Vatsal Ketan Patel
A20458061
CS - 577
Spring 2020

Assignment - 0

(A)

1)

$$2A - B = 2\begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}4\\5\\6\end{bmatrix}$$

$$= \begin{bmatrix}2\\4\\6\end{bmatrix} - \begin{bmatrix}4\\5\\6\end{bmatrix}$$

$$= \begin{bmatrix}-2\\-1\\0\end{bmatrix}$$

2)

$$A = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\|A\| = \sqrt{1^2+2^2+3^2}$$
$$= \sqrt{1+4+9}$$
$$= \sqrt{14}$$

Angle between A & positive X - axis is given by,

$$\cos\theta = \frac{\vec{A}\cdot\vec{x}_1}{|\vec{A}||\vec{x}_1|}$$

$\therefore \cancel{\theta = \cos^{-1}}$

Assume a unit vector say $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in positive x-axis.

$$\therefore \theta = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right)$$

$$= 1.30 \text{ rad} = 74.4845^\circ$$

3) $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\|A\| = \sqrt{14}$$

$$\hat{u} = \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{bmatrix}$$

4) $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Direction cosines of A,

$\alpha = \cos^{-1}\left(\frac{A_x}{\|A\|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) = 1.3002$
$= 74.4959°$

$\beta = \cos^{-1}\left(\frac{A_y}{\|A\|}\right) = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) = 1.0068$
$= 57.6853°$

$\gamma = \cos^{-1}\left(\frac{A_z}{\|A\|}\right) = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) = 0.6405$
$= 36.6979°$

5) $A \cdot B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

$= [1 \times 4 + 2 \times 5 + 3 \times 6]$

$= 32$

$B \cdot A = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$= [4 \times 1 + 2 \times 5 + 6 \times 3]$

$= 32$

6) Angle between A & B is given by,

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{\|A\| \, \|B\|}$$

$$= \frac{32}{\sqrt{14} \cdot \sqrt{77}}$$

$$= \frac{32}{7 \cdot \sqrt{2} \cdot \sqrt{11}}$$

$$= \frac{32}{7\sqrt{22}}$$

$$\theta = \cos^{-1}\left(\frac{32}{7\sqrt{22}}\right)$$

$$= 0.2257 \text{ rad}$$

$$= 12.9316^\circ$$

7)

$$z_1 = A \times B = \langle -3, 6, -3 \rangle$$

Now, for checking if z_1 is perpendicular to A,

$$z_1 \cdot A = -3\times1 + 6\times2 - 3\times3 = 0.$$

Since the dot product of $z_1 \cdot A$ is zero.

$\therefore$ z_1 is perpendicular to A.

8)

$$A \times B = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

$$= i \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} - j \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + k \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= (12 - 15)\, i - (6 - 12)\, j + k\,(5 - 8)$$

$$= (-3)i + 6j - 3k$$

$$= \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix}$$

$$B \times A = \begin{vmatrix} i & j & k \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= i \begin{vmatrix} 5 & 6 \\ 2 & 3 \end{vmatrix} - j \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix} + k \begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix}$$

$$= (15 - 12)\, i - j\,(12 - 6) + k\,(8 - 5)$$

$$= 3i - 6j + 3k$$

$$= \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$$

~~Let $\vec{x}$ be a vector perpendicular to both $\vec{A}$ & $\vec{B}$~~

9)

From question 8, we have

$Z_1 = A \times B = \langle -3, 6, -3 \rangle$

$Z_2 = B \times A = \langle 3, -6, 3 \rangle$

Now, for checking if Z_1 & Z_2 are perpendicular to A & B,

$$Z_1 . A = -3 \times 1 + 2 \times 6 + 3 \times -3 = 0$$
$$Z_2 . A = 3 \times 1 + 2 \times -6 + 3 \times 3 = 0$$

$$Z_1 . B = -3 \times 4 + 6 \times 5 + -3 \times 6 = 0$$
$$Z_2 . B = 3 \times 4 + 5 \times -6 + 3 \times 6 = 0$$

Since the dot products of two vectors is zero.
$\therefore$ They are perpendicular to each other.

$\therefore$ $Z_1 = \langle -3, 6, -3 \rangle$, $Z_2 = \langle 3, -6, 3 \rangle$ are perpendicular to both A & B.

10) $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ $C = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$

$$a_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + a_3 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -6 & 6 & 0 \end{array}\right]$$

$R_2 - 2R_1 \rightarrow R_2$

$R_3 - 3R_1 \rightarrow R_3$

$R_3 - 2R_2 \rightarrow R_3$

$\downarrow$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x + 4y - z = 0.$

$-3y + 3z = 0.$

$y = z$

$x + 4y - y = 0.$

$x = -3y$

Let $y = t.$

$\therefore (-3t, t, t)$ is non-trivial.

A, B, C are linearly dependent.

