# COMBINATIONS AND PERMUTATIONS

Reference books;

1. C L Liu, Elements of Discrete Mathematics.
2. E S Page and L B Wilson, An introduction to Computational Combinatorics.
3. Alan Tucker, Applied Combinatorics.

# COMBINATIONS AND PERMUTATIONS

The number of $r$-permutations of $n$ objects in

- 1. With no repetition: ${}^{n}P_{r}$
- 2. With unlimited repetition : $n^r$
- 3.Permutation with restricted repetition: $\dfrac{n!}{m_1!m_2!\ldots m_k!}$ if $m_1$ objects are of the first kind, $m_2$ are of the second kind, ..., $m_k$ are of the $k^{th}$ kind.

The number of $r$-combinations of $n$ objects in

- 1. With no repetition: ${}^{n}C_{r}$
- 2. With unlimited repetition : ${}^{n+r-1}C_{r}$

# Distribution

Distributing $r$ distinct objects to $n$ distinct cells such that

- 1. Each cell has at most one object: ${}^{n}P_{r}$
- 2. Each cell to hold any number of objects: $n^{r}$

When the $r$ objects to be distributed are not all distinct (identical objects)

- 1. Such that each cell has at most one object: ${}^{n}C_{r}$
- 2. If we allow each cell to hold any number of objects: ${}^{n+r-1}C_{r}$

# Problems

Q1. Find the no of ways in which 3 exams can be scheduled in a 5 day period s.t(i) No two exams are scheduled on the same day?
(ii) There are no restrictions on number of exams conducted on a day?
Q2. Find the no of permutations of the word INSTITUTION?
(i) How many of them begin with I and end with N?
(ii) How many permutations are with 3 $Ts$ not together? How many of these begin with $I$?
Q3. In how many ways 3 integers can be selected from $3n$ consecutive integers such that the sum is a multiple of 3?
Q4. If 5 men A,B,C,D,E intend to speak at a meeting, (i) in how many orders can they do so without B speaking before A?
(ii) how many orders are there in which A speaks immediately before B?
Q5. A new national flag has to be designed with 6 verticl strips in yellow, green, blue and red. In how many ways can this be done s.t no two adjacent strips have same color?

Q6. In how many ways can 2 squares be selected one by one from $8 \times 8$ chess board such that they are not in the same row and same columns?
Q7. Find the sum of all 4 digit numbers that can be obtained using the digits 1,2,3,4 once in each?
Q8. How many positive integers less than one million can be formed using
(i) $7's, 8's, 9's$ only?
(ii) Using $0's, 8's, 9's$ only?
Q9. 6 distinct symbols are transmitted through a communication channel. A total of 12 blanks are to be inserted between the symbols with at least two blank spaces between every pair of symbols. In how many ways can we arrange symbols and blanks?
Q10. Three identical dice are rolled. How many outcomes can be recorded?
Q11. In how many ways can an examiner assign 30 Marks to 8 questions such that no question recieves less than 2 marks?

Q12.In how many ways can two adjacent squares can be selected from an $8 \times 8$ chess board?

Q13.Among all 7 digits numbers, how many of them contain exactly three 9s?

Q14. How many points of intersection are formed by n lines drawn in a plane if no two are parallel and no three concurrent? Into how many regions is the plane divided?

Q15. **Suppose we print all FIVE-digit numbers on slips of paper with one number on each slip.** However, since the digits 0, 1, 6, 8, and 9 become 0, 1, 9, 8, and 6 when they are read upside down, there are pairs of numbers that can share the same slip if the slips are read right side up or upside down. For example, we can make up one slip for the numbers 89166 and 99168. Then **how many distinct slips will we have to make up for all five-digit numbers?**

Solution:

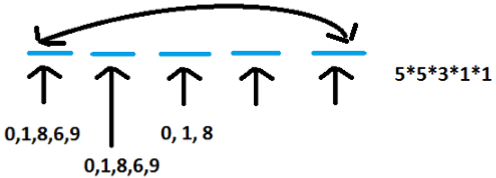
There are $10^5$ distinct five-digit numbers.

Among these numbers, $5^5$ of them can be read either right side up or upside down. (They are made up of the digits 0, 1, 6, 8, and 9.)

Example: 16808 and 80891.

But, there are numbers that read the same either right side up or upside down, for example, 16091,80108,61819.

How many such numbers possible????




There are $3(5^2)$ such numbers which can be read the same even when u read upside down.

Consequently, there are $5^5 - 3(5^2)$ numbers that can be read either right side up or upside down but will read differently. These numbers can be divided into pairs so that every pair of numbers can share one slip.

The total number of distinct slips we need is

$$10^5 - \left( \frac{5^5 - 3(5^2)}{2} \right).$$

Q16. In how many ways can a lady wear five rings on the fingers (not the thumb) of her right hand?

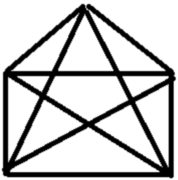
Ans: Initially assume the 5 rings as identical.

Then distribution of 5 identical rings(objects) in to 4 distinct fingers (cells) such that each finger can hold any number of rings is ${}^{4+5-1}C_5 = {}^{8}C_5$.

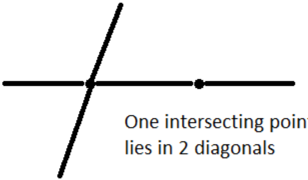
As 5 rings can be arranged in 5! ways the answer is $5!({}^{8}C_5) = 6720$.

Q17. If no three diagonals of a convex decagon meet at the same point inside the pentagon (or decagon), into how many line segments are the diagonals divided by their intersections?

