Q1 b) $\|A\|_{max} = \max_{ij} |a_{ij}|$

① $\|e\| \geq 0 \Rightarrow$ $a_{ij} \in \mathbb{C} \Rightarrow |a_{ij}| \geq 0$

$\therefore$ The maximum of all the absolute values will also be greater than 0.

$\therefore$ $\|A\|_{max} = \max_{ij} |a_{ij}| \geq 0$

② $\|e\| = 0 \Leftrightarrow e = 0$

If $\|A\|_{max} = 0 \Rightarrow \max_{ij} |a_{ij}| = 0$

$\because$ If the maximum absolute value of the elements in a matrix is '0', $\Rightarrow$ all the elements are 0.

$\therefore$ $A = O_{n \times n}$

The other way round $\Rightarrow$ If all the elements of the matrix are 0, $\Rightarrow \max_{ij}(A) = 0 \Rightarrow \|A\|_{max} = 0$

③ $\|\alpha e\| = |\alpha| \|e\|, \forall \alpha \in \mathbb{C}$

$\|A\|_{max} = \max_{ij} |a_{ij}|$

$\|\alpha A\|_{max} = \max_{ij} |\alpha a_{ij}|$

$= |\alpha| \max_{ij} |a_{ij}| = |\alpha| \|A\|_{max}$

④ $\|e_1 + e_2\| \leq \|e_1\| + \|e_2\|, \forall e_1, e_2$

$\|A + B\|_{max} = \max_{ij} |a_{ij} + b_{ij}| \leq \max_{i,j} (|a_{ij}| + |b_{ij}|)$

$\leq \max_{i,j} |a_{ij}| + \max_{ij} |b_{ij}|$

$= \|A\|_{max} + \|B\|_{max}$

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\|A\|_{max} = \max_{i,j} |a_{ij}| = 1$

$\rho(A) = \max(\mathrm{eig}(A))$

$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 - 1$

$= \lambda^2 - 2\lambda = 0$

$\Rightarrow \lambda = 0, 2$

$\therefore \max(\mathrm{eig}(A))$

$= 2$

$\therefore \|A\|_{max} < \rho(A)$

let $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 6 & 8 \\ 6 & 8 \end{bmatrix}$

$\|AB\|_{max} = 8$

$\|A\|_{max} = 1$; $\|B\|_{max} = 5$

$\therefore \|A\|_{max} \times \|B\|_{max} = 1 \times 5 = 5$

$\therefore \|AB\|_{max} > \|A\|_{max} \|B\|_{max}$

(c) $\|G(s)\|_2 \triangleq \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega}$; $Q = e^{-sT}$

For time delays,

(For scalars, $|AB| = |A||B|$)

$$\|QG\|_2 \triangleq \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |QG|^2 d\omega}$$

$$= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|e^{-j\omega T} G(j\omega)\right|^2 d\omega}$$

$$= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|e^{-j\omega T}\right| |G(j\omega)|^2 d\omega}$$

$$= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} 1 \cdot |G(j\omega)|^2 d\omega}$$

$$= \|G(s)\|_2$$

$\therefore$ H_2 norm is time delay invariant

$\|G\|_\infty = \max |G(j\omega)|$; $Q = \frac{s-a}{s+a}$ $(a>0)$

$$\|QG\|_\infty = \max |QG| = \max \left| \left(\frac{s-a}{s+a}\right) G \right|$$

$$= \max \left(\left|\frac{s-a}{s+a}\right| |G| \right)$$

$$= \max \left(\frac{|j\omega - a|}{|j\omega + a|} |G| \right)$$

$$= \max \left(\frac{\sqrt{\omega^2 + a^2}}{\sqrt{\omega^2 + a^2}} |G| \right)$$

$$= \max (1 \cdot |G|)$$

$$= \max (|G|)$$

$$= \|G\|_\infty$$

$\therefore$ H_∞ is all-pass filter invariant.

$$\|G(s)\|_2 \triangleq \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}|G(j\omega)|^2\,d\omega} \quad ; \quad Q = \frac{s-a}{s+a}, \quad a>0$$

$$\|QG\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}|QG|^2\,d\omega}$$

$$= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{s-a}{s+a}\right|^2|G|^2\,d\omega}$$

$$= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|j\omega-a|^2}{|j\omega+a|^2}|G|^2\,d\omega}$$

