Ordering of Permutations

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Lexicographical Order

To find the k^{th}permutation of n marks (letter, symbols), say $a_1, a_2, \ldots, a_n$, when the permutations are sorted lexicographically, proceed as given below: Write $k-1$ in the form

$$k - 1 = c_{n-1}(n-1)! + c_{n-2}(n-2)! + \cdots + c_1 1!$$

where each integer c_i has the maximum possible value, $0 \leq c_i \leq i$. In other words, we divide $k-1$ by $(n-1)!$, and take c_{n-1} as the quotient; then divide the remainder by $(n-2)!$ and take c_{n-2} as the quotient; and so on. This gives us a sequence $c_{n-1}c_{n-2}\cdots c_1$.

Example 1.1

To compute the 35^{th}*permutation of the five marks* $1, 2, 3, 4, 5$*, we note that* $k = 35$ *and* $n = 5$*. Now*

$$35 - 1 = 34 = \underline{1} \times 4! + \underline{1} \times 3! + \underline{2} \times 2! + \underline{0} \times 1!$$

so that the sequence is $\boxed{1120}$.

Next, the sequence $c_{n-1} \cdots c_1$ is treated as a sequence of array indices (the range being 0 to $n-1$). Then the k^{th}permutation is constructed in the following manner. Start with the array of marks $1, 2, \ldots, n$, and pick the element indexed by c_{n-1} as the first element of the permutation. Remove this element from the array to get a new array, and also remove c_{n-1} from the sequence of indices to get the new sequence $c_{n-2} \cdots c_1$. Now continue until the sequence of indices is exhausted. At this point, exactly one mark will remain in the array, and write this down as the last element of the permutation.

Example 1.2

Continuing from the previous example, to compute the 35^{th} *permutation of the five marks* $1, 2, 3, 4, 5$, *we have already obtained the sequence* 1120. *Now, consider the array of marks* 12345.
The first index is 1 *(the first element of* 1120*), and the element of the array indexed by this is* 2. *Thus, the permutation is* $2\_\_\_\_$.
The new array is 1345, *and the new sequence of indices is* 120. *Now, the element indexed by the first index* 1 *is* 3. *Thus the permutation is* $23\_\_\_$.
The new array is 145, *and the indices are* 20. *The element indexed by* 2 *is* 5, *so the permutation is* $235\_\_$.
The array is now 14, *and the only index remaining is* 0. *The corresponding element is* 1, *and the permutation is* $2351\_$.
The only remaining element 4 *is the last element of the permutation, so the complete permutation is* $\boxed{23514}$.

Solved problems

1. Find the 23rd permutation of the four marks $1, 2, 3, 4$ in lexicographical order.

$$23 - 1 = 22 = \underline{3} \times 3! + \underline{2} \times 2! + \underline{0} \times 1! \to 320$$

Index	Marks	Mark
$\underline{3}20$	$123\underline{4}$	$\to 4$
$\underline{2}0$	$12\underline{3}$	$\to 3$
$\underline{0}$	$\underline{1}2$	$\to 1$
	$\underline{2}$	$\to 2$

Thus, the 23rd permutation of $1, 2, 3, 4$ in lexicographical order is $\boxed{4312}$.

2. Find the 18^{th} permutation of the marks a, b, c, d in lexicographical order.

$$18 - 1 = 17 = \underline{2} \times 3! + \underline{2} \times 2! + \underline{1} \times 1! \to 221.$$

$\underline{2}21$	$ab\underline{c}d$	$\to c$
$\underline{2}1$	$ab\underline{d}$	$\to d$
$\underline{1}$	$a\underline{b}$	$\to b$
	$\underline{a}$	$\to a$

Thus, the 18^{th} permutation of the marks a, b, c, d in lexicographical order is $\boxed{cdba}$.

