Lattice:
A partially ordered set is said to be a lattice if every two elements in the set have a unique glb and unique lub. Let $(L, \leq)$ be a lattice. For any two elements a, b, let
$a \vee b$: **lub of** a **and** b and $a \wedge b$: **glb of** a **and** b.
Then $(L, \leq, \vee, \wedge)$ is an algebraic system defined by the lattice $(L, \leq)$.

Example 1.1.

Let $P(S)$ be the power set of a nonempty set S. Then $(P(S), \subseteq)$ is a lattice where $A \vee B = A \cup B$ and $A \wedge B = A \cap B$. This defines the algebraic system $(P(S), \subseteq, \cup, \cap)$.

Example 1.2.

Let N^+ be the set of all positive integers. Then $(N^+, |)$ ($a|b$ if a divides b) is a lattice where $a \vee b = lcm(a, b)$ and $a \wedge b = gcd(a, b)$.

Theorem 1.3.

For any elements a, b in a lattice $(A, \leq)$,

- $a \leq a \vee b$ *and* $b \leq a \vee b$
- $a \wedge b \leq a$ *and* $a \wedge b \leq b$

Proof.

Because the join of a and b is an upper bound of a, $a \leq a \vee b$. Because the meet of a and b is a lower bound of a, $a \wedge b \leq a$. $\square$

Theorem 1.4.

For any elements a, b, c, d in a lattice $(A, \leq)$, if $a \leq b$ and $c \leq d$

- $a \vee c \leq b \vee d$
- $a \wedge c \leq b \wedge d$

Proof.

Given that $a \leq b$ and $c \leq d$. Since $b \leq b \vee d$ and $d \leq b \vee d$ then by transitivity $a \leq b \vee d$ and $c \leq b \vee d$.
In other words, $b \vee d$ is an upperbound of a and c. As $a \vee c$ is the least upper bound of a and c, we have $a \vee c \leq b \vee d$.
Since $a \wedge c \leq a$ and $a \wedge c \leq d$ by transitivity $a \wedge c \leq b$ and $a \wedge c \leq d$.
In other words, $a \wedge c$ is a lower bound of b and d. Since $b \wedge d$ is the greast lower bound of b and d, we have $a \wedge c \leq b \wedge d$ $\square$

Duality Principle

Let $(A, \leq)$ be a poset. Let $\geq$ be a binary relation on A such that for any a, b in A, $a \geq b$ if and only if $b \leq a$. We note that $(A, \geq)$ is a poset.

- If $(A, \leq)$ is a lattice, then so is $(A, \geq)$
- The join operation of the algebraic system defined by the lattice $(A, \leq)$ is the meet operation of the algebraic system defined by $(A, \geq)$ and vice versa.
- Consequenntly, given any valid statement concerning the general properties of the lattices, we can obtain another valid statement by replacing the relation $\leq$ with $\geq$, the meet operation with the join operation and the join operation with the meet operation. This is known as principle of duality for lattices.
- If the statement remains the same after dualism, then such a statement is called self dual.

Properties of aglebraic systems defined by lattices:

Let $(A, \leq, \vee, \wedge)$ be the algebraic system defined by the lattice $(A, \leq)$. For any elements $a, b, c \in A$,

1. Commutative property:
   - $a \vee b = b \vee a$
   - $a \wedge b = b \wedge a$
2. Associative property:
   - $(a \vee b) \vee c = a \vee (b \vee c)$
   - $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
3. Idempotent property:
   - $a \vee a = a$
   - $a \wedge a = a$
4. Absorption property:
   - $a \wedge (a \vee b) = a$
   - $a \vee (a \wedge b) = a$

Note: Proofs for the above properties are available in the book ELEMENTS OF DISCRETE MATHEMATICS BY C.L. Liu (Page numbers:390-392)

Problems:

Q1. Let a and b be two elements in a lattice $(A, \leq)$. Show that $a \wedge b = b$ if and only if $a \vee b = a$.

Sol.

Let

$a \wedge b = b$ $-------------------------(2)$

Consider $a \vee (a \wedge b) = a$ (absorption law)

$a \vee b = a$ from (2)

Conversely, let $a \vee b = a$ $---------------------(3)$

Consider $b \wedge (a \vee b) = b$ (absorption law)

$a \wedge b = b$ $\square$

Q2. Let a, b, c be elements in a lattice $(A, \leq)$. Show that

- i. $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$
- ii. $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$

Sol.

i. $a \leq a \vee b$ and $a \leq a \vee c \implies a \leq (a \vee b) \wedge (a \vee c) - - - - - - - - - -$ (4)
$b \leq a \vee b$ and $c \leq a \vee c \implies b \wedge c \leq (a \vee b) \wedge (a \vee c) - - - - - - - - -$ (5)
From (4) and (5), $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$. (By Theorem 2.4)
ii. $(a \wedge b) \leq a$ and $(a \wedge c) \leq a \implies (a \wedge b) \vee (a \wedge c) \leq a - - - - - -$ (6)
$(a \wedge b) \leq b$ and $(a \wedge c) \leq c \implies (a \wedge b) \vee (a \wedge c) \leq (b \vee c) - - - - - -$ (7)
From (6) and (7), $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$. (By Theorem 2.4)

□

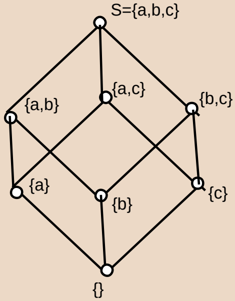
Q3. Let a, b, c be elements in a lattice $(A, \leq)$. Show that if $a \leq b$, then $a \vee (b \wedge c) \leq b \wedge (a \vee c)$.
Q4. Let $(A, \vee, \wedge)$ be an algebraic system where $\vee, \wedge$ are binary operations satisying absorption law. Show that $\vee$ and $\wedge$ also satisfy the idempotent law.
Q5. Let $(A, \vee, \wedge)$ be an algebraic system where $\vee, \wedge$ are binary operations satisying commutative, associative and absorption laws. Define a binary relation $\leq$ on A such that for any x and y in A, $x \leq y$ if and only if $x \vee y = y$. Show that $\leq$ is a partial ordering relation.

Distributive lattice: A lattice is said to be a distributive lattice if the meet operation is distributes over the join operation and the join operation distributes over the meet operation. For any a, b, c

- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Example 2.1.

Let $S = \{a, b, c\}$. Then $(P(S), \subseteq)$ is a distributive lattice.



Theorem 2.2.

If the meet operation is distributive over the join operation in a lattice, then the join operation is also distributive over the meet operation. If the join operation is distributive over the meet operation in a lattice, then the meet operation is also distributive over the join operation.

Proof.

Given that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) - - - - - - - - - - - - - - - (1)$

To prove $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,

Consider
$$\begin{aligned}
(a \vee b) \wedge (a \vee c) &= [(a \vee b) \wedge a] \vee [(a \vee b) \wedge c] && \text{from (1)} \\
&= a \vee [(a \vee b) \wedge c] && \text{(absorption law)} \\
&= a \vee [c \wedge (a \vee b)] && \text{(commutative law)} \\
&= a \vee [(c \wedge a) \vee (c \wedge b)] && \text{from (1)} \\
&= [a \vee (c \wedge a)] \vee (c \wedge b) && \text{(associative law)} \\
&= a \vee (c \wedge b) && \text{(absorption law)} \\
&= a \vee (b \wedge c) && \text{(commutative law)}
\end{aligned}$$

Second part follows from the principle of duality. □