Group → closure, identity, inverse, Associative

subgroup →

cosets → $G \to$ group $\quad H \to$ subgroup

$a \in G$

$aH = \{ah \mid h \in H\}$

$Ha = \{ha \mid h \in H\}$

$G = (Z, +)$ is a group

$H = (Z_{2n}, +)$

$G = \{\ldots -4, -3, -2, -1, 0, 1, 2, 3, 4 \ldots\}$

$H = \{\ldots -4, -2, 0, 2, 4 \ldots\}$

$Ha \quad aH$

$3 \in G$

$3 + H = \{\ldots 3-4, 3-2, \; , 3, 5, 7, \ldots\}$

$H + 3 = \{\ldots -4+3, -2+3, 3, 5, 7, \ldots\}$

$1 + H = \{\ldots 1-4, 1-2, 1+0, 1+2, \ldots 1+4 \ldots\}$

$\Rightarrow \quad G = \{-1, 1, i, -i\} \qquad (G, \cdot)$

$H = \{1, -1\} \quad$ (Inv of $-1 \Rightarrow -1$

$Hi = \{i, -i\} \qquad H_1 = \{1, -1\}$

$H_{-i} = \{-i, i\} \qquad H_{-1} = \{-1, 1\}$

$iH, \; -iH, \; 1H, \; -1H$

$Hi = \{i, -i\}$

$H_1 = \{1, -1\}$

Lemma 1:

Let G be a group and H be a subgroup. Then any two right cosets of H in G are either identical or disjoint.

Similarly,

Any two left cosets of H in G are either identical or disjoint.

Lemma:

Any two right cosets of a subgroup H in a group G are in one-to-one correspondence with each other.

OR

Any two right cosets of H in G have same cardinality.

one to one corresp

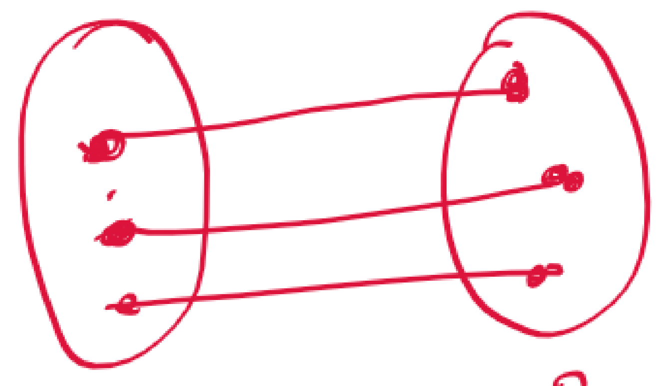
$A = \{1, 2, 3, 4\}$

$B = \{2, 5, 10, 17\}$

A & B

$f(x) \to$ one one onto

$f(x) = x^2 + 1$



one one fn :- If $f(x_1) = f(x_2)$ then $x_1 = x_2$

onto fn :- $\forall x \in B$, $\exists$ an elt $y \in A$ $f(y) = x$

Every elt of B has a pre image

not allowed

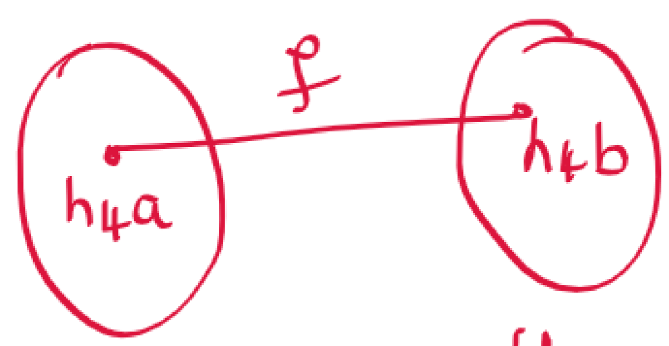
Bijective $\Rightarrow$ one one + onto

one to one corresp

$Ha \longrightarrow Hb$: one to one corresp b/w n

Any 2 right cosets have same cardinality

Define $f : Ha \to Hb$ by $f(ha) = hb$ $\forall h \in H$



i) To show f is one one

If $f(h_1a) = f(h_2a)$

$h_1b = h_2b$

$h_1 = h_2$ ($\because$ right canc law)

$h_1a = h_2a$

$\therefore$ one one

one one $\Rightarrow$ If $f(x_1) = f(x_2)$ $\Rightarrow x_1 = x_2$

ii) onto :-

for all $hb \in Hb$, there exists $ha \in Ha$ s.t $f(ha) = hb$ ($\because h \in H$)

onto

$\forall y \in B$, there exist $x \in A$ $f(x) = y$

f is one one & onto

$\therefore f$ is bijectn

Lemma:

Any two left cosets of a subgroup H in a group G are in one-to-one correspondence with each other.

OR

Any two left cosets of H in G have same cardinality.

