AVD624 - Computer Vision

Assignment - 1

Fundamentals of Maths, Projective Geometry and Camera

Submitted by

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Question - 1

Determine whether $sin(x)$ and $sin(2x)$ might be orthogonal or orthonormal functions.

Two functions f and g are said to be orthogonal if they satisfy (for arbitrary real number L),

$$\langle f, g\rangle = \frac{1}{L}\int\limits_{-L}^{L} f(x)g(x)\,dx = 0$$

Let $f(x) = sin(x)$ and $g(x) = sin(2x)$, then we have,

$$\langle sin(x), sin(2x)\rangle = \frac{1}{L}\int\limits_{-L}^{L} sin(x)sin(2x)\,dx = \frac{2}{L}\int\limits_{-L}^{L} sin^2(x)cos(x)\,dx$$

Since the integrand is an even function, it will be zero only when $L = n\pi$, where n belongs to integers. Hence, we can say that $sin(x)$ and $sin(2x)$ will be orthogonal only in certain intervals of the form $[-n\pi, n\pi]$ or $[0, n\pi]$.

Two functions f and g are said to be orthonormal if and only if they are are orthogonal and normalised. Here, normalised means that the inner product of the function with itself should be one. Now considering that interval is chosen to satisfy the above orthogonality condition, then

$$\langle sin(x), sin(x)\rangle = \frac{1}{L}\int\limits_{-L}^{L} sin(x)sin(x)\,dx = \frac{1}{n\pi}\int\limits_{-n\pi}^{n\pi} sin^2(x)\,dx = \frac{1}{n\pi}\int\limits_{-n\pi}^{n\pi} \frac{1-cos(2x)}{2}\,dx$$

$$= \frac{1}{n\pi}\left[\frac{x}{2} - \frac{sin(2x)}{4}\right]_{-n\pi}^{n\pi} = -\frac{sin(2n\pi) - 2n\pi}{2n\pi} = 1 \ , \quad n \in \mathbb{Z}$$

$$\langle sin(2x), sin(2x)\rangle = \frac{1}{L}\int\limits_{-L}^{L} sin(2x)sin(2x)\,dx = \frac{1}{n\pi}\int\limits_{-n\pi}^{n\pi} sin^2(2x)\,dx = \frac{1}{n\pi}\int\limits_{-n\pi}^{n\pi} \frac{1-cos(4x)}{2}\,dx$$

$$= \frac{1}{n\pi}\left[\frac{x}{2} - \frac{sin(4x)}{8}\right]_{-n\pi}^{n\pi} = -\frac{sin(4n\pi) - 4n\pi}{4n\pi} = 1 \ , \quad n \in \mathbb{Z}$$

Hence, the functions $sin(x)$ and $sin(2x)$ are orthogonal and orthonormal in the interval $[-n\pi, n\pi]$ or $[0, n\pi]$, where $n \in \mathbb{Z}$.

Question - 2

A positive definite matrix has positive eigenvalues. Prove this.

A matrix A is said to be positive definite if $x^T Ax > 0$ for all vectors $x \neq 0$. Let λ be a real eigenvalue of A and let x be a corresponding real eigenvector. Therefore, we have

$$Ax = \lambda x$$

Pre-multiplying x^T on both sides of the equation, we get,

$$x^T Ax = x^T \lambda x \implies x^T Ax = \lambda x^T x \implies x^T Ax = \lambda ||x||^2$$

Here, the left hand side is positive as A is positive definite and x is a non-zero vector as it is an eigenvector. Since the length $||x||^2$ is positive, we must have λ is positive. Hence, we can say that a positive definite matrix has positive eigenvalues.

Question - 3

Does the function $y = xe^{-x}$ have a unique value x that minimises y? If so, can you find it by taking a derivative and setting it equal to zero? Suppose this problem requires gradient descent to solve. Write the algorithm you would use to find x that minimises y.