$$= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}1\cdot|G|^2\,d\omega}$$

$$= \|G\|_2$$

$$\|G(s)\|_\infty \triangleq \max|G(j\omega)| \quad ; \quad Q = e^{-sT}$$

$$\|QG\|_\infty \triangleq \max|QG|$$

$$= \max\left(|e^{-sT}||G|\right)$$

$$= \max\left(|e^{-j\omega T}||G|\right)$$

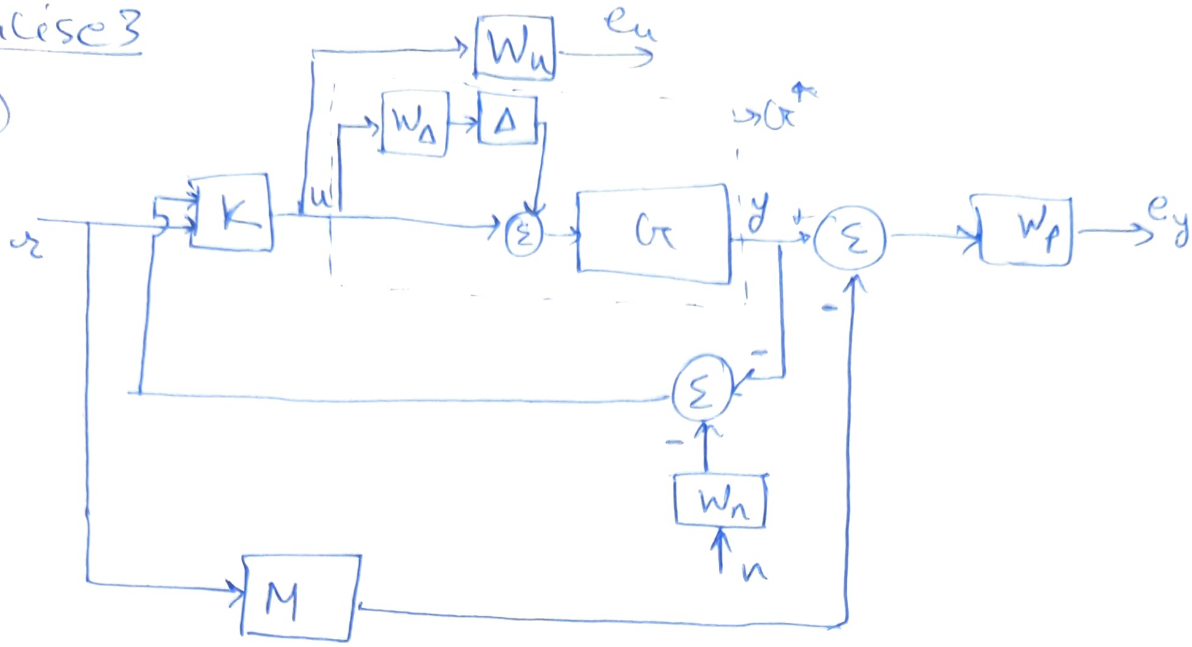
$$= \max\left(1\cdot|G|\right)$$

$$= \max\left(|G|\right)$$

$$= \|G\|_\infty$$

Exercise 3

(a)



Exogenous outputs: $e_u, e_y = z$

Sensed outputs: $-y - W_n n$

$v = -y - W_n n, r$

exogenous inputs, $w = r, n$

$e_u = W_u u$; $e_y = W_p(y - Mr)$

$= W_p(G^* u - Mr)$

$v_1 = -y - W_n n$; $v_2 = r$

$= -G^* u - W_n n$

$$\begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} e_u \\ e_y \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & W_u \\ -W_p M & 0 & W_p G^* \\ 0 & -W_n & -G^* \\ I & 0 & \cancel{W_p G^*}\ 0 \end{bmatrix} \begin{bmatrix} r \\ n \\ u \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} z \\ v \end{bmatrix} = \begin{array}{c|c} P_{11} & P_{12} \end{array} \begin{bmatrix} 0 & 0 & W_u \\ -W_p M & 0 & W_p G(I + W_\Delta \Delta) \\ 0 & -W_n & -G(I + W_\Delta \Delta) \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

P_{21} P_{22}

$$N = P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21}$$

$$= \begin{bmatrix} 0 & 0 \\ -W_p M & 0 \end{bmatrix} + \begin{bmatrix} W_u \\ W_p G (I + W_\Delta \Delta) \end{bmatrix} K \left(I - \begin{bmatrix} G (I + W_\Delta \Delta) \\ 0 \end{bmatrix} K \right)^{-1} \begin{bmatrix} 0 & -W_n \\ I & 0 \end{bmatrix}$$

Q3 (c) $F_l(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$

$$F_l\left(H, \frac{1}{s}\right) = H_{11} + \frac{H_{12}}{s}\left(I - \frac{H_{22}}{s}\right)^{-1} H_{21}$$

$$= H_{11} + \frac{H_{12}}{s} \frac{(sI - H_{22})^{-1}}{s^{-1}} H_{21}$$

$$= H_{12}(sI - H_{22})^{-1} H_{21} + H_{11}$$

Also,

$$F_l\left(H, \frac{1}{s}\right) = C(sI - A)^{-1}B + D$$

Comparing,

$$C = H_{12},\ A = H_{22},\ B = H_{21},\ D = H_{11}$$

$$\therefore\ H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = \begin{bmatrix} D & C \\ B & A \end{bmatrix}$$