ES-221
Mechanics Of Solids
Project Report
Analysis of Stress distribution of a Torsional bar

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0.1 Objectives

This study aims to perform a thorough investigation of the stress distribution inside a circular torsional bar when an applied external moment is applied. Our goal is to visualise the stress state using Mohr's circle representation and to ascertain the stress level at any given place within the bar through theoretical modelling and numerical simulations.

1. Develop an understanding of torsional loading and its effects on circular bars with the application of solid mechanics principles.

2.Determine the state of stress at a specific point within the bar.

3.Visualize the stress distribution within the torsional bar using visualization methods.

4. Mohr's circle to represent the principal stresses and their orientations at the chosen point within the torsional bar for a better understanding of the stress state.

0.2 Problem Statement

Consider a circular torsional bar fixed at its ends with shear modulus G, radius R_o, and length L. The bar is subjected to an external moment M_t applied at a location $z = a$. We define a reference cylindrical coordinate system (r, θ, z). Our objective is to analyze the state of stress at a given point A located at coordinates (r, θ, z) and plot its Mohr's circle using code.
We have 8 unknown attributes-

1. Torque $\mathrm{M}_{\mathbf{t}}$
2. Location of applied moment, a
3. Radius of bar, R
4. Shear modulus G and Length L
5. The location of the point of analysis is specified as (r, θ, z).

We **assume** there is no distortion within the plane of the cross-section, and from symmetry, we can implicitly assume $\sigma_r = \sigma_\theta = \sigma_z = \tau_{r\theta} = \tau_{rz} = 0$.

The Desired Output we are willing to produce is :
1. The state of stress, i.e., $\tau_{\theta z}$, at a general point $A(r, \theta, z)$
2. Mohr's circle analysis shows stress components on a chosen orientation (angle Φ) at that point (r, θ, z).
3. Showing the state of stress at the rotated element using a figure

0.2.1 Shear Stress Distribution

The shear stress (τ) at any point z along the bar can be calculated using the formula:

$$\tau = \frac{Mr}{I}$$

where M_t is the applied torque, r is the radial distance from the center of the bar, and J is the polar moment of inertia given by $I = \frac{\pi R_o^4}{2}$.

0.2.2 Ploting Mohr's Circle

Once the stress components at point A are determined we will construct the stress tensor matrix. Then,compute the principal stresses and their corresponding directions. We will then plot Mohr's circle using the principal stresses as the circle's diameter and the directions of the principal stresses as the circle's axis. This is done using Mathematica for coding portion.

We will then Visualize the stress distribution within the torsional bar, This visualization will provide insights into the variation of stress along the length and radial direction of the bar.

0.3 Methodology

We followed the following steps for the stress analysis:

1. **Fixing the Constants**: We start by defining the constants, such as the length (L) and radius (R) of the torsional bar and the second moment of Area(I_z).

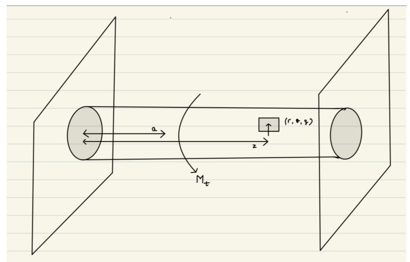


Figure 1: Twisted bar with fixed ends

2. **Shear Stress Distribution and Values of stresses for Mohr's circle**: We calculate the shear stress distribution along the rod's central axis direction(z direction). Since there is no strain $(\epsilon_{\theta r}, \epsilon_{\theta z}, \epsilon_{rz}) = (0, 0, 0)$ and external force in the radial and tangential directions, we assumed σ_θ and σ_z to be 0. Additionally, $\gamma_{r\theta}$ and γ_{rz} were assumed to be zero, and $\tau_{\theta z}$ (shear stress calculated for a particular z) was considered.

3. **Moment Definition and Shear Stress Calculation**: We define the moment at a particular position z along the rod and calculate the shear stress for the rod at that position, which would later be used in the stress analysis of an element at that position. Depending upon the a, the bending moment (M_b) will be different at different z.

$$\begin{aligned} \text{factor} &= \begin{cases} \frac{L-a}{L}, & \text{if } z < a \\ \frac{a}{L}, & \text{otherwise} \end{cases} \\ M_b &= \text{factor} \times M_t \\ \tau_{\theta z} &= \frac{r \times M_b}{I_z} \end{aligned}$$

4. **Mohr's Circle Calculation and Plot**: To plot Mohr's circle, we only needed θ and r since we had a, M_t, and z. We know $\sigma_{\theta\theta}$, σ_{zz} and $\tau_{\theta z}$ We used the equations and manipulated them accordingly.

