Homework - 11.

#6.

(a) $f(x) = \ln\left(\ln \frac{x}{2}\right)$

$$f'(x) = \left(\ln\left(\ln \frac{x}{2}\right)\right)' = \frac{1}{\ln \frac{x}{2}} \cdot \left(\ln \frac{x}{2}\right)' = \frac{1}{\frac{x}{2} \cdot \ln \frac{x}{2}} \cdot \left(\frac{x}{2}\right)' = \boxed{\frac{1}{x \ln \frac{x}{2}}}$$

(b) $f(x) = 2^{\sin x^2}$

$$f'(x) = \left(2^{\sin x^2}\right)' = 2^{\sin x^2} \cdot \ln 2 \cdot (\sin x^2)' = 2^{\sin x^2} \cdot \ln 2 \cdot \cos x^2 \cdot 2x =$$

$$= \boxed{x \cdot \cos x^2 \cdot \ln 2 \cdot 2^{\sin x^2 + 1}}$$

(c) $f(x) = (\sin x)^{\cos x} = e^{\cos x \cdot \ln \sin x}$

$$f'(x) = \left(e^{\cos x \cdot \ln \sin x}\right)' = e^{\cos x \cdot \ln \sin x} \cdot (\cos x \cdot \ln \sin x)' =$$

$$= e^{\cos x \ln \sin x} \cdot \left(-\sin x \cdot \ln \sin x + \cos x \cdot \frac{1}{\sin x} \cdot \cos x\right) =$$

$$= \boxed{(\sin x)^{\cos x} \cdot \left(\frac{\cos^2 x}{\sin x} - \sin x \cdot \ln \sin x\right)}$$

(d) $f(x) = \arccos\left(\frac{x^{2n}-1}{x^{2n}+1}\right)$

$$f'(x) = -\frac{1}{\sqrt{1-\left(\frac{x^{2n}-1}{x^{2n}+1}\right)^2}}\left(\frac{x^{2n}-1}{x^{2n}+1}\right)' =$$

$$= \frac{-1}{\sqrt{\left(1-\frac{x^{2n}-1}{x^{2n}+1}\right)\left(1+\frac{x^{2n}-1}{x^{2n}+1}\right)}} \cdot \frac{2nx^{2n-1}\cdot(x^{2n}+1) - 2nx^{2n-1}(x^{2n}-1)}{(x^{2n}+1)^2} =$$

$$= \frac{-1}{\sqrt{\frac{2\cdot 2x^{2n}}{(x^{2n}+1)^2}}} \cdot \frac{2nx^{2n-1}\cdot 2}{(x^{2n}+1)^2} = \frac{-2nx^{2n-1}}{\sqrt{x^{2n}}\cdot(x^{2n}+1)} = \boxed{\frac{-2nx^{n-1}}{x^{2n}+1}}$$

#5.

a) $\lim\limits_{x\to\frac{\pi}{6}} \frac{\cos\left(\frac{2\pi}{3}-x\right)}{\sqrt{3}-2\cos x}$ Замена: $x-\frac{\pi}{6}=t$; $x = t+\frac{\pi}{6}$

$x\to\frac{\pi}{6} \Rightarrow t\to 0$

$$\frac{\cos\left(\frac{2\pi}{3}-\left(t+\frac{\pi}{6}\right)\right)}{\sqrt{3}-2\cos\left(t+\frac{\pi}{6}\right)} = \frac{\cos\left(\frac{\pi}{2}-t\right)}{\sqrt{3}-2\cos t\cdot\cos\frac{\pi}{6}+2\sin t\cdot\sin\frac{\pi}{6}} = \frac{\sin t}{\sqrt{3}-\sqrt{3}\cos t+\sin t} =$$

$$= \frac{\sin t}{\sqrt{3}(1-\cos t)+\sin t} = \frac{\sin t}{\sqrt{3}(1-1+2\sin^2\frac{t}{2})+\sin t} = \frac{2\cos\frac{t}{2}\cdot\sin\frac{t}{2}}{2\sqrt{3}\sin^2\frac{t}{2}+2\sin\frac{t}{2}\cdot\cos\frac{t}{2}} =$$

