Algebraic Topology

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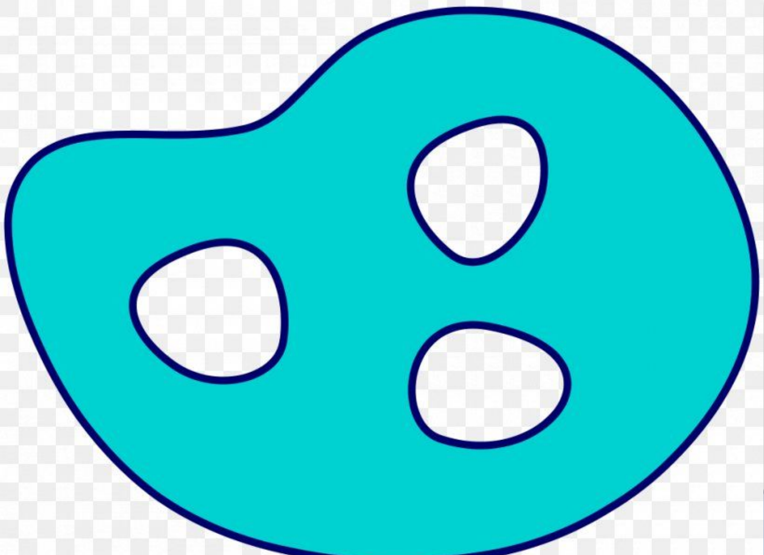
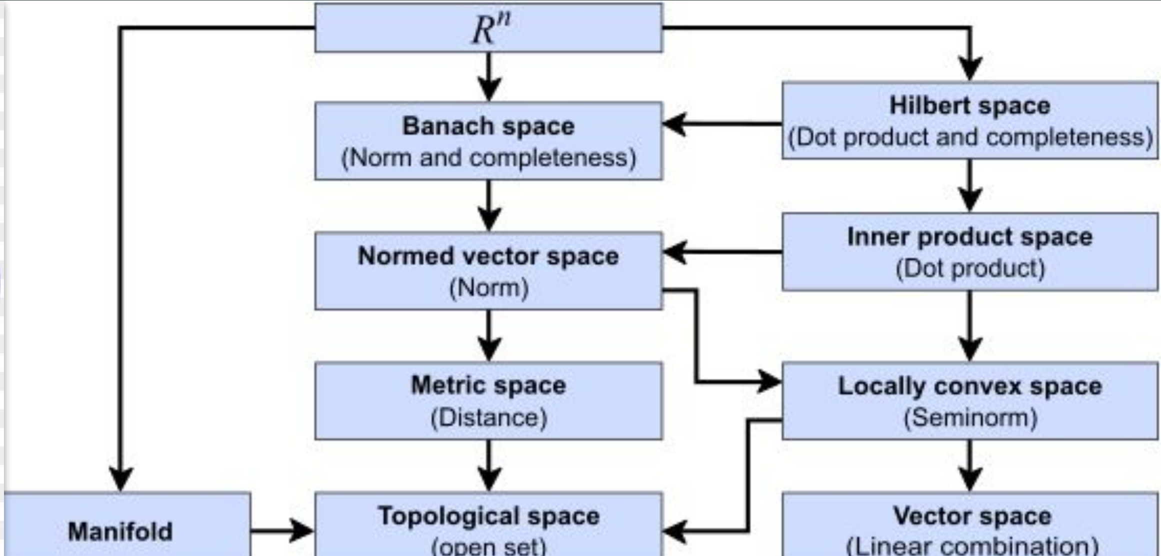
Intro to Topology

