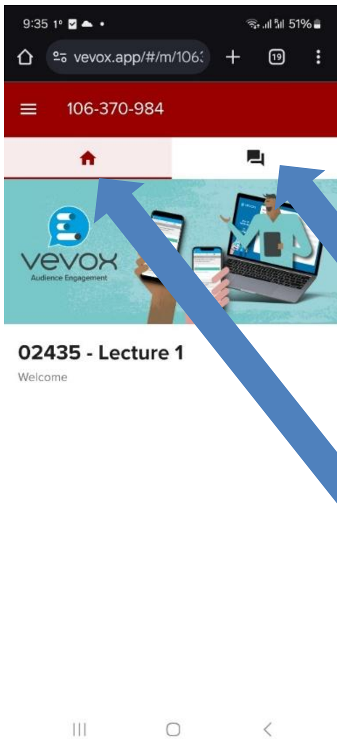
Decision Making under Uncertainty (02435)

Section for Dynamical Systems, DTU Compute.

DTU Compute
Department of Applied Mathematics and Computer Science

Scan Me



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Anonymous Survey (at the end)

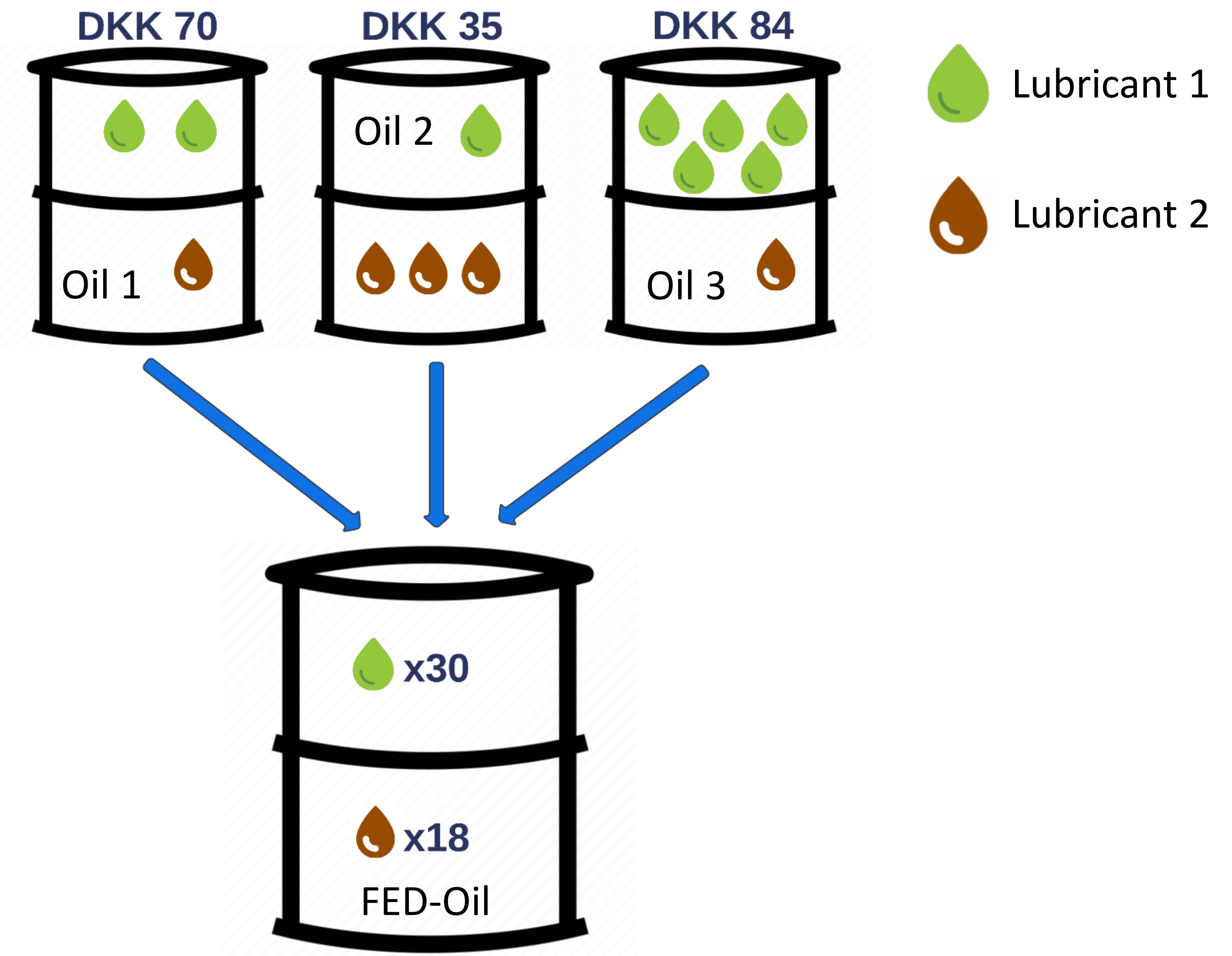
Anonymous Questions (during or after the lecture)

