1. Suppose, a transformation matrix A, transforms the standard basis vectors of $R^3$ as follows :

$$\begin{bmatrix}1\\0\\0\end{bmatrix} => \begin{bmatrix}6\\4\\2\end{bmatrix} ; \begin{bmatrix}0\\1\\0\end{bmatrix} => \begin{bmatrix}1\\2\\8\end{bmatrix} ; \begin{bmatrix}0\\0\\1\end{bmatrix} => \begin{bmatrix}1\\5\\7\end{bmatrix}$$

(a) If the volume of a hypothetical parallelepiped in the un-transformed space is $100 units^3$ what will be volume of this parallelepiped in the transformed space?

**Solution:**

(b) What will be the volume if the transformation of the basis vectors is as follows :

$$\begin{bmatrix}1\\0\\0\end{bmatrix} => \begin{bmatrix}1\\4\\7\end{bmatrix} ; \begin{bmatrix}0\\1\\0\end{bmatrix} => \begin{bmatrix}2\\5\\8\end{bmatrix} ; \begin{bmatrix}0\\0\\1\end{bmatrix} => \begin{bmatrix}3\\6\\9\end{bmatrix}$$

**Solution:**

(c) Comment on the uniqueness of the second transformation.

**Solution:**

2. If $R^3$ is represented by following basis vectors : $\begin{bmatrix}4\\2\\1\end{bmatrix}, \begin{bmatrix}-3\\-1\\5\end{bmatrix}, \begin{bmatrix}0\\2\\7\end{bmatrix}$

(a) Find the representation of the vector $\begin{pmatrix} 2 & -1 & 6 \end{pmatrix}^T$ (as represented in standard basis) in the above basis.

**Solution:**

(b) We know that, orthonormal basis simplifies this to a great extent. What would be the representation of vector $\begin{pmatrix} 2 & -1 & 6 \end{pmatrix}^T$ in the orthogonal basis represented by : $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$

**Solution:**

(c) Comment on the advantages of having orthonormal basis.

**Solution:**

3. Consider a square matrix $A$ such that the sum of the entries of every column of $A$ is the same number $c$. Prove that $c$ is an eigenvalue of transpose of $A$.

**Solution:**

4. Let $C$ be a $2 \times 2$ matrix. If the trace of matrix $C$ is 0, then what can you say about matrix $C^n$ where $n$ is a positive integer?

**Solution:**

5. *If $c$ and $d$ are two real numbers then the exponential of $c + d$ is the product of the exponential of $c$ with the exponential of $d$ i.e

$$e^{c+d} = e^c e^d$$

(a) If we replace them with square matrices $C$ and $D$, does the equality still holds ? Prove it, if yes, else provide a counterexample.

**Solution:**

(b) Are there any special conditions on $C$ and $D$ under which it will always hold ?

**Solution:**

6. Prove that if $u, v$ are nonzero vectors in $R^2$, then $< u, v >=\| u \|\| v \| cos\theta$ where $\theta$ is the angle between $u$ and $v$ (thinking of $u$ and $v$ as arrows with initial point at the origin).

**Solution:**

7. **Linear Dependence Lemma**
   If the list of vectors $(u_1, \ldots, u_n)$ is linearly dependent in some vector space $V$, and $u_1 \neq \mathbf{0}$, then prove that there exists an index $i \in \{2, \ldots, n\}$ such that $u_i \in span(u_1, \ldots, u_{i-1})$, and if $u_i$ is removed, the span of the list remains unchanged.