We are given the function $y = xe^{-x}$. So, differentiating it with respect to x we get,

$$\frac{dy}{dx} = \frac{d}{dx}(xe^{-x}) = -(x-1)e^{-x} \ \ , \ \ \frac{dy}{dx} = 0 \implies x = 1, \infty$$

But, we have one more critical point, i.e. $x = -\infty$. Substituting in $y(x)$ gives that $y(1) = 1/e$ is a point of global maxima, while $y \to 0$ as $x \to \infty$, and $y \to -\infty$ as $x \to -\infty$. Hence, the function does not have a unique minima nor can it be found by setting the derivative equal to zero. We can try using gradient descent algorithm as specified below,

1: Guess $x(0)$, set $k \leftarrow 0$
2: **while** $|y'(x^{(k)})| \geq \epsilon$ **do**
3: $\quad x^{(k+1)} = x^{(k)} - t_k y'(x^{(k)})$
4: $\quad k \leftarrow k + 1$
5: **end while**
6: **return** $x^{(k)}$

Here, $x(0)$ is the initial guess, $x^{(k)}$ is the value at k^{th} iteration, t_k is the step size or learning rate and ϵ is the desired precision. If we choose our initial guess as $x(0) \leq 1$, then the iterations will not converge since the slope will keep on increasing. Whereas, if we choose $x(0) > 1$, then depending upon the desired precision, we will get a certain value of $x^{(k)}$. But this solution will not be unique since it depends upon ϵ, nor will it give the global minima of the function.

Question - 4

Find the minimum of $y = 50sin(x) + x^2$ over $-10 \leq x \leq 10$. Use gradient descent to locate the minimum. Use python to plot this function. Visually observe the multiple local minima and global minimum within the interval of x. Implement Gradient Descent (GD) and test the code under following parameters setup.

1. Pick a starting point at $x = 7$. What is the minimum?
2. Pick a starting point at $x = 1$. What is the minimum?
3. Change the step size (or learning rate) and see what kind of step size will help the local minimum when $x = 7$?
4. (Optional) If possible, animate the function minimization iteration.

We use a similar approach as discussed in previous question. First we find the derivative of the given function,

$$\frac{dy}{dx} = \frac{d}{dx}(50sin(x) + x^2) = 50cos(x) + 2x$$

Now we plot the given function to get to know about the possibility of multiple local minimas.

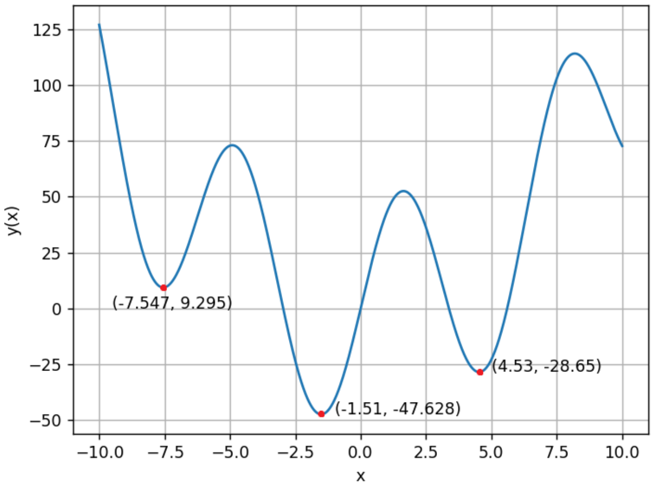

Figure 1: $y(x)$ vs x over $-10 \leq x \leq 10$

We can see that 3 local minimas are possible in the given domain [-10, 10]. The form of gradient descent equation will be

$$x^{(k+1)} = x^{(k)} - t_k y'(x^{(k)})$$

Case 1 : $x_0 = 7$, learning rate $= 0.01$, precision $= 0.001$
The gradient descent algorithm converges on 18^{th} iteration, and the local minimum occurs at $x^{(18)} = 4.530191408249529$, and $y(x^{(18)}) = -28.649760157331393$

