Types of Lattices:

1. Distributive Lattice: Where two distributive laws are satisfied → only one Law is sufficient
2. Complemented Lattice: Where every element of the Lattice has complement. → an elt may have more than one comp
3. Boolean lattice: Complemented and distributive →

If distributive, if an elt has a complement, then unique

Boolean Lattice → Every elt has unique complement $\underbrace{(P(S), \subseteq)}$

Boolean algebra:

The algebraic system $(A, \leq, \vee, \wedge, -)$ defined by the Boolean lattice $(A, \leq)$ Ex: $(P(S), \subseteq, \cup, \cap, \setminus)$

$(A, \vee, \wedge, -)$

$(P(S), \subseteq, \cup, \cap, \setminus)$

universal lower bound '0' → ϕ

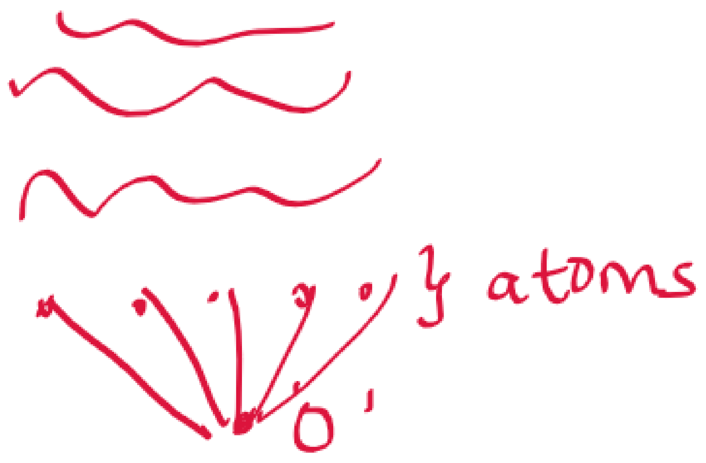
upper '1' → 'S'

Atoms :- "Singletons"

Atom :

Let $(A, \leq)$ be a Boolean lattice with universal lower bound $'0'$. An element is called an atom if it covers 0.

covers '0'



THEOREM: A finite Boolean algebra has exactly 2^n elements for some $n > 0$.

$P(S)$

$S = \{a, b, c, d\}$

$\phi = '0'$

$S = '1'$

atoms = $\{a\}$ $\{b\}$ $\{c\}$ $\{d\}$

Lemma 1:

Let $(A, \leq)$ be a finite lattice with universal lower bound $'0'$. Then for any nonzero element b, there exists at least one atom a such that $a \leq b$.

Lemma 2:

In a distributive lattice, if $b \wedge \bar{c} = 0$, then $b \leq c$.

(prev class)

$\{a, b, c\} \to \{a\}\ \{b\}\ \{c\}$

$\{a, b\} = \{a\}\ \{b\}$.

Lemma 3:

Let $(A, \leq, \vee, \wedge, -)$ be a finite Boolean algebra. Let b be any nonzero element in A and $a_1, a_2, \dots, a_k$ be all the atoms of A such that $a_i \leq b$. Then

atoms

$$b = a_1 \vee a_2 \vee \dots \vee a_k \;:$$

$b = a_1 \vee a_2 \vee \dots \vee a_k$

$\{a, b, c\} = \{a\} \cup \{b\} \cup \{c\}$

Proof:- Denote $c = a_1 \vee a_2 \vee \dots \vee a_k$. P.T $b = c$.

$a_1 \leq b \quad a_2 \leq b \quad \dots \quad a_k \leq b$ (given)

$a_1 \vee a_2 \vee \dots \vee a_k \leq b$ ($\because$ prop of lattice $b \vee b = b$)

$c \leq b$ ——— (1)

Now I should prove $b \leq c$. If I prove $b \wedge \bar{c} = 0$, it is sufficient ($\because$ Lemma 2, if $b \wedge \bar{c} = 0$, the $b \leq c$)

I should prove $b \wedge \bar{c} = 0$

Suppose $b \wedge \bar{c} \neq 0$, (nonzero elt), by Lemma 1, it will be related to atleast one atom : ie $a \leq b \wedge \bar{c}$ $a \to$ some atom

$a \leq b \wedge \bar{c}$ and $b \wedge \bar{c} \leq b \xrightarrow{\text{trans}} a \leq b$ $\because$ (a is an atom & a is $a \leq b$)

Thus a must be one among $a_1 \dots a_k$ $\to$ $a \leq a_1 \vee a_2 \vee \dots a_k$

$a \leq c$ ——— (2)

$a \leq b \wedge \bar{c}$ and $b \wedge \bar{c} \leq \bar{c} \xrightarrow{\text{trans}} a \leq \bar{c}$ ——— (3)

From (2) & (3) $\Rightarrow$ $a \leq c$ & $a \leq \bar{c}$

$a \wedge a \leq c \wedge \bar{c}$

$a \leq 0$, a contradiction

$\therefore$ Our assumption $b \wedge \bar{c} \neq 0$ is wrong.

