COMBINATIONS AND PERMUTATIONS

Reference books;
1. C L Liu, Elements of Discrete Mathematics.
2. E S Page and L B Wilson, An introduction to Computational Combinatorics.
3. Alan Tucker, Applied Combinatorics.

The number of r-permutations of n objects in

- 1. With no repetition: ${}^{n}P_{r}$
- 2. With unlimited repetition : n^r
- 3.Permutation with restricted repetition: $\dfrac{n!}{m_1!m_2!\ldots m_k!}$ if m_1 objects are of the first kind, m_2 are of the second kind, ..., m_k are of the k^{th} kind.

The number of r-combinations of n objects in

- 1. With no repetition: ${}^{n}C_{r}$
- 2. With unlimited repetition : ${}^{n+r-1}C_{r}$

Distribution

Distributing r distinct objects to n distinct cells such that

- 1. Each cell has at most one object: ${}^{n}P_{r}$
- 2. Each cell to hold any number of objects: n^{r}

When the r objects to be distributed are not all distinct (identical objects)

- 1. Such that each cell has at most one object: ${}^{n}C_{r}$
- 2. If we allow each cell to hold any number of objects: ${}^{n+r-1}C_{r}$

Problems

Q1. Find the no of ways in which 3 exams can be scheduled in a 5 day period s.t(i) No two exams are scheduled on the same day?
(ii) There are no restrictions on number of exams conducted on a day?
Q2. Find the no of permutations of the word INSTITUTION?
(i) How many of them begin with I and end with N?
(ii) How many permutations are with 3 Ts not together? How many of these begin with I?
Q3. In how many ways 3 integers can be selected from $3n$ consecutive integers such that the sum is a multiple of 3?
Q4. If 5 men A,B,C,D,E intend to speak at a meeting, (i) in how many orders can they do so without B speaking before A?
(ii) how many orders are there in which A speaks immediately before B?
Q5. A new national flag has to be designed with 6 verticl strips in yellow, green, blue and red. In how many ways can this be done s.t no two adjacent strips have same color?

Q6. In how many ways can 2 squares be selected one by one from 8×8 chess board such that they are not in the same row and same columns?
Q7. Find the sum of all 4 digit numbers that can be obtained using the digits 1,2,3,4 once in each?
Q8. How many positive integers less than one million can be formed using (i) $7's, 8's, 9's$ only? (ii) Using $0's, 8's, 9's$ only?
Q9. 6 distinct symbols are transmitted through a communication channel. A total of 12 blanks are to be inserted between the symbols with at least two blank spaces between every pair of symbols. In how many ways can we arrange symbols and blanks?
Q10. Three identical dice are rolled. How many outcomes can be recorded?
Q11. In how many ways can an examiner assign 30 Marks to 8 questions such that no question recieves less than 2 marks?
Q12.In how many ways can two adjacent squares can be selected from an 8×8 chess board?
Q13.Among all 7 digits numbers, how many of them contain exactly three 9s?

Q14. How many points of intersection are formed by n lines drawn in a plane if no two are parallel and no three concurrent? Into how many regions is the plane divided?

Q15. Suppose we print all FIVE-digit numbers on slips of paper with one number on each slip. However, since the digits 0, 1, 6, 8, and 9 become 0, 1, 9, 8, and 6 when they are read upside down, there are pairs of numbers that can share the same slip if the slips are read right side up or upside down. For example, we can make up one slip for the numbers 89166 and 99168. Then how many distinct slips will we have to make up for all five-digit numbers?

Q16. In how many ways can a lady wear five rings on the fingers (not the thumb) of her right hand?

Q17. If no three diagonals of a convex decagon meet at the same point inside the pentagon (or decagon), into how many line segments are the diagonals divided by their intersections?

Q18. In how many ways can 5 different messages be delivered by 3 messengers (A,B,C) if no messenger is left unemployed. The order in which a messenger delivers his messages is immaterial?