Quizzes

Plan

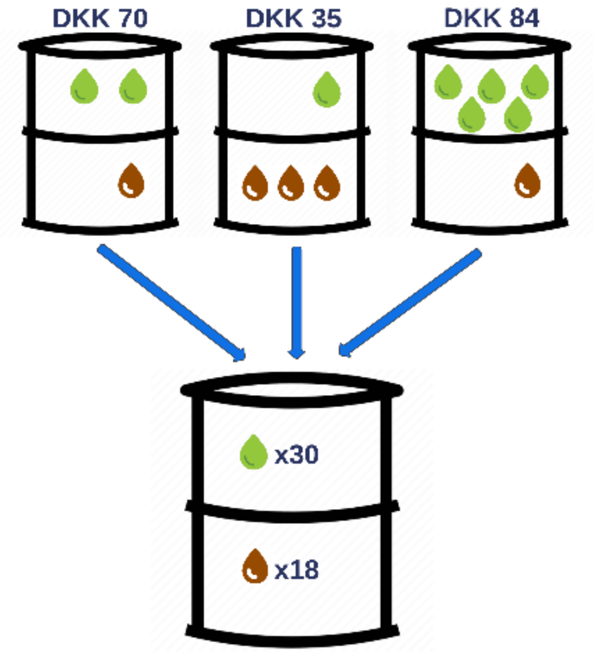
→ ~~Task 0~~
→ ~~Task 1~~
~~Building an evaluation framework for sequential decision-making methods~~
→ ~~Task 2~~
~~Stochastic Programming policy (2-stage)~~
~~+ Expected Value policy a.k.a. MPC~~
→ ~~Task 2~~
~~Multi-stage Stochastic Programming + caveats~~
→ ~~Week 5: Assignment Work for Task 2 and Q&A~~
→ ~~Weeks 6-7: Task 3~~
~~Approximate Dynamic Programming~~
→ ~~Week 8: Assignment Work for Task 3 and Q&A~~
→ Week 9: Recap on LP Duality and introduction to Robust Optimization
→ Week 10: Robust Optimization under Polyhedral Uncertainty Sets (Task 1 of Assignment B)
→ Week 11: Adjustable Robust Optimization (Task 2 of Assignment B)
→ Week 12: Assignment B, Q&A
→ Week 13: Extra Material

Warming-up: Hans' problem,



Warming-up: Hans' problem,

Hans from København has an oil factory. He extracts 3 different types of natural oils at a cost 70, 35 and 84 DKK per unit of oil. Each of these oils contains a different amount of two lubricants (see Table).
Hans wants to produce a mixture of natural oils, which he wants to sell under the name *Fed-Oil*. To ensure the quality, each unit of *Fed-Oil* must contain minimum 30 units of lubricant 1 and 18 units of lubricant 2.



	Type of natural oil		
Lubricant type	1	2	3
1	2	1	5
2	1	3	1

We want to help Hans decide how much of each type of oil he should extract to mix one unit of *Fed-Oil* at a minimum cost.

$$\text{Min } Z = 70x_1 + 35x_2 + 84x_3$$

$$\text{s.t. } 2x_1 + x_2 + 5x_3 \geq 30$$

$$x_1 + 3x_2 + x_3 \geq 18$$

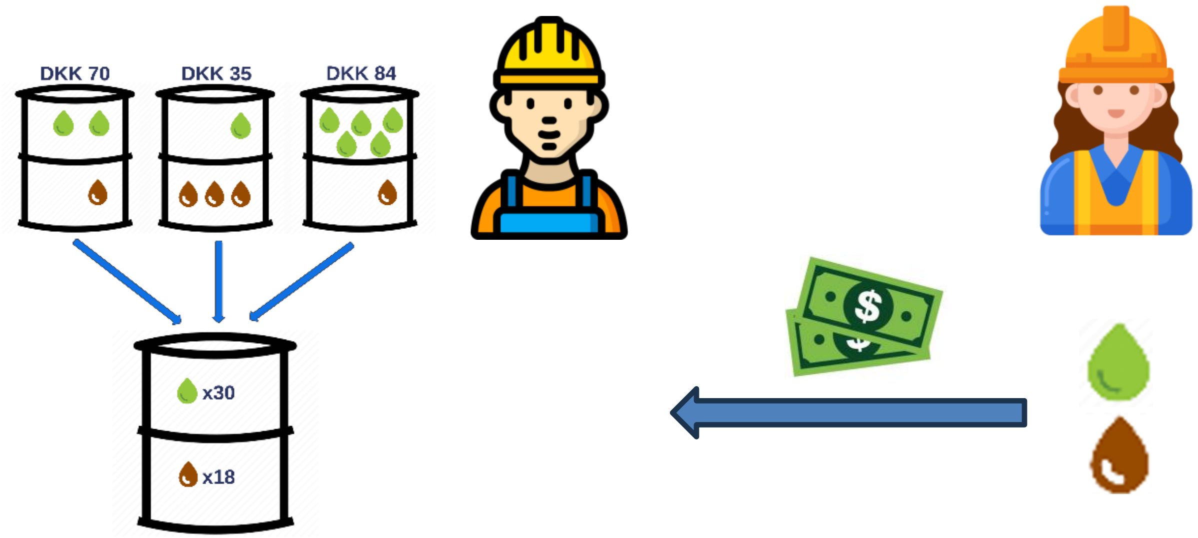
$$x_1, x_2, x_3 \geq 0$$

Variables:

x_1	Units to extract of natural oil 1
x_2	Units to extract of natural oil 2
x_3	Units to extract of natural oil 3

Pepita's problem

Pepita has a factory to produce synthetic lubricants. She manufactures the lubricants 1 and 2 that Hans uses to produce his *Fed-Oil*.