$$= \frac{2\sin\frac{t}{2}}{2\sin\frac{t}{2}}\cdot\frac{\cos\frac{t}{2}}{\sqrt{3}\sin\frac{t}{2}+\cos\frac{t}{2}} = \frac{\cos\frac{t}{2}}{\sqrt{3}\sin\frac{t}{2}+\cos\frac{t}{2}}$$

$$\lim_{t\to 0}\frac{\cos\frac{t}{2}}{\sqrt{3}\sin\frac{t}{2}+\cos\frac{t}{2}} = \frac{1}{0+1} = \boxed{1}$$

b) $\lim\limits_{x\to 0}\frac{e^{7x}-e^{2x}}{\operatorname{tg} x}$

$$e^{7x}-e^{2x} = e^{2x}(e^{5x}-1) = e^{2x}(e^x-1)(e^{4x}+e^{3x}+e^{2x}+e^x+1)$$

$$\lim_{x\to 0}\frac{e^{2x}(e^x-1)(e^{4x}+e^{3x}+e^{2x}+e^x+1)}{\operatorname{tg} x} = \lim_{x\to 0}(e^{4x}+e^{3x}+e^{2x}+e^x+1)e^{2x}\cdot\lim_{x\to 0}\frac{(e^x-1)\cos x}{\sin x} =$$

$$= (1+1+1+1+1)\cdot \lim_{x\to 0} \frac{\overbrace{(e^x-1)}^{}\cdot \overbrace{\cos x}^{\to 1}}{\underbrace{x}_{\to 1}\cdot \underbrace{\frac{\sin x}{x}}_{\to 1}} = 5\cdot 1 = \boxed{5}$$

$$c)\ \lim_{x\to 0}(\cos x)^{\frac{1}{x^2}} = \lim_{x\to 0}\left(\left(1-2\sin^2\frac{x}{2}\right)^{\frac{1}{-2\sin^2\frac{x}{2}}}\right)^{\frac{-2\sin^2\frac{x}{2}}{-x^2}} = \lim_{x\to 0}\left(\overbrace{\left(1-\sin^2\frac{x}{2}\right)^{\frac{-1}{2\sin^2\frac{x}{2}}}}^{\to e}\right)^{\overbrace{\frac{-2\sin^2\frac{x}{2}}{-x^2\cdot\frac{x^2}{4}}\cdot\frac{x^2}{4}}^{\to 2\cdot\frac{1}{4}}} =$$

$$= e^{\frac{1}{2}} = \boxed{\sqrt{e}}$$

$$d)\ \lim_{x\to 0}(\cos x + \operatorname{arctg}^2 x)^{\frac{1}{\operatorname{arctg} x^2}}$$

$$\|$$

$$\left(\cos x\left(1+\frac{\operatorname{arctg}^2 x}{\cos x}\right)\right)^{\frac{1}{\operatorname{arctg} x^2}} = (\cos x)^{\frac{1}{\operatorname{arctg} x^2}}\cdot\left(\left(1+\frac{\operatorname{arctg}^2 x}{\cos x}\right)^{\frac{\cos x}{\operatorname{arctg}^2 x}}\right)^{\frac{1}{\operatorname{arctg} x^2}\cdot\frac{\operatorname{arctg}^2 x}{\cos x}} =$$

$$= \underbrace{\left(\left(1-2\sin^2\frac{x}{2}\right)^{\frac{1}{-2\sin^2\frac{x}{2}}}\right)}_{\to e}^{\frac{-2\sin^2\frac{x}{2}}{\operatorname{arctg} x^2}}\cdot\underbrace{\left(\left(1+\frac{\operatorname{arctg}^2 x}{\cos x}\right)^{\frac{\cos x}{\operatorname{arctg}^2 x}}\right)}_{\to e}^{\underbrace{\frac{x^2}{\operatorname{arctg} x^2}\cdot\frac{\operatorname{arctg}^2 x}{x^2}}_{\to 1}\cdot\underbrace{\frac{1}{\cos x}}_{\to 1}} \to$$

$$\frac{-2\underbrace{\frac{\sin^2\frac{x}{2}}{\frac{x^2}{4}}}_{\to 1}\cdot\frac{x^2}{4}}{\cdot x^2\cdot\underbrace{\frac{x^2}{\operatorname{arctg} x^2}}_{\to 1}} \to -\frac{1}{2}$$

$$\to e^{-\frac{1}{2}}\cdot e = e^{\frac{1}{2}} = \boxed{\sqrt{e}}$$

#3.

