Homework 20.

#1.

a) $\int\limits_0^\infty \operatorname{arctg}(\sqrt[3]{x^2+4})\,dx$

Рассмотрим $f(x)=\operatorname{arctg}(\sqrt[3]{x^2+4})$ и $g(x)=x$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{\operatorname{arctg}(\sqrt[3]{x^2+4})}{x}=\frac{\frac{\pi}{2}}{\infty}=0$$

$$\int\limits_0^\infty g(x)\,dx=\int\limits_0^\infty x\,dx=\frac{x^2}{2}\Big|_0^\infty=\infty \Rightarrow \int\limits_0^\infty g(x)\,dx \text{ расходится}$$

Т.к. $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}=0$, то $\int\limits_0^\infty f(x)\,dx$ расходится.

b) $$\int\limits_0^\infty \frac{x\ln x}{(1+x^2)^2}\,dx=\frac{1}{2}\int\limits_0^\infty\frac{\ln x}{(1+x^2)^2}\,d(x^2)=[x^2=t]=\frac{1}{2}\int\limits_0^\infty\frac{\ln\sqrt{t}}{(1+t)^2}\,dt=$$

$$=\frac{1}{4}\int\limits_0^\infty\frac{\ln t}{(1+t)^2}\,dt=-\frac{1}{4}\int\limits_0^\infty \ln t\,d\left(\frac{1}{1+t}\right)=-\frac{1}{4}\left(\frac{\ln t}{1+t}\Big|_0^\infty-\int\limits_0^\infty\frac{1}{1+t}\,d(\ln t)\right)=$$

$$=-\frac{1}{4}\left(\frac{\ln t}{1+t}-\ln|t|+\ln|t+1|\right)\Big|_0^\infty=-\frac{1}{4}\left(\frac{\ln t-\ln t-(\ln t)\cdot t}{t+1}+\ln(t+1)\right)\Big|_0^\infty=$$

$$=-\frac{1}{4}\left(\ln(t+1)-\frac{t\cdot\ln t}{t+1}\right)\Big|_0^\infty$$

$$\lim_{t\to0}\left(-\frac{1}{4}\left(\underbrace{\ln(t+1)}_{\to0}-\frac{t\cdot\ln t}{t+1}\right)\right)=\lim_{t\to0}\left(-\frac{1}{4}\cdot\frac{t\cdot\ln t}{\underbrace{t+1}_{\to1}}\right)=\frac{\lim\limits_{t\to0}(t\cdot\ln t)}{4}=$$

$$=\frac{1}{4}\lim_{t\to0}\left(\frac{\ln t}{\frac{1}{t}}\right)=\frac{1}{4}\lim_{t\to0}\frac{\frac{1}{t}}{\frac{1}{t^2}}=\frac{1}{4}\lim_{t\to0}t=0$$

$$\lim_{t\to\infty}\left(-\frac{1}{4}\left(\ln(t+1)-\frac{t\cdot\ln t}{t+1}\right)\right)=\lim_{t\to\infty}\left(-\frac{1}{4}\left(\frac{t\ln(t+1)+\ln(t+1)-t\ln t}{t+1}\right)\right)=$$

$$=\lim_{t\to\infty}\left(-\frac{1}{4}\cdot\frac{t\cdot\ln\left(\frac{t+1}{t}\right)+\ln(t+1)}{t+1}\right)=\lim_{t\to\infty}\left(-\frac{1}{4}\cdot\frac{\overbrace{\ln\left(\frac{t+1}{t}\right)}^{\to0}+\overbrace{\frac{\ln(t+1)}{t}}^{\to0}}{1+\frac{1}{t}}\right)=0$$

$$\Rightarrow \int_0^{\infty} \frac{x \ln x}{(1+x^2)^2} dx = 0$$

$$c) \int_0^1 \cos^2(\ln x)\, dx = \frac{1}{2}\int_0^1 (\cos(2\ln x)+1)\, dx = \frac{1}{2}\int_{-\infty}^{0}(1+\cos 2t)\, d(e^t) =$$

