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ELEC 462 # STATISTICAL SIGNAL PROCESSING
TERM PROJECT

a) Derive the maximum likelihood estimator for X.

$$x \longrightarrow \boxed{h[n]} \longrightarrow \underset{\substack{\uparrow \\ w}}{\oplus} \longrightarrow y$$

(additive white Gaussian channel)

$$y[n] = x[n] \circledast h[n] + w[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k] + w[n]$$

(channel equation)

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[256] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 & \cdots & 0 \\ h[1] & h[0] & 0 & \cdots & 0 \\ h[2] & h[1] & h[0] & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ h[256] & h[255] & h[254] & \cdots & h[0] \end{bmatrix} \cdot \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[256] \end{bmatrix} + \begin{bmatrix} w[0] \\ w[1] \\ w[2] \\ \vdots \\ w[256] \end{bmatrix}$$

$$\boxed{y = H \cdot x + w}$$

$x \sim \mathcal{N}(\mu_x, \Sigma_x) \; : \; x \sim \mathcal{N}(\mu_x, \Sigma_x)$

$w \sim \mathcal{N}(0, \sigma^2 I) \; : \; w \sim \mathcal{N}(0, 25 \times 10^{-4} I)$

$y|x \sim \mathcal{N}(Hx, \sigma_w^2 I)$

$$F_{Y|X}(y|x) = P(Y \leq y \mid X = x) = P(Hx + w \leq y \mid X = x)$$
$$= P(W \leq y - Hx)$$
$$F_{Y|X}(y|x) = F_W(y - Hx)$$
$$\frac{\partial}{\partial y}\left(F_{Y|X}(y|x)\right) = \frac{\partial}{\partial y}\left(F_W(y - Hx)\right)$$
$$f_{Y|X}(y|x) = f_W(y - x)$$

likelihood term: $$f_{Y|X}(y|x) = \frac{1}{(2\pi\sigma_w^2)^{256/2}}\, e^{-\frac{1}{2\sigma_w^2}(y-Hx)^T(y-Hx)}$$

Maximum Likelihood: $$\boxed{ML\ \hat{x} = \underset{x}{\arg\max}\left(\log(f(y|x))\right)}$$

$$\hat{J}(x) = \log f(y|x) = -\frac{L}{2}\log(2\pi\sigma_w^2) - \frac{1}{2\sigma_w^2}(y - Hx)^T(y - Hx)$$
$$= \frac{-1}{2\sigma_w^2}\left(y^T y - y^T H x - (Hx)^T y + (Hx)^T Hx\right) + \text{constant}$$
$$= \frac{-1}{2\sigma_w^2}\left(y^T y - y^T H x - x^T H^T y + x^T H^T H x\right) + \text{constant}$$
$$= \frac{-1}{2\sigma_w^2}\left(y^T y - 2x^T y + x^T H^T H x\right) + \text{constant}$$
$$= -\frac{1}{2\sigma_w^2}\left(y^T y - 2x^T y + x^T H^T H x\right)$$

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$$\hat{J}(x) = \frac{-1}{2\sigma_w^2}(y^Ty - 2x^TH^Ty + x^TH^THx) + \text{constant}$$

we want to minimize $J(x)$ term So we'll get derivative and equals it to zero.

$$\frac{\partial \hat{J}(x)}{\partial x} = 0 \quad , \quad \frac{-1}{2\sigma_w^2}(-2H^Ty + 2H^THx) = 0$$

$$H^THx = H^Ty$$

$$\boxed{\hat{X}_{\substack{\text{Maximum}\\ \text{likelihood}}} = (H^TH)^{-1}H^Ty}$$

c) Derive the maximum a-posteriori (MAP) estimator for X

likelihood term : $f(y|x) = \frac{1}{(2\pi\sigma_w)^{L/2}} \exp\left\{\frac{-1}{2\sigma_w^2}(y-Hx)^T(y-Hx)\right\}$

prior term : $f(x) = \frac{1}{(2\pi)^{L/2}\cdot|\Sigma_x|^{1/2}} \exp\left\{\frac{-1}{2}(x-\mu x)^T\cdot\Sigma_x^{-1}(x-\mu x)\right\}$

$$\hat{X}_{\substack{\text{maximum}\\ \text{A Posteriori}}} = \text{argmax} \log f(y|x)\, f(x)$$

Let $\hat{J}(x) = \log[f(y|x)\, f(x)]$

$$= \frac{-L}{2}\log(2\pi\sigma_w^2) - \frac{1}{2\sigma_w^2}(y-Hx)^T(y-Hx) - \frac{L}{2}\log(2\pi)$$

$$- \frac{1}{2}\log|\Sigma_x| - \frac{1}{2}(x-\mu x)^T\Sigma_x^{-1}(x-\mu x)$$

we want to minimize $J(x)$ term. So we'll get derivative and equals it to zero.