Ans:



2 intersecting points
having 3 line segments



One intersecting point
lies in 2 diagonals

Total number of diagonals: ${}^5C_2 - 5$
Total number of intersection points: ${}^5C_4$ (For every subset of 4 vertices, there exists exactly one intersection point)
A diagonal is divided into $(k+1)$ line segments when there are $k$ intersection points on it.
Each intersecting point lies on two diagonals. Thus the total no of line segments are
$({}^5C_2 - 5) + ({}^5C_4)2 = 15$

When it is an $n-$ sided polygon, then the total number of line segments is $\left({}^nC_2 - n\right) + 2\left({}^nC_4\right)$

Q18. In how many ways can 5 different messages be delivered by 3 messengers (A,B,C) if no messenger is left unemployed. The order in which a messenger delivers his messages is immaterial?
Solution:
The options are

- 1 person has 3 and other two have 1 each $(113, 311, 131)$: $3 \times \frac{5!}{3!}$
- 1 person has 1 and other 2 have 2 each $(122, 211, 112)$: $3 \times \frac{5!}{2!2!}$

Ans: $3\dfrac{5!}{3!1!1!} + 3\dfrac{5!}{2!2!1!} = 150.$

Identical forms of combinations with repititions:

- No of ways of selecting $r$ identical objects from $n$ objects with repetition
- Distibute $r$ identical objects into $n$ distinct boxes such that each box can have any number of objects
- Number of non negative solutions of $x_1 + x_2 + \cdots + x_n = r$

# Generating function

Generating function:Let $\{a_r\}$ be a sequence. A function $f(x)$ is said to be a generating function for the sequence $\{a_r\}$ if $f(x) = \sum_{n=0}^{\infty} a_n x^n$.
The co-eff of $x^r$ in the expansion of $f(x)$ is $a_r$.

Exponential generating function:If the terms of the seuence are obtained by taking the co-eff of $\frac{x^r}{r!}$ in the expansion of $f(x)$, then $f(x)$ is said to be a generating function.

# Generating function for Combinations without repetition:

Consider the three distinct objects a, b and c, and form the polynomial $(1+ax)(1+bx)(1+cx) = 1+(a+b+c)x+(ab+bc+ca)x^2+(abc)x^3$.

- Co-eff of $x^1$=ways of selecting one object(a or b or c) out of 3.
- Co-eff of $x^2$=ways of selecting 2 object(ab or bc or ca)
- Co-eff of $x^3$=ways of selecting 3 objects(abc)
- In LHS, $(1+ax)$ symbolically represents 2 things, i.e nonselection and selection of the object $a$.
- Since we want to enumerate rather than the objects, we take $a=b=c=1$ and get $(1+x)^3 = 1+3x+3x^2+x^3$

In general, if there are $n$ objects,

$$(1+a_1x)(1+a_2x)\dots(1+a_nx) = 1 + (a_1+a_2+\dots+a_n)x \\ +(a_1a_2+a_2a_3+\dots)x^2+\dots$$

- Co-eff of $x^r$=ways of selecting $r$ objects
- For the sake of enumeratiopn, we take $a_i = 1$ and get

$$\begin{aligned}(1+x)^n &= 1 + nx + \frac{n(n-1)}{2}x^2 + \dots + x^n \\ &= \sum_{r=0}^{n} {}^nC_r x^r\end{aligned}$$

- Thus $(1+x)^n$ **is the g.f for $r-$ combination of $n$ distinct objects without repetition.**
- A gf used this way is called **enumerator**.

**Example:**

If one object can be selected at most once, the second object at most twice, and the third object at most three times? then the enumerator is $(1+x)(1+x+x^2)(1+x+x^2+x^3) = 1+3x+5x^2+6x^3+5x^4+3x^5+x^6$.

# Generating function for combination with repetition

For the first object, the corresponding factor is $(1 + x + x^2 + x^3 + \dots) = (1 - x)^{-1}$.

**If there are $n$ objects, the enumerator is** $(1 - x)^{-n} = \sum_{r=0} {}^{n+r-1}C_r x^r$.

(verify)

Q11. In how many ways can an examiner assign 30 Marks to 8 questions such that no question recieves less than 2 marks?
Solution:

$(x^2 + x^3 + \dots)^8$ is the enumerator.
The no of ways is the co-eff of $x^{30}$ in the above enumerator is ${}^{21}C_{14}$

HOW TO EVALUATE THE CO-EFICIENTS???

- $1 + x + x^2 + ... = (1 - x)^{-1}$.
- $1 + x + x^2 + \cdots + x^{r-1} = \frac{1-x^r}{1-x}$
- $(1 + x)^n = \sum\limits_{r=0}^{n} {}^nC_r x^r$
- $(1 - x^m)^n = \sum\limits_{r=0}^{n} {}^nC_r (x^m)^r$.
- $(1 - x)^{-n} = \sum\limits_{r=0}^{n} {}^{n+r-1}C_r x^r$
- If $f(x) = a_0 + a_1 x + \cdots + a_r x^r + \ldots$ and $g(x) = b_0 + b_1 x + \cdots + b_r x^r + \ldots$, then the product is given by $f(x)g(x) = \cdots + (a_r b_0 + a_{r-1} b_1 + a_{r-2} b_2 + \cdots + a_r b_0)x^r + \ldots$

19. How many ways are there to select 25 toys from 7 types of toys with between 2 to 6 of each type?

20. Find the no of ways to select 10 balls from a large pile of red, white and blue balls if the selection has

(1) at least two balls of each color?

(2) at most two red balls?

# THANK YOU