CS 161
Autumn 2017

Pre-Lecture Exercise for Lecture 3
Date: October 2, 2017

Pre-lecture exercises will not be collected for credit. However, you will get more out of each lecture if you do them, and they will be referenced during lecture. We recommend **writing out** your answers to pre-lecture exercises before class. Pre-lecture exercises usually should not take you more than 20 minutes.

In this pre-lecture exercise, you'll explore three *recurrence relations.* For each of the following expressions, try to figure out a closed-form expression for $T(n)$, **when n is a power of 2.** (Don't worry about when n isn't a power of 2 for now).

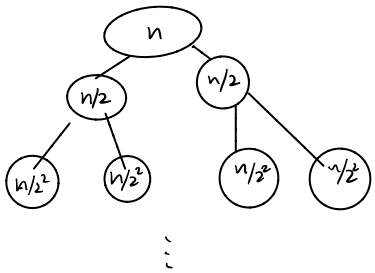
If you are feeling stuck, **we've done the first one for you in two different ways on the next page** to give you some inspiration for how to attack the second and the third.

1. $T(n) = \begin{cases} 2 \cdot T(n/2) + n & n = 2^i, i > 0 \\ T(n) = 1 & n = 1 \end{cases}$.

2. $T(n) = \begin{cases} T(n/2) + n & n = 2^i, i > 0 \\ T(n) = 1 & n = 1 \end{cases}$.

3. $T(n) = \begin{cases} 4 \cdot T(n/2) + n & n = 2^i, i > 0 \\ T(n) = 1 & n = 1 \end{cases}$.

(1) $T(n) = 2T(n/2) + n$
$= 2^2 T(n/2^2) + 2^1 \cdot n/2 + n$
$= 2^3 T(n/2^3) + 2^2 \, n/2^2 + 2^1 \cdot n/2 + n$
$= 2^3 T(n/2^3) + 3n$
$= 2^k T(n/2^k) + kn$ $\quad n/2^k = 1 \Rightarrow k = \log n$
$= 2^{\log_2 n} T(1) + n \cdot \log n$
$= n(\log n + 1) = O(n \log n)$



0 level → n
1 level → n
2 level → n
⋮ ⋮
① ··· ① ① ① ① ··· ① ··· ① logn level → n

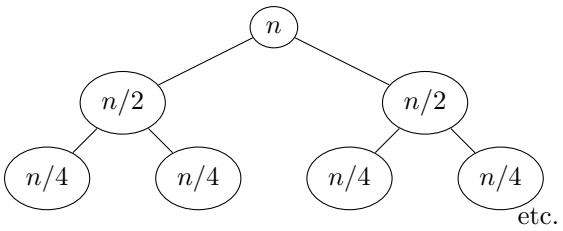
$\Rightarrow$ $(\log n + 1)\, n$ → each level operation
levels

(2) $T(n) = T(n/2) + n$
$= T(n/2^2) + n/2 + n$
$= T(n/2^3) + n/2^2 + n/2 + n$
$= T(n/2^k) + n/2^{k-1} + n/2^{k-2} + \cdots + n/2 + n$
$= T(n/2^k) + \left(2 - \frac{1}{2^{k-1}}\right) n$
$= T(1) + \left(2 - \frac{2}{2^{\log_2 n}}\right) n = 1 + 2n - 2 = 2n - 1 = O(n)$

(3) $T(n) = 4T(n/2) + n$
$= 4^2 T(n/2^2) + 4 \cdot n/2 + n$
$= 4^3 T(n/2^3) + 4^2 \cdot n/2^2 + 4 \cdot n/2 + n$
$= 4^3 T(n/2^3) + 2^2 n + 2n + 2^0 n$
$= 4^k T(n/2^k) + 2^{k-1} n + 2^{k-2} n + \cdots + 2n + 2^0 n$
$= 4^k T(n/2^k) + (2^k - 1) n$
$= 4^{\log_2 n} + (2^{\log_2 n} - 1) n$
$= n^2 - n + n^2 = 2n^2 - n = O(n^2)$

SPOILER ALERT: Here are two solutions to Exercise 1, which you can look at to help you figure out how to do Exercises 2 and 3.

SOLUTION 1. We do as we did with MERGESORT, and imagine a tree with $\log(n) + 1$ levels. The top node is labeled "n", its two children are labeled "$n/2$", and so on.



Consider $T(n) = T(n/2) + T(n/2) + n$. In the context of the tree above, that means that $T(n) = n +$ (stuff contributed by things in the tree lower than the root). That is,

$$T(n) = \text{(label on the root)} + \text{(stuff contributed by things lower than the root)}.$$

We can repeat this logic recursively to figure out what that second term is, all the way down to the bottom of the tree, where we have $T(1) = 1$. We conclude that $T(n)$ is equal to the sum, over all the nodes, of the labels on the nodes[1].

If the root is level 0, then at level $j \leq \log(n)$, there are 2^j nodes, each which have label $n/2^j$. So

$$\sum_{j=0}^{\log(n)} 2^j \cdot \frac{n}{2^j} = n(\log(n) + 1).$$

is our answer.

SOLUTION 2. We can do the exact same calculation without the tree, by repeatedly applying our formula.

$$\begin{aligned} T(n) &= 2T(n/2) + n \\ &= 2\left(2T(n/4) + n/2\right) + n \\ &= 4T(n/4) + 2n \\ &= 4(2T(n/8) + n/4) + 2n \\ &= 8T(n/8) + 3n \end{aligned}$$

and at this point we can spot the pattern: for all $j \leq \log(n)$,

$$T(n) = 2^j T(n/2^j) + jn.$$

In order to formally prove that this is true, we should use a proof by induction; that's called the *substitution method* and we'll talk about it on Wednesday. But for now you can convince yourself that this is true.

Once we have this, we can just plug in $j = \log(n)$, and get

$$T(n) = 2^{\log(n)} T(n/2^{\log(n)}) + n\log(n) = n \cdot T(1) + n\log(n) = n(\log(n) + 1),$$

just as before.

[1]Notice that this is a special consequence of the fact that the term we are adding on is exactly n; if it were, say $11 \cdot n$, we'd have to multiply all the labels by 11 before counting their contribution. Or if it were $\sqrt{n}$, we'd have to take the square root, and so on.