**Solution:**

8. * **Independent Lists Cannot Be Arbitrarily Long**
   Prove that for any finite dimensional vector space, the length of any independent list of vectors is always smaller than or equal to the length of any spanning list of vectors. (Hint: you may want to use the Linear Dependence Lemma)

**Solution:**

9. **Cyclic Differences**
   Consider the two lists of vectors $A = (u_1, u_2, u_3, \ldots, u_{n-1}, u_n)$, and $B = (u_1 - u_2, u_2 - u_3, u_3 - u_4, \ldots, u_{n-1} - u_n, u_n)$ (the last element is the same as $A$) in some vector space $V$. Prove or disprove the following statements:

   1. If $A$ is linearly independent, then so is $B$.
   2. If $A$ is spanning list, then so is $B$.

**Solution:**

10. Compute and compare the L1 norm and Frobenius norm of the matrices given below.

$$\begin{bmatrix} 4 & -2 & 1 \\ 2 & 3 & -6 \\ 1 & 8 & 9 \end{bmatrix}, \begin{bmatrix} 4 & -2 & 1 \\ 8 & 3 & -6 \\ 9 & 2 & 1 \end{bmatrix}$$

**Solution:**

11. * **Induced Matrix Norms**
In case you didn't already know, a norm $\|.\|$ is any function with the following properties:

    1. $\|x\| \geq 0$ for all vectors $x$.
    2. $\|x\| = 0 \iff x = \mathbf{0}$.
    3. $\|\alpha x\| = |\alpha|\|x\|$ for all vectors $x$, and real numbers $\alpha$.
    4. $\|x + y\| \leq \|x\| + \|y\|$ for all vectors $x, y$.

    Now, suppose we're given some vector norm $\|.\|$ (this could be L2 or L1 norm, for example). We would like to use this norm to measure the size of a matrix $A$. One way is to use the corresponding induced matrix norm, which is defined as $\|A\| = \sup_x\{\|Ax\| : \|x\| = 1\}$.

    E.g.: $\|A\|_2 = \sup_x\{\|Ax\|_2 : \|x\|_2 = 1\}$, where $\|.\|_2$ is the standard L2 norm for vectors, defined by $\|x\|_2 = \sqrt{x^T x}$.

    Prove the following properties for an arbitrary induced matrix norm:

    (a) $\|A\| \geq 0$.

**Solution:**

(b) $\|\alpha A\| = |\alpha|\|A\|$ for any real number $\alpha$.

**Solution:**

(c) $\|A + B\| \leq \|A\| + \|B\|$.

**Solution:**

(d) $\|A\| = 0 \iff A = 0$.

**Solution:**

(e) $\|AB\| \leq \|A\|\|B\|$.

**Solution:**

(f) $\|A\|_2 = \sigma_{\max}(A)$, where $\sigma_{\max}$ is the largest singular value.

**Solution:**

12. Prove that the eigen vectors of a real symmetric($S_{n*n}$) matrix are linearly independent and forms a orthogonal basis for $R^n$.

**Solution:**

13. If $A_{n*n}$ is a square symmetric matrix. Prove that solution to the equation $\max_x \{x^T Ax \mid \|x\| = 1\}$ is given by the largest eigen value of A, when $x$ is the eigen vector corresponding to largest eigen value.

**Solution:**

14. Prove that a full rank square matrix $A_{n*n}$ is always similar to some diagonal matrix $D_{n*n}$.

**Solution:**

15. Consider two vectors $x$ and $y$ separated by angle $\theta$. Suppose an orthonormal transformation represented by matrix $A_{n*n}$ is applied to vectors $x$ and $y$. Find the relation between $\theta$ and the angle between the newly transformed vectors $Ax$ and $Ay$.

**Solution:**

16. Let $u_1, u_2, ...., u_n$ be a set of n orthonormal vectors. Similarly let $v_1, v_2, ...., v_n$ be another set of n orthonormal vectors.

(a) Show that $u_1 v_1^T$ is a rank-1 matrix.

| |
|---|
| **Solution:** |

(b) Show that $u_1 v_1^T + u_2 v_2^T$ is a rank-2 matrix.

| |
|---|
| **Solution:** |

(c) Show that $\sum_{i=1}^{n} u_i v_i^T$ is a rank-n matrix.

| |
|---|
| **Solution:** |