5. **Stress Visualization**: Finally, we depicted the state of stresses (values on the element) using the values obtained from Mohr's circle. We visualized them in a 2D plot, which was user-defined, allowing us to vary M_t, a, r, θ, and z. We use the value using the mohrs circle that we plotted in the above section.

0.4 Numerical Implementation

```
L = 1;
R = 0.1;
E = 200e+9;
G = 80e+9;
bar = Rectangle[{0, 0}, {L, 2 R}];
l = Line[{{0, R}, {L, R}}];


σθ=0;
σz=0;
Iz = π*R^4/2;
center[σθ,σz]:={(σθ+σz)/2,0};
radius[σθ,σz,τθz_]:=Sqrt[((σθ-σz)/2)^2+τθz^2];
σθR[σθ,σz,τθz_,θ_]:=(σθ+σz)/2+(σθ-σz)/2 Cos[2θ]+τθz*Sin[2θ];
σzR[σθ,σz,τθz_,θ_]:=(σθ+σz)/2-(σθ-σz)/2 Cos[2θ]-τθz*Sin[2θ];
τθzR[σθ,σz,τθz_,θ_]:=-((σθ-σz)/2)Sin[2θ]+τθz*Cos[2θ];
```

(a) Setting up the parameters

```
Manipulate[
 Mvector = Arrow[{{a, Mt/50 + 2*R}, {a, 2*R}}];
 t = Text[Style["Mt=" <> ToString[Mt], 14, Red], {a + 0.17, 2*R + 0.1}];

 t0 = Text[Style["a=" <> ToString[a], 14, Red], {a/2, 2*R + 0.05}];
 ldash5 = Line[{{0, 2*R + 0.025}, {0, 2*R + 0.075}}];
 ldash6 = Line[{{a - 0.001, 2*R + 0.025}, {a - 0.001, 2*R + 0.025}}];
 v5 = Arrow[{{a/2 - 0.1, 2*R + 0.05}, {0, 2*R + 0.05}}];
 v6 = Arrow[{{a/2 + 0.1, 2*R + 0.05}, {a, 2*R + 0.05}}];
 If[a < 0.1, Clear[t0, v5, v6, ldash5, ldash6]];

 t1 = Text[Style["2R", 10, Black], {L + 0.05, R}];
 ldash1 = Line[{{L + 0.025, 2*R}, {L + 0.075, 2*R}}];
 ldash2 = Line[{{L + 0.025, 0}, {L + 0.075, 0}}];
 v1 = Arrow[{{L + 0.05, R + 0.02}, {L + 0.05, 2*R}}];
 v2 = Arrow[{{L + 0.05, R - 0.02}, {L + 0.05, 0}}];

 t2 = Text[Style["L", 10, Black], {L/2, -0.0375}];
 v3 = Arrow[{{L/2 - 0.02, -0.0375}, {0, -.0375}}];
 v4 = Arrow[{{L/2 + 0.02, -0.0375}, {L, -0.0375}}];
 ldash3 = Line[{{0, -0.05}, {0, -0.025}}];
 ldash4 = Line[{{L, -0.05}, {L, -0.025}}];

 g = Graphics[{FaceForm[White], EdgeForm[Black],
    bar, {Thick, Dashed, l}, , {Mvector}, {t}, {t0}, {Thin,
     ldash5}, {Thin, ldash6}, {v5}, {v6}, {t1}, {Thin, ldash1}, {Thin,
      ldash2}, {v1}, {v2}, {t2}, {Thin, ldash3}, {Thin,
     ldash4}, {v3}, {v4}}];

 fact=If[z<a,(L-a)/L,-a/L];
 Mb=fact*Mt;
 τθz = r*Mb/Iz;
 circle = Circle[center[σθ, σz], radius[σθ, σz, τθz]];
 state = Point[{{σθ, -τθz}, {σz, τθz}}];
 linestate = Line[{{σθ, -τθz}, {σz, τθz}}];

 stateRotated =
  Point[{{σθR[σθ, σz, τθz, π/180 θ], -τθzR[σθ, σz, τθz, π/180 θ]}, {σzR[σθ, σz, τθz, π/180 θ], τθzR[σθ, σz, τθz, π/180 θ]}}]/.
  Point[pts_] :> {PointSize[Large],
    MapThread[{#1, Point[#2]} &, {{Red, Blue}, pts}]};
 lineRotated =
  Line[{{σθR[σθ, σz, τθz, π/180 θ], -τθzR[σθ, σz, τθz, π/180 θ]}, {σzR[σθ, σz, τθz, π/180 θ], τθzR[σθ, σz, τθz, π/180 θ]}}];

 g2 = Graphics[{circle, {PointSize[0.02], state}, {linestate}, {Blue, PointSize[0.02], stateRotated, lineRotated}}, Axes -> True, PlotRange -> {{-15000, 15000}, {-15000, 15000}}
```