Pepita's problem

Pepita has a factory to produce synthetic lubricants. She manufactures the lubricants 1 and 2 that Hans uses to produce his *Fed-Oil*.
She knows that *Fed-Oil* must contain minimum 30 units of lubricant 1 and 18 units of lubricant 2. She also knows the amount of lubricants in the natural oils and the extraction costs (see Table).
Pepita knows that she cannot sell her synthetic lubricants at any price, because if the price is too high, Hans will just use the natural lubricants and not her synthetic ones. Thus, her price should be at most the cost that Hans pays for extracting the lubricants from natural oil.
We want to help Pepita compute the optimal prices for each lubricant she should offer to maximize her profit for selling the lubricants for one unit of *Fed-Oil* to Hans.

Natural oil	1	2	3
Cost	70	35	84
Content Lubricant 1	2	1	5
Content Lubricant 2	1	3	1

Pepita's problem

Pepita has a factory to produce synthetic lubricants. She manufactures the lubricants 1 and 2 that Hans uses to produce his *Fed-Oil*.

She knows that *Fed-Oil* must contain minimum 30 units of lubricant 1 and 18 units of lubricant 2. She also knows the amount of lubricants in the natural oils and the extraction costs (see Table).

Pepita knows that she cannot sell her synthetic lubricants at any price, because if the price is too high, Hans will just use the natural lubricants not her synthetic ones. Thus, her price should be at most the cost that Hans pays for extracting the lubricants from natural oil.

We want to help Pepita compute the optimal prices for each lubricant she should offer to maximize her profit for selling the lubricants for one unit *Fed-Oil* to Hans.

$$\text{Max } W = 30y_1 + 18y_2$$

$$\begin{aligned} \text{s.t.} \quad & 2y_1 + y_2 \leq 70 \\ & y_1 + 3y_2 \leq 35 \\ & 5y_1 + y_2 \leq 84 \\ & y_1, y_2 \geq 0 \end{aligned}$$

Natural oil	1	2	3
Cost	70	35	84
Content Lubricant 1	2	1	5
Content Lubricant 2	1	3	1

Variables:

y_1 Price per unit of lubricant 1

y_2 Price per unit of lubricant 2

Relationship between Hans' and Pepita's problems

Look at both model formulations. Do you notice a relationship?

Hans' problem:

$$\begin{array}{rccccccc}
\text{Min } Z = & 70x_1 & + & 35x_2 & + & 84x_3 & & \\
\text{s.t.} & 2x_1 & + & x_2 & + & 5x_3 & \geq & 30 \\
& x_1 & + & 3x_2 & + & x_3 & \geq & 18 \\
& x_1, & & x_2, & & x_3 \geq & & 0
\end{array}$$

Pepita's problem:

$$\begin{array}{rccccc}
\text{Max } W = & 30y_1 & + & 18y_2 & & \\
\text{s.t.} & 2y_1 & + & y_2 & \leq & 70 \\
& y_1 & + & 3y_2 & \leq & 35 \\
& 5y_1 & + & y_2 & \leq & 84 \\
& y_1, & & y_2 \geq & & 0
\end{array}$$

Relationship Hans' and Pepita's problems

Look at both model formulations. Do you notice a relationship?

Hans' problem:

$$\begin{array}{rlllllll}
\text{Min } Z = & \boxed{70x_1 + 35x_2 + 84x_3} \\
\text{s.t.} & \boxed{2x_1 + x_2 + 5x_3 \geq 30} \\
 & \boxed{x_1 + 3x_2 + x_3 \geq 18} \\
 & x_1,\ x_2,\ x_3 \geq 0
\end{array}$$

Pepita's problem:

$$\begin{array}{rccccccc}
\text{Max } W = & 30y_1 & + & 18y_2 & & \\
\text{s.t.} & 2y_1 & + & y_2 & \leq & 70 \\
 & y_1 & + & 3y_2 & \leq & 35 \\
 & 5y_1 & + & y_2 & \leq & 84 \\
 & y_1, & & y_2 \geq & 0 &
\end{array}$$

Relationship Hans' and Pepita's problems

Look at both model formulations. Do you notice a relationship?

Hans' problem:

$$
\begin{array}{rccccccc}
\text{Min } Z = & \boxed{70x_1 + 35x_2 + 84x_3} \\
\text{s.t.} & \boxed{2x_1 + x_2 + 5x_3 \geq 30} \\
 & \boxed{x_1 + 3x_2 + x_3 \geq 18} \\
 & x_1,\ x_2,\ x_3 \geq 0
\end{array}
$$

Pepita's problem:

$$
\begin{array}{rccccc}
\text{Max } W = & 30y_1 & + & 18y_2 & & \\
\text{s.t.} & 2y_1 & + & y_2 & \leq & 70 \\
 & y_1 & + & 3y_2 & \leq & 35 \\
 & 5y_1 & + & y_2 & \leq & 84 \\
 & y_1, & & y_2 \geq & & 0
\end{array}
$$

Constraint $i \longleftrightarrow$ Variable i

Objective function $\longleftrightarrow$ Right-hand-side

Relationship Hans' and Pepita's problems

Look at both model formulations. Do you notice a relationship?

Hans' problem:

$$\begin{array}{rccccccc}
\text{Min } Z = & \boxed{70x_1 & + & 35x_2 & + & 84x_3 & & } \\
\text{s.t.} & 2x_1 & + & x_2 & + & 5x_3 & \geq & 30 \\
 & x_1 & + & 3x_2 & + & x_3 & \geq & 18 \\
 & x_1, & & x_2, & & x_3 \geq & 0 &
\end{array}$$

Pepita's problem:

$$\begin{array}{rccccc}
\text{Max } W = & 30y_1 & + & 18y_2 & & \\
\text{s.t.} & 2y_1 & + & y_2 & \leq & 70 \\
 & y_1 & + & 3y_2 & \leq & 35 \\
 & 5y_1 & + & y_2 & \leq & 84 \\
 & y_1, & & y_2 \geq & 0 &
\end{array}$$

Pepita's problem is the so-called **dual problem** of Hans' problem.

