Ferrers graph

Ferrers graph

It is a graph to represent a partition by an array of dots.

Ferrers graph

It is a graph to represent a partition by an array of dots.
It has the following property.

Ferrers graph

It is a graph to represent a partition by an array of dots.
It has the following property.
(i) There is one row for each part.
(ii) The number of dots in any row is the size of that part.
(iii) An upper row always contains at least as many dots as a lower row.
(iv) The rows are aligned to the left.

Ferrers graph

It is a graph to represent a partition by an array of dots.
It has the following property.
(i) There is one row for each part.
(ii) The number of dots in any row is the size of that part.
(iii) An upper row always contains at least as many dots as a lower row.
(iv) The rows are aligned to the left.
Example:
Consider the partition 5 3 2 2.

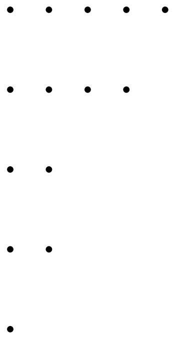
$$\begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \end{array}$$

The partition obtained by reading the Ferrers graph by column is called *conjugate partition*.

The partition obtained by reading the Ferrers graph by column is called *conjugate partition*.
The conjugate partition of 5 3 2 2 is 4 4 2 1 1.

The partition obtained by reading the Ferrers graph by column is called *conjugate partition*.
The conjugate partition of 5 3 2 2 is 4 4 2 1 1.
A partition whose Ferrers graph reads the same by rows and by columns is called self-conjugate.

Example: 5 4 2 2 1, 4 3 2 1.



Problems

Problems

Q1. Show that the number of partitions of n is equal to number of partitions of $2n$ with exactly n parts.

Problems

Q1. Show that the number of partitions of n is equal to number of partitions of $2n$ with exactly n parts.

Soln: Consider a Ferrers graph of a partition of n. Add a column of n dots on the left of the graph.

Problems

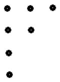
Q1. Show that the number of partitions of n is equal to number of partitions of $2n$ with exactly n parts.

Soln: Consider a Ferrers graph of a partition of n. Add a column of n dots on the left of the graph. Then the graph represents a partition of 2n with exactly n parts.

Problems

Q1. Show that the number of partitions of n is equal to number of partitions of $2n$ with exactly n parts.

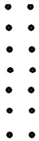
Soln: Consider a Ferrers graph of a partition of n. Add a column of n dots on the left of the graph. Then the graph represents a partition of 2n with exactly n parts.



Partition of n=7

Partition of 14 with 7 parts.

Partition of n=7

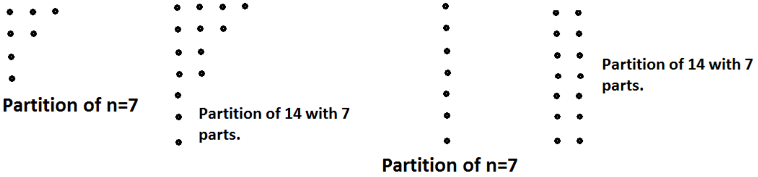


Partition of 14 with 7 parts.

Problems

Q1. Show that the number of partitions of n is equal to number of partitions of $2n$ with exactly n parts.

Soln: Consider a Ferrers graph of a partition of n. Add a column of n dots on the left of the graph. Then the graph represents a partition of 2n with exactly n parts.

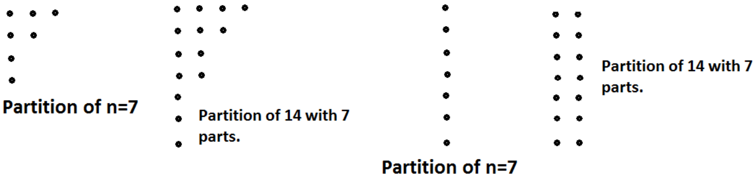


Consider a partition of 2n with exactly n parts. Then the left most column contains n dots.

Problems

Q1. Show that the number of partitions of n is equal to number of partitions of $2n$ with exactly n parts.

Soln: Consider a Ferrers graph of a partition of n. Add a column of n dots on the left of the graph. Then the graph represents a partition of 2n with exactly n parts.

