Lecture 14

Dual-Spin Stabilization

STABILITY of a rigid platform equipped with a spinning wheel is considered. A brief geometric interpretation in terms of the energy and momentum ellipsoids is provided. The equilibrium cases, including one with a non-spinning carrier, are considered, and linear stability analysis is performed to draw conclusions about the system's stability conditions.

Overview

A "dual-spin" system (also known as a "gyrostat"), shown in Figure 14.1, consists of:

- A *platform* (or *carrier*) with a body-fixed frame, $\mathscr{F}_P$, and absolute angular velocity $\underset{\rightarrow}{\omega}$ with respect to an inertial frame, $\mathscr{F}_I$.

- A *wheel* (or *rotor*), assumed to be inertially axisymmetrical and spinning about its axis of symmetry that is fixed in $\mathscr{F}_P$.

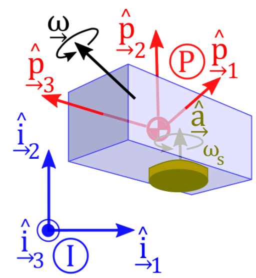

Figure 14.1: Gyrostat

We let $\mathscr{F}_P$ be the principal axes frame (with its origin on the composite centre of mass) of the platform-wheel combination. Resolving the vectors in this frame (such as $\underset{\rightarrow}{h} = \mathscr{F}_P^{\mathsf{T}}\boldsymbol{h}$) and noting the contribution of both the carrier and the wheel to the total angular momentum, $\boldsymbol{h}$, we have:

$$\boldsymbol{h} = \boldsymbol{I}\boldsymbol{\omega} + I_s\omega_s\hat{\boldsymbol{a}} \quad \Rightarrow \quad \dot{\boldsymbol{h}} = \boldsymbol{I}\dot{\boldsymbol{\omega}} + I_s\dot{\omega}_s\hat{\boldsymbol{a}} \tag{14.1}$$

where $\hat{\boldsymbol{a}}$ represents the wheel's axis of symmetry (expressed in $\mathscr{F}_P$) and I_s and ω_s are its moment of inertia and spin rate about this axis. The total moment of inertia matrix $\boldsymbol{I}$ is that of the composite system, consisting of both the platform and the rotor contributions. Introducing $\boldsymbol{h}_s \triangleq I_s\omega_s\hat{\boldsymbol{a}}$ and $\dot{\boldsymbol{h}}_s \triangleq I_s\dot{\omega}_s\hat{\boldsymbol{a}}$ as the wheel's angular momentum and its rate of change, using Euler's equations from DYNAMICS we have:

$$\dot{\boldsymbol{h}} + \boldsymbol{\omega}^\times\boldsymbol{h} = \boldsymbol{\tau} \quad \Rightarrow \quad \boldsymbol{I}\dot{\boldsymbol{\omega}} + \dot{\boldsymbol{h}}_s + \boldsymbol{\omega}^\times(\boldsymbol{I}\boldsymbol{\omega} + \boldsymbol{h}_s) = \boldsymbol{\tau} \tag{14.2}$$

where $\boldsymbol{\tau}$ consists of the $\mathscr{F}_P$ components of the external torque vector.

Stability of Torque-Free Pure Spin

Consider, similarly to SPIN STABILIZATION, Euler's equations in the absence of external torques with a tri-inertial platform, with distinct I_1, I_2, and I_3 as its principal moments of inertia. Assume, for simplicity, the wheel's symmetry axis about which it spins is aligned with the 2-axis of the platform-wheel system, *i. e.* $\hat{\boldsymbol{a}} = [0\ 1\ 0]^\mathsf{T}$. Furthermore, assume $\dot{h}_s = 0$, as required by the special case of a so-called *Kelvin's gyrostat* in which the wheel's spin rate is constant, $\dot{\omega}_s = 0$. With these assumptions, Eq. (14.2) becomes:

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 - h_s\omega_3 = \cancelto{0}{\tau_1} \tag{14.3a}$$

$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 = \cancelto{0}{\tau_2} \tag{14.3b}$$

