Q1 b) $\|A\|_{max} = \max_{ij} |a_{ij}|$

① $\|e\| \geq 0 \Rightarrow$ $a_{ij} \in \mathbb{C} \Rightarrow |a_{ij}| \geq 0$

$\therefore$ The maximum of all the absolute values will also be greater than 0.

$\therefore$ $\|A\|_{max} = \max_{ij} |a_{ij}| \geq 0$

② $\|e\| = 0 \Leftrightarrow e = 0$

If $\|A\|_{max} = 0 \Rightarrow \max_{ij} |a_{ij}| = 0$

$\because$ If the maximum absolute value of the elements in a matrix is '0', $\Rightarrow$ all the elements are 0.

$\therefore$ $A = 0_{n \times n}$

The other way round $\Rightarrow$ If all the elements of the matrix are 0, $\Rightarrow$ $\max_{ij}(A) = 0 \Rightarrow \|A\|_{max} = 0$

③ $\|\alpha e\| = |\alpha| \|e\|, \forall \alpha \in \mathbb{C}$

$\|A\|_{max} = \max_{ij} |a_{ij}|$

$\|\alpha A\|_{max} = \max_{ij} |\alpha a_{ij}|$

$= |\alpha| \max_{ij} |a_{ij}| = |\alpha| \|A\|_{max}$

④ $\|e_1 + e_2\| \leq \|e_1\| + \|e_2\|, \forall e_1, e_2$

$\|A + B\|_{max} = \max_{ij} |a_{ij} + b_{ij}| \leq \max_{i,j} (|a_{ij}| + |b_{ij}|)$

$\leq \max_{i,j} |a_{ij}| + \max_{ij} |b_{ij}|$

$= \|A\|_{max} + \|B\|_{max}$

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\|A\|_{max} = \max_{i,j} |a_{ij}| = 1$

$\rho(A) = \max(\text{eig}(A))$

$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 - 1$

$= \lambda^2 - 2\lambda = 0$

$\Rightarrow \lambda = 0, 2$

$\therefore \max(\text{eig}(A))$

$= 2$

$\therefore \|A\|_{max} < \rho(A)$

let $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 6 & 8 \\ 6 & 8 \end{bmatrix}$

$\|AB\|_{max} = 8$

$\|A\|_{max} = 1$; $\|B\|_{max} = 5$

$\therefore \|A\|_{max} \times \|B\|_{max} = 1 \times 5 = 5$

$\therefore \|AB\|_{max} > \|A\|_{max} \|B\|_{max}$

(c)

$$\|G(s)\|_2 \triangleq \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega} \quad ; Q = e^{-sT}$$

For time delays,

(for scalars, $|AB| = |A||B|$)

$$\|QG\|_2 \triangleq \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |QG|^2 d\omega}$$

$$= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |e^{-j\omega T} G(j\omega)|^2 d\omega}$$

$$= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |e^{-j\omega T}||G(j\omega)|^2 d\omega}$$

$$= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} 1 \cdot |G(j\omega)|^2 d\omega}$$

$$= \|G(s)\|_2$$

$\therefore$ H_2 norm is time delay invariant

$$\|G\|_\infty = \max |G(j\omega)| \quad ; Q = \frac{s-a}{s+a} \quad (a>0)$$

$$\|QG\|_\infty = \max |QG| = \max \left|\left(\frac{s-a}{s+a}\right) G\right|$$

$$= \max\left(\left|\frac{s-a}{s+a}\right||G|\right)$$

$$= \max\left(\frac{|j\omega - a|}{|j\omega + a|} |G|\right)$$

$$= \max\left(\frac{\sqrt{\omega^2+a^2}}{\sqrt{\omega^2+a^2}} |G|\right)$$

$$= \max(1 \cdot |G|)$$

$$= \max(|G|)$$

$$= \|G\|_\infty$$

$\therefore$ H_∞ is all-pass filter invariant.