The idea behind duality

How much revenue can Pepita make?

Hans' problem (primal problem):

$$
\begin{aligned}
\text{Min } Z = & 70x_1 + 35x_2 + 84x_3 \\
\text{s.t. } & 2x_1 + x_2 + 5x_3 \geq 30 \\
& x_1 + 3x_2 + x_3 \geq 18 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Interview Question

Pepita asks you: Can you give me a lower bound on the profit I can make, just by looking at Hans's problem?

The idea behind duality

How much revenue can Pepita make?

Hans' problem (primal problem):

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$$
\begin{aligned}
\text{Min } Z = & 70x_1 + 35x_2 + 84x_3 \\
\text{s.t. } & 2x_1 + x_2 + 5x_3 \geq 30 \\
& x_1 + 3x_2 + x_3 \geq 18 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Pepita asks you: Can you give me a lower bound on the profit I can make, just by looking at Hans's problem?
Hint: To produce one unit of FedOil, Hans currently pays $70x_1 + 35x_2 + 84x_3$.
So, we can charge him up to this much.
Can we somehow say that his cost $70x_1 + 35x_2 + 84x_3$ is at least as high as some value C?

The idea behind duality

Hans' problem (primal problem):

$$\begin{aligned} \text{Min } Z = & 70x_1 + 35x_2 + 84x_3 \\ \text{s.t. } & 2x_1 + x_2 + 5x_3 \geq 30 \qquad (1) \\ & x_1 + 3x_2 + x_3 \geq 18 \qquad (2) \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

How small can the objective value be?

(1): $$70x_1 + 35x_2 + 84x_3 \geq 2x_1 + x_2 + 5x_3 \geq 30$$

Can we do better (give Pepita a lower-bound that is higher than 30)?

The idea behind duality

Hans' problem (primal problem):

$$\begin{aligned} \text{Min } Z = & 70x_1 + 35x_2 + 84x_3 \\ \text{s.t. } & 2x_1 + x_2 + 5x_3 \geq 30 \qquad (1) \\ & x_1 + 3x_2 + x_3 \geq 18 \qquad (2) \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

How small can the objective value be?

(1): $70x_1 + 35x_2 + 84x_3 \geq 2x_1 + x_2 + 5x_3 \geq 30$

10 × (2): $70x_1 + 35x_2 + 84x_3 \geq 10x_1 + 30x_2 + 10x_3 \geq 180$

The idea behind duality

Hans' problem (primal problem):

$$
\begin{aligned}
\text{Min } Z = & 70x_1 + 35x_2 + 84x_3 \\
\text{s.t. } & 2x_1 + x_2 + 5x_3 \geq 30 && (1) \\
& x_1 + 3x_2 + x_3 \geq 18 && (2) \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

How small can the objective value be?

(1):	$70x_1 + 35x_2 + 84x_3 \geq 2x_1 + x_2 + 5x_3 \geq 30$
10 × (2):	$70x_1 + 35x_2 + 84x_3 \geq 10x_1 + 30x_2 + 10x_3 \geq 180$
5 × (1) + 10 × (2):	$70x_1 + 35x_2 + 84x_3 \geq 20x_1 + 35x_2 + 35x_3 \geq 330$

The idea behind duality

Hans' problem (primal problem):

$$\begin{aligned} \text{Min } Z = & 70x_1 + 35x_2 + 84x_3 \\ \text{s.t. } & 2x_1 + x_2 + 5x_3 \geq 30 && (1) \\ & x_1 + 3x_2 + x_3 \geq 18 && (2) \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

How small can the objective value be?

(1):	$70x_1 + 35x_2 + 84x_3 \geq 2x_1 + x_2 + 5x_3 \geq 30$
$10 \times$ (2):	$70x_1 + 35x_2 + 84x_3 \geq 10x_1 + 30x_2 + 10x_3 \geq 180$
$5 \times (1) + 10 \times (2)$:	$70x_1 + 35x_2 + 84x_3 \geq 20x_1 + 35x_2 + 35x_3 \geq 330$
$15.5 \times (1) + 6.5 \times (2)$:	$70x_1 + 35x_2 + 84x_3 \geq 37.5x_1 + 35x_2 + 84x_3 \geq 582$

The idea behind duality

Hans' problem (primal problem):

$$\begin{aligned} \text{Min } Z = & 70x_1 + 35x_2 + 84x_3 \\ \text{s.t. } & 2x_1 + x_2 + 5x_3 \geq 30 \qquad (1) \\ & x_1 + 3x_2 + x_3 \geq 18 \qquad (2) \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

How small can the objective value be?