11)

$$A^TB = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times4 + 2\times5 + 3\times6 \end{bmatrix}$$

$$= 32$$

$$AB^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$$

(B)

1) $2A - B$

$$= 2\begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 6 \\ 8 & -4 & 6 \\ 0 & 10 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 5 \\ 6 & -5 & 10 \\ -3 & 12 & -3 \end{bmatrix}$$

2) $AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1\times1 + 2\times2 + 3\times3 & 1\times2 + 2\times1 + \overset{-6}{6} & 1\times1 - 2\times4 + 3\times1 \\ 4\times1 - 2\times2 + 3\times3 & 4\times2 - 2\times1 + 3\times-2 & 4\times1 + 8 + 3 \\ 0 + 10 + (-3) & 0 + 5 + 2 & 0 - 20 - 1 \end{bmatrix}$

$= \begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1\times1 + 2\times4 + 1\times0 & 1\times2 + 2\times-2 + 1\times5 & 1\times3 + 2\times3 + 1\times-1 \\ 2\times1 + 1\times4 + 0 & 2\times2 + 1\times-2 + -4\times5 & 2\times3 + 1\times3 + -4\times-1 \\ 3\times1 + 4\times-2 + 0 & 3\times2 + -2\times-2 + 1\times5 & 3\times3 - 2\times3 + 1\times-1 \end{bmatrix}$

$= \begin{bmatrix} 9 & 3 & 8 \\ 6 & -18 & 13 \\ -5 & 15 & 2 \end{bmatrix}$

3) $(AB)^T = \begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix}^T = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}$

$$B^T A^T = (AB)^T = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ -4 & 15 & -21 \end{bmatrix}$$

4)

$$|A| = 1 \begin{vmatrix} -2 & 3 \\ 5 & -1 \end{vmatrix} - 2 \begin{vmatrix} 4 & 3 \\ 0 & -1 \end{vmatrix} + 3 \begin{vmatrix} 4 & -2 \\ 0 & 5 \end{vmatrix}$$

$$= (2-15) - 2(-4) + 3(20)$$

$$= -13 + 8 + 60$$

$$|A| = 55$$

$$|C| = 1 \begin{vmatrix} 5 & 6 \\ 1 & 3 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ -1 & 3 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ -1 & 1 \end{vmatrix}$$

$$= (15-6) - 2(12+6) + 3(4+5)$$

$$= 9 - 36 + 27$$

$$= 0$$

5) Out of the three matrices, matrix **B** is the only one in which the row vectors form an orthogonal set. i.e. the dot product is 0.

$$\vec{R_1} \cdot \vec{R_2} = 0, \quad \vec{R_2} \cdot \vec{R_3} = 0, \quad \vec{R_1} \cdot \vec{R_3} = 0.$$

$$\vec{R_1} \cdot \vec{R_2} = 2+2-4 = 0.$$
$$\vec{R_2} \cdot \vec{R_3} = 6-2-4 = 0.$$
$$\vec{R_1} \cdot \vec{R_2} = 3-4+1 = 0.$$

6) $\rightarrow$ A^{-1}

$|A| = 55$ [from B-4]

$$adj.(A) = \begin{bmatrix} 2-15 & -(-4-0) & 20-0 \\ -(-2-15) & -1-0 & -(5-0) \\ 6+6 & -(3-12) & -2-8 \end{bmatrix}^T$$

$$= \begin{bmatrix} -13 & 4 & 20 \\ 17 & -1 & -5 \\ 12 & 9 & -10 \end{bmatrix}^T$$

$$= \begin{bmatrix} -13 & 17 & 12 \\ 4 & -1 & 9 \\ 20 & -5 & -10 \end{bmatrix}$$

$$A^{-1} = \frac{adj.(A)}{|A|}$$

$$= \begin{bmatrix} \frac{-13}{55} & \frac{17}{55} & \frac{12}{55} \\ \frac{4}{55} & \frac{-1}{55} & \frac{9}{55} \\ \frac{20}{55} & \frac{-5}{55} & \frac{-10}{55} \end{bmatrix}$$

$\rightarrow \quad B^{-1}$

$$|B| = 1(1-8) - 2(2+12) + 1(-4-3)$$
$$|B| = -42$$

$$\text{adj.}(B) = \begin{bmatrix} 1\cancel{+}-8 & -(2+12) & -2-3 \\ -(2+2) & 1-3 & -(-2-6) \\ -8-1 & -(-4-2) & 1-4 \end{bmatrix}^T$$

$$= \begin{bmatrix} -7 & -(14) & -7 \\ -(4) & -2 & -(-8) \\ -9 & -(-6) & -3 \end{bmatrix}^T$$

$$= \begin{bmatrix} -7 & -4 & -9 \\ -14 & -2 & 6 \\ -7 & 8 & -3 \end{bmatrix}$$

$$B^{-1} = \frac{\text{adj.}(B)}{|B|}$$

$$= \begin{bmatrix} \frac{1}{6} & \frac{2}{21} & \frac{3}{14} \\ \frac{1}{3} & \frac{1}{21} & \frac{-1}{7} \\ \frac{1}{6} & \frac{-4}{21} & \frac{1}{14} \end{bmatrix}$$

(c)

1) $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$

Eigenvalues of A.