3. Find the 268$^{\text{th}}$permutation of `LISTEN` in lexicographical order.

$$268 - 1 = 267 = \underline{2} \times 5! + \underline{1} \times 4! + \underline{0} \times 3! + \underline{1} \times 2! + \underline{1} \times 1! \to 21011$$

$\underline{2}1011$	$\mathtt{LI\underline{S}TEN}$	$\to$ S
$\underline{1}011$	$\mathtt{L\underline{I}TEN}$	$\to$ I
$\underline{0}11$	$\mathtt{\underline{L}TEN}$	$\to$ L
$\underline{1}1$	$\mathtt{T\underline{E}N}$	$\to$ E
$\underline{1}$	$\mathtt{T\underline{N}}$	$\to$ N
	$\mathtt{\underline{T}}$	$\to$ T

Thus, the 268$^{\text{th}}$permutation of `LISTEN` in lexicographical order is $\boxed{\mathtt{SILENT}}$.

Reverse Lexicographical Order

To obtain the k^{th}permutation of n marks $a_1, a_2, \ldots, a_n$ in reverse lexicographical order, first reverse the order of marks to get $a_n, a_{n-1}, \ldots, a_1$, compute the k^{th}permutation of these marks in *lexicographical order*, and then reverse the resulting permutation.

1. Find the 50^{th}permutation of the five marks $0, 1, 2, 3, 4$ in reverse lexicographical order.

$$50 - 1 = 49 = \underline{2} \times 4! + \underline{0} \times 3! + \underline{0} \times 3! + \underline{1} \times 1! \to 2001$$

$\underline{2}001$	$43\underline{2}10$	$\to 2$
$\underline{0}01$	$\underline{4}310$	$\to 4$
$\underline{0}1$	$\underline{3}10$	$\to 3$
$\underline{1}$	$1\underline{0}$	$\to 0$
	$\underline{1}$	$\to 1$

Thus, the 50^{th}permutation of $0, 1, 2, 3, 4$ in reverse lexicographical order is $\boxed{10342}$.

2. Find the 100^{th}permutation of the marks $1, 2, 3, 4, 5$ in reverse lexicographical order.

$$100 - 1 = 99 = \underline{4} \times 4! + \underline{0} \times 3! + \underline{2} \times 1! + \underline{1} \times 1! \to 4011$$

$\underline{4}011$	$5432\underline{1}$	$\to 1$
$\underline{0}11$	$\underline{5}432$	$\to 5$
$\underline{1}1$	$4\underline{3}2$	$\to 3$
$\underline{1}$	$4\underline{2}$	$\to 2$
	4	$\to 4$

Thus, the 100^{th}permutation of $1, 2, 3, 4, 5$ in reverse lexicographical order is $\boxed{42351}$.

Fike's order

To obtain the k^{th}permutation of n marks $a_1, a_2, \ldots, a_n$ in Fike's order, proceed as follows.
First, we must generate *Fike's sequence*, using which the permutation is to be computed. To find the sequence, first write $k-1$ in the form

$$k-1 = c_1 \times n(n-1)\cdots 3 + c_2 \times n(n-1)\cdots 4 + \cdots + c_{n-2} \times n + c_{n-1} \times 1.$$

That is, the place values are $\frac{n!}{2!}$, $\frac{n!}{3!}$, ..., $\frac{n!}{n!} = 1$. Now, **subtract this sequence** $c_1 c_2 \cdots c_{n-1}$ from the sequence $1\ 2 \cdots (n-1)$ to get the sequence $d_1 d_2 \cdots d_{n-1}$. That is, $d_i = i - c_i$, $i = 1, \ldots, n-1$. This is Fike's sequence.