$$\|G(s)\|_2 \triangleq \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}|G(j\omega)|^2\,d\omega} \quad ; \quad Q = \frac{s-a}{s+a}, \quad a>0$$

$$\|QG\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}|QG|^2\,d\omega}$$

$$= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{s-a}{s+a}\right|^2|G|^2\,d\omega}$$

$$= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|j\omega-a|^2}{|j\omega+a|^2}|G|^2\,d\omega}$$

$$= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}1\cdot|G|^2\,d\omega}$$

$$= \|G\|_2$$

$$\|G(s)\|_\infty \triangleq \max|G(j\omega)| \quad ; \quad Q = e^{-sT}$$

$$\|QG\|_\infty \triangleq \max|QG|$$

$$= \max\left(|e^{-sT}||G|\right)$$

$$= \max\left(|e^{-j\omega T}||G|\right)$$

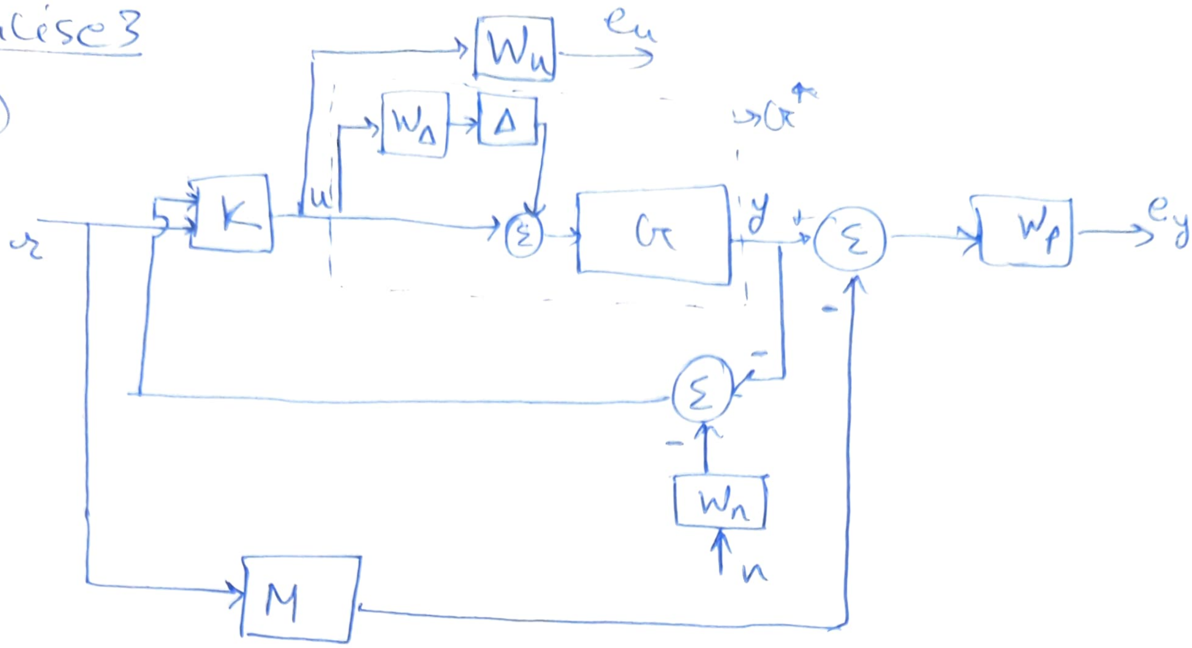
$$= \max\left(1\cdot|G|\right)$$

$$= \max\left(|G|\right)$$

$$= \|G\|_\infty$$

Exercise 3

(a)



Exogenous outputs: $e_u, e_y = z$

Sensed outputs: $-y - W_n n$

$v = -y - W_n n, r$

~~w =~~ exogenous inputs, $w = r, n$

$e_u = W_u u$; $e_y = W_p(y - Mr)$

$= W_p(G^* u - Mr)$

$v_1 = -y - W_n n$; $v_2 = r$

$= -G^* u - W_n n$

$$\begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} e_u \\ e_y \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & W_u \\ -W_pM & 0 & W_pG^* \\ 0 & -W_n & -G^* \\ I & 0 & \cancel{W_pG^*}\ 0 \end{bmatrix} \begin{bmatrix} r \\ n \\ u \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} z \\ v \end{bmatrix} = \begin{array}{c} \begin{array}{cc} P_{11} & P_{12} \end{array} \\ \left[\begin{array}{cc|c} 0 & 0 & W_u \\ -W_pM & 0 & W_p\, G(I + W_\Delta \Delta) \\ 0 & -W_n & -G(I + W_\Delta\Delta) \\ I & 0 & 0 \end{array}\right] \\ \begin{array}{cc} P_{21} & P_{22} \end{array} \end{array} \begin{bmatrix} w \\ u \end{bmatrix}$$

$$N = P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21}$$

$$= \begin{bmatrix} 0 & 0 \\ -W_p M & 0 \end{bmatrix} + \begin{bmatrix} W_u \\ W_p G (I + W_\Delta \Delta) \end{bmatrix} K \left(I - \begin{bmatrix} G (I + W_\Delta \Delta) \\ 0 \end{bmatrix} K \right)^{-1} \begin{bmatrix} 0 & -W_n \\ I & 0 \end{bmatrix}$$

