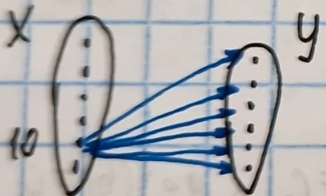
Homework - 4a

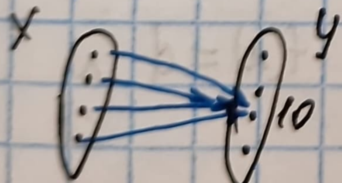
#1.

a) not functional, injective, not total, surjective



$$\boxed{P = \{(x, y) \mid x = 10,\ y \in \mathbb{R}\}}$$

b) functional, not injective, total, not surjective



$$\boxed{P = \{(x, y) \mid x \in \mathbb{R},\ y = 10\}}$$

#6.

a) $Q^{3} = \{(0, \frac{3}{7}), (1, 5), (2, \frac{4}{3})\}$

b) $\mathbb{R}^{\mathbb{Q}} = \{(x, y) \mid x \in \mathbb{Q},\ y = x + \pi\}$

c) $\mathbb{R}^{\mathbb{R} \times \mathbb{Z}} = \{((x, y), z) \mid (x, y) \in \mathbb{R} \times \mathbb{Z},\ z = x \cdot y\}$

#3.

$f : A \to B$ $\quad f = \{(x_i, y_i) \mid x_i \in A,\ y_i \in B,\ x_i \neq x_j \wedge i \neq j\}$

$g : A \to B$ $\quad g = \{(x_i, z_i) \mid x_i \in A,\ z_i \in B,\ x_i \neq x_j \wedge i \neq j\}$

$f \cup g : A \to B \overset{?}{\iff} f = g$

□ „$\Rightarrow$" if $f \cup g$, then $f \cup g = \{(x_i, y_i), (x_i, z_i)\}$

but $f \cup g$ is a function, so there is functionality:

$\forall i\ (x_i, y_i) = (x_i, z_i) \Rightarrow y_i = z_i \Rightarrow f = \{(x_i, y_i)\} = g$

$\Downarrow$

$f = g$

„$\Leftarrow$" $f = g \Rightarrow f = \{(x_i, y_i) \mid x_i \in A, y_i \in B, x_i \neq x_j \wedge i \neq j\} = g \Rightarrow$

$\Rightarrow f \cup g = \{(x_i, y_i) \mid x_i \in A, y_i \in B, x_i \neq x_j \wedge i \neq j\} \Rightarrow f \cup g : A \to B$ ■

#4.

$f : A \to B$

$g : B \to C$

$g \circ f\text{-inj} \overset{?}{\Rightarrow} f\text{-inj}$

□ if $g \circ f$ is injection, then $\forall a, b \in A\ \ g \circ f(a) = c \wedge g \circ f(b) = c$

$\Rightarrow a = b$

Пп: f isn't injection $\Rightarrow \exists a, b \in A : a \neq b \wedge f(a) = d \wedge$

$\wedge f(b) = d$.

But $g \circ f(a) = g(d) = c \wedge g \circ f(b) = g(d) = c$

$\Rightarrow g \circ f(a) = g \circ f(b) \Rightarrow a = b$ (w) $\blacksquare$

#5

$f: A \to B$ - inj $\overset{?}{\Leftrightarrow} (\forall C, \forall g, h: C \to A: f \circ g \overset{(*)}{=} f \circ h \Rightarrow g = h)$

$\square$ "$\Rightarrow$" $f: A \to B$ inj $\Leftrightarrow \forall a, b \in A \quad f(a) \neq f(b) \wedge a \neq b$

fix C and $g, h: C \to A$, then

$$\left.\begin{array}{l} f \circ g = \{(a,b) \in C \times B \mid \exists c \quad g(a) = c \wedge f(c) = b\} \\ f \circ h = \{(a,b) \in C \times B \mid \exists c \quad h(a) = c \wedge f(c) = b\} \end{array}\right\} \overset{(*)}{\Rightarrow}$$

$\Rightarrow \forall x \in C \quad f \circ g(x) = f \circ h(x)$

$\sqcap\!\sqcap$: $g \neq h \Rightarrow \begin{cases} f \circ g(x) = f \circ h(x) \\ g(x) \neq h(x) \end{cases} \Rightarrow \exists \alpha, \beta \in A : \alpha \neq \beta \wedge$

$\wedge \alpha = g(x) \wedge \beta = h(x)$ But $f(\alpha) = f(\beta) \wedge \alpha \neq \beta$ (w)

"$\Leftarrow$" $(\forall C, \forall g, h: C \to A \quad f \circ g = f \circ h \Rightarrow g = h) \Leftrightarrow$

$\Leftrightarrow (\forall x \in C, \forall g, h \quad f \circ g(x) = f \circ h(x) \Rightarrow g(x) = h(x))$

$\sqcap\!\sqcap$: $f: A \to B$ isn't injective $\Rightarrow \exists \alpha, \beta \in A : f(\alpha) = f(\beta), \alpha \neq \beta$

Let $\alpha = g(x)$ and $\beta = h(x)$, then $f \circ g(x) = f \circ h(x)$

$g(x) = h(x) \qquad f(\alpha) = f(\beta)$

$\alpha = \beta$ (w) $\alpha \neq \beta$

#2.

$R \subseteq A \times B$ $(*)$ – functional $\iff \forall_{x,y,z}\ ((xRy \wedge xRz) \Rightarrow y = z)$

a) □ $\forall X \quad R[R^{-1}[X]] \overset{?}{\subseteq} X$

$R^{-1}[X] = \{x \mid z \in X \wedge zR^{-1}x\}$

$R[R^{-1}[X]] = \{y \mid x \in R^{-1}[X] \wedge xRy\}$

$\Downarrow$

$R[R^{-1}[X]] = \{y \mid z \in X \wedge xRy \wedge zR^{-1}x\} = \{y \mid z \in X \wedge xRy \wedge xRz\}$

$(*) \Rightarrow ((xRy \wedge xRz) \Rightarrow y = z) \wedge z \in X \Rightarrow y \in X$

$\Downarrow$

$R[R^{-1}[X]] \subseteq X$ ■

b) $X \overset{?}{\subseteq} R[R^{-1}[X]]$

□ Counterexample: $A = \{0, 1\}$ $B = \{2, 3\}$

$R = \{(0,2), (1,3)\}$ $R^{-1} = \{(2,0), (3,1)\}$

Let $X = \{8\}$, then $R^{-1}[X] = R^{-1}[\{8\}] = \emptyset$ and $R[\emptyset] = \emptyset$

But $\{8\} \not\subseteq \emptyset$

$\boxed{\text{Answer: no.}}$ ■