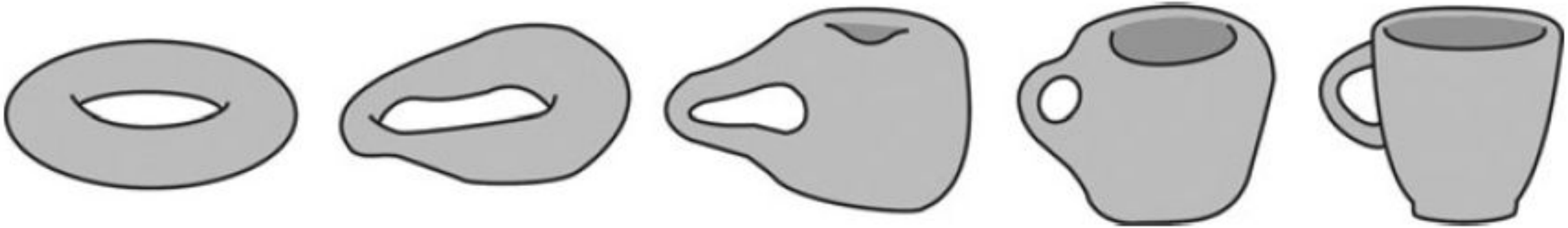
- Mathematical structure representing a space
- Represents "geometric" information, but locally/deformably
- T = (X, O) s.t. X = {"points"}, O = {"open" sets in X}





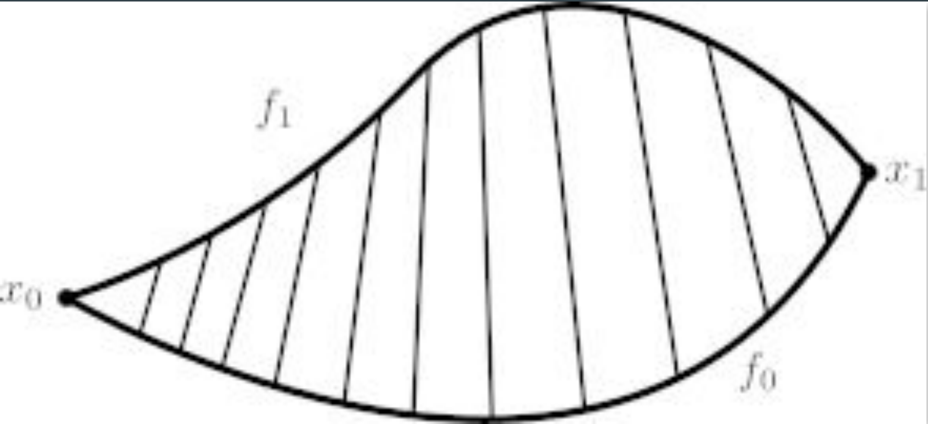
Homeomorphisms: the Gold Standard

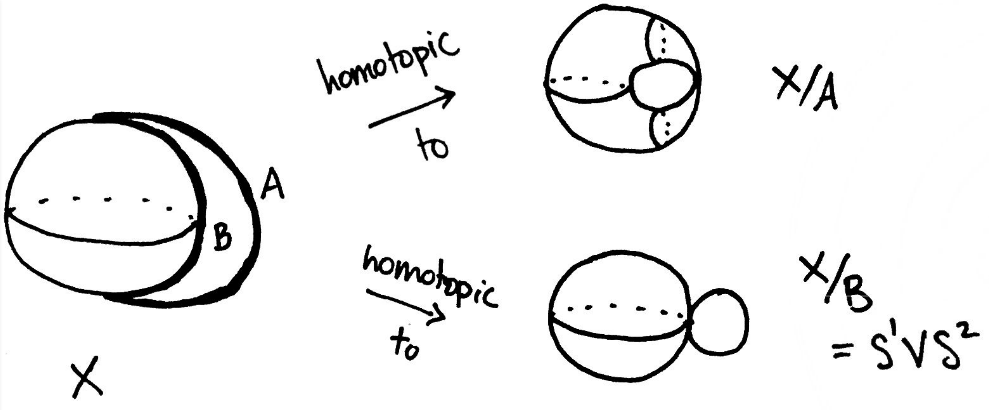
- How can we tell if two top. spaces T1, T2 are "the same"?
- Homeomorphism: bijection f: T1 → T2 "preserving open sets"
- "Homeomorphic to" preserves all topological info
  - Topology ignores embedding information