Q3 (c) $F_l(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$

$$F_l\left(H, \frac{1}{s}\right) = H_{11} + \frac{H_{12}}{s}\left(I - \frac{H_{22}}{s}\right)^{-1} H_{21}$$

$$= H_{11} + \frac{H_{12}}{s} \frac{(sI - H_{22})^{-1}}{s^{-1}} H_{21}$$

$$= H_{12}(sI - H_{22})^{-1} H_{21} + H_{11}$$

Also,

$$F_l\left(H, \frac{1}{s}\right) = C(sI - A)^{-1}B + D$$

Comparing,

$$C = H_{12}, \quad A = H_{22}, \quad B = H_{21}, \quad D = H_{11}$$

$$\therefore \quad H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = \begin{bmatrix} D & C \\ B & A \end{bmatrix}$$

Q1(a)

```
A = [0 1;3 -2];
B = [0 1;3 0];

% 1-norm
A1 = max(sum(abs(A)))
```

```
A1 = 3
```

```
B1 = max(sum(abs(B)))
```

```
B1 = 3
```

```
% inf-norm
Ainf = max(sum(abs(A')))
```

```
Ainf = 5
```

```
Binf = max(sum(abs(B')))
```

```
Binf = 3
```

```
% frobinius-norm
Afro = sqrt(sum(diag(A'* A)))
```

```
Afro = 3.7417
```

```
Bfro = sqrt(sum(diag(B'* B)))
```

```
Bfro = 3.1623
```

```
% 2-norm
A2 = max(sqrt(eig(A'* A)))
```

```
A2 = 3.6503
```

```
B2 = max(sqrt(eig(B'* B)))
```

```
B2 = 3
```

```
%spectral radius
A_sr = max(abs(eig(A)))
```

```
A_sr = 3
```

```
B_sr = max(abs(eig(B)))
```

```
B_sr = 1.7321
```

Q2(a)

Initialize transfer function

```
s = tf('s');
G = [10*(s+2)/(s^2+0.2*s+100),1/(s+5);(s+2)/(s^2+0.1*s+10),5*(s+1)/((s+2)*(s+3))]
```

```
G =

  From input 1 to output...
           10 s + 20
   1:  -----------------
       s^2 + 0.2 s + 100

            s + 2
   2:  ----------------
       s^2 + 0.1 s + 10

  From input 2 to output...
         1
   1:  -----
       s + 5

          5 s + 5
   2:  -------------
       s^2 + 5 s + 6

Continuous-time transfer function.
```