Thus $b \wedge \bar{c} = 0$

Ths $b \leq c$ (Lemma 2) ——— (4)

From (4) & (1)

$b \leq c$ $\quad$ $c \leq b$

$$\boxed{b = c}$$

Lemma 4:

$\{a, b, c\} = \underbrace{\{a\} \cup \{b\} \cup \{c\}}_{\text{unique}}$

Let $(A, \le, \vee, \wedge, -)$ be a finite Boolean algebra. Let b be any nonzero element in A and $a_1, a_2, \ldots, a_k$ be all the atoms of A such that $a_i \le b$. Then

$\underbrace{b = a_1 \vee a_2 \vee \ldots \vee a_k}$ is the unique way to represent b as a join of atoms — ①

Proof

Suppose there is alternate rep $b = a_1' \vee a_2' \vee \ldots \vee a_t'$ — ②

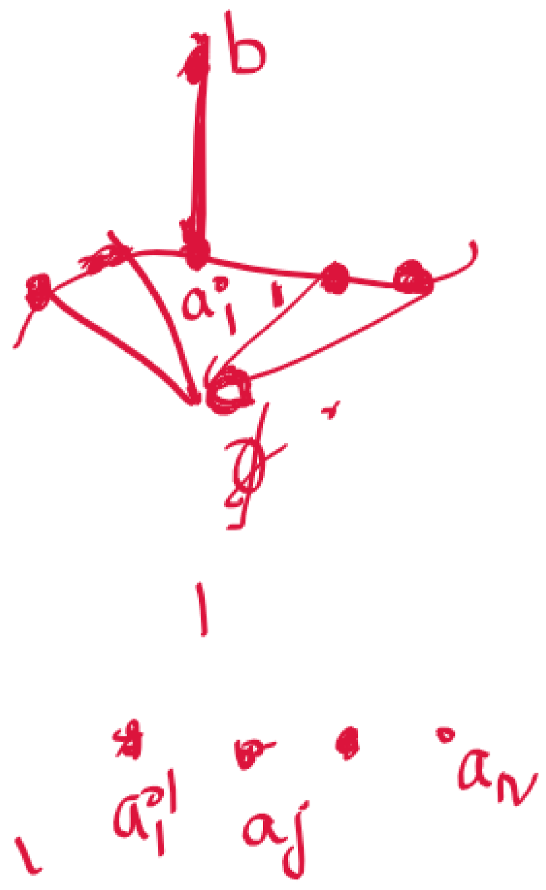
[For every a_i' in alternate rep, there is a_i in original
For every a_j in original, there is a_j' in alternate

$a_1' \le b \quad a_2' \le b \quad \ldots \quad a_t' \le b$ (from ②)

$a_i' \wedge b = a_i'$

$a_i' \wedge (a_1 \vee a_2 \vee \ldots \vee a_k) = a_i'$

$\underbrace{(a_i' \wedge a_1)} \vee \underbrace{(a_i' \wedge a_2)} \vee \ldots \vee \underbrace{(a_i' \wedge a_k)} = a_i'$



atleast $a_i' \wedge a_j \ne 0$

$a_i' = a_j$ (Bcz a_i' & a_j both are atoms & their meet is not '0')

a_i' is equal to some a_j

For every elt in alternate rep, there exists one elt in original

$\underbrace{a_i'}_{\text{in ②}} \longrightarrow \underbrace{a_j}_{\text{in ①}}$

otherway :