Example 1.3

To compute the 65^{th}*permutation of the five marks* $1, 2, 3, 4, 5$ *in Fike's order, we note that* $k = 65$ *and* $n = 5$*. First, compute the place values* $\frac{n!}{2!}$*, . . . ,* $\frac{n!}{n!}$*. For* $n = 5$*, these are 60, 20, 5, 1. Then,*

$$65 - 1 = 64 = \underline{1} \times 60 + \underline{0} \times 20 + \underline{0} \times 5 + \underline{4} \times 1 \to 1004.$$

Now, Fike's sequence is

$$\begin{array}{l} 1234 - \\ 1004 = \\ \hline 0230. \end{array}$$

Using the Fike's sequence, the permutation is generated from the initial permutation $12\cdots n$ by a sequence of interchanges, in the following manner. For the sequence $d_1d_2\cdots d_{n-1}$, first the element of the permutation index 1 is interchanged with the element at index d_1. Similarly, at each stage, the element at index i is interchanged with the element at index d_i, until the sequence is exhausted. The resulting permutation is the k^{th}permutation in Fike's order.

For the sequence 0230 obtained in the previous example, we start with the original arrangement of the marks: 12345. Now, the element at index 1 is 2, and the element at index $d_1 = 0$ is 1. Therefore, interchanging 2 and 1, we get 21345. Next, the element at index 2 is 3, and the element at index $d_2 = 2$ is 3 (the same). "Interchanging" these, we get 21345 (i.e., the permutation remains the same). The element at index 3 is 4, and that at index $d_3 = 3$ is again the same, so once more, the permutation is 21345. Lastly, the element at index 4 is 5, and that at index $d_4 = 0$ is 2. Interchanging these, we get 51342. Thus, the 65^{th}permutation of $1, 2, 3, 4, 5$ in Fike's order is $\boxed{51342}$. We can write this succinctly as given below. First, Fike's sequence is written as a column. Then we write the original permutation in the first row, and underline the element at index 1, which is to be interchanged with the element at index d_1.

$$\begin{array}{ll} 0 & 1\underline{2}345 \\ 2 & \\ 3 & \\ 0 & \end{array}$$

The interchange is performed and the result is written in the next row, and this process is repeated until the sequence is exhausted.

$$\begin{array}{lll} 0 & 1\underline{2}345 & \rightarrow \\ 2 & 21\underline{3}45 & \rightarrow \\ 3 & 213\underline{4}5 & \rightarrow \\ 0 & 2134\underline{5} & \rightarrow \\ & \boxed{51342} & \end{array}$$

Problems

1. Obtain the 40^{th}permutation of the five marks $0, 1, 2, 3, 4$ in Fike's order. Since $n = 5$, the place values are 60, 20, 5, 1.

$$40 - 1 = 39 = \underline{0} \times 60 + \underline{1} \times 20 + \underline{3} \times 5 + \underline{4} \times 1 \rightarrow 0134$$

Then Fike's sequence is

$$\begin{array}{l} 1234 - \\ 0134 = \\ \hline 1100. \end{array}$$

The permutation is then obtained as follows.

$$\begin{array}{ccc} 1 & 0\underline{1}234 & \rightarrow \\ 1 & 01\underline{2}34 & \rightarrow \\ 0 & 021\underline{3}4 & \rightarrow \\ 0 & 3210\underline{4} & \rightarrow \\ & \boxed{42103} & \end{array}$$

2. Obtain the 50$^{\text{th}}$permutation of the five marks $1, 2, 3, 4$ in Fike's order. Since $n = 5$, the place values are 60, 20, 5, 1.

$$50 - 1 = 49 = \underline{0} \times 60 + \underline{2} \times 20 + \underline{1} \times 5 + \underline{4} \times 1 \rightarrow 0214$$

Then Fike's sequence is

$$\begin{array}{r} 1234\ - \\ 0214\ = \\ \hline 1020. \end{array}$$

The permutation is then obtained as follows.

$$\begin{array}{ccc} 1 & 1\underline{2}345 & \rightarrow \\ 0 & 12\underline{3}45 & \rightarrow \\ 2 & 321\underline{4}5 & \rightarrow \\ 0 & 3241\underline{5} & \rightarrow \\ & \boxed{52413} & \end{array}$$