Set the high frequency value to achieve the approximate desired loop shape

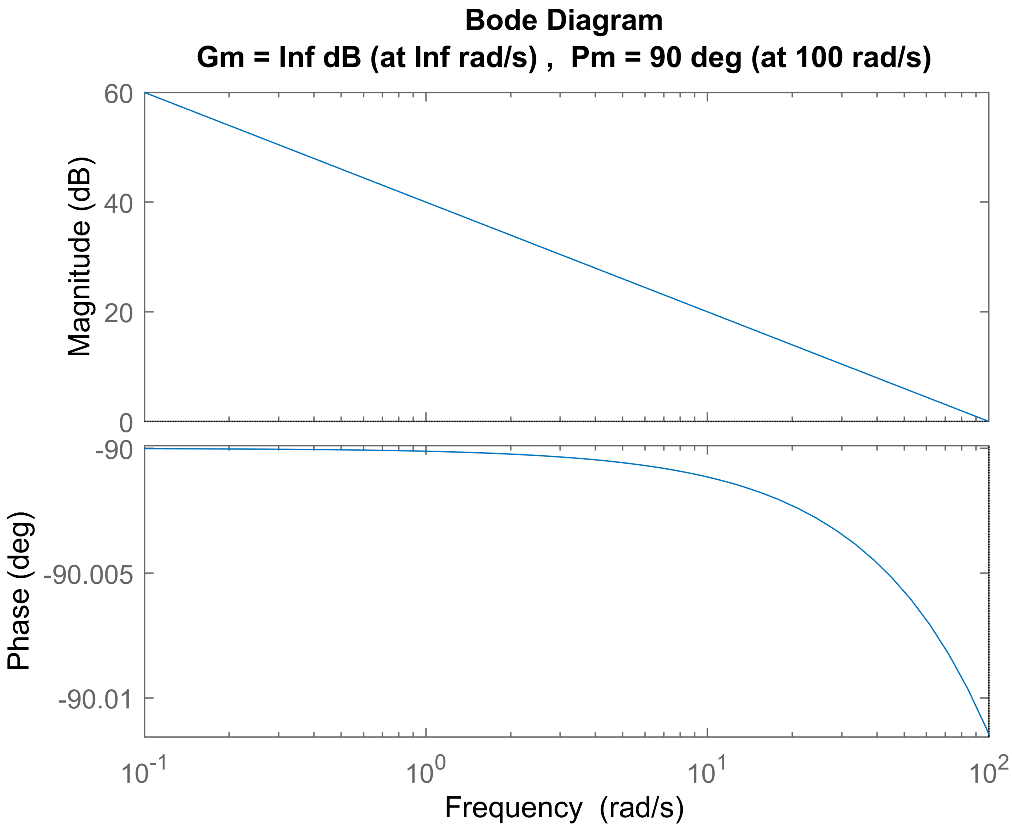
```
tau = 500000;
```

Calculating L

```
G_hat = inv(G,'min')/(1+s/tau);
L = [100/s 0;0 100/s];
K = L*G_hat;

l = G*K;
```

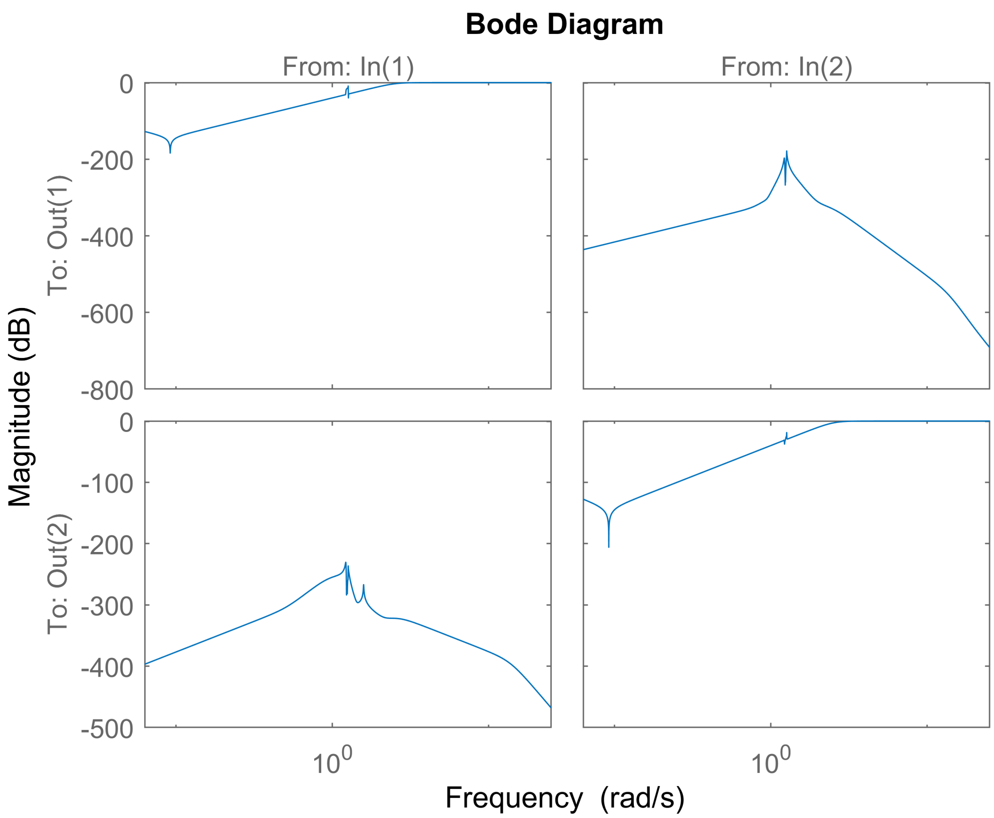
Performance plot of l(1,1)

```
L11 = l(1,1);
margin(L11)
```