Case 2 : $x_0 = 1$, learning rate $= 0.01$, precision $= 0.001$
The gradient descent algorithm converges on 19^{th} iteration, and the local minimum occurs at $x^{(19)} = -1.510327618697581$, and $y(x^{(19)}) = -47.627526717942786$

Case 3 : $x_0 = 7$, learning rate $= 0.001$, precision $= 0.001$
The gradient descent algorithm converges on 232^{nd} iteration, and the local minimum occurs at $x^{(232)} = 4.530194568289868$, and $y(x^{(232)}) = -28.649760154476844$

Case 4 : $x_0 = 7$, learning rate = 0.0001, precision = 0.001
The gradient descent algorithm converges on 2367^{th} iteration, and the local minimum occurs at $x^{(2367)} = 4.530194798264702$, and $y(x^{(2367)}) = -28.649760154249154$

It can be seen that decreasing the learning rate helps in achieving a more accurate value for the function minima, but there is a trade-off with the number of iterations required to get to the solution. Hence, we need to appropriately choose the learning rate and precision, considering the tolerances and error margin allowed.

Question - 5

You were introduced to the following notation $[\mathbf{a}]_\times$ that was claimed to be a 3×3 matrix. Write various properties of this matrix. Evaluate the product $[\mathbf{a}]_\times \mathbf{b}$, where $\mathbf{b}$ is a 3×1 vector. Compare the results of this operation with the result of computing cross product of $\mathbf{a}$ with $\mathbf{b}$.

Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors defined as $[a_1\ a_2\ a_3]^T$ and $[b_1\ b_2\ b_3]^T$ respectively. Then their cross product is defined as

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

$$= (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$$

Now, consider the product of a 3 x 3 matrix $[\mathbf{a}]_\times$ and a 3 x 1 vector $\mathbf{b}$. The matrix $[\mathbf{a}]_\times$ is a *skew-symmetric* matrix, i.e. its diagonal elements are zero and $[\mathbf{a}]_\times^T = -[\mathbf{a}]_\times$. The non-diagonal elements are the elements of the 3 x 1 vector $\mathbf{a}$.

$$[\mathbf{a}]_\times \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}_{3\times 3} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{3\times 1} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}_{3\times 1}$$

We have verified that the result from vector cross product is same as matrix multiplication. Hence, to simplify the cross product calculation, we can write the vector cross product as the product of a 3 x 3 skew-symmetric matrix $[\mathbf{a}]_\times$ and a 3 x 1 vector $\mathbf{b}$.

Question - 6

Derive camera calibration matrix from scratch (from the pinhole idea). Clearly describe what are extrinsic parameters and what are intrinsic parameters. How do you derive these parameters?

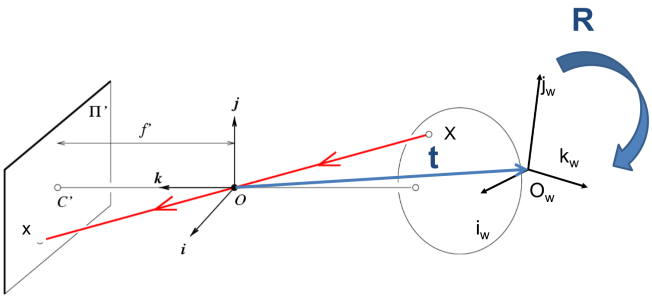

Figure 2: Alignment of Camera setup and the Image plane

Using the concept of similar triangles, we can write

$$\frac{f}{z} = \frac{u}{x} = \frac{v}{y}$$

This gives us $u = fx/z$ and $v = fy/z$. Then, we can use homogeneous coordinates for writing the expression for x.

$$x = K \begin{bmatrix} R & t \end{bmatrix} X$$

where, x: Image Coordinates (u, v, 1)
K: Intrinsic Matrix (3 x 3)
R: Rotation (3 x 3)
t: Translation (3 x 1)
X: World Coordinates (X, Y, Z, 1)

As a preliminary case, we first consider some assumptions

- Intrinsic assumptions: Unit aspect ratio, Optical centre at (0,0), No skew
- Extrinsic assumptions: No rotation, Camera at (0,0,0)

Hence, our expression becomes

$$x = K \begin{bmatrix} I & 0 \end{bmatrix} X \implies w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

where, f is the focal length.
We can generalise the expression by removing our assumptions one by one.