Дано: $f(x) = O(g(x)) \iff |f(x)| \le C\cdot|g(x)|,\quad C>0$

Док-ть: $2^{f(x)} = O(2^{g(x)})$

$2^{|f(x)|} \le 2^{C\cdot|g(x)|} = 2^C\cdot 2^{|g(x)|}$, тогда при $K = 2^C$

$2^{|f(x)|} \le K\cdot 2^{|g(x)|} \Rightarrow 2^{f(x)} = O(2^{g(x)})$ ч.т.д.

#1.

a) $f(x) = \sqrt{x^4+x^3} - \sqrt{x^4-x^3}$

$$① \lim_{x\to+\infty}\frac{f(x)}{x} = \frac{\sqrt{x^4+x^3}-\sqrt{x^4-x^3}}{x} \overset{\lim_{x\to+\infty}}{=} \frac{x^4+x^3-x^4+x^3}{x(\sqrt{x^4+x^3}+\sqrt{x^4-x^3})} \overset{\lim_{x\to+\infty}}{=} \frac{2x^2}{\sqrt{x^4+x^3}+\sqrt{x^4-x^3}} =$$

$$\lim_{x\to+\infty} \frac{2}{\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{1}{x}}} = \frac{2}{1+1} = 1 = k_1$$

$$b_1 = \lim_{x\to+\infty} (f(x) - 1\cdot x) = \lim_{x\to+\infty} \Big(\overbrace{\frac{f(x)}{x}}^{\to 1}\cdot x - x\Big) = x - x = 0$$

$y = x$

② $$\lim_{x\to-\infty} \frac{f(x)}{x} = \frac{\sqrt{x^4+x^3}-\sqrt{x^4-x^3}}{x} = \lim_{x\to-\infty} \frac{x^4+x^3-x^4+x^3}{x\left(\sqrt{x^4+x^3}+\sqrt{x^4-x^3}\right)} =$$

$$= \lim_{x\to-\infty} \frac{2x^2}{\sqrt{x^4+x^3}+\sqrt{x^4-x^3}} = \lim_{x\to-\infty} \frac{2}{\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{1}{x}}} = \frac{2}{2} = 1$$

$$b_2 = \lim_{x\to-\infty} (f(x) \;\; x) = \lim_{x\to-\infty} \Big(\overbrace{\frac{f(x)}{x}}^{\to 1}\cdot x \;\; x\Big) = x + x = 0$$

$y = x$

$$D_{f(x)}: \begin{cases} x^4+x^3 \geq 0 \\ x^4-x^3 \geq 0 \end{cases} \qquad \begin{cases} x = 0 \\ x \geq 1 \\ x \leq -1 \end{cases}$$

$$\boxed{\text{Ответ: } D_{f(x)}: (-\infty; -1] \cup \{0\} \cup [1; +\infty);\ \text{Асимптоты: } x = y}$$

б) $f(x) = |x+2|\, e^{-\frac{1}{x}}$

$x \neq 0$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0^-} |x+2|\, e^{-\frac{1}{x}} = 2\cdot e^{\infty} = \infty \Rightarrow x = 0 \text{ — вертикальная асимптота}$$

$$b_1 = \lim_{x\to+\infty}\left(|x+2|e^{-\frac{1}{x}} - x\right) = \lim_{x\to+\infty}\left(\frac{1}{e^{\frac{1}{x}}}\left(|x+2| - x\cdot e^{\frac{1}{x}}\right)\right) =$$

$$= \lim_{x\to+\infty}\frac{1}{e^{\frac{1}{x}}}\cdot\lim_{x\to+\infty}\left(x+2-xe^{\frac{1}{x}}\right) = 1\cdot\lim_{x\to+\infty}\left(2-\frac{e^{\frac{1}{x}}-1}{\frac{1}{x}}\right) =$$