Sensitivity function magnitude bode plot

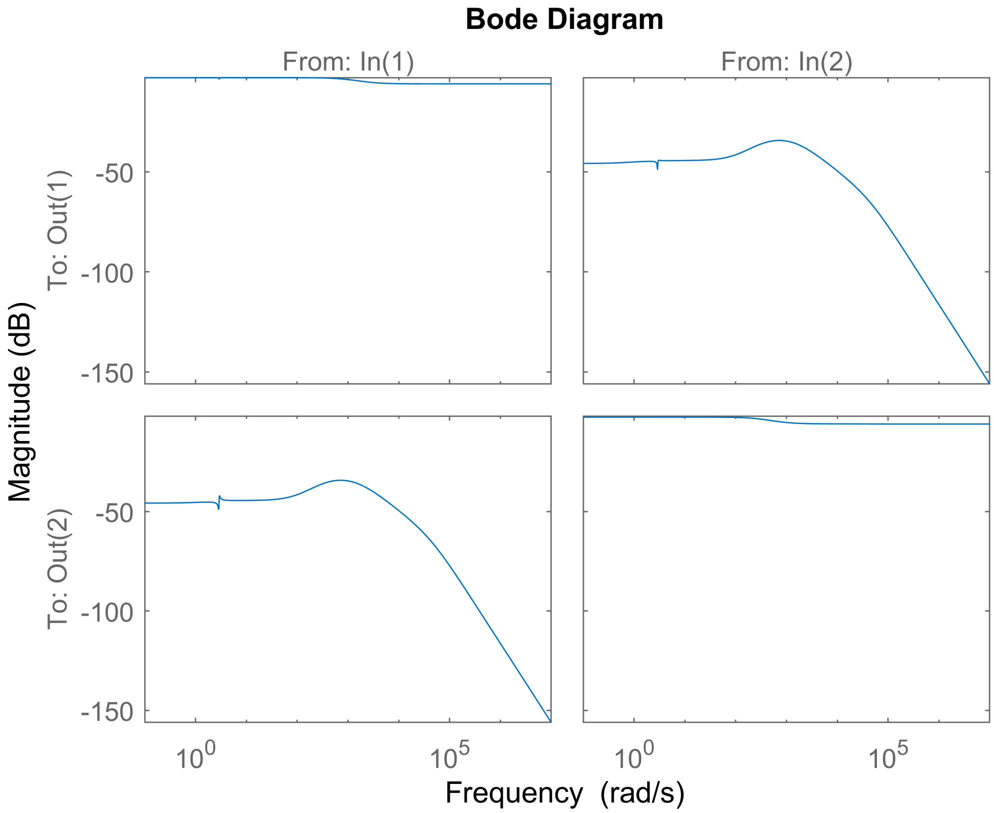
```
T = feedback(G*K,eye(2));
S = eye(2) - T;
bodemag(S)
```



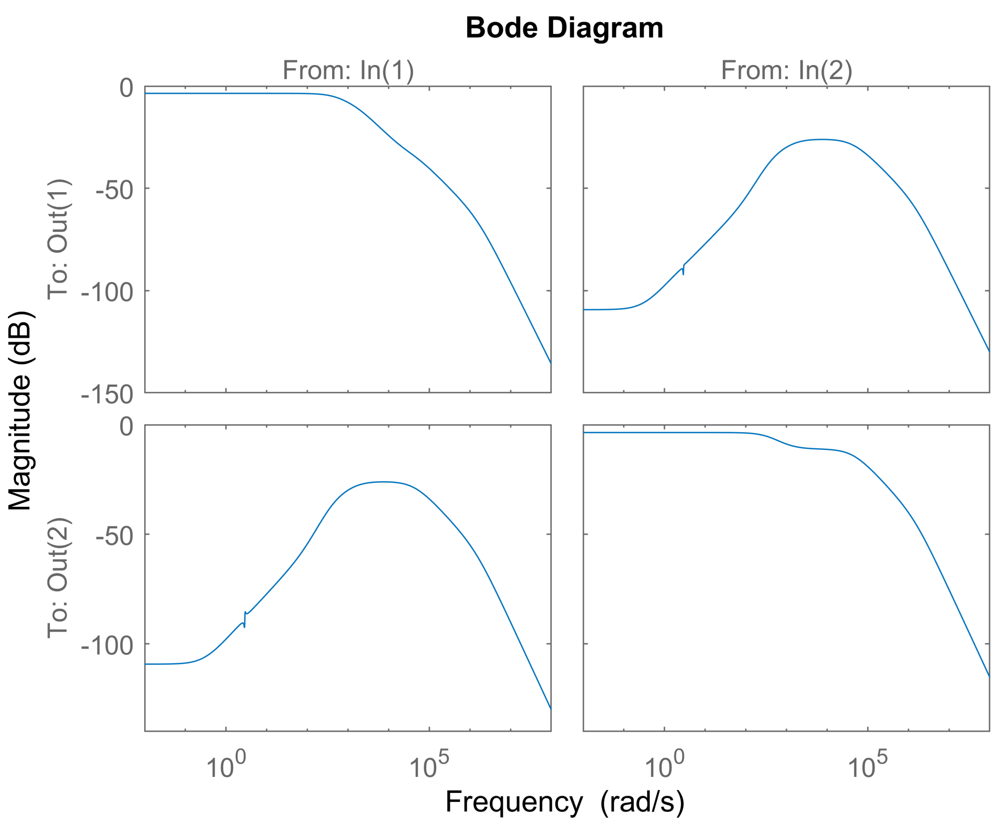
Q2(b)