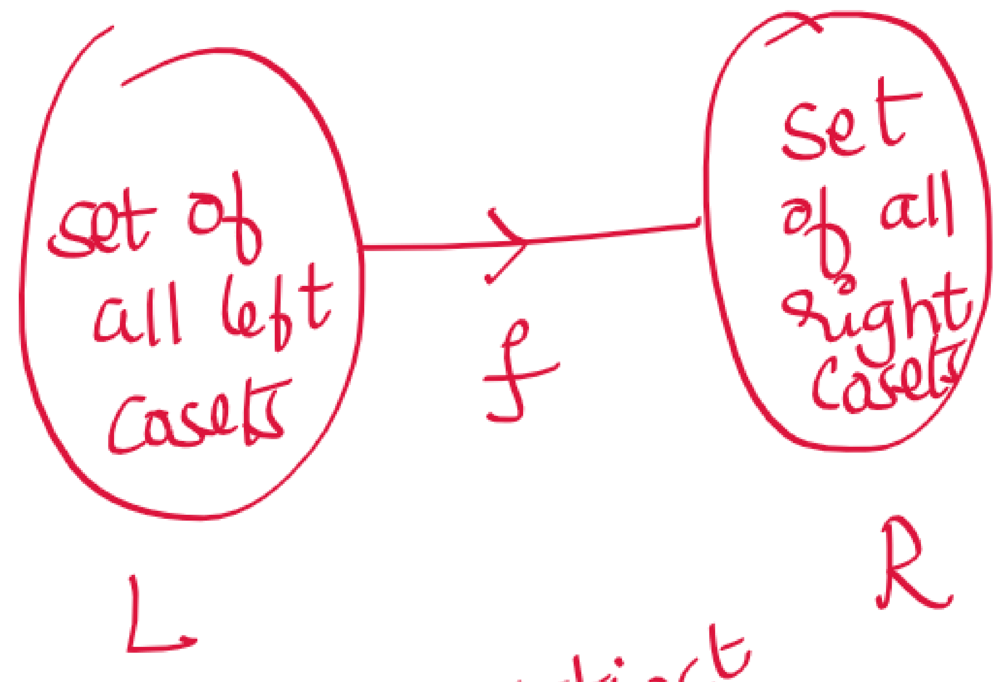
$(G, *)$ $(H, *)$

Lemma:

Suppose H is a subgroup of a group G. Then the number of distinct left cosets of H in G is equal to the number of distinct right cosets of H is G.

OR

There exists one-to-one correspondence between the set of all left cosets of H in G and the set of all right cosets of H is G.



proof:-

$$L = \{aH \mid a \in G\}$$

$$R = \{Ha \mid a \in G\}$$

Define $f : L \longrightarrow R$ by $f(aH) = Ha^{-1} \; \forall a \in G$

No of distinct left cosets = No of distinct right cosets

i) f is well defined:

well defined

if $x = y$, then $f(x) = f(y)$

not allowed

$x, y \in L$

If $aH = bH$

$b^{-1}aH = H$ (pre operatn of left by b^{-1})

$\Rightarrow b^{-1}a \in H$

$\Rightarrow (b^{-1}a)^{-1} \in H$ ($\because H$ is a subgp & inverse law is true)

$\Rightarrow a^{-1}b \in H$ ($\because (xy)^{-1} = y^{-1}x^{-1}$)

$Ha^{-1} = Hb^{-1}$

$\Rightarrow Ha^{-1}b = H$

post op on right by b^{-1}

$$Ha^{-1} = Hb^{-1}$$

ii) f is one one

$f(aH) = f(bH)$ $\qquad aH, bH \in L$

If $f(x) = f(y)$

$\Rightarrow x = y$

$x, y \in L$

$Ha^{-1} = Hb^{-1}$

post op by a on right

$aH = bH$

$Ha^{-1}a = Hb^{-1}a$

$H = Hb^{-1}a$

$b^{-1}a \in H$

$\Rightarrow (b^{-1}a)^{-1} \in H$ (Inverse law holds in H as its a gp

$\Rightarrow a^{-1}b \in H$ $\quad$ ($\because (b^{-1}a)^{-1} = a^{-1}b$

$a^{-1}b \in H$

$a^{-1}bH = H$

pre op on left by a

$bH = aH$

ii) onto :-

F.r any $Ha \in R$, $\exists$ $(a^{-1}H) \in L$ s.t

$f(a^{-1}H) = Ha$

| $\forall y \in R$, $\exists$ $x \in L$ s.t $f(x) = y$.

One to one corresp L & R

$\therefore$ $|L| = |R|$

No of distinct left cosets = No of distinct right cosets

Lemma:

If H is any subgroup of G, then G is equal to the union of all right cosets of H on G.

$\boxed{G = \bigcup Ha}$

Proof

$G \subseteq \bigcup Ha$

$\bigcup Ha \subseteq G$

$Ha \to$ subset of G
$\to$ may not be a subgp

$H_i = \{i, -i\}$ X

$H_{-i} = \{-i, i\}$

$H_1 = \{1, -1\}$

$H_{-1} = \{1, -1\}$

$G = \{1, -1, -i, i\}$

Each right coset is subset of G

$Ha \cup Hb \cup \ldots \subseteq G$

$$\bigcup_{a \in G} Ha \subseteq G \quad \text{--- (1)}$$

To prove $G \subseteq \bigcup Ha$

(take elt from G, prove that it belongs to $\bigcup Ha$ then $G \subseteq \bigcup Ha$)

$H = \{\ldots e, \ldots\}$

$ea = a$
$\downarrow$
Ha

Let $a \in G$

$a = ea \in Ha$

$$a \in \bigcup_{a \in G} Ha$$

$$G \subseteq \bigcup Ha \quad \text{--- (2)}$$

from (1) & (2)

$$\boxed{G = \bigcup_{a \in G} Ha}$$

Lagrange's theorem:

Let G be a finite group and H be a subgroup of G. Then the order of H divides the order of G.

$$o(H) \mid o(G)$$

$H \to$ subgp $\quad G \to$ gp

$$o(H) \mid o(G)$$

order $\Rightarrow$ cardinality

Soln

$$G = \bigcup_{a \in G} Ha$$

$H = \{1, -1\}$

$G = \{1, -1, i, -i\}$

$o(H) = 2$

$o(G) = 4$

$2 \mid 4$

$$o(G) = o(Ha) + o(Hb) + \dots + o(H_k)$$

$$= \underbrace{o(H) + o(H) + \dots + o(H)}_{K \text{ times}}$$

$$= \{o(H + o(H) + \dots + o(H)\}$$

$$o(G) = c\, o(H)$$

$$o(H) \mid o(G)$$