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 + h_s\omega_1 = \cancelto{0}{\tau_3} \tag{14.3c}$$

where ω_1, ω_2, and ω_3 are the components of $\boldsymbol{\omega}$, the platform's angular velocity in the body-fixed frame, $\mathscr{F}_P$. We study, as in TORQUE-FREE MOTION, the constant of motion before proceeding with its stability analysis:

$$T_p = \frac{1}{2}\boldsymbol{\omega}^\mathsf{T}\boldsymbol{I}\boldsymbol{\omega} \;\Rightarrow\; \dot{T}_P = \boldsymbol{\omega}^\mathsf{T}\boldsymbol{I}\overset{\bullet}{\boldsymbol{\omega}} = \boldsymbol{\omega}^\mathsf{T}\left[\cancelto{\mathbf{0}}{\boldsymbol{\tau}} - \cancelto{\mathbf{0}}{\overset{\bullet}{\boldsymbol{h}}_s} - \boldsymbol{\omega}^\times(\boldsymbol{I}\boldsymbol{\omega} + \boldsymbol{h}_s)\right] = (\boldsymbol{I}\boldsymbol{\omega} + \boldsymbol{h}_s)^\mathsf{T}\cancelto{\mathbf{0}}{\boldsymbol{\omega}^\times\boldsymbol{\omega}} \;\Rightarrow\; \dot{T}_p = 0 \tag{14.4a}$$

$$h^2 = \boldsymbol{h}^\mathsf{T}\boldsymbol{h} \;\Rightarrow\; \cancel{2}h\dot{h} = \cancel{2}\boldsymbol{h}^\mathsf{T}\overset{\bullet}{\boldsymbol{h}} \;\Rightarrow\; h\dot{h} = \boldsymbol{h}^\mathsf{T}(\cancelto{\mathbf{0}}{\boldsymbol{\tau}} - \boldsymbol{\omega}^\times\boldsymbol{h}) = -\boldsymbol{h}^\mathsf{T}\boldsymbol{\omega}^\times\boldsymbol{h} = -\boldsymbol{\omega}^\mathsf{T}\cancelto{\mathbf{0}}{\boldsymbol{h}^\times\boldsymbol{h}} \;\Rightarrow\; \dot{h} = 0 \tag{14.4b}$$

where anti-symmetry of the cross operator and the scalar triple product identity are used for the simplifications. It must be noted that T_p as defined above *only* accounts for the rotational kinetic energy of the platform, and not that of the wheel. The total rotational kinetic energy, including the wheel's rotational energy about its own axis and its contribution to the platform's angular velocity, is given by:

$$T = T_p + \frac{1}{2}I_s\omega_s^2 + \boldsymbol{h}_s^\mathsf{T}\boldsymbol{\omega} \;\Rightarrow\; \dot{T} = \cancelto{0}{\dot{T}_p} + I_s\omega_s\cancelto{0}{\dot{\omega}_s} + \boldsymbol{h}_s^\mathsf{T}\overset{\bullet}{\boldsymbol{\omega}} = I_s\omega_s\hat{\boldsymbol{a}}^\mathsf{T}\overset{\bullet}{\boldsymbol{\omega}} = \tau_a\omega_s \tag{14.5}$$

where τ_a is the torque applied by the platform on the wheel about its symmetry axis to keep it spinning.

We conclude that T_p and h^2 are constants of motion, and analogously to rigid body rotation, we can construct angular momentum and energy ellipsoids, $\mathscr{H}$ and $\mathscr{T}_p$; however, in this dual-spin case, setting $\boldsymbol{h} = \mathbf{0}$ results in $\boldsymbol{I}\boldsymbol{\omega} + I_s\omega_s\hat{\boldsymbol{a}} = \mathbf{0}$, so $\mathscr{H}$ will now be centred at $\boldsymbol{\omega}_0 = -h_s\boldsymbol{I}^{-1}\hat{\boldsymbol{a}}$, as shown in Figure 14.2, instead of at the origin of $\mathscr{F}_P$, namely O_P. For an equilibrium, in order to satisfy the $\overset{\bullet}{\boldsymbol{\omega}} = \mathbf{0}$ condition, the size of $\mathscr{T}_p$ and $\mathscr{H}$ can be varied, but in addition, the location of the centre of $\mathscr{H}$ may also be manipulated by varying the magnitude of the wheel's angular momentum, namely h_s.