$a_j \wedge b = a_j$

$a_j \wedge (a_1' \vee a_2' \vee \ldots a_t') = a_j$

$(a_j \wedge a_1') \vee \ldots \vee (a_j \wedge a_t') = a_j$

atleast $a_j \wedge a_s' \ne 0$

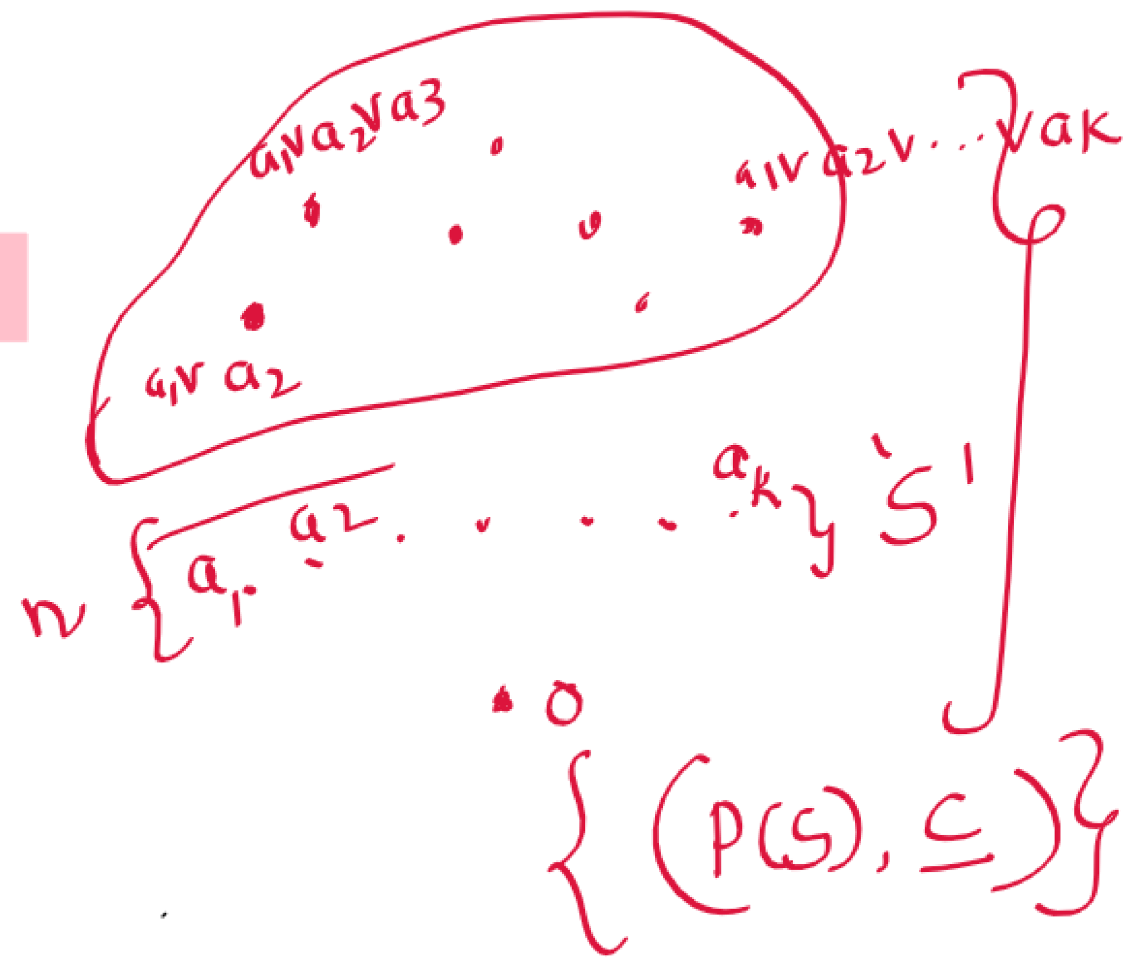
$a_j = a_s'$

Thus for every a_j in ①, there is a_s' in ②

It is clear that there is one to one correspondence between the elements of a Boolean lattice and subset of atoms. As a matter of fact, there is one to one correspondence from $(A, \leq)$ to $(P(S), \subseteq)$, where S is the set of all atoms.

Theorem:

Let $(A, \leq, \vee, \wedge, -)$ be a finite Boolean algebra. Let S be the set all the atoms. Then $(A, \leq, \vee, \wedge, -)$ is isomorphic to the algebraic system defined by the lattice $(P(S), \subseteq)$.



It follows immediately from the above theorem that there exists a unique finite Boolean algebra of 2^n elements for any $n > 0$. Further, there are no finite Boolean algebra.

Boolean functions and Boolean expressions:

Boolean expression:

Let $(A, \vee, \wedge, -)$ be a Boolean algebra. A Boolean expression over $(A, \vee, \wedge, -)$ is defined as follows:

i. Any element of A is a Boolean expression.
ii. Any variable name is a Boolean expression.
iii. If E_1 and E_2 are Boolean expressions, then $\overline{E_1}$, $E_1 \vee E_2$ and $E_1 \wedge E_2$ are also Boolean expressions

(2)

Assignment of values:

Let $E(x_1, x_2, \dots, x_n)$ be a Boolean expression of n variables over a Boolean algebra $(A, \vee, \wedge, -)$. By assignment of values to the variables $x_1, x_2, \dots, xn$, we mean an assignment of elements of A to be the values of the variables. For an assignment of values to the variables, we can evaluate $E(x_1, x_2, \dots, x_n)$ by substituting the variables in the expression by their values.