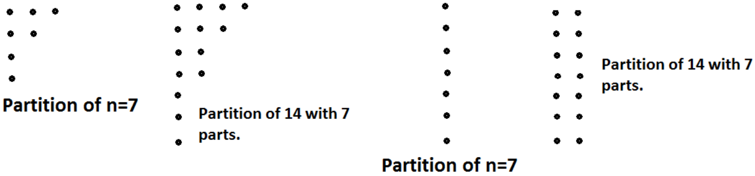


Consider a partition of 2n with exactly n parts. Then the left most column contains n dots. Eliminating the first column, results in a partition of n.

Problems

Q1. Show that the number of partitions of n is equal to number of partitions of $2n$ with exactly n parts.

Soln: Consider a Ferrers graph of a partition of n. Add a column of n dots on the left of the graph. Then the graph represents a partition of 2n with exactly n parts.

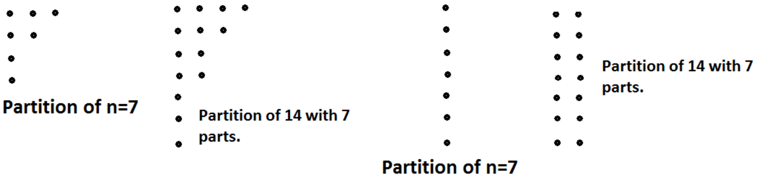


Consider a partition of 2n with exactly n parts. Then the left most column contains n dots. Eliminating the first column, results in a partition of n. Thus, for every partition of n dots there corresponds a partition of 2n with exactly n parts and vice versa.

Problems

Q1. Show that the number of partitions of n is equal to number of partitions of $2n$ with exactly n parts.

Soln: Consider a Ferrers graph of a partition of n. Add a column of n dots on the left of the graph. Then the graph represents a partition of 2n with exactly n parts.



Consider a partition of 2n with exactly n parts. Then the left most column contains n dots. Eliminating the first column, results in a partition of n. Thus, for every partition of n dots there corresponds a partition of 2n with exactly n parts and vice versa. Hence, the number of partitions of n is equal to number of partitions of $2n$ with exactly n parts.

Q2. Show that number of partitions of an integer n with no part greater than k is equal to the number of partition of n with atmost k parts.

Q2. Show that number of partitions of an integer n with no part greater than k is equal to the number of partition of n with atmost k parts.
Soln: Consider the Ferrers graph representation of a partition of n with no part greater than k.

Q2. Show that number of partitions of an integer n with no part greater than k is equal to the number of partition of n with atmost k parts.
Soln: Consider the Ferrers graph representation of a partition of n with no part greater than k.
The number of dots in each row is less than or equal to k.

Q2. Show that number of partitions of an integer n with no part greater than k is equal to the number of partition of n with atmost k parts.

Soln: Consider the Ferrers graph representation of a partition of n with no part greater than k.

The number of dots in each row is less than or equal to k.

Read the partition column-wise.

Q2. Show that number of partitions of an integer n with no part greater than k is equal to the number of partition of n with atmost k parts.

Soln: Consider the Ferrers graph representation of a partition of n with no part greater than k.

The number of dots in each row is less than or equal to k.

Read the partition column-wise. Then the number of parts is $\leq k$.

Q2. Show that number of partitions of an integer n with no part greater than k is equal to the number of partition of n with atmost k parts.

Soln: Consider the Ferrers graph representation of a partition of n with no part greater than k.

The number of dots in each row is less than or equal to k.

Read the partition column-wise.Then the number of parts is $\leq k$.

Thus, for a partition of an integer n with no part greater than k is equal to the number of partition of n with atmost k parts.

Q2. Show that number of partitions of an integer n with no part greater than k is equal to the number of partition of n with atmost k parts.

Soln: Consider the Ferrers graph representation of a partition of n with no part greater than k.

The number of dots in each row is less than or equal to k.

Read the partition column-wise.Then the number of parts is $\leq k$.

Thus, for a partition of an integer n with no part greater than k is equal to the number of partition of n with atmost k parts.

Conversely, consider a partition with atmost k parts, then the number of rows is $\leq k$ in the corresponding Ferrors graph.

Q2. Show that number of partitions of an integer n with no part greater than k is equal to the number of partition of n with atmost k parts.