Initialize parameters

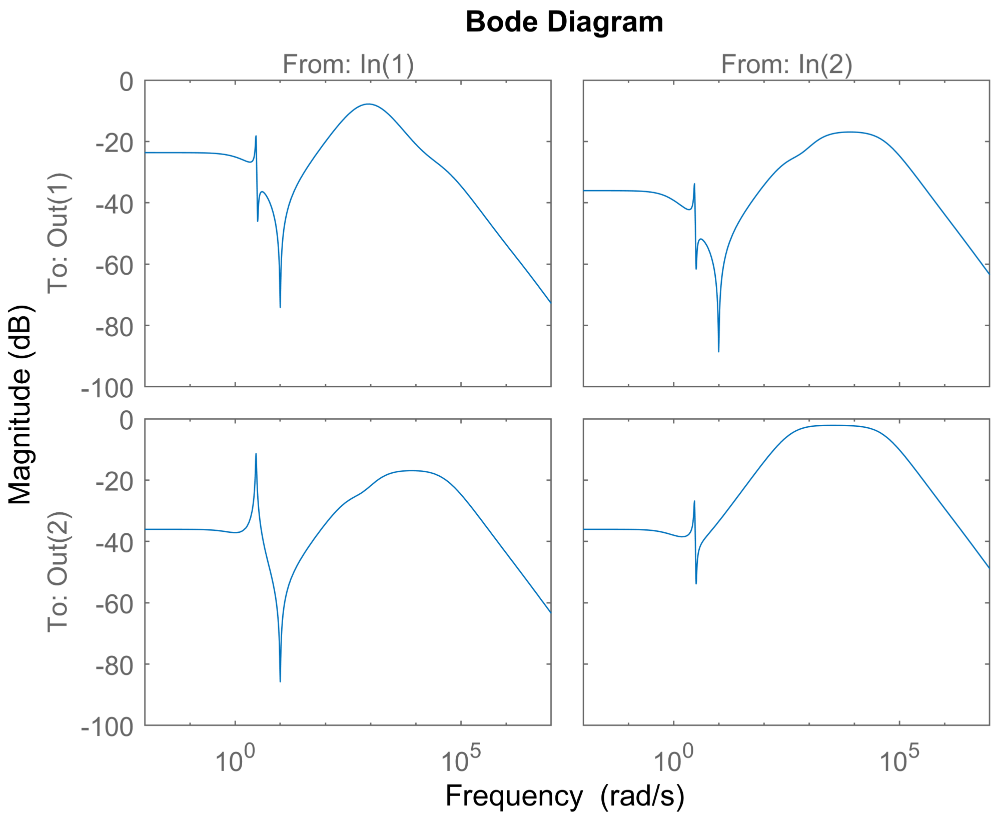
```
BW = 100;
Wu = [1/100 0;0 1/100];

% A while loop to maximize the bandwidth
while 1
    Wp = makeweight(1000,BW,1/2)*eye(2);
    Wt = makeweight(1/1.5,3*BW,1000)*eye(2);
    [K,CL,GAM,info] = mixsyn(G,Wp,Wu,Wt);
    if GAM>1
        break
    end
    BW = BW + 5;
end

% Computing the Sensitivity, Complementary Sensitivity, and the controller weight compensator
L = G*K;
T = feedback(L,eye(2));
S = eye(2) - T;

bodemag(Wp*S)
```


