Football

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(Based on a talk by Po-Shen Loh)

1 Football

Football is a game played by points in the plane. Each point has a certain maximum movement speed.

One point is the offense; their goal is to advance as far as possible to the right before being tackled. If possible, they want to advance infinitely far to the right without being tackled.

The other points are the defense. Their goal is to prevent the offense from advancing. If at any point the offense point and a defense point are at the same point in space, then the offense point is tackled.

2 Constant speed

To get a feel for this game, let's first look at the case where all points (offense and defense) move at the same speed.

2.1 Single defender

Even with this, the picture is pretty messy. With multiple defenders, there are tons of possible strategies for the defense coordinated among several independently moving parts. That sounds hard so let's not do that.

Instead, let's consider the case where there's exactly 1 defender, moving at the same speed as the offense. What are optimal strategies here?

Some possible ideas:

1. The offense can just charge at the defender. With this, no matter how the defender reacts, the offense will get at least halfway to the defender.
2. Since they players are the same speed, the defender could use a mirroring strategy. Using this, the offense can't advance farther than the line between the two players.

3. All points on the offense's side of the dividing line are closer to the offense than the defense. Thus, if the offense runs straight for any point on their side, they can get there before being tackled.

With this, we conclude that the offense can get to exactly all points on their side of the dividing line.

However, what if the offense wants to go on forever without getting tackled? Just getting to a point arbitrarily far to the right isn't the same thing as going on forever. Seeing that as points get arbitrarily far our direction approaches parallel, we try going parallel and it totally works hooray!

2.2 Multiple defenders

With this idea in mind, the multiple defenders case suddenly seems pretty doable. Let's blindly apply our previous strategy and see what happens.

With every defender using the mirror strategy, the offense can only get to the points on their side of every dividing line. Furthermore, the points on their side of every dividing line are closer to them than every other player, so they can get to any of those points by charging straight for them.

Thus, this strategy is the best that both the offense and defense can do. The offense can advance exactly as far as the furthest point to the right in the region cut out by the offense's side of each dividing line.

3 Variable speed

Now that we've solved one version of football, we can look at some more complex variants. One variant that jumps out is when the players' speeds are no longer fixed. For simplicity, we'll say the defenders all move at speed 1.

3.1 Lower offense speed

First, what happens when the offense moves at speed $s < 1$?

Since the defense can use the same old mirroring strategy, this case isn't that interesting. The only change is that the locus of points that the offense can get to against any 1 defender is now a circle instead of a half-plane, so the space the offense can get to in the end will be the intersection of a bunch of circles.

3.2 Higher offense speed

Now, say the offense moves at speed $1 + s > 1$. Things get interesting here.

We haven't yet dealt with a case where the offense starts out surrounded, so we'll start by assuming all defenders start to the right of the offense. What are some possible strategies here?

After experimenting for a while, we see that the defense will have a *very* hard time actually catching the offense without being able to mirror. So let's see if the offense has a way to escape.

The only new thing the offense has to work with is the speed boost, so we want to use that to as much advantage as possible. First, note that the offense can run arbitrarily far away from all the defenders to execute their master plan without being in a hurry.

Now, we need some way to deal with the massive variability of the defenders' strategies. To do this, we'll stop thinking of the defenders as points and start thinking of them as an area where the defender could be at any given time. This region is a circle with radius t (the amount of time passed). Now we can plan against a deterministic defense rather than a strategic one, so this will make things a lot easier.

However, we're still dealing with a bunch of circles. To get rid of that, we'll lump all the circles together in one giant circle with radius increasing at speed 1. Since the offense is faster, they can escape from this circle after some finite time.

Now, note that the offense only needs speed 1 to escape from the expanding circle of doom. Furthermore, if the offense can get to the other side of the circle, then they can keep going forever. So let's try running away at speed 1 and inching around the circle at speed s.

Set the angle between the offense and the center of the circle to $\theta = 0$; we want to increase it to $\theta = \pi$. Assume that the offense starts distance r away from the center of the circle.

Each step, the offense will run 1 unit away from the center of the circle, then s units around the circle. To calculate the change in θ per step, we can use the formula $s = 2\pi r\theta$, where s is speed = arc length. Since the radius in round i will be $r + i$, we can calculate the net change in θ after n rounds with the following sum:

$$\theta_n = \sum_{i=1}^{n} \frac{s}{2\pi} * \frac{1}{r+i}$$

This is basically the Harmonic Series, which diverges as $n \to \infty$. That means after some number of rounds, the offense will get to $\theta = \pi$ and escape to the right, regardless of the initial value of r.

4 Hard things

This is stuff that, at the time of my typing these notes, I don't know the answers to. So yeah, these problems are pretty hard.

Let's stay with the assumption that the offense moves with speed $1 + s > 1$ because that's the hard case.

4.1 Surrounded

In our proof that a fast offense can beat a finite number of defenders, we assumed that the offense has a direction to "run away" free of defenders. What if the offense starts out totally surrounded?