(b) Manipulate the parameters and generate the Mohr's circle

```
element = Graphics[{FaceForm[None], EdgeForm[Black], Rectangle[], Red,
   Arrow[{{1, -0.1}, {0, -0.1}}], Red, Arrow[{{1.1, 0}, {1.1, 1}}],
   Red, Arrow[{{0, 1.1}, {1, 1.1}}], Red,
   Arrow[{{-0.1, 1}, {-0.1, 0}}], Blue,
   Arrow[{{0.5, 0.5}, {1.5, 0.5}}], Blue,
   Arrow[{{0.5, 0.5}, {0.5, 1.5}}],
   Text[Style["σθ", 16], {1.6, 0.5}],
   Text[Style["σz", 16], {0.5, 1.6}], Red,
   Text[Style["τθz", 16], {1.2, 1}]},
  PlotRange -> {{-2, 2}, {-2, 2}}, Axes -> False, ImageSize -> 500, ImagePadding -> 40];

L = 1;
R = 0.1;
Iz = π*R^4/2;

σθθ'[σθ, σz, τθz_, θ_] := (σθ + σz)/2 + ((σθ - σz)/2 )*Cos[2 θ] + τθz*Sin[2 θ];
σzz'[σθ, σz, τθz_, θ_] := (σθ + σz)/2 - ((σθ - σz)/2)* Cos[2 θ] - τθz*Sin[2 θ];
τθz'[σθ, σz, τθz_, θ_] := -(((σθ - σz)/2)) *Sin[2 θ] + τθz*Cos[2 θ];

Manipulate[
 τθz = If[z < a, (L - a)/L, a/L]*Mt;
 element1 = Graphics[{FaceForm[None], EdgeForm[Black], Rectangle[], Red,
   Arrow[{{1, -0.1}, {0, -0.1}}], Red, Arrow[{{1.1, 0}, {1.1, 1}}],
   Red, Arrow[{{0, 1.1}, {1, 1.1}}], Red,
   Arrow[{{-0.1, 1}, {-0.1, 0}}], Blue,
   Arrow[{{0.5, 0.5}, {1.5, 0.5}}], Blue,
   Arrow[{{0.5, 0.5}, {0.5, 1.5}}],
   Text[Style["σθθ'=" <> ToString[NumberForm[σθθ'[0, 0, τθz, θ], {Infinity, 2}]],16], {1.8, 0.5}],
   Text[Style["σzz'=" <> ToString[NumberForm[σzz'[0, 0, τθz, θ], {Infinity, 2}]], 16], {0.5, 1.6}],
   Red, Text[Style["τθz'=" <> ToString[NumberForm[τθz*Cos[2 θ],{Infinity,2}]], 16], {1.4, 1.1}]},

  PlotRange -> {{-2, 2}, {-2, 2}}, Axes -> False, ImageSize -> 500, ImagePadding -> 40];
 Grid[{{element, Spacer[50], Rotate[element1, θ, {0.5, 0.5}],
   Spacer[50],
   Column[{Style[StringForm["θ:`` rad", θ], 16, "Label"],
   Style[StringForm["σθθ':`` ", NumberForm[σθθ'[0, 0, τθz, θ],{Infinity,2}]], 16, "Label"],
   Style[StringForm["σzz':`` ", NumberForm[σzz'[0, 0, τθz, θ],{Infinity,2}]], 16, "Label"],
   Style[StringForm["τθz':`` ", NumberForm[τθz*Cos[2 θ],{Infinity,2}]], 16, "Label"]},
   Alignment->Center]}}, Spacings -> {0, 100}],
{{θ, 0}, 0, 2 Pi, Pi/10}, {{Mt, 0}, 0, 20, 0.01}, {{a, 0}, 0, L, 0.01}, {{r, 0}, 0, R, 0.01}, {{z, 0}, 0, L, 0.01}]
```

(c) Make a Small element and rotate it at given orientation

1. **Representation of the Torsional bar**: Firstly, the parameters such as the length (L), radius (R), Young's modulus (E) and shear modulus (G) are initialized at the beginning of the code. Graphical representations of the Torsional bar and its neutral axis are created using geometric primitives like `Rectangle` and `Line`.

