3)} =$$

$$= 1 + x\left(1 - \frac{x^2}{2} + \bar{o}(x^3)\right) + \frac{x^2}{2}\left(1 - \frac{x^2}{2} + \bar{o}(x^3)\right)^2 +$$

$$+ \frac{x^3}{6}\left(1 - \frac{x^2}{2} + \bar{o}(x^3)\right)^3 + \bar{o}(x^3) =$$

$$= 1 + x - \frac{x^3}{2} + \bar{o}(x^3) + \frac{x^2}{2}\left(1 - \frac{2x^2}{2} + \frac{x^4}{4} + \bar{o}(x^3)\right) + \frac{x^3}{6}\left(1 - \frac{3x^2}{2} + ?\cdot x^4 + ?\cdot x^6 + \bar{o}(x^3)\right) =$$

$$= 1 + x + \frac{x^2}{2} + x^3\left(-\frac{1}{2} + \frac{1}{6}\right) + \bar{o}(x^3) =$$

$$= \boxed{1 + x + \frac{x^2}{2} - \frac{x^3}{3} + \bar{o}(x^3)}$$

$\bar{o}(\sin^4 x) = \bar{o}(x^4)$

④ Представить ф. Маклорена с $\bar{o}(x^6)$.

$\frac{\sin x}{x} \xrightarrow[x\to 0]{} 1$

$\sin x \sim x$

$$f(x) = \frac{x^2}{1 + \sin x} = x^2\left(1 - \sin x + \sin^2 x - \sin^3 x + \sin^4 x + \bar{o}(\sin^4 x)\right) =$$

$$= x^2\left(1 - x + \frac{x^3}{6} + \left(x - \frac{x^3}{6} + \bar{o}(x^4)\right)^2 + \bar{o}(x^4) - \left(x - \frac{x^3}{6} + \bar{o}(x^4)\right)^3 +\right.$$

$$\left. + \left(x - \frac{x^3}{6} + \bar{o}(x^4)\right)^4 + \bar{o}(\sin^4 x)\right) =$$

$$= x^2\left(1 - x + \frac{x^3}{6} + \bar{o}(x^4) + x^2 - \frac{2x^4}{6} + \bar{o}(x^4) - x^3 + \bar{o}(x^4) + x^4 + \bar{o}(x^4)\right) =$$

$$= x^2\left(1 - x + x^2 - \frac{5}{6}x^3 + \frac{2}{3}x^4 + \bar{o}(x^4)\right) =$$

$$= \boxed{x^2 - x^3 + x^4 - \frac{5}{6}x^5 + \frac{2}{3}x^6 + \bar{o}(x^6)}$$

⑤ Представить ф. Тейлора в окрестности $x_0 = -1$ с $\bar{o}((x+1)^{2n})$

$$f(x) = \frac{3x + 3}{\sqrt{3 - 2x - x^2}}$$

$x + 1 = y$

$x = y - 1$

$$\| $$

$$\frac{3(y-1) + 3}{\sqrt{3 - 2(y-1) - (y-1)^2}} = \frac{3y}{\sqrt{4 - y^2}} = \frac{3y}{2\sqrt{1 - \frac{y^2}{4}}} = 3y \cdot \frac{1}{2}\left(\sqrt{1 - \frac{y^2}{4}}\right)^{-1} \ominus$$

$$\left[\frac{1}{2}\left(1-\frac{y^2}{4}\right)^{-\frac{1}{2}} = \frac{1}{2}\sum_{k=0}^{n} C_{-\frac{1}{2}}^{k}\left(-\frac{y^2}{4}\right)^{n} + \bar{o}(y^{2n})\right]$$

$$= \boxed{\frac{3y}{2}\sum_{k=0}^{n-1} C_{-\frac{1}{2}}^{k}\left(-\frac{y^2}{4}\right)^{n} + \bar{o}(y^{2n+1})}$$