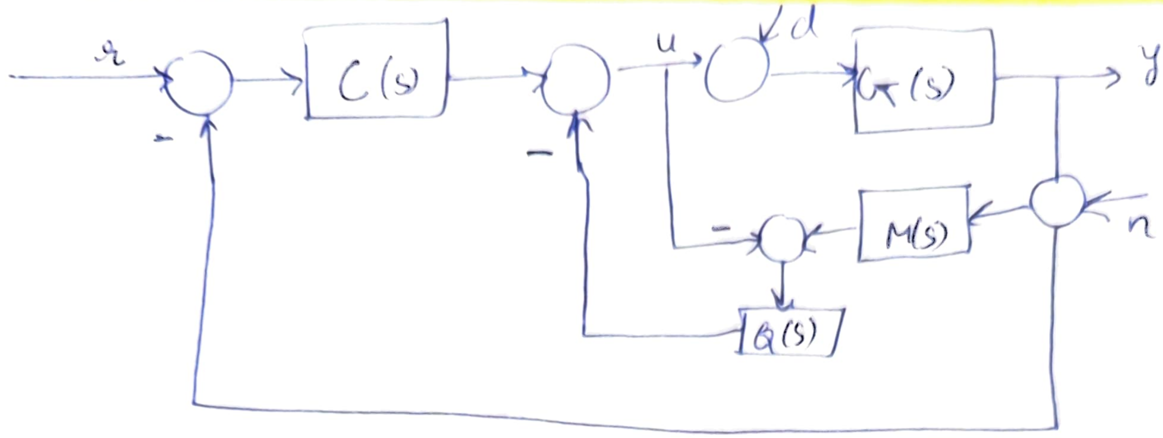
Q1



$$y = Gu + Gd \Rightarrow u = \frac{y}{G} - d \quad — (1)$$

$$C\left(r - (y+n)\right) - Q\left(M(y+n) - u\right) = u$$

$$C(r - y - n) - Q\left(My + Mn + d - \frac{y}{G}\right) = \frac{y}{G} - d$$

$$Cr - Cy - Cn - QMy - QMn - Qd + \frac{Qy}{G} = \frac{y}{G} - d$$

a) To find Transfer function, $T = \frac{y}{R}$

assume $d = 0$, $n = 0$,

$$Cr - Cy - QMy + \frac{Qy}{G} = \frac{y}{G}$$

$$Cr = y\left(\frac{1}{G} + C + QM - \frac{Q}{G}\right)$$

$$\boxed{\frac{y}{r} = \frac{CG}{I + GC + QMG - Q} = T} \quad — (2)$$

For sensitivity function $S = \frac{E}{R}$

$$\frac{y}{r} - I = \frac{CG}{I + GC + QMG - Q} - I = \frac{Q - QMG - I}{I + GC + QMG - Q} = \frac{-E}{R}$$

$$\Rightarrow \boxed{\frac{E}{R} = \frac{I + QMG - Q}{I + GC + QMG - Q} = S}$$

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b) Input sensitivity function $\frac{Y}{D}$,

assume, $n = 0$, $r = 0$

$$-Cy - QMy - Qd + \frac{Qy}{G} = \frac{y}{G} - d$$

$$\cancel{d(I+Q} \; d(I - Q) = y\left(\frac{1}{G} + C + QM - \frac{Q}{G}\right)$$

$$\cancel{\frac{y}{d} = \frac{G - QG}{I + CG + QMG - Q}}$$

$$\boxed{\frac{y}{d} = \frac{G - QG}{I + CG + QMG - Q}}$$

when $Q = I$, $M = G^{-1}$

$$\frac{y}{d} = 0$$

Q acts as a filter to account for the imperfections in the block M (to remove the disturbance).

Q2

$$\ddot{\theta} + \frac{g}{l}\sin\theta = \frac{\tau}{ml^2}$$

Assume states $\{x_1, x_2\}$ to be $\{\theta, \dot{\theta}\}$

$$\Rightarrow \quad x_1 = \theta$$

$$x_2 = \dot{\theta} = \dot{x}_1$$

$$\Rightarrow \quad \dot{x}_2 + \frac{g}{l}\sin x_1 = \frac{\tau}{ml^2}$$

$$\Rightarrow \quad \dot{x}_2 = \frac{\tau}{ml^2} - \frac{g}{l}\sin x_1$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{\tau}{ml^2} - \frac{g}{l}\sin x_1 \end{bmatrix}$$

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & 0 \end{bmatrix}_{n\times n = 2\times 2} ; \quad B = \frac{\partial f}{\partial \tau} = \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}$$

Assume $C = [1 \quad 0]$ [Output as encoder angle]

At $x_1 = \frac{\pi}{2}$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

To check system stability, check eigen values of A

$$\Rightarrow \quad \lambda_1 = \lambda_2 = 0$$

To check whether A is defective, $m = n - \text{rank}(\lambda I - A)$

geometric multiplicity

$$= 2 - \text{rank}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \cancel{2} - \cancel{\text{rank}}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$$

$$= \cancel{2 - \text{rank}}(0 \; 2 \; -1$$

$$m = 2 - 1 = 1$$

$\therefore$ A is defective $\Rightarrow$ The system is unstable

linearized form,

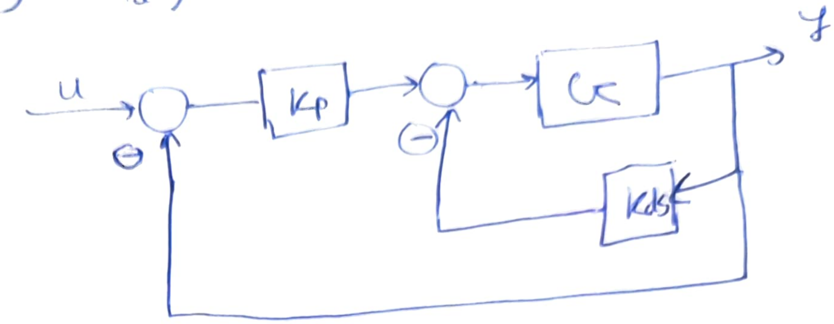
$$\boxed{\begin{aligned} \delta\dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\delta x + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}\delta\tau \\ \delta y &= [1 \quad 0]\,\delta x \end{aligned}}$$