(1):	$70x_1 + 35x_2 + 84x_3 \geq 2x_1 + x_2 + 5x_3 \geq 30$
$10 \times$ (2):	$70x_1 + 35x_2 + 84x_3 \geq 10x_1 + 30x_2 + 10x_3 \geq 180$
$5 \times (1) + 10 \times (2)$:	$70x_1 + 35x_2 + 84x_3 \geq 20x_1 + 35x_2 + 35x_3 \geq 330$
$15.5 \times (1) + 6.5 \times (2)$:	$70x_1 + 35x_2 + 84x_3 \geq 37.5x_1 + 35x_2 + 84x_3 \geq 582$

$\rightarrow$ every linear combination of constraints that leads to coefficients less or equal than the objective function coefficients gives us a lower bound for the optimal objective value

The idea behind duality

Hans' problem (primal problem):

$$
\begin{aligned}
\text{Min } Z = & 70x_1 + 35x_2 + 84x_3 \\
\text{s.t. } & 2x_1 + x_2 + 5x_3 \geq 30 && (3) \\
& x_1 + 3x_2 + x_3 \geq 18 && (4) \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

How can we find all lower bounds?

The idea behind duality

Hans' problem (primal problem):

$$\begin{aligned} \text{Min } Z = & 70x_1 + 35x_2 + 84x_3 \\ \text{s.t. } & 2x_1 + x_2 + 5x_3 \geq 30 \quad (3) \\ & x_1 + 3x_2 + x_3 \geq 18 \quad (4) \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

How can we find all lower bounds?

Introduce variables y_1 and y_2 representing the coefficients of the linear combination of constraints (1) and (2):

$Z \geq (2x_1 + x_2 + 5x_3)y_1 + (x_1 + 3x_2 + x_3)y_2 \geq 30y_1 + 18y_2$

$\Leftrightarrow (2y_1 + y_2)x_1 + (y_1 + 3y_2)x_2 + (5y_1 + y_2)x_3 \geq 30y_1 + 18y_2$

and ensure that the combined coefficients are less or equal than the objective coefficients:

$2y_1 + y_2 \leq 70$

$y_1 + 3y_2 \leq 35$

$5y_1 + y_2 \leq 84$

The idea behind duality

Hans' problem (primal problem):

$$\begin{aligned} \text{Min } Z = & 70x_1 + 35x_2 + 84x_3 \\ \text{s.t. } & 2x_1 + x_2 + 5x_3 \geq 30 && (5) \\ & x_1 + 3x_2 + x_3 \geq 18 && (6) \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

How can we find best (maximum) lower bound?

$\rightarrow$ build an LP to find it

$$\begin{aligned} \text{Max } & 30y_1 + 18y_2 \\ \text{s.t. } & 2y_1 + y_2 \leq 70 \\ & y_1 + 3y_2 \leq 35 \\ & 5y_1 + y_2 \leq 84 \\ & y_1, y_2 \geq 0 \end{aligned}$$

$\rightarrow$ Pepita's problem (dual problem)

Primal-Dual Transformation

Primal		**Dual**
Max Z	$\leftrightarrow$	Min W
Constraints i		*Variables* y_i
$\leq$ constraint	$\leftrightarrow$	$y_i \geq 0$
$=$ constraint	$\leftrightarrow$	$y_i \in \mathbb{R}$ (free)
$\geq$ constraint	$\leftrightarrow$	$y_i \leq 0$
Variables x_j		*Constraints* j
$x_j \geq 0$	$\leftrightarrow$	$\geq$ constraint
$x_j \in \mathbb{R}$ (free)	$\leftrightarrow$	$=$ constraint
$x_j \leq 0$	$\leftrightarrow$	$\leq$ constraint

Primal problem:

$$\text{Max} \quad Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \forall i = 1, 2, \ldots, m$$

$$x_j \geq 0 \quad \forall j = 1, 2, \ldots, n$$

Dual problem:

$$\text{Min} \quad W = \sum_{i=1}^{m} b_i y_i$$

$$\text{s.t.} \quad \sum_{i=1}^{m} a_{ij} y_i \geq c_i \quad \forall i = 1, 2, \ldots, n$$

$$y_i \geq 0 \quad \forall i = 1, 2, \ldots, m$$

Primal-Dual Transformation

The SOB principle to remember how to transform:

Label	Primal		Dual
	Max Z	$\leftrightarrow$	Min W
	Constraints i		*Variables* y_i
Sensible	$\leq$ constraint	$\leftrightarrow$	$y_i \geq 0$
Odd	$=$ constraint	$\leftrightarrow$	$y_i \in \mathbb{R}$ (free)
Bizarre	$\geq$ constraint	$\leftrightarrow$	$y_i \leq 0$
	Variables x_j		*Constraints* j
Sensible	$x_j \geq 0$	$\leftrightarrow$	$\geq$ constraint
Odd	$x_j \in \mathbb{R}$ (free)	$\leftrightarrow$	$=$ constraint
Bizarre	$x_j \leq 0$	$\leftrightarrow$	$\leq$ constraint

Hans' and Pepita's problems - Solutions

Hans

Variable values:

Natural oil	Units to extract
1	0.000
2	4.286
3	5.143

Minimal costs per unit Fed-Oil: 582 DKK

Pepita

Variable values:

Lubricant	Price per unit
1	15.50
2	6.50

Maximal profit by selling lubricants for one unit Fed-Oil: 582 DKK

Let's look at the dual values of both programs.
In Julia, you can get the dual value via calling the command `dual` on the constraint:

```
@printf "Natural Oil 1:  %0.3f\n" dual.(NaturalOil1)
```

Duality

Strong duality

If x^* is an optimal solution of the primal problem and y^* is an optimal solution of the dual problem, then

$$cx^* = by^*$$

(The objective values are the same)

Hans' problem extended

Hans' chemist has pointed out an error in the minimum amount of lubricant 1 that the Fed-Oil must contain. It seems now that each unit of Fed-Oil should contain at least 29 units of lubricant 1 and not 30.