Remove assumption: Known optical centre

The optical centre is no longer at origin. Let it be translated to (u_0, v_0), we get

$$x = K \begin{bmatrix} I & 0 \end{bmatrix} X \implies w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & u_0 & 0 \\ 0 & f & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Remove assumption: Square pixels

Since, the pixels are no longer in square shape, hence focal length in x-direction (α) will be different from focal length in y-direction (β). So we get

$$x = K \begin{bmatrix} I & 0 \end{bmatrix} X \implies w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & u_0 & 0 \\ 0 & \beta & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Remove assumption: Non-skewed pixels

The skew parameter (s) defines the skewness in the intrinsic system. It defines the property of x and y not being linearly related, i.e. x may be varying at a different rate as compared to y. We get

$$x = K \begin{bmatrix} I & 0 \end{bmatrix} X \implies w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 & 0 \\ 0 & \beta & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Allow camera translation

Now we consider that the 3D world coordinates are not perfectly aligned to image plane and there can be translation along x, y, and z axis.

$$x = K \begin{bmatrix} I & t \end{bmatrix} X \implies w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Allow camera rotation

We assume that we know the rotation along all 3 axes, defined by 3 matrices as

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\alpha & -sin\alpha \\ 0 & sin\alpha & cos\alpha \end{bmatrix}, \; R_y(\beta) = \begin{bmatrix} cos\beta & 0 & sin\beta \\ 0 & 1 & 0 \\ -sin\beta & 0 & cos\beta \end{bmatrix}, \; R_z(\gamma) = \begin{bmatrix} cos\gamma & -sin\gamma & 0 \\ sin\gamma & cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying these rotations matrices in a certain order (since matrix multiplication is not commutative), we get simplified expression considering the camera rotation.

$$x = K \begin{bmatrix} I & t \end{bmatrix} X \implies w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha & s & u_0 \\ 0 & \beta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Hence, we have relaxed all the assumptions and obtained a generalised expression for mapping a 3D world point onto a 2D image plane using a projection matrix P, defined as the matrix multiplication of intrinsic (5 degrees of freedom) and extrinsic (6 dof) matrix, simplified as

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = P_{3\times4} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

We can obtain the elements of the camera calibration matrix using various methods, such as Direct Linear Transformation (DLT) and Zhang's method which works by applying DLT on different perspective views to obtain the intrinsic parameters. This gives us the 5 unknowns of the intrinsic matrix. Then, we can apply camera pose estimation methods, such as Perspective-n-Point (PnP) method, to obtain the 6 extrinsic parameters. This way we can completely specify the camera calibration, pose and orientation.

Question - 7

The continuous convolution of two functions $f(x)$ and $g(x)$ is given by

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(y)\, g(x-y)\; dy. \tag{1}$$

The Gaussian function at scale s is defined as

$$G_s(x) = \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{x^2}{2s}\right), \tag{2}$$

and has the property that

$$\int_{-\infty}^{+\infty} G_s(x)\; dx = 1. \tag{3}$$

Prove that this class of functions satisfies the *semigroup property*: the convolution of one Gaussian with another produces a third Gaussian with scale equal to their sum, or

