Groups $\rightarrow$ $(G, *)$ — closure
associative
identity
inverse law

Subgroup :- $(G, *)$, H be a subset of G. H is a subgp of G if 'H' itself is a group w.r.to the same operatn :- $H \leq G$

ex:- $(\mathbb{Z}, +)$, $(\mathbb{R}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{C}, +)$

$(\mathbb{Z}, +)$ is subgp of $(\mathbb{Q}, +)$
$(\mathbb{Q}, +)$ " $(\mathbb{R}, +)$
$(\mathbb{R}, +)$ is " $(\mathbb{C}, +)$

ex:- $(\mathbb{Q}-\{0\}, \cdot)$ is group
Eventhough $\mathbb{Q} \subseteq \mathbb{R}$, $(\mathbb{Q}-\{0\}, \cdot)$ is the subgroup of $(\mathbb{R}, +)$. Bcz the operatns are different
$(\mathbb{Q}-\{0\}, \cdot)$ is a subgp of $(\mathbb{R}-\{0\}, \cdot)$

ex:- $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle/ \underbrace{ad - bc \neq 0}_{\text{To satisfy inverse law}} \right\}$

$(S, \cdot)$ is a group $(\cdot \rightarrow x^n)$ ✓ $e = I$

associative
$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in W$
$I \in W$ when $a=1$ $b=0$ $d=1$

$W = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle/ ad \neq 0 \right\}$

$(W, \cdot)$ is a subgp of $(S, \cdot)$

Theorem :

A non-empty subset H of a group $(G,*)$ is a subgroup of G if and only if the follow conditions are satisfied

i. $a * b \in H$ for all $a, b \in H$ → closure
ii. $a^{-1} \in H$ for all $a \in H$ → inverse law

Statement A $\Longleftrightarrow$ Statement B

$A \Rightarrow B$

$B \Rightarrow A$

Proof :-

Let $(H,*)$ be a subgp of $(G,*)$. I have to prove (i) (ii) are true

$\Rightarrow (H,*)$ itself is a group

$\Rightarrow$ All 4 laws are satisfied

$\Rightarrow \forall\ a,b \in H,\ a*b \in H$ (By closure law) → (i) holds good

$\Rightarrow \forall\ a \in H,\ a^{-1} \in H$ (Inverse law) → (ii) holds good

$\therefore$ (i) & (ii) are true

Converse :-

Let (i) & (ii) are true. I've prove that $(H,*)$ subgroup of $(G,*)$

$\Rightarrow$ I've prove $(H,*)$ itself is a group

From (i) $\Rightarrow$ closure law is true ✓

(ii) $\Rightarrow$ Inverse law is true ✓

Associativity is true since H is a subset of G ✓

From (i) $\Rightarrow \forall\ a, b \in H$, $a*b \in H$

$a \in H,\ a^{-1} \in H \Rightarrow a * a^{-1} \in H$ (chosen $b = a^{-1}$)

$e \in H$

$\therefore$ Identity law holds ✓

$\therefore (H,*)$ is subgp of $(G,*)$

Theorem :

A non-empty subset H of a group $(G,*)$ is a subgroup of G if and only if $a * b^{-1} \in H$ for all $a, b \in H$ (1)

$(H,*)$ subgp $\Longleftrightarrow \forall a,b \in H$ $a*b^{-1} \in H$

Proof:-

Let $(H,*)$ be a subgp of $(G,*)$. I've to prove that (1) is true

$(H,*)$ satisfies all 4 laws ($\because$ it's a group)

$\forall a, b \in H \Rightarrow b^{-1} \in H$ (Inverse law)

$a, b^{-1} \in H$, By closure law, it follows that $a*b^{-1} \in H$

$\therefore$ (1) is true

Converse:-

Given (1) is true. I've to P.T $(H,*)$ is a subgroup of $(G,*)$

$\forall a,b \in H$, its true that $a*b^{-1} \in H$. I've prove all 4 laws

Since H is nonempty subset, there is atleast one elt $a \in H$

i) $a \in H$, $a \in H$, its true that $a * a^{-1} \in H$ (choose $b=a$)

$e \in H$ $\therefore$ Identity elt exists in H

$\therefore$ Identity law holds

ii) $e \in H$, $a \in H$, it's true that $e * a^{-1} \in H$

we already prove identity elt exists in H,

$a^{-1} \in H$

$\therefore$ inverse law is true

iii) Associativity is true as H is subset of G

iv) $a \in H$, $b^{-1} \in H$, its true that $a*(b^{-1})^{-1} \in H$

we already proved inverse of 'b' exists

$a * b \in H$

$\therefore$ closure law satisfied.