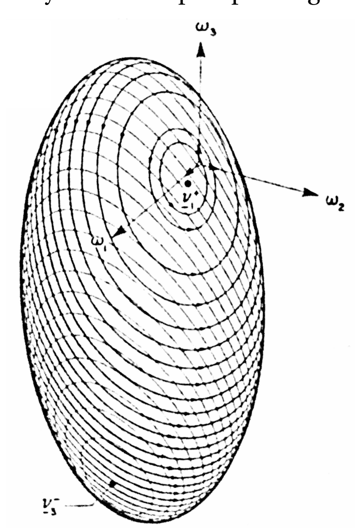

Figure 14.2: $\mathscr{H}$ with off-set centre [Hughes] (used with permission)

Note: For dual-spins $\mathscr{H}$ is more fundamental than $\mathscr{T}$. Unlike a single body's Poinsot construction in which $\boldsymbol{h}^\mathsf{T}\boldsymbol{\omega} = 2T$ was constant and the tip of $\boldsymbol{\omega}$ remained on a polhode on $\mathscr{T}$, now $\boldsymbol{h}^\mathsf{T}\boldsymbol{\omega} = 2T_p + \boldsymbol{h}_s^\mathsf{T}\boldsymbol{\omega}$, which is no longer constant. Also, when treating energy dissipation later, we will hold h^2 constant and slowly reduce T_p.

Evoking the constant h_s assumption in Eq. (14.2), it is evident that an equilibrium with $\overset{\bullet}{\boldsymbol{\omega}}_0 = \mathbf{0}$ satisfies:

$$\boldsymbol{\omega}_0^{\times}(\boldsymbol{I}\boldsymbol{\omega}_0 + \boldsymbol{h}_s) = \mathbf{0} \tag{14.6}$$

which can be viewed as two cases of zero and non-zero values for the platform's angular velocity.

Nominally Non-Spinning Platform

Since $\boldsymbol{\omega}_0 \equiv \mathbf{0}$, perturbations $\delta\boldsymbol{\omega}$ will be the only contributions to $\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \delta\boldsymbol{\omega}$. From Eq. (14.3), we have:

$$I_1\delta\dot{\omega}_1 + (I_3 - I_2)\cancelto{\approx 0}{\delta\omega_2\delta\omega_3} - h_s\delta\omega_3 = 0 \quad\Rightarrow\quad \delta\dot{\omega}_1 = \frac{+h_s}{I_1}\delta\omega_3 \tag{14.7a}$$

$$I_2\delta\dot{\omega}_2 + (I_1 - I_3)\cancelto{\approx 0}{\delta\omega_3\delta\omega_1} = 0 \quad\Rightarrow\quad \delta\dot{\omega}_2 = 0 \tag{14.7b}$$

$$I_3\delta\dot{\omega}_3 + (I_2 - I_1)\cancelto{\approx 0}{\delta\omega_1\delta\omega_2} + h_s\delta\omega_1 = 0 \quad\Rightarrow\quad \delta\dot{\omega}_3 = \frac{-h_s}{I_3}\delta\omega_1 \tag{14.7c}$$

which lead to the same conclusions as those obtained in the linear stability analysis of SPIN STABILIZATION: from Eq. (14.7b), $\delta\omega_2$ is constant, and differentiating Eqs. (14.7a) and (14.7c) with respect to time yields:

$$\delta\ddot{\omega}_1 = \frac{+h_s}{I_1}\delta\dot{\omega}_3 = \frac{-h_s^2}{I_1 I_3}\delta\omega_1 \quad\Rightarrow\quad \delta\ddot{\omega}_1 = \beta^2\delta\omega_1 \tag{14.8a}$$

$$\delta\ddot{\omega}_3 = \frac{-h_s}{I_1}\delta\dot{\omega}_1 = \frac{-h_s^2}{I_1 I_3}\delta\omega_3 \quad\Rightarrow\quad \delta\ddot{\omega}_3 = \beta^2\delta\omega_3 \tag{14.8b}$$

where $\beta^2 \triangleq -h_s^2/(I_1 I_3)$. Since I_1, I_3, and h_s^2 are all positive, we have $\beta^2 < 0$, implying that $e^{\pm\beta t} = \cos(\mp i\beta t) + i\sin(\mp i\beta t)$ remains bounded as $t \to \infty$. Thus, a gyrostat with a non-spinning platform, which is frequently used for spacecraft stabilization purposes, is *directionally stable*, although not asymptotically so because of the periodic terms. We define *precessional frequency* as:

$$\Omega_{p_0}^2 \triangleq -\beta^2 \quad\Rightarrow\quad \Omega_{p_0} = \frac{h_s}{\sqrt{I_1 I_3}}$$