$E(x_1, x_2) = x_1 \vee x_2$

$x_1 = 0,\ x_2 = 1$ $\qquad E(x_1, x_2) = 1$

$0 \vee 1 = 1 \qquad 1 \vee 1 = 1$

$0 \wedge 1 = 0 \qquad 1 \wedge 1 = 1$

Equivalent Boolean expressions:

Two Boolean expressions of n variables are said to be equivalent if they assume the same values for every assignment of values to the n variables. If $E_1(x_1, x_2, \dots, x_n)$ and $E_2(x_1, x_2, \dots, x_n)$ are equivalent, then we write $E_1(x_1, x_2, \dots, x_n) = E_2(x_1, x_2, \dots, x_n)$

$x_1 \vee (x_2 \wedge x_3) = (x_1 \vee x_2) \wedge (x_1 \vee x_3)$

Boolean function:

A function $f: A^n \to A$ is said to be a Boolean function if it can be specified by a Boolean expression of n variables.

$f: A^n \to A$ by $E(x_1 \dots x_n)$

Example:

Let $f: A^n \to A$ where $A = \{0,1\}$ defined by the Boolean expression is $E(x_1, x_2, x_3) = \overline{x_1} \wedge x_2 \wedge \overline{x_3}$

x_1	x_2	x_3	f
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

$\{0,1\}^3 \to \{0,1\}$

000 →
001
010
011
100
101
110
111

$A = \{0,1\}$

$f_1: A^3 \to A$ by $E = x_1 \vee x_2 \vee x_3$

$(0,0,1) \to 1$

$(1,0,0) \to 1$

$(0,0,0) \to 0$

$f_2: A^3 \to A$ by $E = x_1 \wedge x_2 \wedge x_3$

$(0,0,0) \to 0$

$(0,1,1) \to 0$

$0 \wedge 0 = 0$

$1 \wedge 1 = 1$

$0 \wedge 1 = 0$

$1 \vee 1 = 1$

$0 \vee 1 = 1$

$\bar{0} = 1$

$\bar{1} = 0$

$010 \to \bar{0} \wedge 1 \wedge \bar{0} = 1 \wedge 1 \wedge 1$

$= 1$

$x_1\ x_2 \cdots x_n$

Minterm:
A Boolean expression of n variables $x_1, x_2, \dots, xn$ is said to be a minterm if it is of the form

$$\widetilde{x_1} \wedge \widetilde{x_2} \wedge \dots \wedge \widetilde{x_n} \text{ where } \widetilde{x_1} \text{ is either } x_i \text{ or } \overline{x_i}$$

$\longrightarrow \tilde{x}_1 \wedge \tilde{x}_2 \wedge \tilde{x}_3 \wedge \cdots \wedge \tilde{x}_n$

$\tilde{x}_i = x_i \ \& \ \bar{x}_i$

Disjunctive Normal Form (DNF):
A Boolean expression over $(\{0,1\}, \vee, \wedge, -)$ is said to be in disjunctive normal form if it is join of minterms.

join of minterms

Maxterm:
A Boolean expression of n variables $x_1, x_2, \dots, xn$ is said to be a maxterm if it is of the form

$$\widetilde{x_1} \vee \widetilde{x_2} \vee \dots \vee \widetilde{x_n} \text{ where } \widetilde{x_1} \text{ is either } x_i \text{ or } \overline{x_i}$$

$\tilde{x}_1 \vee \tilde{x}_2 \vee \cdots \vee \tilde{x}_n$

Conjunctive Normal Form (CNF):
A Boolean expression over $(\{0,1\}, \vee, \wedge, -)$ is said to be in conjunctive normal form if it is meet of maxterms.

meet of maxterms

How to obtain DNF?
Given a function $\{0,1\}^n \to \{0,1\}$, we can obtain a Boolean expression in DNF corresponding to this function by having a minterm corresponding to each ordered n tuple of 0s and 1s for which the value of the function is 1. For each n tuple with the functional value is 1, we have

the minterm $\widetilde{x_1} \wedge \widetilde{x_2} \wedge \dots \wedge \widetilde{x_n}$ where $\widetilde{x_i} = \begin{cases} x_i & \textit{if } i^{th} \textit{component is } 1 \\ \overline{x_i} & \textit{if } i^{th} \textit{component is } 0 \end{cases}$