$$= 1\cdot(2-1) = 1$$

$$y = k_1x + b_1 = x+1$$

$$\lim_{x\to-\infty}\left(\frac{|x+2|e^{-\frac{1}{x}}}{x}\right) = \lim_{x\to-\infty}\left(-\left(1+\frac{2}{x}\right)e^{-\frac{1}{x}}\right) = -(1+0)e^{0} = -1 = k_2$$

$$b_2 = \lim_{x\to-\infty}\left(|x+2|e^{-\frac{1}{x}} + x\right) = \lim_{x\to-\infty}\left(\frac{1}{e^{\frac{1}{x}}}\cdot\left(|x+2| + x\cdot e^{\frac{1}{x}}\right)\right) =$$

$$= \lim_{x\to-\infty}\frac{1}{e^{\frac{1}{x}}}\cdot\lim_{x\to-\infty}\left(-x-2+xe^{\frac{1}{x}}\right) = 1\cdot\lim_{x\to-\infty}\left(-2+\frac{e^{\frac{1}{x}}-1}{\frac{1}{x}}\right) = 1\cdot(-2+1) = -1$$

$$y = -x-1$$

Ответ: $D_{f(x)}: \mathbb{R}\setminus\{0\}$. Асимптоты: $y = x+1$
$y = -x-1$
$x = 0$.

#4.

$f(y) = 1 + 3y - y^2 + o(y^2)$ при $y \to 0$

$f(x + 4x^2) = 1 + 6x + 12x^2 - 4x^2 - 16x^3 - 16x^4 + \bar{o}((2x + 4x^2)^2) =$

$= 1 + 6x + 8x^2 - 16x^3 - 16x^4 + \bar{o}((2x + 4x^2)^2) =$

$= [\ 16x^3 = \bar{o}(x^2);\ 16x^4 = \bar{o}(x^2);\ \bar{o}(4x^2 + 16x^3 + 16x^4) = \bar{o}(x^2)\ ] =$
при $x \to 0$

$= 1 + 6x + 8x^2 + \bar{o}(x^2) + \bar{o}(x^2) + \bar{o}(x^2) = \boxed{1 + 6x + 8x^2 + \bar{o}(x^2)}$

$\lim\limits_{x \to 0} \frac{f(x) - 1}{x} = \lim\limits_{x \to 0} \frac{1 + 6x + 8x^2 + \bar{o}(x^2) - 1}{x} = \lim\limits_{x \to 0} \frac{6x + 8x^2 + \bar{o}(x) \cdot x}{x} =$

$= \lim\limits_{x \to 0} (6 + 8x + \bar{o}(x)) = 6 + 8 \cdot 0 + 0 = \boxed{6}$

#2.

(a) $f(x) = O(x^2)$ при $x \to 0$

Т.к. $|f(x)| \leq C \cdot |g(x)| = C \cdot |x^2|$, то $|f(x)| \leq C \cdot |x^2| \cdot |x| = C \cdot |x^3|$

$\Downarrow$

$\frac{|f(x)|}{|x^3|} \leq C$ при $x \to 0$

$\Downarrow$

$f(x) = O(x^3)$ при $x \to 0$

ч.т.д.

(b) $f(x) = o(x^3)$ при $x \to 0$

$\lim\limits_{x \to 0} \frac{f(x)}{x^3} = 0$ Пусть $f(x) = x^4$

$\lim\limits_{x \to 0} \frac{x^4}{x^3} = \lim\limits_{x \to 0} x = 0$ $\lim\limits_{x \to +\infty} \frac{x^4}{x^2} = \lim\limits_{x \to +\infty} x^2 = \infty$ W

$x^4 \neq O(x^3)$ при $x \to +\infty$

(c) $f(x) = O(x^3)$ при $x \to +\infty$

Пусть $f(x) = \frac{1}{x}$, тогда $\lim\limits_{x \to +\infty} \frac{f(x)}{x^3} = \lim\limits_{x \to +\infty} \frac{1}{x^4} = 0$

$\lim\limits_{x \to 0} \frac{f(x)}{x^3} = \lim\limits_{x \to 0} \frac{1}{x^4} = \infty \Rightarrow \frac{1}{x} \neq O(x^3)$ при $x \to 0$ (w)

Ответ: a) Верно
b) Неверно
c) Неверно.