We want to use the mathematical problem to find out the impact of this change on the cost. What do we notice?
Let us test also 31 and 131 instead of 30.

$$\begin{aligned}
\text{Min } Z = & 70x_1 + 35x_2 + 84x_3 \\
\text{s.t. } & 2x_1 + x_2 + 5x_3 \geq 29 \\
& x_1 + 3x_2 + x_3 \geq 18 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

Hans' problem extended - Solution

Now (29 units):

Costs: 566.5 DKK

Variable values:

x_j	Value
1	0.0
2	4.357
3	4.929

Dual values:

y_i	Value
1	15.5
2	6.5

Before (30 units):

Costs: 582 DKK

Variable values:

x_j	Value
1	0.000
2	4.286
3	5.143

Dual values:

y_i	Value
1	15.5
2	6.5

Hans' problem extended - Solution

Now (29 units):

Costs: 566.5 DKK

Variable values:

x_j	Value
1	0.0
2	4.357
3	4.929

Dual values:

y_i	Value
1	15.5
2	6.5

Before (30 units):

Costs: 582 DKK

Variable values:

x_j	Value
1	0.000
2	4.286
3	5.143

Dual values:

y_i	Value
1	15.5
2	6.5

$$582 - 15.5 = 566.5$$

Hans' problem extended

Hans' chemist has pointed out an error in the minimum amount of lubricant 1 that the Fed-Oil must contain. It seems now that each unit of Fed-Oil should contain at least 29 units of lubricant 1 and not 30.

We want to use the mathematical problem to find out the impact of this change on the cost. What do we notice?

What if, the chemist meant 31 and not 29?

$$\begin{aligned}
\text{Min } Z = & 70x_1 + 35x_2 + 84x_3 \\
\text{s.t. } & 2x_1 + x_2 + 5x_3 \geq 31 \\
& x_1 + 3x_2 + x_3 \geq 18 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

Hans' problem extended - Solution

Now (31 units):

Costs: 597.5 DKK

Variable values:

x_j	Value
1	0.000
2	4.214
3	5.357

Dual values:

y_i	Value
1	15.5
2	6.5

Before (30 units):

Costs: 582 DKK

Variable values:

x_j	Value
1	0.000
2	4.286
3	5.143

Dual values:

y_i	Value
1	15.5
2	6.5

Hans' problem extended - Solution

Now (31 units):

Costs: 597.5 DKK

Variable values:

x_j	Value
1	0.000
2	4.214
3	5.357

Dual values:

y_i	Value
1	15.5
2	6.5

Before (30 units):

Costs: 582 DKK

Variable values:

x_j	Value
1	0.000
2	4.286
3	5.143

Dual values:

y_i	Value
1	15.5
2	6.5

$$582 + 15.5 = 597.5$$

Marginal values

The values of the dual variables are also called **marginal values** or **shadow prices**.
The values specifies the amount Hans needs to pay / receives when one additional unit of lubricant is needed / not needed.

Interpretation of dual values

The dual variable y_i is interpreted as the additional profit per unit of resource i.

This is how prices are determined in electricity markets:
If Demand increases by one unit, how much will be the additional cost to satisfy it?

Hans' problem extended

Hans' chemist has pointed out an error in the minimum amount of lubricant 1 that the Fed-Oil must contain. Now each unit of Fed-Oil should contain at least 29 units of lubricant 1 and not 30.

We want to use the mathematical problem to find out the impact of this change on the cost. What do we notice?

What if, the chemist meant 131 and not 31?

$$\begin{aligned}
\text{Min } Z =& 70x_1 + 35x_2 + 84x_3 \\
\text{s.t. } & 2x_1 + x_2 + 5x_3 \geq 131 \\
& x_1 + 3x_2 + x_3 \geq 18 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

Hans' problem extended - Solution

Now (131 units):

Costs: 2200.8 DKK

Variable values:

x_j	Value
1	0.0
2	0.0
3	26.2

Dual values:

y_i	Value
1	16.8
2	0.0

Before (30 units):

Costs: 582 DKK

Variable values:

x_j	Value
1	0.000
2	4.286
3	5.143

Dual values:

y_i	Value
1	15.5
2	6.5

Hans' problem extended - Solution

Now (131 units):

Costs: 2200.8 DKK

Variable values:

x_j	Value
1	0.0
2	0.0
3	26.2

Dual values:

y_i	Value
1	16.8
2	0.0

Before (30 units):

Costs: 582 DKK

Variable values:

x_j	Value
1	0.000
2	4.286
3	5.143

Dual values:

y_i	Value
1	15.5
2	6.5

NOTE! $2200.8 \neq 582 + (101 * 15.5) = 2147.5$

$\rightarrow$ dual values apply only for marginal changes

Focus of Part B (and Assignment B)

Robust Optimization is about ensuring that the constraints will be satisfied, even in the worst-case uncertainty realization.

Didn't Stochastic Programming already do that?
It does enforce the constraints for *all* scenarios.

Motivation

Cases where robust optimization can help:

- Data is often uncertain or not known exactly.

  Examples: measurement/estimation errors, implementation errors
- Maybe a small deviation in the data makes the optimal solution of a deterministic problem completely meaningless (e.g. infeasible).
- Need for a methodology that generates a **robust solution**, i.e., the solution is immunized against the effect of data uncertainty.