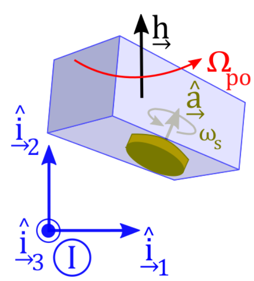

Figure 14.3: Precession

As illustrated in Figure 14.3, the motion of the dual-spin can then be geometrically described by precession of the wheel's spin axis, along $\underset{\rightarrow}{\hat{\boldsymbol{a}}}$, about the body's inertially-fixed angular momentum vector, $\underset{\rightarrow}{\boldsymbol{h}}$, on the surface of an elliptic cone. The frequency of precession is Ω_{p_0}.

Linear Mechanical Systems Perspective

Similarly to SPIN STABILIZATION, we have $\boldsymbol{\omega} \approx \delta\overset{\bullet}{\boldsymbol{\theta}} + \left(\mathbf{1} - \delta\boldsymbol{\theta}^{\times}\right)\boldsymbol{\omega}_0$, but for this special case of a non-spinning platform, $\boldsymbol{\omega}_0 \equiv \mathbf{0}$. The perturbed angular velocity reduces to $\boldsymbol{\omega} \approx \delta\overset{\bullet}{\boldsymbol{\theta}}$, substituting which into Eq. (14.3) yields:

$$I_1\delta\ddot{\theta}_1 + (I_3 - I_2)\cancelto{\approx 0}{\delta\dot{\theta}_2\delta\dot{\theta}_3} - h_s\delta\dot{\theta}_3 = 0 \quad\Rightarrow\quad I_1\delta\ddot{\theta}_1 - h_s\delta\dot{\theta}_3 = 0 \tag{14.9a}$$

$$I_2\delta\ddot{\theta}_2 + (I_1 - I_3)\cancelto{\approx 0}{\delta\dot{\theta}_3\delta\dot{\theta}_1} = 0 \quad\Rightarrow\quad I_2\delta\ddot{\theta}_2 = 0 \quad\Rightarrow \delta\dot{\theta}_2(t) = \delta\dot{\theta}_2(0) \tag{14.9b}$$

$$I_3\delta\ddot{\theta}_3 + (I_2 - I_1)\cancelto{\approx 0}{\delta\dot{\theta}_1\delta\dot{\theta}_2} + h_s\delta\dot{\theta}_1 = 0 \quad\Rightarrow\quad I_3\delta\ddot{\theta}_3 + h_s\delta\dot{\theta}_1 = 0 \tag{14.9c}$$

where Eq. (14.9b) implies an *attitude unstable* system with unbounded growth in the 2-component. Following a similar approach to SPIN STABILIZATION and focusing on the remaining two components (that are decoupled from θ_2 and its rates), Eqs. (14.9a) and (14.9c) can be written in the form of a gyric system:

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{G}\dot{\boldsymbol{q}} = \boldsymbol{0} \tag{14.10}$$

$$\boldsymbol{q} \triangleq \begin{bmatrix}\delta\theta_1 \\ \delta\theta_3\end{bmatrix} \quad , \quad \boldsymbol{M} \triangleq \begin{bmatrix} I_1 & 0 \\ 0 & I_3\end{bmatrix} \quad , \quad \boldsymbol{G} \triangleq \begin{bmatrix} 0 & -1 \\ 1 & 0\end{bmatrix} h_s \quad ,$$

where $\boldsymbol{M} = \boldsymbol{M}^{\mathsf{T}} > 0$ and $\boldsymbol{G} = -\boldsymbol{G}^{\mathsf{T}}$ as required. Although the complete system (with all 3 $\delta\theta_i$'s) would have had $det(\boldsymbol{G}) = 0$, confirming the system's *attitude instability* as mentioned in STABILITY, the "reduced" system in Eq. (14.10) has $\det(\boldsymbol{G}) = \mathrm{h}_\mathrm{s}^2 \neq 0$ for any non-zero wheel spin, suggesting *directional stability*. Further insight is gained by examining the characteristic equation, $\det\big(\boldsymbol{M}\mathrm{r}^2 + \boldsymbol{G}\mathrm{r}\big) = 0$:

$$b_0 r^4 + b_1 r^2 = r^2(b_0 r^2 + b_1) = 0 \tag{14.11}$$

$$b_0 \triangleq I_1 I_3 \quad , \quad b_1 \triangleq h_s^2$$

To have r^2 on the negative real axis, stability is achieved if and only if $b_0 > 0$ and $b_1 > 0$, both of which are satisfied. In this case, the characteristic equation can be easily solved, with the non-trivial solution, $r^2 = -h_s^2/(I_1 I_3) = -\Omega_{p_0}^2$, providing the precession previously seen.