$$G = C(sI - A)^{-1}B + D$$

$$= [1 \quad 0]\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} + 0$$

$$= [1 \quad 0]\begin{bmatrix} 1/s & 1/s^2 \\ 0 & 1/s \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix}$$

$$G = \frac{1}{s^2 m l^2}$$



New transfer function, G''

$$G' = \frac{G}{1 + G K_d s}$$

$$G'' = \frac{K_p G/(1 + G K_d s)}{1 + \frac{K_p G}{(1 + G K_d s)}} = \frac{K_p G}{1 + K_p G + G K_d s} = \frac{\frac{K_p}{s^2 m l^2}}{1 + \frac{K_p}{s^2 m l^2} + \frac{K_d s}{s^2 m l^2}}$$

$$\Rightarrow G'' = \frac{K_p}{s^2 m l^2 + K_p + K_d s} \Rightarrow G'' = \frac{K_p}{16 s^2 + K_d s + K_p}$$

Characteristic eqn. $\Rightarrow 16 s^2 + K_d s + K_p = 0$

Overshoot, $M_p = \exp\left(\frac{-\pi \zeta}{\sqrt{1 - \zeta^2}}\right) = 0.25 \Rightarrow \zeta = 0.403$

Settling time, $t_s = \frac{-\ln 0.02}{\zeta \omega_n} = 2 \Rightarrow \omega_n = 4.85$

$$\frac{K_d}{16} = 2\zeta\omega_n = 2 \times 0.403 \times 4.85 \Rightarrow K_d = 62.54$$

$$\frac{K_p}{16} = \omega_n^2 \Rightarrow K_p = (4.85)^2 \times 16 = 376.36$$

(c) The system is stable for all initial conditions from $[-\pi, \pi]$. The system is able to achieve steady state for all possible configurations.

Q3

$$\dot{x} = \underbrace{\begin{bmatrix} -0.75 & -0.25 \\ -0.5 & -0.5 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} 4 & 2 \end{bmatrix}}_{C} x$$

To check observability, $Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} [4 \quad 2] \\ [4 \quad 2] \begin{bmatrix} -0.75 & -0.25 \\ -0.5 & -0.5 \end{bmatrix} \end{bmatrix}$

$$= \begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix} \Rightarrow \text{Rank}(Q) = 1 < 2$$

$\therefore$ The system is not observable

To check controllability, $P = [B \quad AB]$

$$= \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -0.75 & -0.25 \\ -0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \Rightarrow \text{Rank}(P) = 1 < 2$$

$\therefore$ The system is not controllable

For minimal realization, first perform controllable decomposition,

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \hat{A} = M^{-1}AM, \quad \hat{B} = M^{-1}B, \quad \hat{C} = CM$$

$\hookrightarrow$ arbitrary vector

$$\Rightarrow \hat{A} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -0.75 & -0.25 \\ -0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -0.5 \\ 0 & -0.25 \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\hat{C} = [4 \quad 2] \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = [6 \quad 4]$$

The controllable modes are,

$$\hat{\dot{x}}_c = A_c x_c + B_c u_c = -1 x_c + 1 u_c$$

$$y_c = C_c x_c = 6 x_c \quad \text{—— (A)}$$

Check observability of (A), $Q = \begin{bmatrix} 6 \\ 6 \times (-1) \end{bmatrix} \Rightarrow \text{Rank}(Q) = 1 \Rightarrow$ Observable

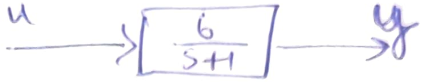
$\therefore$ (A) is the minimal decomposition as it is both controllable and observable

We know that

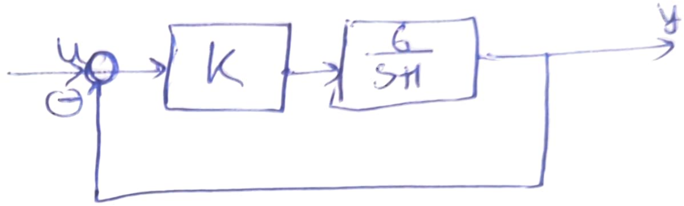
$$G = C(SI - A)^{-1}B + D$$

$$= 6\,(s-(-1))^{-1}\,1 + 0$$

$$G = \frac{6}{s+1}$$



To design a controller, the loop will look like the following diagram,



for the TF, $G' = \dfrac{b}{\tau s + 1}$

Time constant $= \tau$

Compare it with, $$G'' = \frac{\frac{6k}{s+1}}{1 + \frac{6k}{s+1}} = \frac{6k}{s+1+6k}$$

$$\frac{1}{\tau} = 1 + 6k \Rightarrow \cancel{\frac{1}{1000}}\ 1 + 6k = 1000$$

$$\Rightarrow \cancel{1 = 1000 + 600k}$$

$$\Rightarrow k = \frac{999}{6} = \frac{333}{2} = 166.5$$

$\therefore$ A proportional gain $\boxed{K = 166.5}$ will give us a time constant $\tau = 0.001\,s$.