Farmer's problem (from lecture 02)

$$
\begin{aligned}
\text{Min } Cost = & -170z^W - 150z^C - 36z^S - 10v^S \\
& + 150x^W + 230x^C + 260x^S + 238y^W + 210y^C \\
\text{s.t. } & x^W + x^C + x^S \leq 500 \\
& a^W x^W - y^W + z^W \leq -200 \\
& a^C x^C - y^C + z^C \leq -240 \\
& z^S \leq 6000 \\
& z^S + v^S + a^S x^S \leq 0 \\
& x^W, x^C, x^S, y^W, y^C, z^W, z^C, z^S, v^S \geq 0
\end{aligned}
$$

Instead of scenarios now assume the following uncertainty sets for the yield: $a^W \sim U(-3.0, -2.0), a^C \sim U(-3.5, -2.5), a^S \sim (-24.0, -16.0)$

Example: a^W can take any value between -3.0 and -2.0.

Farmer's problem - uncertainty sets

$$a^W \sim U(-3.0, -2.0), a^C \sim U(-3.5, -2.5), a^S \sim (-24.0, -16.0)$$

Let's **reformulate** the uncertainty sets using the mean value and range:

For a^W:
$a^W = \bar{a}^W + P^W \zeta^W$ with $|\zeta^W| \leq 1$, $\bar{a}^W = -2.5$ and $P^W = 0.5$

Farmer's problem - uncertainty sets

$a^W \sim U(-3.0, -2.0), a^C \sim U(-3.5, -2.5), a^S \sim (-24.0, -16.0)$

Let's **reformulate** the uncertainty sets using the mean value and range:

For a^W:
$a^W = \bar{a}^W + P^W \zeta^W$ with $|\zeta^W| \leq 1$, $\bar{a}^W = -2.5$ and $P^W = 0.5$ For a^C:
$a^C = \bar{a}^C + P^C \zeta^C$ with $|\zeta^C| \leq 1$, $\bar{a}^C = -3.0$ and $P^C = 0.5$

For a^S:
$a^S = \bar{a}^S + P^S \zeta^S$ with $|\zeta^S| \leq 1$, $\bar{a}^S = -20.0$ and $P^S = 4.0$

Now the only uncertainties are $\zeta^W, \zeta^C, \zeta^S$ that can vary between
$-1 \leq \zeta^W, \zeta^C, \zeta^S \leq 1$.
These are so-called **box uncertainty sets**.

Robust Farmer's problem - with box uncertainty

Replacing a^W, a^C, a^S with the affine functions of the box-uncertainty sets :

$$
\begin{aligned}
\text{Min } Cost = & -170z^W - 150z^C - 36z^S - 10v^S \\
& + 150x^W + 230x^C + 260x^S + 238y^W + 210y^C \\
\text{s.t. } & x^W + x^C + x^S \leq 500 \\
& (\bar{a}^W + P^W\zeta^W)x^W - y^W + z^W \leq -200 && \forall|\zeta^W| \leq 1 \\
& (\bar{a}^C + P^C\zeta^C)x^C - y^C + z^C \leq -240 && \forall|\zeta^C| \leq 1 \\
& z^S \leq 6000 \\
& z^S + v^S + (\bar{a}^S + P^S\zeta^S)x^S \leq 0 && \forall|\zeta^S| \leq 1 \\
& x^W, x^C, x^S, y^W, y^C, z^W, z^C, z^S, v^S \geq 0
\end{aligned}
$$

Can we code and solve this problem?

Robust Farmer's problem

Let's look at the constraint for wheat:

$$\Leftrightarrow \qquad \bar{a}^W x^W + P^W \zeta^W x^W - y^W + z^W \leq -200 \qquad \forall |\zeta^W| \leq 1$$

Problem: We have infinitely many constraints $\rightarrow$ reformulation

We are interested in the **worst-case realization of the uncertainty**. What is the worst case here?

Robust Farmer's problem

Let's look at the constraint for wheat:

$$\Leftrightarrow \qquad \bar{a}^W x^W + P^W \zeta^W x^W - y^W + z^W \leq -200 \qquad \forall |\zeta^W| \leq 1$$

Problem: We have infinitely many constraints $\rightarrow$ reformulation
We are interested in the **worst-case realization of the uncertainty**. What is the worst case here?
The maximum, due to a "$\leq$" constraint.

$$\bar{a}^W x^W + \max_{-1 \leq \zeta^W \leq 1} \{P^W \zeta^W x^W\} - y^W + z^W \leq -200$$

Which value of ζ maximizes this expression?

Robust Farmer's problem

Let's look at the constraint for wheat:

$$\Leftrightarrow \qquad \bar{a}^W x^W + P^W \zeta^W x^W - y^W + z^W \leq -200 \qquad \forall |\zeta^W| \leq 1$$

Problem: We have infinitely many constraints $\rightarrow$ reformulation

We are interested in the **worst-case realization of the uncertainty**. What is the worst case here?

The maximum, due to a "$\leq$" constraint.

$$\bar{a}^W x^W + \max_{-1 \leq \zeta^W \leq 1} \{P^W \zeta^W x^W\} - y^W + z^W \leq -200$$

Which value of ζ maximizes this expression?

$$\Leftrightarrow \qquad \bar{a}^W x^W + |P^W x^W| - y^W + z^W \leq -200$$

The maximum is either at $\zeta^W = -1$ or $\zeta^W = 1$ (because, more generally, x^W may be negative). Therefore, we can take the absolute value, which can be easily linearized.