$$(G_{s_1} * G_{s_2})(x) = G_{s_1+s_2}(x). \tag{4}$$

The Fourier transform of a Gaussian function can be written as,

$$\mathcal{F}\{G_s(x)\} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}} e^{2\pi i f x}\, dx = \frac{1}{\sqrt{2\pi s}} \int_{-\infty}^{\infty} e^{-\frac{1}{2s}[x^2 - 4\pi i s x f]}\, dx$$

$$= \frac{1}{\sqrt{2\pi s}} \left[\int_{-\infty}^{\infty} e^{-\frac{1}{2s}(x-2\pi i s f)^2}\, dx\right] e^{\frac{1}{2s}(2\pi i s f)^2} = \frac{1}{\sqrt{2\pi s}} \left[\sqrt{2\pi s}\right] e^{-2\pi^2 s f^2} = e^{-2\pi^2 s f^2}$$

Using convolution theorem of Fourier transform,

$$\mathcal{F}\{G_{s_1} * G_{s_2}\} = \mathcal{F}\{G_{s_1}\} \times \mathcal{F}\{G_{s_2}\} = e^{-2\pi^2 s_1 f^2} \times e^{-2\pi^2 s_2 f^2} = e^{-2\pi^2 (s_1+s_2) f^2} = \mathcal{F}\{G_{s_1+s_2}\}$$

We know that the Fourier transform is a one-to-one preserving map, hence it can be concluded that $(G_{s_1} * G_{s_2})(x) = G_{s_1+s_2}$.

Question - 8

The convolution theorem states that the Fourier transform of the convolution of two functions is the product of their Fourier transforms:

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\}\mathcal{F}\{g\}, \tag{5}$$

where

$$\mathcal{F}\{f\}(k) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi xk} dx \tag{6}$$

is the Fourier transform of a continuous function $f(x)$. Using the definitions of the convolution operation (Equation (1)) and Fourier transform (Equation (6)), prove that the convolution theorem holds true.

To prove the convolution theorem, we start by taking the Fourier transform of a convolution. We want to show that this is equivalent to the product of the two individual Fourier transforms. In the following equation, the convolution integral is taken over the variable x to give a function of u. The Fourier transform then involves an integral over the variable u.

$$\mathcal{F}\{f(x) * g(x)\} = \mathcal{F}\left(\int_{-\infty}^{+\infty} f(x) g(u-x)\, dx\right) = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x) g(u-x)\, dx\right) e^{-i2\pi ku}\, du$$

Substitute w for $u - x$. The infinite integration limits don't change. Then we expand the exponential of a sum into the product of exponentials and rearrange to bring together expressions in x and expressions in w.

$$\mathcal{F}\{f(x) * g(x)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x) g(w) e^{-i2\pi k(x+w)}\, dx\, dw = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x) e^{-i2\pi kx} g(w) e^{-i2\pi kw}\, dx\, dw$$

Expressions in x can be taken out of the integral over w so that we have two separate integrals.

$$\mathcal{F}\{f(x) * g(x)\} = \int_{-\infty}^{+\infty} f(x) e^{-i2\pi kx}\, dx \int_{-\infty}^{+\infty} g(w) e^{-i2\pi kw}\, dw$$

Now replace w with x in the second integral, hence we have the result of the convolution theorem.

$$\mathcal{F}\{f(x) * g(x)\} = \int_{-\infty}^{+\infty} f(x) e^{-i2\pi kx}\, dx \int_{-\infty}^{+\infty} g(x) e^{-i2\pi kx}\, dx = \mathcal{F}\{f(x)\}\mathcal{F}\{g(x)\}$$

Question - 9 (1)

What are all the points in the representational space $\mathbb{R}^3$ that are the homogeneous coordinates of the origin in the physical space $\mathbb{R}^2$?

Let us consider the points in $\mathbb{R}^3$ to be of the form $[u\ v\ w]^T$. So, $x = u/w = 0$ and $y = v/w = 0 \implies u = v = 0$ and $w \in \mathbb{R} \mid 0$ or $w \neq 0$. Hence, all the points forming the equivalence class represented by

$$\begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix} = w \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3 \quad , \quad w \in \mathbb{R} \mid 0 \ \ or \ \ w \neq 0$$

represent the origin $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathbb{R}^2$.