Wheel Aligned with System Principal Axis

Once again, we assume the wheel's spin axis is parallel to one of the principal axes of the platform-wheel combination, such as its 2-axis. With $\hat{\boldsymbol{a}} = [0\ 1\ 0]^{\mathsf{T}}$, the equilibrium condition in Eq. (14.6) expands as:

$$I_1\omega_{0_1} = \lambda\omega_{0_1} \tag{14.12a}$$

$$I_2\omega_{0_2} + h_s = \lambda\omega_{0_2} \tag{14.12b}$$

$$I_3\omega_{0_3} = \lambda\omega_{0_3} \tag{14.12c}$$

where λ is some scalar. These equations imply $\omega_{0_1} = \omega_{0_3} = 0$ and $\omega_{0_2} = \nu$ for some constant ν, and $\lambda = I_2 + h_s/\nu$. This correspond to a "pure spin" case of the gyrostat. We now consider a perturbation away from the equilibrium:

$$\boldsymbol{\omega}_0 = \begin{bmatrix} 0 \\ \nu \\ 0 \end{bmatrix} \quad , \quad \boldsymbol{\omega} = \boldsymbol{\omega}_0 + \delta\boldsymbol{\omega} = \begin{bmatrix} \delta\omega_1 \\ \nu + \delta\omega_2 \\ \delta\omega_3 \end{bmatrix} \tag{14.13}$$

where $\delta\omega_i$'s are small perturbations. With the perturbed $\boldsymbol{\omega}$, the equations of motion in Eq. (14.3) become:

$$I_1\delta\dot{\omega}_1 + (I_3 - I_2)(\nu\delta\omega_3 + \overset{\approx 0}{\cancel{\delta\omega_2\delta\omega_3}}) - h_s\delta\omega_3 = 0 \quad\Rightarrow\quad 0 = I_1\delta\dot{\omega}_1 + \big[(I_3 - I_2)\nu - h_s\big]\delta\omega_3 \tag{14.14a}$$

$$I_2\delta\dot{\omega}_2 + (I_1 - I_3)(\overset{\approx 0}{\cancel{\delta\omega_3\delta\omega_1}}) = 0 \quad\Rightarrow\quad 0 = I_2\delta\dot{\omega}_2 \tag{14.14b}$$

$$I_3\delta\dot{\omega}_3 + (I_2 - I_1)(\nu\delta\omega_1 + \overset{\approx 0}{\cancel{\delta\omega_1\delta\omega_2}}) + h_s\delta\omega_1 = 0 \quad\Rightarrow\quad 0 = I_3\delta\dot{\omega}_3 + \big[(I_2 - I_1)\nu + h_s\big]\delta\omega_1 \tag{14.14c}$$

where $\delta\omega_2 = \delta\omega_2(0)$ is observed to be constant, and using $\lambda = I_2 + h_s/\nu$ as introduced in Eq. (14.12), we have the following for $\delta\omega_1(t)$ and $\delta\omega_3(t)$:

$$I_1\delta\dot{\omega}_1 + (I_3 - \lambda)\nu\delta\omega_3 = 0 \tag{14.15a}$$

$$I_3\delta\dot{\omega}_3 + (\lambda - I_1)\nu\delta\omega_1 = 0 \tag{14.15b}$$

which are the same as the relationships obtained in SPIN STABILIZATION, but with λ taking the role that I_2 played in the single-spin case. Using the same arguments as previously made, we conclude that $\boldsymbol{\omega}$-*stability* is achieved in one of two cases:

- $I_1 > \lambda$ and $I_3 > \lambda$: the value of $I_2 + h_s/\nu$ made sufficiently small
- $I_1 < \lambda$ and $I_3 < \lambda$: the value of $I_2 + h_s/\nu$ made sufficiently large

Simple spin motion of a torque-free gyrostat with $I_1 < \lambda < I_2$ is, however, *unstable*.

