Q6.(Cancellation laws) Let $(A, \leq)$ be a distributive lattice. Show that if $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ for some a, then $x = y$.

Q7. Show that a lattice is distributive if and only if for any elements a, b, c in lattice $(a \vee b) \wedge c \leq a \vee (b \wedge c)$.

Universal lower and upper bounds: An element a in a lattice $(A, \leq)$ is called a universal lower bound if for every element $b \in A,\ a \leq b$. We use '0'to denote universal lower bound.
An element a in a lattice $(A, \leq)$ is called a universal upper bound if for every element $b \in A,\ b \leq a$. We use '1' to denote universal upper bound.
If a lattice has a universal lower (upper) bound, then it is unique.
In the lattice $(P(S), \subseteq)$, the nullset ϕ and the set S are the universal lower and upper bounds respectively.

Theorem 1.1.

Let $(A, \leq)$ be a lattice with universal upper and lower bounds 1 and 0. For any elements $a \in A$

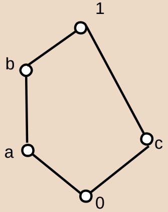
- $a \vee 1 = 1$
- $a \wedge 1 = a$
- $a \vee 0 = a$
- $a \wedge 0 = 0$

Complement of an element: Let $(A, \leq)$ be a lattice with universal upper and lower bounds 1 and 0. For any element $a \in A$, an element b is said to be a complement of a if $a \vee b = 1$ and $a \wedge b = 0$

An element in a lattice may have more than one complement. Not all the elements in a lattice have complements. It's evident that '0'is the unque complement of '1'and vice versa.

Complemented lattice: A lattice is said to be a complemented lattice if every element in the lattice has a complement. Clearly, a complemented lattice has a universal lower and upper bounds.

Example 1.2.



Complement of a and b is c. Complements of c are a, b.

Theorem 1.3.

In a distributive lattice, if an element has a complement then this complement is unique.

Proof.

Suppose an element a has two complements b and c. i.e.
$a \vee b = a \vee c = 1$ and $a \wedge b = a \wedge c = 0$.

Consider
$$\begin{aligned} b &= b \wedge 1 \\ &= b \wedge (a \vee c) \\ &= (b \wedge a) \vee (b \wedge c) \\ &= 0 \vee (b \wedge c) \\ &= (a \wedge c) \vee (b \wedge c) \\ &= c \wedge (a \vee b) \\ &= c \wedge 1 \\ &= c \end{aligned}$$

Thus $b = c$ □

Boolean lattice: A complemented and distributive lattice is called a boolean lattice.

Example 2.1.

$(P(S), \subseteq)$ is a boolean lattice.

Let $(A, \leq)$ be a boolean lattice. Since every element a has a unique complement $\bar{a}$, we have another unary operation known as complementation and denoted by $^{-}$. Thus we can say that the lattice $(A, \leq)$ defines an algebraic system $(A, \vee, \wedge, -)$ where $\vee$, $\wedge$ and - are the join, meet and complementation operations respectively. The algebraic system defined by a boolean lattice is known as a **boolean algebra.**

Theorem 2.2.

DeMorgan's laws: For any a and b in a boolean algebra

- $\overline{a \vee b} = \bar{a} \wedge \bar{b}$
- $\overline{a \wedge b} = \bar{a} \vee \bar{b}$

Proof.

We have to prove that $(a \vee b) \vee (\bar{a} \wedge \bar{b}) = 1$ and $(a \vee b) \wedge (\bar{a} \wedge \bar{b}) = 0$.

Consider
$$
\begin{aligned}
(a \vee b) \vee (\bar{a} \wedge \bar{b}) &= [(a \vee b) \vee \bar{a}] \wedge [(a \vee b) \vee \bar{b}] \text{ (distributive law)} \\
&= [\bar{a} \vee (a \vee b)] \wedge [a \vee (b \vee \bar{b})] \text{ (associative law)} \\
&= [(\bar{a} \vee a) \vee b] \wedge [a \vee 1] \text{ (associative law)} \\
&= [1 \vee b] \wedge [a \vee 1] \\
&= 1 \wedge 1 = 1
\end{aligned}
$$

Similarly,
$$
\begin{aligned}
(a \vee b) \wedge (\bar{a} \wedge \bar{b}) &= (\bar{a} \wedge \bar{b}) \wedge (a \vee b) \text{ (commutative law)} \\
&= [(\bar{a} \wedge \bar{b}) \wedge a] \vee [(\bar{a} \wedge \bar{b}) \wedge b] \text{ (distributive law)} \\
&= [a \wedge (\bar{a} \wedge \bar{b})] \vee [(\bar{a} \wedge \bar{b}) \wedge b] \text{ (commutative law)} \\
&= [(a \wedge \bar{a}) \wedge \bar{b}] \vee [\bar{a} \wedge (\bar{b} \wedge b)] \text{ (associative law)} \\
&= [0 \wedge \bar{b}] \vee [(\bar{a} \wedge 0] \\
&= 0 \vee 0 = 0
\end{aligned}
$$

The second part follows from principle of duality. □