Question - 9 (2)

Are all points at infinity in the physical plane $\mathbb{R}^2$ the same? Justify your answer.

No, all points at infinity in the physical plane $\mathbb{R}^2$ are not the same. Consider a point in $\mathbb{R}^2$. It can be represented in $\mathbb{R}^3$ to be of the form $[u\ v\ w]^T$. Lets assume $w \in \mathbb{R} \mid 0$ or $w \neq 0$, $x = u/w$ and $y = v/w$.

As w goes to zero, the physical points in $\mathbb{R}^2$ are $x = \infty$, $y = \infty$ which means the x and y coordinates of the physical point moves moves away from the origin towards infinity in a particular direction. However, the direction in which the point approaches infinity in $\mathbb{R}^2$ is different for different points.

So all points at infinity are not same, but can be connected in a single straight line whose equation is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Question - 9 (3)

Argue that the matrix rank of a degenerate conic can never exceed 2.

The homogeneous representation of a degenerate conic C formed by lines l and m is given as $C = lm^T + ml^T$. Let $l = (l_1\ l_2\ l_3)^T$ and $m = (m_1\ m_2\ m_3)^T$. Therefore,

$$lm^T = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} \begin{pmatrix} m_1 & m_2 & m_3 \end{pmatrix} = \begin{bmatrix} l_1m_1 & l_1m_2 & l_1m_3 \\ l_2m_1 & l_2m_2 & l_2m_3 \\ l_3m_1 & l_3m_2 & l_3m_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$$

Here, a_1, a_2, a_3 represent $1^{st}, 2^{nd}, 3^{rd}$ columns of the product matrix lm^T, respectively. But, $a_2 = \frac{m_2}{m_1}a_1$ and $a_3 = \frac{m_3}{m_1}a_1$. Hence, the matrix lm^T, has only one independent column and its rank is 1. Similarly, ml^T also has only one independent column and its rank is 1. So,

$$rank(A + B) \le rank(A) + rank(B) \implies rank(lm^T + ml^T) \le rank(lm^T) + rank(ml^T)$$

$$\implies rank(C) \le 1 + 1 \implies rank(C) \le 2$$

Question - 9 (4)

Derive in just 3 steps the intersection of two lines l_1 and l_2 with l_1 passing through the points $(0, 0)$ and $(2, 3)$, and with l_2 passing through the points $(-3, 3)$ and $(-1, 2)$. How many steps would take you if the second line passed through $(-4, -5)$ and $(4, 5)$?

Line passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is $l_1 = x_1 \times x_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}$

Line passing through $\begin{pmatrix} -3 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is $l_2 = x_3 \times x_4 = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

Their point of intersection $= l_1 \times l_2 = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -6 \\ -9 \\ -8 \end{pmatrix}$

Therefore, the point of intersection in the physical plane $\mathbb{R}^2$ is given by $x = \frac{-6}{-8} = \frac{3}{4}$ and $y = \frac{-9}{-8} = \frac{9}{8}$. Now,

Line passing through $\begin{pmatrix} -4 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ is $l_3 = x_5 \times x_6 = \begin{pmatrix} -4 \\ -5 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ 8 \\ 0 \end{pmatrix}$

The constant term c of the line l_3 is 0, i.e. it passes through origin. Also, the constant term for l_1 is 0 and it passes through origin. Hence, the two lines intersect at the origin. In this case, **two steps** are needed to find the point of intersection.

Question - 9 (5)

Consider that there are two lines. The first line is passing through points $(0, 0)$ and $(2, -2)$. The second line is passing through points $(-3, 0)$ and $(0, -3)$. Find the intersection between these two lines. Comment on your answer.

