Novel methodology for shape analysis

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1 Introduction

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2.1 Scientific Motivation and Scope

The three-dimensional shape of a dark matter halo encodes information about its dynamical state, formation history, and degree of relaxation. In numerical simulations, halo shape is most commonly quantified through axis ratios derived from inertia tensors, which provide a compact description of triaxiality and are straightforward to compare across simulations and with existing literature.

During a major merger, however, the physical assumptions underlying this description are no longer strictly valid. The interacting system is not in equilibrium, the mass distribution is highly asymmetric, and large fractions of material may be temporarily unbound or arranged in tidal features. In such circumstances, forcing the halo into a single ellipsoidal description can obscure physically relevant structure.

For this reason, we adopt a two-component methodology. We first apply a standard, mass-weighted inertia-tensor analysis to maintain consistency with established approaches. We then supplement this with a purely geometric characterization based on the convex hull of the particle distribution, which is sensitive to merger-driven distortions and does not rely on assumptions of symmetry or relaxation.

2.2 Particle Selection and Reference Frame

The analysis is performed independently for each simulation snapshot. Dark matter particles associated with the Milky Way (MW) and Andromeda (M31) systems are read from their respective data files and combined into a single particle ensemble representing the merged halo.

Particle *positions* refer to their Cartesian coordinates in the simulation frame, while particle *masses* are the simulation-assigned dark matter particle masses. Prior to any shape measurement, the combined system is recentered using the Milky Way center of mass. The center of mass is defined as the mass-weighted mean position of particles and provides a physically meaningful origin that tracks the dominant gravitational potential well.

Recentering removes bulk translational motion of the system. Without this step, apparent shape evolution could arise purely from coordinate drift rather than intrinsic structural change, making physical interpretation unreliable.

2.3 Mass-Weighted Inertia Tensor Analysis

2.3.1 Definition of the Inertia Tensor

The inertia tensor is a second-rank tensor that summarizes how mass is distributed relative to a chosen origin. In this work, we use the unreduced inertia tensor, defined as

$$I_{ij} = \sum_k m_k \left(r_k^2, \delta_{ij} - x_{k,i} x_{k,j} \right), \tag{1}$$

where m_k is the mass of particle k, $\mathbf{x} * k = (x * k, 1, x_{k,2}, x_{k,3})$ is its position relative to the center of mass, $r_k = |\mathbf{x} * k|$ is its radial distance, and $\delta * ij$ is the Kronecker delta.

This definition weights each particle by both its mass and the square of its distance from the center. As a result, particles at larger radii contribute more strongly, making the tensor sensitive to the global mass distribution rather than only the central region.

2.3.2 Eigenvalues and Axis Ratios

The inertia tensor is diagonalized to obtain three real, non-negative eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$. These eigenvalues correspond to the principal moments of inertia and define the principal axes of the mass distribution.

To express halo shape in a dimensionless and easily comparable form, we define the axis ratios

$$b/a = \sqrt{\lambda_2/\lambda_1}, \qquad c/a = \sqrt{\lambda_3/\lambda_1}. \tag{2}$$

Here, a, b, and c represent the lengths of the major, intermediate, and minor axes of the best-fitting ellipsoid implied by the inertia tensor. A perfectly spherical halo corresponds to $b/a = c/a = 1$, while deviations from unity indicate triaxiality or flattening.

2.3.3 Physical Meaning and Exigency

The inertia-tensor axis ratios provide a mass-weighted measure of halo shape and are well suited for describing relaxed systems where an ellipsoidal approximation is physically meaningful. Their widespread use in the literature makes them essential for benchmarking and comparison.

However, during a major merger, this approach has clear limitations. The inertia tensor cannot distinguish between a genuinely extended, coherent halo and a compact core surrounded by low-density tidal debris. Moreover, it enforces an ellipsoidal interpretation even when the true particle distribution is strongly non-ellipsoidal. These limitations motivate the introduction of an additional, geometry-based diagnostic.

2.4 Convex Hull–Based Geometric Analysis

2.4.1 Concept of the Convex Hull

The convex hull of a set of points in three-dimensional space is defined as the smallest convex volume that contains all points. Equivalently, it is the shape that would be obtained by stretching an elastic membrane around the outermost particles.

Unlike the inertia tensor, the convex hull depends only on particle positions and not on particle masses. It therefore provides a purely geometric description of the halo's spatial extent and outer morphology.

2.4.2 Construction and Interpretation

For each snapshot, the convex hull of the centered particle positions is computed. The resulting hull is defined by a subset of particles, known as hull vertices, that trace the outer boundary of the distribution.

The convex hull volume, denoted $V_{\rm hull}$, provides a direct measure of the geometric size of the halo. Changes in this volume reflect merger-driven expansion, tidal heating, and the formation of extended debris, even in cases where the bound mass distribution changes little.

2.4.3 Geometric Axis Ratios

To characterize the shape of the halo envelope, we compute the covariance matrix of the convex-hull vertices. Diagonalizing this covariance matrix yields eigenvalues $\mu_1 \geq \mu_2 \geq \mu_3$, which describe the spatial extent of the hull along its principal directions.

Analogous to the inertia-tensor case, we define geometric axis ratios as

$$b/a = \sqrt{\mu_2/\mu_1}, \qquad c/a = \sqrt{\mu_3/\mu_1}. \tag{3}$$

These ratios describe the anisotropy of the halo's outer boundary rather than its mass distribution, making them particularly sensitive to tidal features and large-scale asymmetries.

2.4.4 Volume Inflation Factor

To connect the geometric and mass-based descriptions, we construct an effective ellipsoidal volume from the inertia-tensor eigenvalues,

$$V_{\rm ellipsoid} = \frac{4\pi}{3} abc, \tag{4}$$

where a, b, and c are proportional to $\sqrt{\lambda_1}$, $\sqrt{\lambda_2}$, and $\sqrt{\lambda_3}$, respectively.

We then define the volume inflation factor

$$\mathcal{I} = \frac{V_{\text{hull}}}{V_{\text{ellipsoid}}}. \tag{5}$$

This dimensionless quantity measures the degree to which the halo geometry deviates from an ideal ellipsoid. Values of $\mathcal{I} \approx 1$ indicate that the ellipsoidal approximation is adequate, while larger values indicate strong geometric irregularity associated with merger activity.

2.5 Methodological Complementarity

The inertia-tensor and convex-hull analyses probe distinct but complementary aspects of halo structure. The inertia tensor captures mass-weighted triaxiality and relaxation, while the convex hull captures geometric extent and irregularity. Applying both methods to the same snapshots allows us to disentangle genuine dynamical relaxation from transient, merger-driven distortions in halo morphology.