Homotopy Equivalence: the Silver Standard

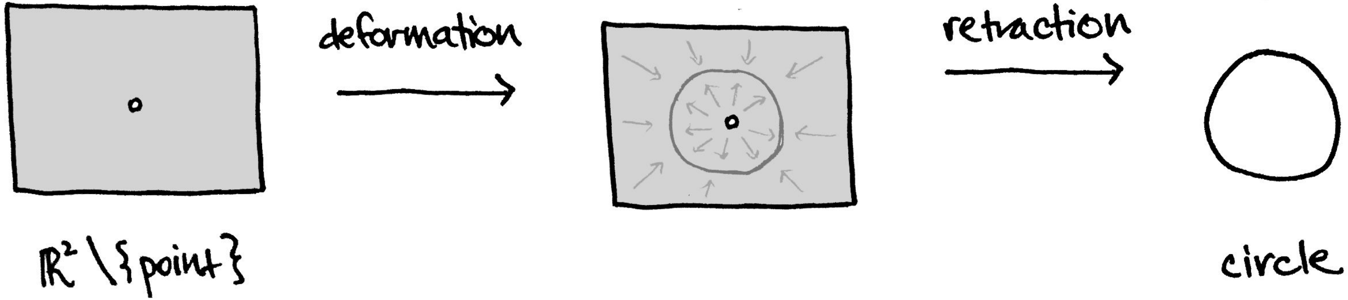
- Two functions f, g: X → Y are *homotopic*, f ~ g
    - h: X x [0, 1] → Y s.t. h(x, 0) = f(x), h(x, 1) = g(x)
- Top. spaces T1, T2 are *homotopy equivalent*, T1 ~ T2
    - f: T1 → T2, g: T2 → T1 s.t. g o f ~ ID_{T1}, f o g ~ ID_{T2}

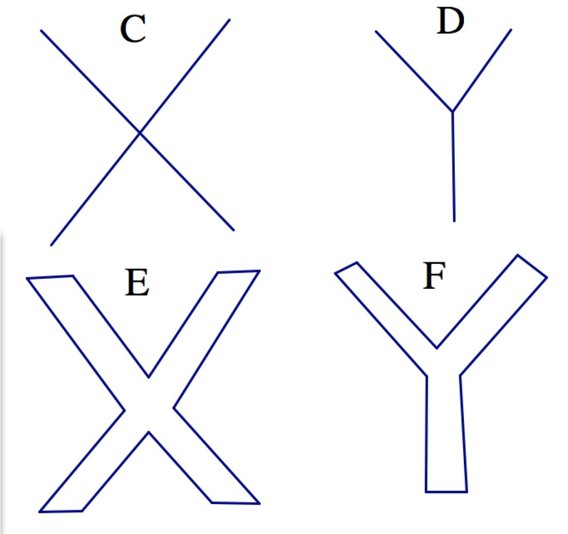




Differentiating between Homotopy Types

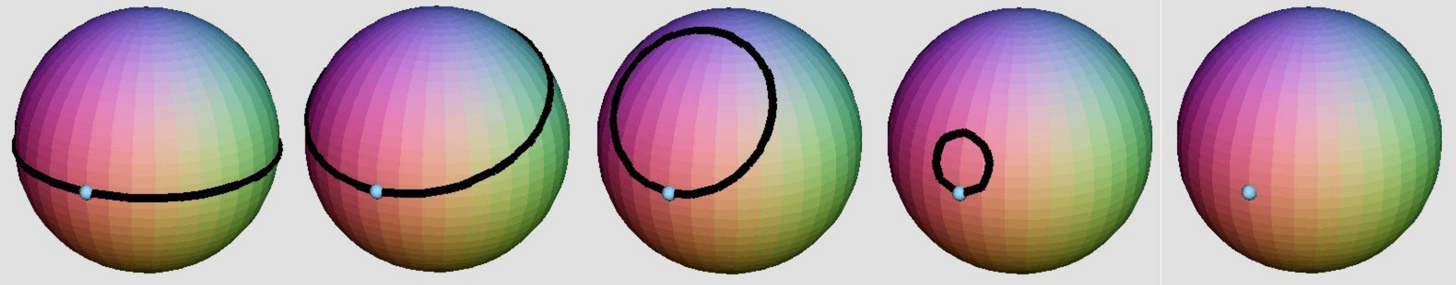
- ~ is an equiv. relation on topological spaces
  - Call the equiv. class *homotopy type*
- How to tell if same htpy type: find a homotopy equivalence
- How to tell if different htpy type: ???