The important conclusion is that the dual-spin system's stability may be manipulated by adjusting λ via $h_s = I_s\omega_s$. Analogously to the inertia ratios defined for a single body, namely $k_1 \triangleq (I_2 - I_3)/I_1$ and $k_3 \triangleq (I_2 - I_1)/I_3$, we define the following parameters, with the same significance, for the dual-spin system:

$$k_{1h} \triangleq \frac{\lambda - I_3}{I_1} = \frac{I_2 - I_3}{I_1} + \frac{h_s/I_1}{\nu} = k_1 + \hat{\Omega}_{p0}\sqrt{\frac{1-k_1}{1-k_3}}$$

$$k_{3h} \triangleq \frac{\lambda - I_1}{I_3} = \frac{I_2 - I_1}{I_3} + \frac{h_s/I_3}{\nu} = k_3 + \hat{\Omega}_{p0}\sqrt{\frac{1-k_3}{1-k_1}}$$

where $\hat{\Omega}_{p_0} \triangleq h_s/(\nu\sqrt{I_1 I_3})$ is a non-dimensional form of Ω_{p_0} seen before. For a given value of $\hat{\Omega}_{p_0}$, the above definitions can be used to generate $k_1 - k_3$ stability diagrams. One such example using $\hat{\Omega}_{p_0} = 0.5$ is shown in Figure 14.4. Since $I_1 I_3 > 0$, the necessary and sufficient stability condition is $k_{1h}k_{3h} > 0$, which can be categorized as:

- $k_{1h} > 0, k_{3h} > 0$: *static stability*, increased by having $\hat{\Omega}_{p_o} > 0$ when the platform and the wheel spin in the same direction
- $k_{1h} < 0, k_{3h} < 0$: *gyric stability*, decreased by having $\hat{\Omega}_{p_o} < 0$ as a result of the platform and the wheel spinning in opposite directions

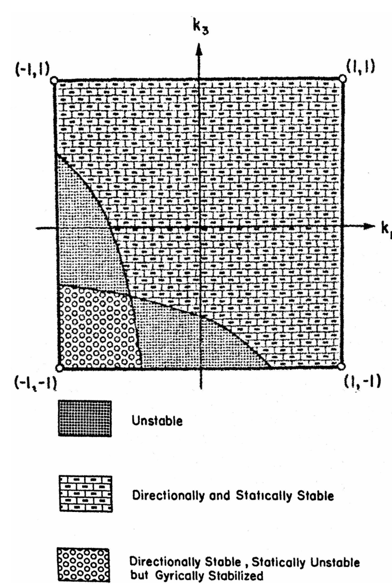

Figure 14.4: Dual-Spin $\hat{\Omega}_{po} = 0.5$ [Hughes] (used with permission)

All of the results of this lesson have been obtained disregarding any internal energy dissipation. Accounting for these effects is the subject of next lecture.

Linear Mechanical Systems Perspective

Once again, we have $\boldsymbol{\omega} \approx \delta\dot{\boldsymbol{\theta}} + \big(\mathbf{1} - \delta\boldsymbol{\theta}^{\times}\big)\boldsymbol{\omega}_0$, but now with $\boldsymbol{\omega}_0 \triangleq [0 \ \nu \ 0]^{\mathsf{T}}$. The perturbed angular velocity is provided, similarly to SPIN STABILIZATION, by:

$$\boldsymbol{\omega} \approx \delta\dot{\boldsymbol{\theta}} + \big(\mathbf{1} - \delta\boldsymbol{\theta}^{\times}\big)\boldsymbol{\omega}_0 = \begin{bmatrix} \delta\dot{\theta}_1 + \nu\delta\theta_3 \\ \delta\dot{\theta}_2 + \nu \\ \delta\dot{\theta}_3 - \nu\delta\theta_1 \end{bmatrix} \tag{14.16}$$

substituting which into Eq. (14.3) yields:

$$I_1(\delta\ddot{\theta}_1 + \nu\delta\dot{\theta}_3) + (I_3 - I_2)(\cancelto{\approx 0}{\delta\dot{\theta}_2\delta\dot{\theta}_3} + \nu\delta\dot{\theta}_3 - \cancelto{\approx 0}{\nu\delta\dot{\theta}_2\delta\theta_1} - \nu^2\delta\theta_1) - h_s(\delta\dot{\theta}_3 - \nu\delta\theta_1) = 0 \tag{14.17a}$$