① table
② pick the rows for which $f=1$
③ write corresp row

$101 \to 1$
$110 \to 1$

$(x_1 \wedge \bar{x}_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge \bar{x}_3)$

How to obtain CNF?
Given a function $\{0,1\}^n \to \{0,1\}$, we can obtain a Boolean expression in DNF corresponding to this function by having a minterm corresponding to each ordered n tuple of 0s and 1s for which the value of the function is 0. For each n tuple with the functional value is 1, we have

the maxterm $\widetilde{x_1} \vee \widetilde{x_2} \vee \dots \vee \widetilde{x_n}$ where $\widetilde{x_i} = \begin{cases} x_i & \textit{if } i^{th} \textit{component is } 0 \\ \overline{x_i} & \textit{if } i^{th} \textit{component is } 1 \end{cases}$

① table
② pick rows for which $f=0$
③ write corresp row

$001 \to 0$

$x_1 \vee x_2 \vee \bar{x}_3$

Problem:

1. Let $E(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (\overline{x_2} \wedge x_3)$ be a Boolean expression defined on $(\{0,1\}, \vee, \wedge, -)$. Write the Boolean expression in both DNF and CNF.
Solution:

DNF $\to$ join of minterms ($\wedge$)
CNF $\to$ meet of maxterms ($\vee$)

x_1	x_2	x_3	$(x_1 \wedge x_2)$	$(x_1 \wedge x_3)$	$(\overline{x_2} \wedge x_3)$	f
0	0	0	0	0	0	0 ✓
0	0	1	0	0	1	1
0	1	0	0	0	0	0 ✓
0	1	1	0	0	0	0 ✓
1	0	0	0	0	0	0 ✓
1	0	1	0	1	1	1
1	1	0	1	0	0	1
1	1	1	1	1	0	1

DNF $\Rightarrow (\bar{x}_1 \wedge \bar{x}_2 \wedge x_3) \vee (x_1 \wedge \bar{x}_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge \bar{x}_3) \vee (x_1 \wedge x_2 \wedge x_3)$

CNF $\Rightarrow (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3)$

$\overline{a \vee b} = \bar{a} \wedge \bar{b}$

$\overline{a \wedge b} = \bar{a} \vee \bar{b}$

① $E = \overline{(\overline{x_1 \vee x_2}) \vee (\overline{x_1} \wedge x_3)}$

Soln

$E = (\overline{\overline{x_1 \vee x_2}}) \wedge (\overline{\overline{x_1} \wedge x_3})$ (∵ D'Morgan's)

$= (x_1 \vee x_2) \wedge (x_1 \vee \overline{x_3})$ (∵ D'Morgans)

$= \Big[\underbrace{x_1 \vee x_2}_{a} \vee (\underbrace{x_3}_{b} \wedge \underbrace{\overline{x_3}}_{c})\Big] \wedge \Big[\underbrace{x_1 \vee \overline{x_3}}_{a} \vee (\underbrace{x_2}_{b} \wedge \underbrace{\overline{x_2}}_{c})\Big]$ (∵ Missing variables)

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$= (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \overline{x_3}) \wedge (x_1 \vee \overline{x_3} \vee x_2) \wedge (x_1 \vee \overline{x_3} \vee \overline{x_2})$

CNF $= \boxed{(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \overline{x_3}) \wedge (x_1 \vee \overline{x_3} \vee \overline{x_2})}$

000, 001, 011 $\longrightarrow$ 0

DNF $= \begin{pmatrix} 100 \\ 101 \\ 110 \\ 111 \\ 010 \end{pmatrix} \longrightarrow \boxed{(x_1 \wedge \overline{x_2} \wedge \overline{x_3}) \vee (x_1 \wedge \overline{x_2} \wedge x_3) \vee (x_1 \wedge x_2 \wedge \overline{x_3}) \vee (x_1 \wedge x_2 \wedge x_3) \vee (\overline{x_1} \wedge x_2 \wedge \overline{x_3})}$

Prob

① $E(x_1, x_2, x_3, x_4) = (x_1 \wedge x_2 \wedge \overline{x_3}) \vee (x_1 \wedge \overline{x_2} \wedge x_4) \wedge (x_2 \wedge \overline{x_3} \wedge \overline{x_4})$

write the corresp CNF & DNF

② $E(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (\overline{x_2} \wedge x_3)$ Express in CNF & DNF

③ Express the boolean expression $E(x_1, x_2, x_3) = (\overline{x_1} \wedge x_2 \wedge \overline{x_3}) \vee (x_1 \wedge \overline{x_2}) \vee (x_1 \wedge x_3)$ as a conjunctive normal form and disjunctive normal form over $\{0,1\}$.