Soln: Consider the Ferrers graph representation of a partition of n with no part greater than k.

The number of dots in each row is less than or equal to k.

Read the partition column-wise. Then the number of parts is $\leq k$.

Thus, for a partition of an integer n with no part greater than k is equal to the number of partition of n with atmost k parts.

Conversely, consider a partition with atmost k parts, then the number of rows is $\leq k$ in the corresponding Ferrors graph.

Hence, in the conjugate partition, size of any column is $\leq k$.

Q2. Show that number of partitions of an integer n with no part greater than k is equal to the number of partition of n with atmost k parts.

Soln: Consider the Ferrers graph representation of a partition of n with no part greater than k.

The number of dots in each row is less than or equal to k.

Read the partition column-wise.Then the number of parts is $\leq k$.

Thus, for a partition of an integer n with no part greater than k is equal to the number of partition of n with atmost k parts.

Conversely, consider a partition with atmost k parts, then the number of rows is $\leq k$ in the corresponding Ferrors graph.

Hence, in the conjugate partition, size of any column is $\leq k$.

Thus, for every partition with atmost k parts, there corresponds a partition with no part greater than k.

Q2. Show that number of partitions of an integer n with no part greater than k is equal to the number of partition of n with atmost k parts.

Soln: Consider the Ferrers graph representation of a partition of n with no part greater than k.

The number of dots in each row is less than or equal to k.

Read the partition column-wise.Then the number of parts is $\leq k$.

Thus, for a partition of an integer n with no part greater than k is equal to the number of partition of n with atmost k parts.

Conversely, consider a partition with atmost k parts, then the number of rows is $\leq k$ in the corresponding Ferrors graph.

Hence, in the conjugate partition, size of any column is $\leq k$.

Thus, for every partition with atmost k parts, there corresponds a partition with no part greater than k.

Hence, number of partitions of an integer n with no part greater than k is equal to the number of partition of n with atmost k parts.

Q3. Show that the number of partitions of n in which no part is smaller than 3 is equal to the number of partitions of $n+3$ in which the three largest parts are consecutive integers.

Soln: Consider the Ferrers diagram of a partition of n with no part smaller than 3. Then the last row contains at least 3 dots, and therefore in the conjugate of this diagram the first three rows have the same length. Now, add 2 dots to the first row, and 1 dot to the second row. The resulting diagram has n + 3 dots and the first three rows have consecutive lengths. Thus, it represents a partition of $n+3$ in which the three largest parts are consecutive integers. Both the operations of conjugations and adding 2 dots to the first row and 1 dot to the second are invertible, and therefore there is a bijection between the partitions of the two kinds, showing that they are equinumerous.

Q4. Obtain a generating function for the partition of n with exactly k parts.

Q4. Obtain a generating function for the partition of n with exactly k parts.

Solution:

Q4. Obtain a generating function for the partition of n with exactly k parts.

Solution: Enumerator for partitions of n with exactly k parts $=$ (Enumerator with atmost k parts) $-$ (Enumerator with atmost $(k-1)$ parts).

Q4. Obtain a generating function for the partition of n with exactly k parts.

Solution: Enumerator for partitions of n with exactly k parts $=$ (Enumerator with atmost k parts) $-$ (Enumerator with atmost $(k-1)$ parts).

$=$ (Enumerator with no part greater than k) $-$ (Enumerator with no part greater than $(k-1)$)

Q4. Obtain a generating function for the partition of n with exactly k parts.

Solution: Enumerator for partitions of n with exactly k parts $=$ (Enumerator with atmost k parts) $-$ (Enumerator with atmost $(k-1)$ parts).

$=$ (Enumerator with no part greater than k) $-$ (Enumerator with no part greater than $(k-1)$)

$$=\{(1-x)^{-1}(1-x^2)^{-1}...(1-x^k)^{-1}\}-\{(1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}\}$$

Q4. Obtain a generating function for the partition of n with exactly k parts.

Solution: Enumerator for partitions of n with exactly k parts $=$ (Enumerator with atmost k parts) $-$ (Enumerator with atmost $(k-1)$ parts).