$$\frac{\partial \hat{J}(x)}{\partial x} = 0$$

$$\frac{1}{\sigma_w^2}H^T(y-Hx) - \Sigma_x^{-1}(x-\mu x) = 0$$

$$\left(\frac{1}{\sigma_w^2}H^TH + \Sigma_x^{-1}\right)x = \frac{1}{\sigma_w^2}H^Ty + \Sigma_x^{-1}\mu x$$

$$\boxed{\hat{X}_{\substack{\text{maximum}\\ \text{A-Posteriori}}} = (H^TH + \sigma_w^2\Sigma_x^{-1})^{-1}\cdot(H^Ty + \sigma_w^2\Sigma_x^{-1}\mu x)}$$

$\mu w = 0$
$\mu x = 0$
$\sigma_w^2 = 25\times10^{-4}$
$\Sigma_x = (\lambda L^TL)^{-1}$

$$X_{\substack{\text{maximum}\\ \text{A Posteriori}}} = [H^TH + 25\times10^{-4}\cdot(\lambda L^TL)]^{-1}\cdot[H^Ty + 25\times10^{-4}\cdot(\lambda L^TL)\cdot 0]$$

$$\boxed{X_{\substack{\text{maximum}\\ \text{A-Posteriori}}} = [H^TH + 25\times10^{-4}\cdot(\lambda L^TL)]^{-1}\cdot[H^Ty]}$$

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e) Derive the minimum mean square error (MMSE) Linear estimator for for X.

For Gaussian case, the Linear MMSE estimator is the same as MMSE estimator. So, I'll calculate the MMSE estimator, because It is more easy for me and I'll find the same result with Linear MMSE (orthogonality principled theorem).

$$\hat{X}_{MMSE} = E\{X \mid Y=y\} \quad \text{(expected value of } X \text{ given } Y=y)$$

Bayesian rule: $f(x|y) = \dfrac{f(y|x)\,f(x)}{\int_{-\infty}^{+\infty} f(y|x')\,f(x')\,dx'} = \dfrac{\text{likelihood} \times \text{prior}}{\text{normalization term}}$

(we know observation is Gaussian)

likelihood $f(y|x) = \dfrac{1}{(2\pi \hat{\sigma}_w^2)^{L/2}} \exp\left\{ \dfrac{-1}{2\hat{\sigma}_w^2} (y-Hx)^T (y-Hx) \right\}$

prior $f(x) = \dfrac{1}{(2\pi)^{\frac{L}{2}} \cdot |\Sigma_x|^{1/2}} \exp\left\{ \dfrac{-1}{2} (x-\mu_x)^T \Sigma_x^{-1} (x-\mu_x) \right\}$

we'll calculate the $f(x|y)$ posterior.

$f(x)$: Gaussian, $f(y|x)$ Gaussian : and W and X are independent.

joint pdf of X, W:

$$\begin{bmatrix} X \\ W \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_x \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_x & 0 \\ 0 & \sigma_w^2 I \end{bmatrix} \right)$$

μ_x: mean of X.
μ_w: mean of W.
Σ_x: covariance matrix of X,
and crosses terms are zero because of W and X are independent.

joint pdf of Y, X:

$$\begin{bmatrix} Y \\ X \end{bmatrix} \sim \mathcal{N}(d, e) \quad , d: \text{mean}, \ e: \text{variance}.$$

$$d = \begin{bmatrix} H & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \mu_x \\ 0 \end{bmatrix} = \begin{bmatrix} H\mu_x \\ \mu_x \end{bmatrix}$$

$$e = \begin{bmatrix} H & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \Sigma_x & 0 \\ 0 & \sigma_w^2 I \end{bmatrix} \begin{bmatrix} H^T & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} H^T & I \\ I & 0 \end{bmatrix} \begin{bmatrix} \Sigma_x H^T & \Sigma_x \\ \sigma_w^2 I & 0 \end{bmatrix} = \begin{bmatrix} H\Sigma_x H^T + \sigma_w^2 I & H\Sigma_x \\ \Sigma_x H^T & \Sigma_x \end{bmatrix}$$

$$\begin{bmatrix} Y \\ X \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} H\mu_x \\ \mu_x \end{bmatrix}, \begin{bmatrix} H\Sigma_x H^T + \sigma_w^2 I & H\Sigma_x \\ \Sigma_x H^T & \Sigma_x \end{bmatrix} \right)$$

conditional pdf:

$$X|\{Y=y\} \sim \mathcal{N}\Big(\underbrace{\Sigma_x H^T (H\Sigma_x H^T + \sigma_w^2 I)^{-1} (y - H\mu_x) + \mu_x}_{\text{mean}} , \underbrace{\Sigma_x - \Sigma_x H^T (H\Sigma_x H^T + \sigma_w^2 I)^{-1} \cdot (H\Sigma_x)}_{\text{variance}} \Big)$$

$$\boxed{\hat{X}_{\text{Linear MMSE}} = \hat{X}_{MMSE} = \Sigma_x H^T (H\Sigma_x H^T + \sigma_w^2 I)^{-1} (y - H\mu_x) + \mu_x}$$

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9) Calculate the peak signal-to-noise ratio PSNR of each estimate such that

$$PSNR = 20\log_{10}\left(\frac{\max(x)\sqrt{256^2}}{\|x-\hat{x}\|}\right)$$

	ML	MAP (λ = 100)	NMSE (λ = 1000)
PSNR (dB	-11,2520dB	-9,5841 dB	-11,1809 dB

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