$\therefore$ $(H,*)$ is a subgp

Problem :

Let $(G,*)$ be a group and let H_1, H_2 be subgroups of G. Check if

i. $H_1 \cap H_2$ is a subgroup

ii. $H_1 \cup H_2$ is a subgroup

$(G,*)$ is a gp

$(H_1,*)$ and $(H_2,*)$ are subgroups

Soln

i) since H_1 & H_2 are subgps. All 4 laws are true in $(H_1,*)$ & $(H_2,*)$

$\forall\, a,b \in H$, $a * b^{-1} \in H$

obviously $H_1 \cap H_2$ is nonempty. $e \in H_1$ & $e \in H_2 \Rightarrow e \in H_1 \cap H_2$

To check $H_1 \cap H_2$ is a subgp, check $a,b \in H_1 \cap H_2$

check if $a * b \in H_1 \cap H_2$

consider $a,b \in H_1 \cap H_2 \Rightarrow a,b \in H_1$ & $a,b \in H_2$

$\underbrace{a * b^{-1} \in H_1}_{(H_1 \text{ is a subgp})}$ & $a * b^{-1} \in H_2$ (H_2 is a subgp)

$a * b^{-1}$ belongs to both H_1 & H_2

$\therefore a * b^{-1} \in H_1 \cap H_2$

$\therefore H_1 \cap H_2$ is a subgp

(ii) $H_1 \cup H_2$ is not a subgp of $(G,*)$

$(\mathbb{Z},+)$ is a group

$H_1 = \mathbb{Z}_{2n} = \{\ldots -4, -2, 0, 2, 4, \ldots\}$ $(H_1,+)$ is a group

$H_2 = \mathbb{Z}_{3n} = \{\ldots -6, -3, 0, 3, 6, \ldots\}$ $(H_2,+)$ is group

$\therefore (H_1,+)$ & $(H_2,+)$ are subgps of $(\mathbb{Z},+)$

$H_1 \cup H_2 = \{\ldots -6, -4, -3, -2, 0, 2, 3, 4, 6, \ldots\}$ $(H_1 \cup H_2, +)$

$2, 3 \longrightarrow 2+3 = 5 \notin H_1 \cup H_2$

closure fails

$H_1 \cup H_2$ is not a subgp

Theorem :

Let $(H,.)$ And $(K,.)$ be two subgroups of the group $(G,.)$. Define

$$HK = \{hk \mid h \in H, k \in K\}$$

Then HK is a subgroup of $(G,.)$ if and only if $HK = KH$.

HK subgp $\Longleftrightarrow$ $HK = HK$.

Proof

Let HK is a subgp of $(G,*)$. I've to P.T $HK = KH$

($HK \subseteq KH$
$HK \supseteq KH$
Then $HK = KH$)

Let $x \in KH \Rightarrow x = k_1 h_1$ where $h_1 \in H$, $k_1 \in K$

$$x^{-1} = (k_1 h_1)^{-1} = h_1^{-1} k_1^{-1} \in HK$$

$(a*b)^{-1} = b^{-1} * a^{-1}$

k_1^{-1} ↳ some elt of K

h_1^{-1} ↳ some elt of H

↳ H is a subgp

$x^{-1} \in HK$

since HK is subgp $\Rightarrow (x^{-1})^{-1} \in HK$

ie $(x^{-1})^{-1} = x = k_1 h_1 \in HK$

$\therefore$ Let $x \in KH \Longrightarrow x \in HK \Longrightarrow \boxed{KH \subseteq HK}$

Illarly we prove $\boxed{HK \subseteq KH}$ $\therefore \boxed{HK = KH}$

converse :- Let $HK = KH$. I've to P.T HK is a subgp

$(a, b \in HK \Rightarrow a*b^{-1} \in HK)$

Since $e \in H$, $e \in K \Rightarrow e \cdot e \in HK$

$\therefore$ HK is nonempty

$a, b \in HK \Rightarrow a = h_1 k_1$ & $b = h_2 k_2$ where $h_1 h_2 \in H$, $k_1 k_2 \in K$

consider $a \cdot b^{-1} = h_1 k_1 \cdot (h_2 k_2)^{-1} = h_1 \underbrace{k_1 k_2^{-1} h_2^{-1}}$