$=$ (Enumerator with no part greater than k) $-$ (Enumerator with no part greater than $(k-1)$)

$$=\{(1-x)^{-1}(1-x^2)^{-1}...(1-x^k)^{-1}\}-\{(1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}\}$$

$$= \{(1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}\}((1-x^k)^{-1}-1)$$

Q4. Obtain a generating function for the partition of n with exactly k parts.
Solution: Enumerator for partitions of n with exactly k parts $=$ (Enumerator with atmost k parts) $-$ (Enumerator with atmost $(k-1)$ parts).
$=$ (Enumerator with no part greater than k) $-$ (Enumerator with no part greater than $(k-1)$)

$$=\{(1-x)^{-1}(1-x^2)^{-1}...(1-x^k)^{-1}\}-\{(1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}\}$$

$$= \{(1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}\}((1-x^k)^{-1}-1)$$

$$= (1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}x^k(1-x^k)^{-1}$$

Q4. Obtain a generating function for the partition of n with exactly k parts.
Solution: Enumerator for partitions of n with exactly k parts $=$ (Enumerator with atmost k parts) $-$ (Enumerator with atmost $(k-1)$ parts).
$=$ (Enumerator with no part greater than k) $-$ (Enumerator with no part greater than $(k-1)$)

$$=\{(1-x)^{-1}(1-x^2)^{-1}...(1-x^k)^{-1}\}-\{(1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}\}$$

$$= \{(1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}\}((1-x^k)^{-1}-1)$$

$$= (1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}x^k(1-x^k)^{-1}$$

Coefficient of x^n in $(1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}x^k(1-x^k)^{-1}$ is the number of partitions of n with exactly k parts.

Q5. Prove that the number of partition of n with exactly k parts is equal to the number of partition of $n-k$ with no part greater than k.

Q5. Prove that the number of partition of n with exactly k parts is equal to the number of partition of $n - k$ with no part greater than k.
Solution:

Q5. Prove that the number of partition of n with exactly k parts is equal to the number of partition of $n-k$ with no part greater than k.

Solution:

The number of partition of $n-k$ with no part greater than k is coefficient of x^{n-k} in $(1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}(1-x^k)^{-1}$.

Q5. Prove that the number of partition of n with exactly k parts is equal to the number of partition of $n-k$ with no part greater than k.
Solution:
The number of partition of $n-k$ with no part greater than k is coefficient of x^{n-k} in $(1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}(1-x^k)^{-1}$.

The number of partition of n with exactly k parts is equal to coefficient of x^n in $x^k(1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}(1-x^k)^{-1}$

Q5. Prove that the number of partition of n with exactly k parts is equal to the number of partition of $n-k$ with no part greater than k.
Solution:
The number of partition of $n-k$ with no part greater than k is coefficient of x^{n-k} in $(1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}(1-x^k)^{-1}$.

The number of partition of n with exactly k parts is equal to coefficient of x^n in $x^k(1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}(1-x^k)^{-1}$

which is equal to the coefficient of x^{n-k} in $(1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}(1-x^k)^{-1}$.

Q5. Prove that the number of partition of n with exactly k parts is equal to the number of partition of $n-k$ with no part greater than k.
Solution:
The number of partition of $n-k$ with no part greater than k is coefficient of x^{n-k} in $(1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}(1-x^k)^{-1}$.

The number of partition of n with exactly k parts is equal to coefficient of x^n in $x^k(1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}(1-x^k)^{-1}$

which is equal to the coefficient of x^{n-k} in
$(1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}(1-x^k)^{-1}$.

Which is equal to the number of partition of $n-k$ with no part greater than k.

Q5. Prove that the number of partition of n with exactly k parts is equal to the number of partition of $n-k$ with no part greater than k.
Solution:
The number of partition of $n-k$ with no part greater than k is coefficient of x^{n-k} in $(1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}(1-x^k)^{-1}$.

The number of partition of n with exactly k parts is equal to coefficient of x^n in $x^k(1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}(1-x^k)^{-1}$

which is equal to the coefficient of x^{n-k} in $(1-x)^{-1}(1-x^2)^{-1}...(1-x^{k-1})^{-1}(1-x^k)^{-1}$.

Which is equal to the number of partition of $n-k$ with no part greater than k.
Hence, the number of partition of n with exactly k parts is equal to the number of partition of $n-k$ with no part greater than k.