Robust Farmer's problem - with box uncertainty

Inserting the specific parameters:

$$
\begin{aligned}
\text{Min } Cost = & -170z^W - 150z^C - 36z^S - 10v^S \\
& + 150x^W + 230x^C + 260x^S + 238y^W + 210y^C \\
\text{s.t. } & x^W + x^C + x^S \leq 500 \\
& -2.5x^W + \max_{-1 \leq \zeta^W \leq 1} \{0.5\zeta^W x^W\} - y^W + z^W \leq -200 \\
& -3.0x^C + \max_{-1 \leq \zeta^C \leq 1} \{0.5\zeta^C x^C\} - y^C + z^C \leq -240 \\
& z^S \leq 6000 \\
& z^S + v^S - 20.0x^S + \max_{-1 \leq \zeta^S \leq 1} \{4.0\zeta^S x^S\} \leq 0 \\
& x^W, x^C, x^S, y^W, y^C, z^W, z^C, z^S, v^S \geq 0
\end{aligned}
$$

Example

Robust Farmer's problem - with box uncertainty

Reformulation of the box uncertainty to absolute values:

$$\begin{aligned}
\text{Min } Cost = &-170z^W - 150z^C - 36z^S - 10v^S \\
&+150x^W + 230x^C + 260x^S + 238y^W + 210y^C \\
\text{s.t. } & x^W + x^C + x^S \le 500 \\
&-2.5x^W + |0.5x^W| - y^W + z^W \le -200 \\
&-3.0x^C + |0.5x^C| - y^C + z^C \le -240 \\
&z^S \le 6000 \\
&z^S + v^S - 20.0x^S + |4.0x^S| \le 0 \\
&x^W, x^C, x^S, y^W, y^C, z^W, z^C, z^S, v^S \ge 0
\end{aligned}$$

Robust Farmer's problem - with box uncertainty

x^W, x^C, x^S are always positive, therefore we can omit the absolute value in this specific case:

$$
\begin{aligned}
\text{Min } Cost = & -170z^W - 150z^C - 36z^S - 10v^S \\
& + 150x^W + 230x^C + 260x^S + 238y^W + 210y^C \\
\text{s.t. } & x^W + x^C + x^S \leq 500 \\
& -2.5x^W + 0.5x^W - y^W + z^W \leq -200 \\
& -3.0x^C + 0.5x^C - y^C + z^C \leq -240 \\
& z^S \leq 6000 \\
& z^S + v^S - 20.0x^S + 4.0x^S \leq 0 \\
& x^W, x^C, x^S, y^W, y^C, z^W, z^C, z^S, v^S \geq 0
\end{aligned}
$$

Robust Farmer's problem - with box uncertainty

x^W, x^C, x^S are always positive, therefore we can omit the absolute value in this case:

$$\begin{aligned}
\text{Min } Cost = & -170z^W - 150z^C - 36z^S - 10v^S \\
& + 150x^W + 230x^C + 260x^S + 238y^W + 210y^C \\
\text{s.t. } & x^W + x^C + x^S \leq 500 \\
& -2.0x^W - y^W + z^W \leq -200 \\
& -2.5x^C - y^C + z^C \leq -240 \\
& z^S \leq 6000 \\
& z^S + v^S - 16.0x^S \leq 0 \\
& x^W, x^C, x^S, y^W, y^C, z^W, z^C, z^S, v^S \geq 0
\end{aligned}$$

Robust counterpart

$$
\begin{aligned}
&\text{Min } c^T x \\
&\text{s.t. } \mathbf{a}_i^{\mathrm{T}} x \leq b_i \qquad\qquad \forall i = 1, ..., m, \forall a_i \in \mathcal{U}_i
\end{aligned}
$$

An uncertainty set $\mathcal{U}_i$ characterizes the set of possible values for an uncertain parameter.

In general, $\mathcal{U}$ can be any arbitrary subset of $\mathbb{R}^n$.
$\rightarrow$ **Problem**: the uncertainty sets $\mathcal{U}_i$ are continuous and leading to an infinite number of constraints

To still have a computationally tractable problem, some assumptions can be made.

The uncertainty sets themselves must be computationally tractable.

Uncertainty sets and computational tractability

If the uncertainty if formulated as box uncertainty set, we can reformulate the robust counterpart to a linear problem.

Box uncertainty set

(as in the farmer example):
The parameters vary in a given interval defined by deviations P

$$(a + P\zeta)^T x \leq b \qquad \forall -1 \leq \zeta \leq 1$$

Linear formulation of box uncertainty

Consider constraint i:

$$(a_i + P_i\zeta_i)^{\mathrm{T}} x \leq b_i \qquad \forall -1 \leq \zeta_i \leq 1$$

$$\Leftrightarrow a_i^{\mathrm{T}} x + (P_i\zeta_i)^{\mathrm{T}} x \leq b_i \qquad \forall -1 \leq \zeta_i \leq 1$$

This constraint must be feasible for all outcomes of uncertainty $-1 \leq \zeta_i \leq 1$.

$$a_i^{\mathrm{T}} x + \max_{-1 \leq \zeta \leq 1} \{(P_i\zeta_i)^{\mathrm{T}} x\} \leq b_i$$

$$\Leftrightarrow a_i^{\mathrm{T}} x + |P_i^{\mathrm{T}} x| \leq b_i$$

Resulting in the following linear formulation for the the worst case:

$$-u_i \leq P_i^{\mathrm{T}} x \leq u_i$$

$$a_i^{\mathrm{T}} x + u_i \leq b_i$$

$$u_i \geq 0$$

Assignment B

1. Will be announced next week
2. No coding
3. Re-establish groups

Questions and Survey

Join at:
vevox.app

ID:
140-487-820