4.2 Infinite Defenders

This is the weird one.

If a finite number of defenders isn't enough to stop a sped-up offense, then could an infinite number manage?

The obvious first thought is "ok, let's place a defender at every point in the plane." While that works, it sounds more like a witty comeback than the solution to a math problem. While there can be infinite defenders, there shouldn't be *that* many.

Clearly, the problem as stated isn't very interesting. To fix the issue, let's make our infinity a little bit smaller. Specifically, only countably many defenders will be allowed. For reference, countably infinite = can be paired up with natural numbers; uncountably infinite = can't be (things that look like real numbers).

4.3 Differentiable (Smooth) Path

Intuitively, the defense should win here. for example, consider the setup with defenders at every point with 2 rational coordinates (lemmas: rationals are countable, ordered pairs of countable sets are countable). A picture of this setup would be infinite defenders everywhere.

5 HOS - finite defenders

In the previous proof that a speed $1 + s$ offense can beat a finite number of defenders, we assumed that the offense wasn't surrounded and they could get outside of the defender circles. This ignores the case where the offense starts with defenders on all sides. This section will (attempt to) prove the more general case, that a speed $1+s$ offense can beat any number of defenders in any starting configuration.

5.1 ϵ - moves

First, we consider the 1v1 case. Since the offense is faster, they can use the strategy of copying the moves of the defense with speed 1 while rotating around the defense with speed s. If the offense and defense start distance r apart, it will take time $\frac{\pi s}{r}$ for the offense to get around the defense with this strategy.

Now, consider the following strategy for 1v1 for some $\epsilon > 0$:

- The offense starts blindly moving forward
- If the defense moves within distance ϵ of the offense, they rotate around the defense with the above strategy
- Once the offense gets to the other side of the defense, they resume going straight ahead

Note that to get around the defender, the offense only required $\frac{\pi \epsilon}{s}$ added time. That mean the offense can get around a defender by taking an arbitrarily small detour.

5.2 Defender induction

With this in mind, we want to set up an induction argument looking something like this:

Let d be the number of defenders. We claim by induction that the offense can escape to the right (past all the defenders) in some finite time for all values of d.

Base Case ($d = 0$): It works yay

Inductive step: Assume the offense has some winning strategy for d defenders. The offense then uses that strategy against the first d defenders, doding the $d+1$th defender with arbitrarily little movement and thus "basically" sticking to the d defenders strategy.

This argument works in principle, but there are a couple flaws in this informal reasoning. First of all, "basically" sticking to the strategy might not be good enough. Perhaps the defense can gain some marginal advantage from the $d + 1$th defender, letting the others snag the offense. Additionally, we have no idea how many times the $d + 1$th defender will threaten the offense (possibly infinitely many), so ϵ-moves might not be good enough.

5.3 Better induction

Let's fix the problems!

First, if the offense moves in continuous curves, then the $d+1$th defender can interfere infinitely many times. To get rid of that, we'll force the offense to move in some finite number of straight lines, the last of which goes straight right (signifying passing all the defenders).

Also, the notion of "basically sticking to the d-defender strategy" needs to be formalized. As it stands, the defense can do clever tricks like forcing the offense into successively more dangerous situations with the $d+1$th defender.

So let's do a better induction.

Let d be the number of defenders. We claim that for all starting configurations of d defenders with distance to the offense at least 1, the offense can escape in N straight-line moves with no defender getting within distance ϵ of the offense.

Base case ($d = 0$): $N = 1$, $\epsilon = 1$.

Inductive step:

Ignoring the $d+1$th defender for the moment, we use our inductive hypothesis-generated strategy to escape with at most ϵ distance of breathing room. Note that we can change our path by $\frac{\epsilon}{2}$ and still be safe here, so we can use $\frac{\epsilon}{2}$ distance as extra to deal with the $d+1$th defender.

To deal with the $d+1$th defender, we have to modify ϵ-moves to use a finite number of line moves, which we can do by adding some (say $\frac{\epsilon}{2}$) error.

Crucially, the ϵ-moves put the offense on the opposite side of the defender in a straight line path, so the $d+1$th defender can only force 1 ϵ-move per line segment. In total, the $d+1$th defender can force N ϵ-moves.

Since we only have $\frac{\epsilon}{2}$ error distance to work with in total, we set ϵ in the ϵ-moves such that the first one causes max $\frac{\epsilon}{4}$ error, the second one causes max $\frac{\epsilon}{8}$ error, etc. Thus, the total error will be less than $\frac{\epsilon}{2}$ and the smallest ϵ-move was of $\frac{\epsilon}{2^{N+1}}$ error.

Zooming out, we see that the closest a defender got was during the smallest ϵ-move, which is some constant times $\frac{\epsilon}{2^{N+1}}$. Furthermore, finite ϵ-moves were used and thus the total number of line moves was finite, thus the new N is finite.

By this argument, a speed $1+s$ offense can always beat a finite number of defenders. This section is complicated and messy and seriously abuses the letter ϵ; sorry. Have a cookie.