The line l_1 passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ is given as $x_1 \times x_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$

The line l_2 passing through $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$ is given as $x_3 \times x_4 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 9 \end{pmatrix}$

Their point of intersection $= l_1 \times l_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 3 \\ 3 \\ 9 \end{pmatrix} = \begin{pmatrix} 18 \\ -18 \\ 0 \end{pmatrix}$

The point of intersection in the physical plane $\mathbb{R}^2$ is given by, $x = \frac{18}{0} = \infty$ and $y = \frac{-18}{0} = \infty$, hence the lines l_1 and l_2 do not intersect. It can also be observed that the ratio of $-\frac{a}{b}$, i.e the negative ratio of first two elements of the resultant vector is same for both the lines. So, we can say that the lines are parallel.

Question - 9 (6)

As you know, when a point x is on a conic, the tanget to the conic at that point is given by $l = Cx$. That raises the question of what Cx corresponds to when x is, say, outside the conic. As you'll see later in class, when x is outside the conic, Cx is the line that joins the two points of contact if you draw tangents to C from the point x. This line is referred to as the *polar line*. Now consider for our conic a circle of radius 1 that is centered at the coordinates $(-6, -6)$ and let x be the origin of the $\mathbb{R}^2$ physical plane. Where does the polar line intersect the x and y axes in this case?

We have been given that the conic is a circle of radius 1 and centred at (-6, -6). Therefore, its equation is given by,

$$(x+6)^2 + (y+6)^2 = 1 \implies x^2 + 0xy + y^2 + 12x + 12y + 71 = 0$$

Substitute $x = x_1/x_3$ and $y = x_2/x_3$ in the equation. The implicit representation in Homogeneous Coordinates in $\mathbb{R}^3$ is:

$$x_1^2 + x_2^2 + 12x_1x_3 + 12x_2x_3 + 71x_3^2 = 0$$

Write this as a matrix product,

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 6 \\ 6 & 6 & 71 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

Given that, x is the origin in $\mathbb{R}^2 \implies x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$. So, the polar line l can be written as,

$$l = Cx = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 6 \\ 6 & 6 & 71 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 71 \end{pmatrix}$$

In $\mathbb{R}^3$, the x-axis is l_x and the y-axis is l_y so,

$$x\text{-intercept is } l \times l_x = \begin{pmatrix} 6 \\ 6 \\ 71 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -71 \\ 0 \\ 6 \end{pmatrix}$$

$$y\text{-intercept is } l \times l_y = \begin{pmatrix} 6 \\ 6 \\ 71 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 71 \\ -6 \end{pmatrix}$$

Therefore, the polar line l cuts the x-axis at (-71/6, 0) and the y-axis at (0, -71/6).

Appendix

Python Code for Question - 4

```python
import math
import numpy as np
import matplotlib.pyplot as plt

### Plotting the given function

x = np.linspace(-10, 10, 1000)
y = 50*np.sin(x) + x**2
fig, ax = plt.subplots()
ax.plot(x, y)
plt.xlabel("x")
plt.ylabel("y(x)")
plt.annotate("(4.53, -28.65)", (4.53, -28.65))
plt.annotate("(-1.51, -47.628)", (-1.51, -47.628))
plt.annotate("(-7.547, 9.295)", (-7.547, 9.295))
plt.grid()
plt.show()

### Gradient descent algorithm

current_x = 7 # initial guess
learning_rate = 0.01 # learning rate
precision = 0.001 # desired precision
max_iter = 10000 # maximum number of iterations
iter = 0 # iteration counter
derivative = lambda x: 50*math.cos(x)+2*x # gradient/derivative
while abs(derivative(current_x)) > precision and iter < max_iter and abs(
    current_x) < 10:
    previous_x = current_x
    current_x = current_x - learning_rate * derivative(previous_x) # GD
    iter = iter + 1 # increment iteration
    print("Iteration Number:", iter, "\nValue of x is", current_x)
print("The local minima occurs at", current_x)
```

■■■