Bode Diagram
From: In(1)
From: In(2)
Magnitude (dB)
To: Out(1)
To: Out(2)
-50
-100
-150
10^0
10^5
Frequency (rad/s)

```
bodemag(Wt*T)
```


Bode Diagram
From: In(1)
From: In(2)
Magnitude (dB)
To: Out(1)
To: Out(2)
0
-50
-100
-150
10^0
10^5
Frequency (rad/s)

```
bodemag(Wu*K*S)
```



Q3. (b)

Defining summing junctions

```
Sum1 = sumblk('V = w-y',2);
Sum2 = sumblk('yh = w-y',2);
s = tf('s');
```

Defining the transfer function blocks' inout and output signals

```
Wu = Wu*tf(1,1);
G.u = 'u';
G.y = 'y';

Wu.u = 'u';
Wu.y = 'z2';
Wp.u = 'yh';
Wp.y = 'z1';
Wt.u = 'y';
Wt.y = 'z3';
```

Connecting the blocks

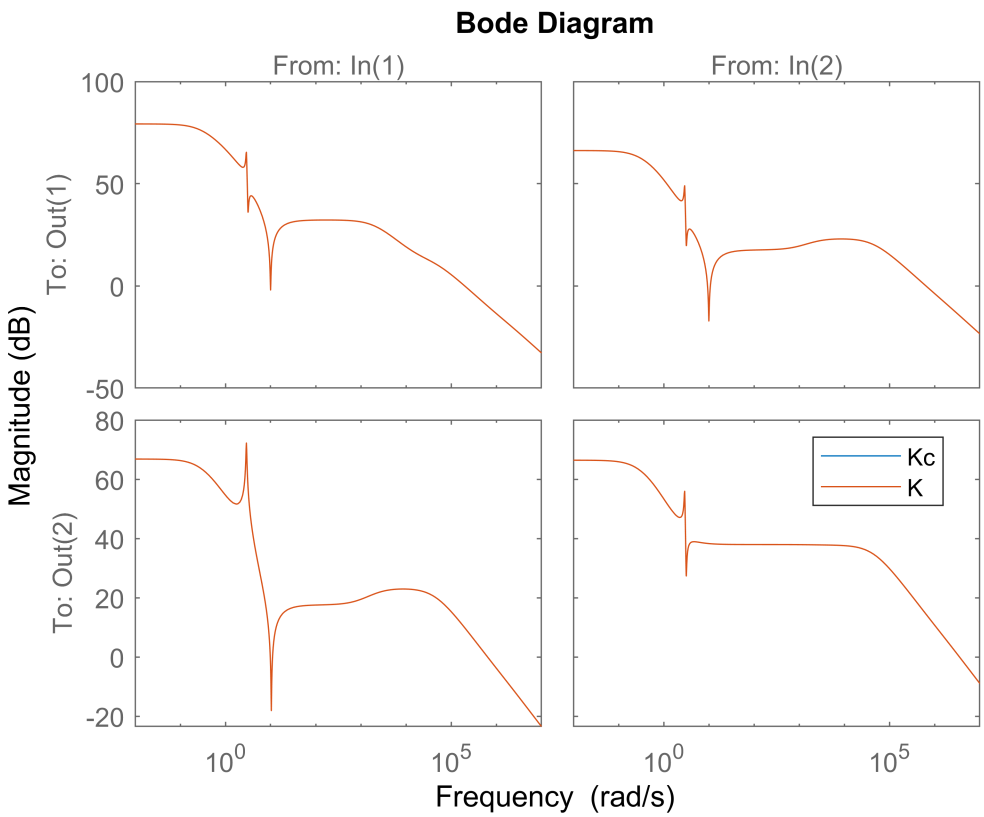
```
P = connect(G,Wp,Wu,Wt,Sum1,Sum2,{'w','u'},{'z1','z2','z3','V'});
```

Perform hinfsyn for finding K

```
Kc = hinfsyn(P,2,2);
```

Plotting magnitude bode graph

```
bodemag(Kc,K)
legend('Kc','K','Location',"best")
```



It can be observed that the controllers designed by both the methods are exactly equal and overlap each other on the magnitude bode plot.