$= \underbrace{h_1 h_3} k_3$

$= h_4 k_3$

($k_1 k_2^{-1} h_2^{-1} \in KH$
Since $KH = HK$
$k_1 k_2^{-1} h_2^{-1} \in HK$
$\therefore k_1 k_2^{-1} h_2^{-1} = h_3 k_3$)

$\therefore ab^{-1} \in HK$

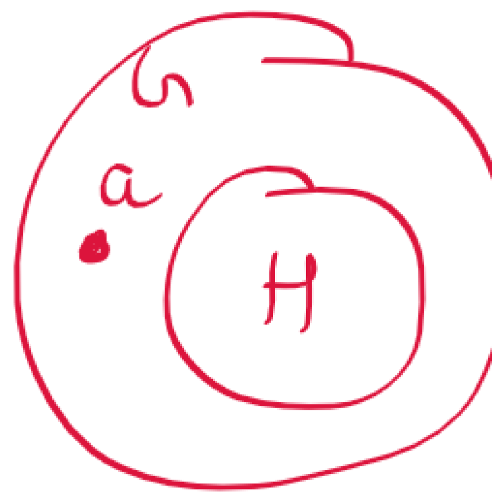
Cosets:

Let $(G,.)$ be a group and H be a subgroup of G. For any $a \in G$, the set

$Ha = \{ha \mid h \in H \text{ and } a \in G\}$ ------ is called as the right coset of H in G

$aH = \{ah \mid h \in H \text{ and } a \in G\}$ ------ is called as the right coset of H in G

$(G,\cdot)$ is gp
$(H,\cdot)$ is subgp of $(G,\cdot)$
$a \in G$



$$Ha = \{ha \mid h \in H\}$$
$$aH = \{ah \mid h \in H\}$$

ex:- $G = \{1, -1, i, -i\}$ $\Rightarrow$ $(G,\cdot)$ is a group

$e \Rightarrow 1$

$\text{inv } 1 = 1$

$\text{inv}(-1) = -1$

$\text{inv}(i) = -i$

$\text{inv}(-i) = i$

$-1 \times -1 = 1$

$i \times -i = 1$

$H = \{1, -1\}$ we note that $(H,\cdot)$ is a subgp of $(G,\cdot)$

$i \in G$

$Hi = \{i, -i\}$ is right coset

② $(\mathbb{Z},+)$ is a gp

$(\mathbb{Z}_{2n},+)$ is subgp of $(\mathbb{Z},+)$

$H = \mathbb{Z}_{2n} = \{\ldots -4, -2, 0, 2, 4, \ldots\}$

$a=3$

$H+3 = \ldots\{\ldots, -4+3, -2+3, 0+3, 2+3, 4+3, \ldots\}$

$3+H = \ldots\{\ldots, 3-4, 3-2, 3+0, \ldots\}$

Left coset $\Rightarrow$ $a*H = \{a*h \mid h \in H\}$

right coset $\Rightarrow$ $H*a = \{h*a \mid h \in H\}$

H is subgroup

Theorem:

Let $(G, .)$ be a group and H be a subgroup of G. Then any two right cosets of H in G are either identical or disjoint.

$Ha \quad Hb$

Either $Ha = Hb$

or $Ha \cap Hb = \phi$

Proof :-

Let $a, b \in G$

Ha, Hb be two right cosets of H in G

If Ha & Hb are disjoint, there is nothing to prove

If Ha & Hb are not disjoint $\Rightarrow$ $Ha \cap Hb \neq \phi$ — I've prove they are identical

$x \in Ha \cap Hb$

$\Rightarrow x \in Ha$ & $x \in Hb$

$\Rightarrow x = h_1 a$ for some $h_1 \in H \longrightarrow (\because x \in Ha$

$x = h_2 b$ for some $h_2 \in H$ $(\because x \in Hb$

$h_1 a = h_2 b$

operate on left by h_2^{-1}

$\boxed{h_2^{-1} h_1 a = b}$

prove that $Ha = Hb \Rightarrow (Ha \subseteq Hb$ & $Hb \subseteq Ha)$

Let $y \in Hb$

$y = h_3 b$ where $h_3 \in H$

$= h_3 h_2^{-1} h_1 a$ $(\because b = h_2^{-1} h_1 a$

$= h_4 a$ where $h_4 = h_3 h_2^{-1} h_1 \in H$

$y \in Ha$

$\therefore Hb \subseteq Ha$

similarly $Ha \subseteq Hb$

$\therefore \underline{Ha = Hb}$