- $(1+x)^n$ is the g.f for $r-$ combination of n identical objects without repetition.

$$(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$$

coefl of x^r is ${}^nc_r$

- $(1+x)^n$ **is the g.f for** $r-$ **combination of** n **identical objects without repetition.**

$$(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$$

- $(1-x)^{-n}$ **is the g.f for** $r-$ **combination of** n **identical objects with repetition.**

$$(1-x)^{-n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r x^r.$$

- $1 + x + x^2 + ... = (1-x)^{-1}$.
- $1 + x + x^2 + \cdots + x^{r-1} = \frac{1-x^r}{1-x}$
- $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$
- $(1-x^m)^n = \sum_{r=0}^{n} {}^nC_r (x^m)^r (-1)^r$
- $(1-x)^{-n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r x^r$
- If $f(x) = a_0 + a_1 x + \cdots + a_r x^r + \ldots$ and $g(x) = b_0 + b_1 x + \cdots + b_r x^r + \ldots$, then the product is given by $f(x)g(x) = \cdots + (a_r b_0 + a_{r-1} b_1 + a_{r-2} b_2 + \cdots + \underbrace{a_r b_0}_{a_0 b_r}) x^r + \ldots$

19. How many ways are there to place an order of 12 scoops of ice cream if there are 5 types of flavors and at most 4 scoops of each flavour is allowed?

Soln

5 flavours → select 12 scoops

1st :- $(1+x+x^2+x^3+x^4)$

$gf = f(x) = (1+x+x^2+x^3+x^4)^5$

coeff of x^{12}

$$f(x) = (1+x+\cdots+x^4)^5$$

$$= \left(\frac{1-x^5}{1-x}\right)^5 = (1-x^5)^5(1-x)^{-5}$$

$$= \sum {}^5C_r(-x^5)^r \sum {}^{5+r-1}C_r x^r$$

$$= \sum (-1)^r \, {}^5C_r x^{5r} \sum {}^{5+r-1}C_r x^r$$

coeff of x^{12}

$$f(x) = \sum (-1)^r \, {}^5C_r \, x^{5r} \sum {}^{5+r-1}C_r \, x^r$$

$$a_{12}b_0 + a_{11}b_1 + \cdots + a_0 b_{12}$$

$a_i \to$ coeff of x^r

$$\underbrace{a_0}_{r=0}\underbrace{b_{12}}_{r=12} + \underbrace{a_5}_{r=1}\underbrace{b_7}_{r=7} + \underbrace{a_{10}}_{r=2}\underbrace{b_2}_{r=2}$$

$$a_0 b_{12} + a_5 b_7 + a_{10} b_2$$

$$^5C_0 \; ^{5+12-1}C_{12} + {^5C_1} (-1)^7 \; ^{7+5-1}C_7$$

$$+ {^5C_2} (-1)^2 \; ^{5+2-1}C_2$$

$$^5C_0 \, {^{16}C_{12}} - {^5C_1} \, {^{11}C_7} + {^5C_2} \, {^6C_2}$$

20. Find the no of ways to select 10 balls from a large pile of red, white and blue balls if the selection has
(1) at least two balls of each color?
(2) at most two red balls?

Soln

:- select 10 ball from red + blue + white

-

coeff of x^{10}

(1) red $\rightarrow (x^2 + x^3 + \dots)$

1) $f(x) = (x^2 + x^3 + \dots)^3$

coeff of x^{10}

$$f(x) = (x^2)^3 (1 + x + x^2 + \dots)^3$$

$$= x^6 (1-x)^{-3}$$

$$= \underset{\sim}{x^6} \underbrace{\sum {}^{3+r-1}C_r x^r}_{\text{coeff of } x^4}$$

$\therefore$ coeff is $\Rightarrow {}^{3+4-1}C_4 = {}^{6}C_4$

(2) atmost 2 red balls

$\times$ blue & white

$$f(x) = (1+x+x^2)(1+x+x^2+\ldots\ldots)^2$$

$$= (1+x+x^2)(1-x)^{-2}$$

$$= (1+x+x^2)(1-x)^{-2}$$

$$= (1-x)^{-2} + x\,(1-x)^{-2} + x^2(1-x)^{-2}$$

$$= \sum {}^{2+r-1}C_r\, x^r + x\sum {}^{2+r-1}C_r\, x^r$$

$$+ x^2 \sum {}^{2+r-1}C_r\, x^r$$

coeff of x^{10}

$${}^{2+10-1}C_{10} + {}^{2+9-1}C_9 + {}^{2+8-1}C_8$$

enumerator $\Rightarrow$ coeff x^r

a_r

exponential gf $\Longrightarrow$ coeff of $\dfrac{x^r}{r!}$

a_r

Exponential generating function for permutation with no repetition

We know that

$$(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r \qquad \text{(gf of } {}^nC_r\text{)}$$

$$= \sum_{r=0}^{n} \frac{{}^nP_r}{r!} x^r \qquad {}^nC_r = \frac{{}^nP_r}{r!}$$

Co efficient of $\frac{x^r}{r!}$ is ${}^nP_r$

$$\frac{x}{1!} = {}^nP_1 \quad \Big| \quad \frac{x^2}{2!} \Rightarrow {}^nP_2$$

Thus $(1+x)^n$ is the exponential gf for ${}^nP_r$, permutation with repetition

Exponential generating function for permutation with no repetition

We know that

$$(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$$

Co efficient of $\frac{x^r}{r!}$ is ${}^nP_r$

Thus $(1+x)^n$ is the exponential gf for ${}^nP_r$, permutation with repetition

Exponential generating function for permutation with no repetition

We know that

$$
\begin{aligned}
(1+x)^n &= \sum_{r=0}^{n} {}^nC_r x^r \\
&= \sum_{r=0}^{n} \frac{{}^nP_r}{r!} x^r
\end{aligned}
$$