$$Ax = \lambda x$$
$$Ax - \lambda I x = 0.$$
$$(A - \lambda I)x = 0.$$
$$\det(A - \lambda I) = 0.$$

$$\begin{vmatrix} (1-\lambda) & 2 \\ 3 & (2-\lambda) \end{vmatrix} = 0.$$

$$(1-\lambda)(2-\lambda) - 6 = 0.$$
$$\lambda^2 - 3\lambda - 4 = 0.$$
$$\lambda^2 - 4\lambda + \lambda - 4 = 0.$$
$$\lambda(\lambda - 4) + 1(\lambda - 4) = 0.$$
$$(\lambda + 1)(\lambda - 4) = 0.$$

$\boxed{\lambda = -1}$ and $\boxed{\lambda = 4.}$

Eigenvectors of A.

$$\lambda = -1.$$

$$\begin{bmatrix} (1+1) & 2 \\ 3 & (2-1) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x + 2y = 0.$
$3x + 3y = 0.$

$$\boxed{x = -y}$$

$x = -1, \; y = 1.$

$\therefore$ Eigen vector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda = 4$

$$\begin{bmatrix} (1-\lambda) & 2 \\ 3 & (2-\lambda) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-3x + 2y = 0.$
$3x - 2y = 0$

$$\boxed{3x = 2y}$$

$x = \frac{2}{3}, \; y = 1.$

$\therefore$ Eigen vector is $\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$

2)

$$V = \begin{bmatrix} -1 & 2/3 \\ 1 & 1 \end{bmatrix}$$

$$V^{-1} = \frac{Adj(V)}{|V|}$$

$$|V| = -1(1) - 1\left(\frac{2}{3}\right)$$

$$= \frac{-5}{3}$$

$$V^{-1} = \frac{-3}{5} \begin{bmatrix} 1 & -2/3 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3/5 & 2/5 \\ 3/5 & 3/5 \end{bmatrix}$$

$$V^{-1}AV = \begin{bmatrix} -3/5 & 2/5 \\ 3/5 & 3/5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2/3 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3/5 & -2/5 \\ 12/5 & 12/5 \end{bmatrix} \begin{bmatrix} -1 & 2/3 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$$

3) Dot product of eigen vectors of A,

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$$

$$= -1\left(\frac{2}{3}\right) + 1$$

$$= \frac{1}{3}.$$

4) Eigen values of B,

$Bx = \lambda x.$

$\det(B - \lambda I) = 0.$

$$\begin{vmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = 0.$$

$$10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0.$$

$$\lambda(\lambda-6) - 1(\lambda-6) = 0.$$

$$\boxed{\lambda = 1, 6}$$

Eigenvectors of B,

$\lambda = 1$

$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 0.$$

$$x - 2y = 0.$$
$$-2x + 4y = 0.$$

$$x = 2y.$$

$$x = 2, \quad y = 1$$

$\therefore$ Eigen vector is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$$\lambda = 6$$

$$\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

$$-4x - 2y = 0.$$
$$-2x - y = 0.$$

$$x = \frac{-2y}{4}$$

$$x = \frac{-1}{2}, \quad y = 1$$

$\therefore$ Eigen vector is $\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$

Dot product of eigen vectors of B,

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

$$= 2(-1/2) + 1(1)$$
$$= -1 + 1$$
$$= 0.$$

5) Both vectors are perpendicular to each other i.e. they form an orthogonal pair.

Now,

$$\vec{x} \cdot \vec{y} = |\vec{x}| \, |\vec{y}| \cos\theta.$$

$\because$ $|\vec{x}| \, |\vec{y}| \neq 0$

$\therefore$ $\vec{x} \cdot \vec{y}$ can be zero if $\cos\theta = 0$.

$\therefore \theta = 90^\circ$.

$\therefore$ Angle b/w x & y is 90°.

(D) $f(x) = x^2 + 3$, $g(x, y) = x^2 + y^2$.

i) $f'(x)$ & $f''(x)$.

$$f'(x) = \frac{d}{dx} x^2 + \frac{d}{dx}(3)$$

$$= 2x.$$

$$f''(x) = \frac{d^2}{dx^2} 2x$$

$$= 2.$$

$$f'(x) = 2x, \quad f''(x) = 2.$$

2) $\frac{dg}{dx}$ & $\frac{dg}{dy}$.

$$\frac{dg}{dx} = \frac{d}{dx} x^2 + \frac{d}{dx} y^2$$

$$= 2x.$$

$$\frac{dg}{dy} = \frac{d}{dx} x^2 + \frac{d}{dx} y^2$$

$$= 2y.$$

$$\frac{dg}{dx} = 2x, \quad \frac{dg}{dy} = 2y$$

3) $\nabla g\,(x, y)$

$$\frac{dg}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2)$$

$$= 2x.$$

$$\frac{dg}{dy} = 2y.$$

$$\nabla g(x, y) = \begin{bmatrix} 2x \\ \hline 2y \end{bmatrix}$$

4)

$$pdf = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-1/2\left(\frac{x-\mu}{\sigma}\right)}$$

where, μ = mean

$\sigma^2 \geq 0$ = Variance.