$$I_2\delta\ddot{\theta}_2 + (I_1 - I_3)(\cancelto{\approx 0}{\delta\dot{\theta}_1\delta\dot{\theta}_3} + \cancelto{\approx 0}{\nu\delta\theta_3\delta\dot{\theta}_3} - \cancelto{\approx 0}{\nu\delta\dot{\theta}_1\delta\theta_1} - \cancelto{\approx 0}{\nu^2\delta\theta_1\delta\theta_3}) = 0 \tag{14.17b}$$

$$I_3(\delta\ddot{\theta}_3 - \nu\delta\dot{\theta}_1) + (I_2 - I_1)(\cancelto{\approx 0}{\delta\dot{\theta}_1\delta\dot{\theta}_2} + \cancelto{\approx 0}{\nu\delta\theta_3\delta\dot{\theta}_2} + \nu\delta\dot{\theta}_1 + \nu^2\delta\theta_3) + h_s(\delta\dot{\theta}_1 + \nu\delta\theta_3) = 0 \tag{14.17c}$$

which, upon factoring the like terms, can be rewritten as:

$$I_1\delta\ddot{\theta}_1 + \Big[I_1 + I_3 - \cancelto{\lambda}{\big(I_2 + \frac{h_s}{\nu}\big)}\Big]\nu\delta\dot{\theta}_3 + \Big[\cancelto{\lambda}{\big(I_2 + \frac{h_s}{\nu}\big)} - I_3\Big]\nu^2\delta\theta_1 = 0 \tag{14.18a}$$

$$I_2\delta\ddot{\theta}_2 = 0 \quad \Rightarrow \quad \delta\dot{\theta}_2(t) = \delta\dot{\theta}_2(0) \tag{14.18b}$$

$$I_3\delta\ddot{\theta}_3 - \Big[I_1 + I_3 - \cancelto{\lambda}{\big(I_2 + \frac{h_s}{\nu}\big)}\Big]\nu\delta\dot{\theta}_1 + \Big[\cancelto{\lambda}{\big(I_2 + \frac{h_s}{\nu}\big)} - I_1\Big]\nu^2\delta\theta_3 = 0 \tag{14.18c}$$

where Eq. (14.18b), similarly to Eq. (14.9b), implies an *attitude unstable* system. Setting up Eqs. (14.18a) and (14.18c) in linear mechanical system form as before, we have:

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{G}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \mathbf{0} \tag{14.19}$$

$$\boldsymbol{q} \triangleq \begin{bmatrix} \delta\theta_1 \\ \delta\theta_3 \end{bmatrix} \ , \ \boldsymbol{M} \triangleq \begin{bmatrix} I_1 & \mathbf{0} \\ 0 & I_3 \end{bmatrix} \ , \ \boldsymbol{G} \triangleq \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}(I_1 + I_3 - \lambda)\nu \ , \ \boldsymbol{K} \triangleq \begin{bmatrix} \lambda - I_3 & 0 \\ 0 & \lambda - I_1 \end{bmatrix}\nu^2$$

where $\boldsymbol{M} = \boldsymbol{M}^{\mathsf{T}} > 0$, $\boldsymbol{G} = -\boldsymbol{G}^{\mathsf{T}}$, and $\boldsymbol{K} = \boldsymbol{K}^{\mathsf{T}}$. We thus, once again, reach the same results as the single spin case, but with λ taking on the role of I_2. The system in Eq. (14.19):

- is *statically stable* if $\boldsymbol{K} > 0$, satisfied if and only if $\lambda > I_1$, $\lambda > I_3$.
- may be *gyrically stable* even if $\boldsymbol{K} \not> 0$, satisfied if and only if $\lambda < I_1$, $\lambda < I_3$.

Note: Both of the above cases conform to the aforementioned $k_{1h}k_{3h} > 0$ condition, with $k_{1h} \triangleq (\lambda - I_3)/I_1$ and $k_{3h} \triangleq (\lambda - I_1)/I_3$, that was introduced as a necessary and sufficient condition for stability.

References

[Hughes] Hughes, P. C., *Spacecraft Attitude Dynamics*, Dover Publications Inc., New York, Chap. 6.