Co efficient of $\frac{x^r}{r!}$ is ${}^nP_r$

Exponential generating function for permutation with no repetition

We know that

$$
\begin{aligned}
(1+x)^n &= \sum_{r=0}^{n} {}^nC_r x^r \\
&= \sum_{r=0}^{n} \frac{{}^nP_r}{r!} x^r
\end{aligned}
$$

Co efficient of $\frac{x^r}{r!}$ is ${}^nP_r$

Thus $(1+x)^n$ is the exponential gf for ${}^nP_r$, permutation with repetition

Exponential generating function for permutation without repetition

If repetition is allowed, the factor ccorresponding to one object is

$$\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)=e^x$$

n obj's $\Rightarrow$ $(e^x)^n$

$$e^{nx}=\sum\frac{(nx)^r}{r!}$$

Exponential generating function for permutation without repetition

If repetition is allowed, the factor ccorresponding to one object is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots =$$

Exponential generating function for permutation without repetition

If repetition is allowed, the factor ccorresponding to one object is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x$$

Exponential generating function for permutation without repetition

If repetition is allowed, the factor ccorresponding to one object is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x$$

Thus, for n objects, it is e^{nx}

Exponential generating function for permutation ~~without~~ with repetition

If repetition is allowed, the factor ccorresponding to one object is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x$$

Thus, for n objects, it is e^{nx}
That is,

$$e^{nx} = 1 + \frac{(nx)}{1!} + \frac{(nx)^2}{2!} + \cdots = \sum_{r=0}^{\infty} \frac{(nx)^r}{r!}$$

Exponential generating function for permutation without repetition

If repetition is allowed, the factor ccorresponding to one object is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x$$

Thus, for n objects, it is e^{nx}
That is,

$$e^{nx} = 1 + \frac{(nx)}{1!} + \frac{(nx)^2}{2!} + \cdots = \sum_{r=0}^{\infty} \frac{(nx)^r}{r!}$$

Thus e^{nx} is the exponential gf n^r, permutation with repetition

Generating functions for Combinations (Enumerators)

- Without repetition: $(1+x)^n$ $\rightarrow (1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$
- With repetition: $(1-x)^{-n}$ $\rightarrow (1-x)^{-n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r x^r$

Generating functions for Permutations (Exponential gf)

- Without repetition: $\rightarrow (1+x)^n$ $\rightarrow (1+x)^{n} = \sum_{r=0}^{n} \frac{{}^nP_r}{r!} x^r$
- With repetition: e^{nx} $\rightarrow e^{nx} = \cancel{\sum_{r=0}^{\infty} \frac{x^r}{r!}}$

$$e^{nx} = \sum_{r=0}^{\infty} \frac{(nx)^r}{r!}$$

Q21. Find how many $r-$ digit ternary sequences (3 dig sequences made of 0, 1 and 2) are there with (i). Even n of 0s?
(ii). Even no of 0s and even no of 1s?

Soln

i) even no of 0s & 1,2 have no restn

$$f(x) = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)^2$$

coeff of $\dfrac{x^r}{r!}$

$$f(x) = \quad 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots = \frac{e^x+e^{-x}}{2} \quad \dots$$

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots = e^x$$

$$f(x) = \left(\frac{e^x+e^{-x}}{2}\right)\left(e^x\right)^2 = \frac{e^{2x}}{2}\left[e^x+e^{-x}\right]$$

$$f(x) = \frac{1}{2}\left[e^{3x} + e^{x}\right]$$

$$= \frac{1}{2}\left[\sum \frac{(3x)^n}{n!} + \sum \frac{x^n}{n!}\right]$$

coeff $\frac{x^n}{n!}$

$$e^x = \sum \frac{x^n}{n!}$$

$$\text{Ans} = \frac{1}{2}\left[3^n + 1\right]$$

ii) Even no of 0's, Even no of 1's

$$f(x) = \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots\right)^2\left(1+x+\frac{x^2}{2!}+\dots\right)$$