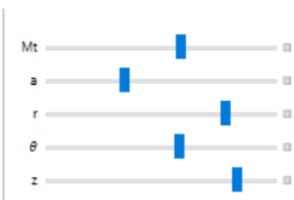
2. **Definition of Stress Transformation Functions**: Functions such as `stateRotated` and `lineRotated` calculates the transformed coordinates of stress points and stress lines after applying a rotation transformation. These functions incorporate the initial stresses (σ_θ, σ_z, and $\tau_{\theta z}$) and the rotation angle (θ), and provides the stress variations with rotation.

3. **Adjusting the parameters**: The `Manipulate` function is used to create an interface where we can adjust parameters such as applied moment (M_t), distance from the neutral axis (a), rotation angle (θ), and vertical position (z). This allows us to dynamically visualize stress distributions and transformations in response to changing parameters.

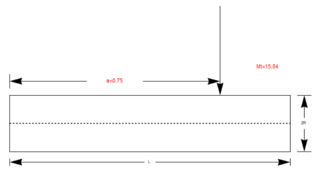
4. **Representing the state of stress of a small element**: To represent the state of stress for a small element, we first defined $\sigma_{\theta\theta}$, σ_{zz}, and $\tau_{\theta z}$ to calculate the normal and shear stress components acting on the square element. Blue arrows represent the normal stresses ($\sigma_{\theta\theta}$ and σ_{zz}), while Red arrows represent the shear stress ($\tau_{\theta z}$).

5. **Rotation of the Stress components**: The stress components can be rotated around the square element's center using the `Rotate` function.

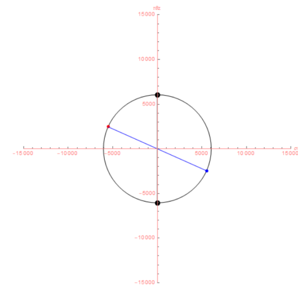
0.5 Result and Discussions



(a) Manipulate the parameters

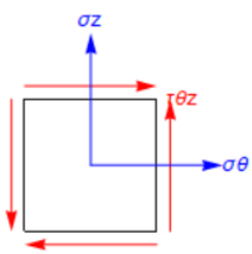


(b) Represntation of the Torsional Shaft

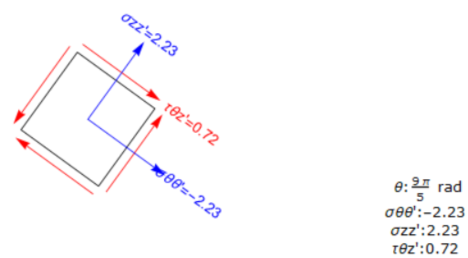


(c) Representation of the Mohr's circle

1. **Interactive Manipulation of Parameters**: One of the key part is the use of the `Manipulate` function where we can adjust parameters and observe their effects on stress states.
2. **Representation of Mohr's Circle**: The code also includes visualization of Mohr's circle, which provides valuable insights into the relationship between normal and shear stresses.



(a) Initial stress states



(b) Rotated stress states

3. **Checking the correctness of results**: For validation of the correctness of our results, one can check that $\tau_{\Theta z}$ is 0 at $a = 0$.
   At M_t=16.09, a=0.4, r=0.05, z=0.75, the following table represents the stress states:

Angle	σ_θ	σ_z	$\tau_{\theta z}$
0	0	0	6.40
$\pi/5$	6.09	-6.09	1.98
$2\pi/5$	3.76	-3.76	-5.18
$3\pi/5$	3.76	-3.76	-5.18
$4\pi/5$	6.09	-6.09	1.98
π	0	0	6.40

0.6 Learning Outcomes

1. We gained an understanding of torsional loading and its effects on circular torsional bars, including the distribution of shear stress within the bar.

2. We also learned to use solid mechanics concepts to analyse stress levels in intricate structural parts by applying different kind of theories.

3. By plotting Mohr's circle and visualizing stress distributions we learned effective visualization techniques to interpret stress states in three-dimensional structures.

4. Got clearity in understanding of symmetry and boundary conditions.