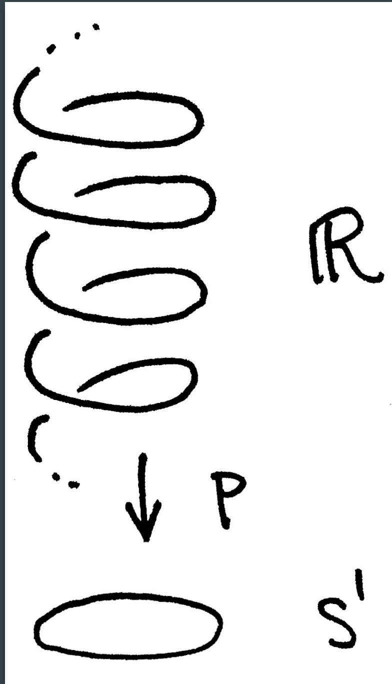




Idea: the Fundamental Group π_1

- Describes "noncontractible loops" in a top. space T
  - Continuous paths $p: [0, 1] \to T$ s.t. $p(0) = p(1) = x$ (base pt)
  - $\pi_1 = (\{\text{paths}\} / \sim, \text{concatenation})$
- Invariant under htpy equivalence
  - Non-isomorphic $\pi_1 \to$ not htpy equivalent

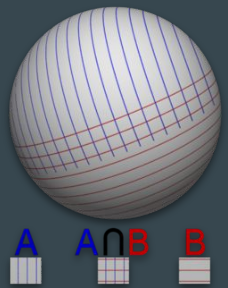


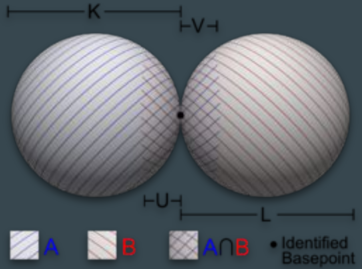


Homotopy Groups π_n

- Generalization of the fundamental group π_1
  - Uses maps $[0, 1]^n \to T$ (equivalently $S^n \to T$)
  - “nth homotopy group”, denoted π_n
- Example: $\pi_n(S^n) = Z$
  - Can’t “unwrap” S^n around itself

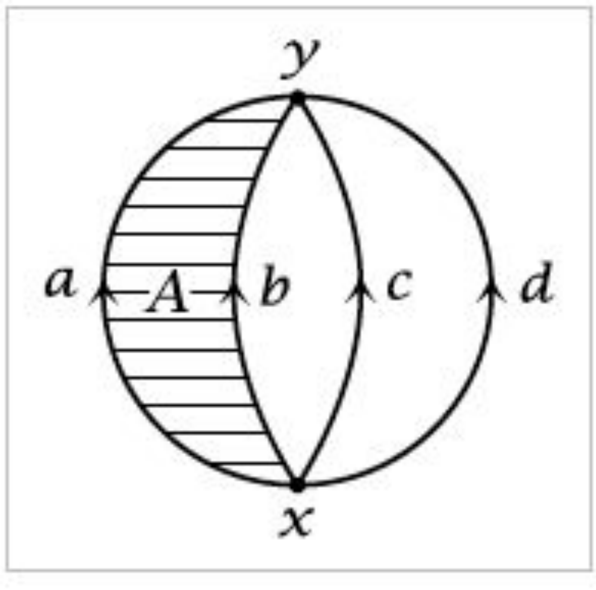
	S^0	S^1	S^2	S^3	S^4	S^5	S^6	S^7	S^8
π_1	0	$\mathbb{Z}$	0	0	0	0	0	0	0
π_2	0	0	$\mathbb{Z}$	0	0	0	0	0	0
π_3	0	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0	0
π_4	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	0
π_5	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
π_6	0	0	$\mathbb{Z}_{12}$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
π_7	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
π_8	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
π_9	0	0	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
π_{10}	0	0	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_2$	0	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$
π_{11}	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\mathbb{Z}_{24}$
π_{12}	0	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	0	0
π_{13}	0	0	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_2$	0





Homology Groups H_n

- Instead of top. spaces, we consider cell complexes
  - Singular homology extends theory to top. spaces
- H_n measures S^n-sized "holes" in the space
  - $H_k(S^n) = Z$ iff $k = n$
    (or $k = 0 \to$ reduced homology)
  - Formal definition is technical
    uses chain complexes



Thank you for listening!

- Any questions?