$$= \left(\frac{e^x+e^{-x}}{2}\right)^2 e^x$$

$$= \frac{e^x}{4}\left[(e^x)^2+(e^{-x})^2+2e^xe^{-x}\right]$$

$$= \frac{1}{4}\left[e^{3x}+e^{-x}+2e^x\right]$$

$$= \frac{1}{4}\left[\sum \frac{(3x)^r}{r!} + \sum \frac{(-x)^r}{r!} + 2\sum \frac{(x)^r}{r!}\right]$$

coeff of $\frac{x^r}{r!}$

$$\text{Ans} = \frac{1}{4}\left[3^r + (-1)^r + 2\right]$$

22. How many 10 letter words are there with each of the letters e, n, r, s occur (i). at most once
(ii). at least once

Soln

i) $\underbrace{(1+x)^4}\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^{22}$

$(1+x)^2(1+x)^2\left(e^x\right)^{22}$

$(1+x^2+2x)(1+x^2+2x)e^{22x}$

$$= (x^4+4x^3+6x^2+4x+1)\sum \frac{(22x)^n}{n!}$$

coeff of $\frac{x^{10}}{10!}$

$$\frac{22^6}{6!}\times 10! + \frac{4(22)^7}{7!}\times 10!$$

$$+ \frac{6(22)^8}{8!}\times 10! + \frac{4(22)^9}{9!}\times 10!$$

$$+ (22)^{10}$$

$$\underbrace{x^4}_{x^4}\ \underbrace{\sum \frac{(22x)^n}{n!}}_{\text{coeff } \frac{x^6}{10!}}$$

$$\Downarrow$$

$$\frac{(22)^6}{6!}\times 10!$$

ii) atleast once

$$\underbrace{\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right)^4}\left(1+x+\frac{x^2}{2!}+\dots\right)^{22}$$

$$(e^x-1)^4\, e^{22x}$$

$$e^{22x}\left[(e^x-1)^2(e^x-1)^2\right]$$

$$f(x) = e^{22x}\left[e^{4x} + 6e^{2x} - 4e^{3x} - 4e^{x} + 1\right]$$

$$= e^{26x} + 6e^{24x} - 4e^{25x} - 4e^{23x} + e^{22x}$$

coeff of $\frac{x^{10}}{10!}$

$$= \sum \frac{(26x)^n}{n!} + 6\sum \frac{(24x)^n}{n!} - 4\sum \frac{(25x)^n}{n!}$$

$$- 4\sum \frac{(23x)^n}{n!} + \sum \frac{(22x)^n}{n!}$$

Ans =

$$(26)^{10} + 6(24)^{10} - 4(25)^{10} - 4(23)^{10} + (22)^{10}$$

23. How many ways are there to select 25 toys from 7 types of toys with between 2 and 6 of each type?

25 items frm $\underbrace{\text{Large pile}}_{\text{7 types}}$

Ist:- $(x^2+x^3+x^4+x^5+x^6)$

$$g(x) = (x^2+x^3+x^4+x^5+x^6)^7$$

$$g(x) = (x^2)^7 \underbrace{(1+x+\dots\dots+x^4)}^7$$

$$= x^{14}\left\{\frac{(1-x^5)}{1-x}\right\}^7$$

$$= \underbrace{x^{14}}(1-x^5)^7(1-x)^{-7}$$

$$= x^{14} \underbrace{\sum {}^7C_r (-x)^{5r} \sum {}^{7+r-1}C_r \, x^r}_{\text{coeff } x^{11}}$$

$$\Rightarrow a_0 b_{11} + a_1 b_{10} + \cdots + a_{11} b_0$$

$$\Rightarrow \underbrace{a_0}_{r=0} \underbrace{b_{11}}_{r=11} + \underbrace{a_5}_{r=1} \underbrace{b_6}_{r=6} + \underbrace{a_{10}}_{r=2} \underbrace{b_1}_{r=1}$$

Ans $\Rightarrow$

$$^{7}C_0 \; ^{7+11-1}C_{11} - {}^{7}C_1 \; ^{7+6-1}C_6 + {}^{7}C_2 \; ^{7+1-1}C_1$$

Quiz1

Permutat^n &

Comb^n

No gf ✗

24. In how many ways can 4 letters of the word ENGINE be arranged?

25. Find the number of ways to collect 15 dollas from 20 distinct people if each of the first 19 people can give one dollar or nothing and 20^{th} person can give 1 dollar or 5 dollar or nothing

THANK YOU