$$= \frac{1}{2}\int_{-\infty}^{0} e^t dt + \frac{1}{2}\int_{-\infty}^{0} \cos 2t\, d(e^t) = \frac{e^t}{2}\Big|_{-\infty}^{0} + \frac{1}{2}\int e^t\, d\left(\frac{\sin 2t}{2}\right) =$$

$$= \frac{e^t}{2}\Big|_{-\infty}^{0} + \frac{1}{2}\left(\frac{e^t \cdot \sin 2t}{2} - \frac{1}{2}\int \sin 2t\, d(e^t)\right) =$$

$$= \frac{e^t}{2}\Big|_{-\infty}^{0} + \frac{e^t \cdot \sin 2t}{4} - \frac{1}{4}\int_{-\infty}^{0} e^t \sin 2t\, dt = \frac{e^t}{2}\Big|_{-\infty}^{0} + \frac{e^t \cdot \sin 2t}{4} ~~- \frac{1}{4}\int_{-\infty} e^t d(~~$$

$$= \frac{e^t}{2}\Big|_{-\infty}^{0} + \frac{e^t \cdot \sin 2t}{4} + \frac{6}{5}\left(\frac{e^t \cdot \cos 2t}{2}\right) \xrightarrow[t\to 0]{} 0 + 0 + \frac{3}{5} = \frac{3}{5}$$

$$e) \int_0^{\infty} \frac{dx}{(x^2+x+1)^2} = -\int_0^{\infty} \frac{d\left(\frac{1}{x^2+x+1}\right)}{2x+1} = \left(\frac{-1}{(2x+1)(x^2+x+1)} + \int \frac{d(2x+1)}{x^2+x+1}\right)\Big|_0^{\infty} =$$

$$= \frac{-1}{(2x+1)(x^2+x+1)} + 4\int \frac{d(2x+1)}{(2x+1)^2+3} = \frac{-1}{(2x+1)(x^2+x+1)} + \frac{4}{\sqrt{3}} \operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) = f(x)$$

$$\lim_{x\to\infty}(f(x) - f(0)) = \lim_{x\to\infty} f(x) - 1 + \frac{2\sqrt{3}\pi}{9} = -1 + \frac{4\sqrt{3}\pi}{9}$$

$$\lim_{x\to\infty}\left(\frac{4}{\sqrt{3}} \operatorname{arctg} \frac{2x+1}{\sqrt{3}}\right) = \frac{2\sqrt{3}\pi}{9}$$

#2.

$$a) \int_0^{\infty} \frac{e^{-3x}}{1+x^2} dx < \int_0^{\infty} \frac{dx}{1+x^2} < \int_0^{\infty} \frac{dx}{x^2} \text{ – сходится} \Rightarrow \int_0^{\infty} \frac{e^{-3x}}{1+x^2} dx \text{ сходится}$$

$$b) \int_0^{\infty} \frac{dx}{\sqrt{e^x - 1}} = 2\int_0^{\infty} \frac{d(\sqrt{e^x-1})}{e^x} \overset{t=\sqrt{e^x-1}}{=} 2\int_0^{\infty} \frac{dt}{t^2+1} = 2\operatorname{arctg}\sqrt{e^x-1} \text{ – сходится}$$

$$c) \int_0^{\infty} \frac{\arctan 3x - \arctan 5x}{x} dx = \int_0^{\infty} \frac{\operatorname{arctg} 3x}{x} dx - \int_0^{\infty} \frac{\operatorname{arctg} 5x}{x} dx$$

$\frac{\arctan 3x}{x} > \frac{\frac{\pi}{4}}{x}$ - расходится $\Rightarrow$ интеграл расходится

d) $\int\limits_1^{\infty} \frac{x^{\frac{5}{2}}}{(1+x^2)^2} dx < \int\limits_1^{\infty} \frac{x^{\frac{5}{2}}}{x^4} = \int\limits_1^{\infty} x^{-\frac{3}{2}}$ - сходится

f) $\int\limits_2^{\infty} \frac{dx}{x^2 + \sqrt[3]{x^4+1}} < \int\limits_2^{\infty} \frac{1}{x^2} dx$ - сходится